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1 Extending Drawings of Graphs to Arrangements 2 of Pseudolines

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12 — Abstract —

13 In the recent study of crossing numbers, drawings of graphs that can be extended to an
14 arrangement of pseudolines (pseudolinear drawings) have played an important role as they are a
15 natural combinatorial extension of rectilinear (or straight-line) drawings. A characterization of the
16 pseudolinear drawings of K_n was found recently. We extend this characterization to all graphs, by
17 describing the set of minimal forbidden subdrawings for pseudolinear drawings. Our characterization
18 also leads to a polynomial-time algorithm to recognize pseudolinear drawings and construct the
19 pseudolines when it is possible.

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23 lines, crossing numbers, stretchability.

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29 **1 Introduction**

30 Since 2004, geometric methods have been used to make impressive progress for determining
31 the crossing number of (certain classes of drawings of) the complete graph K_n . In particular,
32 drawings that extend to straight lines, or, more generally, arrangements of pseudolines, have
33 been central to this work, spurring interest in such drawings for arbitrary graphs, not just
34 complete graphs [2, 4, 5, 6, 12].

In particular, for pseudolinear drawings, it is now known that, for $n \geq 10$, a pseudolinear
drawing of K_n has more than

$$H(n) := \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor$$

35 crossings [1, 14]. The number $H(n)$ is conjectured by Harary and Hill to be the smallest
36 number of crossings over all topological drawings of K_n ; that is, the crossing number $\text{cr}(K_n)$
37 is conjectured to be $H(n)$.

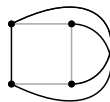
38 A *pseudoline* is the image ℓ of a continuous injection from the real numbers \mathbb{R} to the
39 plane \mathbb{R}^2 such that $\mathbb{R}^2 \setminus \ell$ is not connected. An *arrangement of pseudolines* is a set Σ of

40 pseudolines such that, if ℓ, ℓ' are distinct elements of Σ , then $|\ell \cap \ell'| = 1$ and the intersection
 41 is a crossing point. Informally, a *crossing point* or *crossing* is an intersection point between
 42 two pseudolines that locally looks like a crossing point between two non parallel lines (a
 43 formal definition of crossing will be given when we introduce the notion of string). More on
 44 pseudolines and their importance for studying geometric drawings of graphs can be found in
 45 [10, 11].

46 A drawing D of a graph G is *pseudolinear* if there is an arrangement of pseudolines
 47 consisting of a different pseudoline ℓ_e for each edge e of G and such that $D[e] \subseteq \ell_e$.

48 In the study of crossing numbers, restricting the drawing to either straight lines or
 49 pseudolines yields the rectilinear crossing number $\overline{\text{cr}}(K_n)$ or the pseudolinear crossing number
 50 $\tilde{\text{cr}}(K_n)$, respectively. Clearly $\overline{\text{cr}}(K_n) \geq \tilde{\text{cr}}(K_n)$ and the geometric methods prove that
 51 $\tilde{\text{cr}}(K_n) > H(n)$, for $n \geq 10$.

52 A *good drawing* is one where no edge self-intersects and any two edges share at most
 53 one point—either a crossing or a common end point—and no three edges share a common
 54 crossing. One somewhat surprising result is from Aichholzer et al.: *a good drawing of K_n
 55 in the plane is homeomorphic to a pseudolinear drawing if and only if it does not contain a
 56 non-planar drawing of K_4 whose crossing is incident with the unbounded face of the K_4* [2]
 57 (see Figure 1). By ignoring the grey edges from Figure 1, we see that any such drawing of K_4
 58 contains a *B-configuration*, depicted as the third drawing of the first row of Figure 2. Based
 59 on our Theorem 2, Theorem 2.5.1 from [3] shows that any non-pseudolinear drawing contains
 60 a *B-configuration*. Thus, either Fig. 1 or the *B-configuration* can be used to characterize
 61 pseudolinear drawings of K_n . In [4] pseudolinear drawings of K_n are characterized as *f-convex*,
 62 and in [5] are characterized as monotone and free of a specific drawing of K_4 .



■ **Figure 1** Non-pseudolinear K_4 with its crossing incident with the outer face.

63 Twenty-five years earlier, Thomassen [19] proved a similar theorem for a 1-planar drawing
 64 (that is, a drawing in which each edge is crossed at most once). The *B-* and *W-*configurations
 65 are shown as the third and fourth drawings in the first row of Figure 2. Thomassen's theorem
 66 is: *if D is a 1-planar drawing of graph G , then D is homeomorphic to a rectilinear drawing
 67 of G if and only if D contains no *B-* or *W-*configuration.*

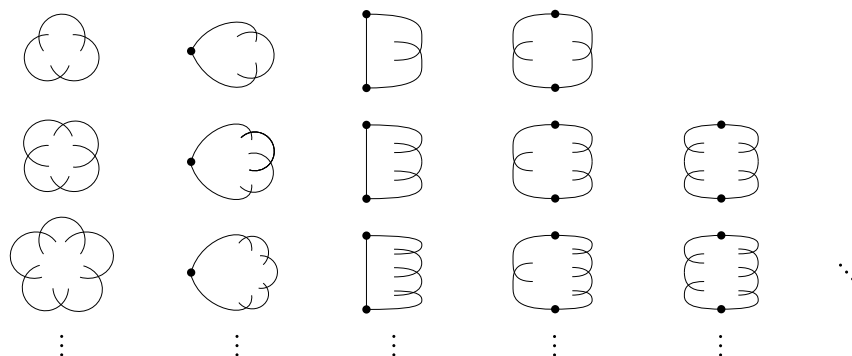
68 Thomassen presented in [19] the *clouds* (first column in Figure 2) as an infinite family of
 69 drawings that are minimally non-pseudolinear.

70 Shortly after Thomassen's paper, Bienstock and Dean proved that if $\text{cr}(G) \leq 3$, then
 71 $\overline{\text{cr}}(G) = \text{cr}(G)$ [7]. They also exhibited examples based on overlapping *W-*configurations to
 72 show the result fails for $\text{cr}(G) = 4$; such graphs can have arbitrarily large rectilinear crossing
 73 number.

74 Despite the existence of infinitely many obstructions to pseudolinearity, we characterize
 75 them all.

76 ► **Theorem 1.** A good drawing of a graph G is pseudolinear if and only if it does not contain
 77 one of the infinitely many obstructions shown in Figure 1.

78 The drawings in Figure 2 are obtained from the *clouds* (first column) by replacing at most
 79 two crossings by vertices. The formal statement of Theorem 1 is Theorem 15 in Section 6; also
 80 a more general version of this statement, Theorem 2, is discussed below. Our result draws a



■ **Figure 2** Obstructions to pseudolinearity.

81 line between the class of pseudolinear drawings and the class of rectilinear drawings: Our
 82 result shows that recognizing pseudolinear drawings is a combinatorial/topological problem
 83 and implies a polynomial-time algorithm to detect pseudolinear drawings (Theorem 14).
 84 This contrast with the rather real algebraic geometry problem of deciding the *stretchability*
 85 of a drawing, defined as the problem of deciding whether a given drawing is homeomorphic
 86 to a rectilinear drawing. Mnëv [16, 17] showed that deciding the stretchability of an
 87 arrangement of pseudolines is $\exists\mathbb{R}$ -hard, implying the $\exists\mathbb{R}$ -hardness for the problem of deciding
 88 the stretchability of a graph drawing. Since $\text{NP} \subseteq \exists\mathbb{R}$ [15, 18, 8], this in particular shows that
 89 the stretchability problem is NP-hard. We refer to Matoušek’s survey [15] for an approachable
 90 introduction to the complexity class $\exists\mathbb{R}$.

91 The natural setting for our characterization is strings embedded in the plane. An *arc* σ
 92 is the image $f([0, 1])$ of the compact interval $[0, 1]$ under a continuous map $f : [0, 1] \rightarrow \mathbb{R}^2$.
 93 Let $S(\sigma) = \{p \in \sigma : |f^{-1}(p)| \geq 2\}$ be the set of self-intersections of σ . A *string* is an arc σ
 94 for which $S(\sigma)$ is finite. If $S(\sigma) = \emptyset$, then σ is *simple*. If σ' is a string and $\sigma' \subseteq \sigma$, then σ' is
 95 a substring of σ .

96 Suppose that σ and σ' whose intersection $\sigma \cap \sigma'$ is a finite set and let $p \in \sigma \cap \sigma'$. The *rotation*
 97 *at* p is a cyclic sequence of substrings determined by a small neighbourhood homeomorphic
 98 to the plane in which p is origin and the substrings incident with p are rays emanating from
 99 p [13, Thm. 3.1]. The strings σ_1, σ_2 cross at p if they each have two substrings that alternate
 100 $\sigma_1 - \sigma_2 - \sigma_1 - \sigma_2$ in the rotation at p .

101 An intersection point between of two strings σ and σ' is *ordinary* if it is either an endpoint
 102 of σ or σ' , or is a *crossing*. A set Σ of strings is *ordinary* if Σ is finite and any two strings
 103 in Σ have only finitely many intersections, all of which are ordinary. All the sets of strings
 104 considered in this paper are ordinary.

105 If Σ is an ordinary set of strings, then its *planarization* $G(\Sigma)$ is the plane graph obtained
 106 from Σ by inserting vertices at each crossing between strings and also at the endpoints of
 107 every string in Σ . To keep track of the information given by the strings, we will always
 108 assume that each string Σ has a different color and that each edge in $G(\Sigma)$ inherits the color
 109 of the string including it.

110 If Σ is an ordinary set of strings, then, for a cycle C in $G(\Sigma)$ (which is a simple closed
 111 curve in \mathbb{R}^2) the *edges inside* C are those drawn in the closed disk bounded by C (this
 112 includes the edges of C). A vertex $v \in V(C)$ is a *rainbow* for C if all the edges incident with
 113 v and drawn inside C have different colours. The reader can verify that, for each drawing in
 114 Figure 2, if we let Σ be the edges of the drawing, then the unique cycle in $G(\Sigma)$ has at most
 115 two rainbows. Our main result characterizes these cycles as the only possible obstructions:

116 ► **Theorem 2.** *An ordinary set of strings Σ can be extended to an arrangement of pseudolines*
 117 *if and only if every cycle C of $G(\Sigma)$ has at least three rainbows.*

118 Henceforth, we define any cycle C in $G(\Sigma)$ with at most two rainbows as an *obstruction*.
 119 A set of strings is *pseudolinear* if it has an extension to an arrangement of pseudolines.

120 Theorem 2 is our main contribution. In the next section, we show that the presence
 121 of an obstruction implies the set of ordinary strings is not pseudolinear. The converse is
 122 proved in Section 4 by extending, one small step at a time, the strings in Σ to get closer
 123 to an arrangement of pseudolines. After each extension, we must show that no obstruction
 124 has been introduced. This involves dealing with cycles in $G(\Sigma)$ that have precisely three
 125 rainbows (that we refer as *near-obstructions*). In Section 3 we show the key lemma that if G
 126 has two such near-obstructions that intersect nicely at a vertex v , then G has an obstruction.
 127 In Section 5 we present a polynomial-time algorithm for detecting obstructions and we argue
 128 why the proof of Theorem 2 implies a polynomial-time algorithm for extending a pseudolinear
 129 set of strings. Finally, in Section 6, we show how Theorem 1 follows from Theorem 2 and we
 130 present some concluding remarks.

131 2 A set of strings with an obstruction is not extendible

132 Let us start by showing the easy direction of Theorem 2:

133 ► **Lemma 3.** *If the underlying graph $G(\Sigma)$ of a set Σ of strings has an obstruction, then Σ*
 134 *is not pseudolinear.*

135 Suppose that C is a cycle of $G(\Sigma)$ for some set of strings Σ . We define $\delta(C)$ as the set of
 136 vertices of C for which their two incident edges in C have different colours. In a set Σ of
 137 simple strings where no two intersect twice, $|\delta(C)| \geq 3$ for every cycle C of $G(\Sigma)$.

138 ► **Lemma 4.** *Let Σ be a set of simple strings where every pair intersect at most once. Suppose*
 139 *that C is an obstruction with $|\delta(C)|$ as small as possible. Let $S = x_0, x_1, \dots, x_\ell$ be a path of*
 140 *$G(\Sigma)$ representing a substring of some string $\sigma \in \Sigma$ such that $x_0x_1 \in E(C)$, $x_1 \in \delta(C)$ and*
 141 *x_1 is not a rainbow of C . Then $V(C) \cap V(S) = \{x_0, x_1\}$.*

142 **Proof.** By way of contradiction, suppose that there is a vertex $x_r \in V(C) \cap V(S)$ with $r \geq 3$.
 143 Assume that $r \geq 3$ is as small as possible. Let P be the subpath of S connecting x_1 to x_r .
 144 The facts $x_0x_1 \in E(C)$, $x_1 \in \delta(C)$, and $P \subseteq \sigma$ imply that $x_1x_2 \notin E(C)$. Because x_1 is not a
 145 rainbow for C and no two strings tangentially intersect at x_1 , the edge x_1x_2 is drawn in the
 146 closed disk bounded by C . By choice of r , P is an arc connecting x_1 to x_r in the interior of
 147 C .

148 Let C_1 and C_2 be the two cycles of $C \cup P$ containing P , labelled so that $x_0x_1 \in E(C_1)$.
 149 We shall use the minimality of $|\delta(C)|$ to show that C_1 and C_2 are not obstructions. Then, we
 150 will count rainbows in C_1 and C_2 to obtain the contradiction that C is not an obstruction.

151 For a cycle X , let $\rho(X)$ be the set of rainbows of X . For $i = 1, 2$, let $Q_i = V(C_i) \setminus V(P)$.
 152 As the edges of S are included in the same string, we see that $\rho(C_1) \setminus Q_1 \subseteq \{x_r\}$ and
 153 $\rho(C_2) \setminus Q_2 \subseteq \{x_1, x_r\}$. Likewise, $\delta(C_1) \setminus Q_1 \subseteq \{x_r\}$ and $\delta(C_2) \setminus Q_2 \subseteq \{x_1, x_r\}$.

154 Let us show that C_1 and C_2 are not obstructions. Because $|\delta(C_2)| \geq 3$ and $\delta(C_2) \setminus Q_2 \subseteq$
 155 $\{x_1, x_r\}$, $|\delta(C) \cap Q_2| \geq 1$. Since $\delta(C_1) \setminus Q_1 \subseteq \{x_r\}$ and $x_1 \in \delta(C)$, $|\delta(C_1)| \leq |\delta(C_1) \cap Q_1| +$
 156 $|\{x_r\}| \leq |\delta(C)| - 2 + |\{x_r\}| < |\delta(C)|$. Because $|\delta(C_1)| \geq 3$ and $|\delta(C_1) \setminus Q_1| \leq 1$, $|\delta(C) \cap Q_1| \geq 2$.
 157 Since $x_1 \in \delta(C) \cap \delta(C_2)$, $|\delta(C_2)| \leq |\delta(C) \cap Q_2| + |\{x_1, x_r\}| \leq |\delta(C)| - 3 + |\{x_1, x_r\}| < |\delta(C)|$.
 158 Thus, neither C_1 nor C_2 is an obstruction.

159 Finally, as $|\rho(C_1)| \geq 3$ and $|\rho(C_1) \setminus Q_1| \leq 1$, $|\rho(C) \cap Q_1| = |\rho(C_1) \cap Q_1| \geq 2$. Because
 160 $|\rho(C_2)| \geq 3$ and $|\rho(C_2) \setminus Q_2| \leq 2$, $|\rho(C) \cap Q_2| = |\rho(C_2) \cap Q_2| \geq 1$. Thus $|\rho(C)| \geq 3$, a
 161 contradiction. ◀

162 **Proof of Lemma 3.** By way of contradiction, suppose that Σ is pseudolinear and that $G(\Sigma)$
 163 has an obstruction C .

164 Consider an extension of Σ to an arrangement of pseudolines, and then cut off the two
 165 infinite ends of each pseudoline to obtain a set of strings Σ' extending Σ , and in which every
 166 pair of strings in Σ' cross once. In $G(\Sigma')$, there is a cycle C' that represents the same simple
 167 closed curve as C . Because every rainbow of C' is a rainbow of C , C' has fewer than three
 168 rainbows. Therefore, we may assume that $\Sigma = \Sigma'$ and $C = C'$. Now, the ends of every string
 169 in Σ are degree-1 vertices in the outer face of $G(\Sigma)$.

170 As every string in Σ is simple and no two strings intersect more than once, $|\delta(C)| \geq 3$.
 171 We will assume that C is chosen to minimize $|\delta(C)|$.

172 Since C is an obstruction, there exists $x_1 \in \delta(C)$ such that x_1 is not a rainbow in C .
 173 Consider a neighbour x_0 of x_1 in C . Let $S = x_0, x_1, \dots, x_\ell$ be the path obtained by traversing
 174 the string σ extending x_0x_1 , such that x_ℓ is an end of σ . By Lemma 4, $V(S) \cap V(C) = \{x_0, x_1\}$,
 175 and because x_ℓ is in the outer face of C , the segment of σ from x_1 to x_ℓ has its relative
 176 interior in the outer face of C .

177 However, since x_1 is not a rainbow, there exists a string $\sigma' \in \Sigma$ including two edges at x_1
 178 drawn inside C . Thus, σ and σ' tangentially intersect at x_1 , a contradiction. ◀

179 **3 The key lemma**

180 In this section we present the key lemma used in the proof of Theorem 2.

181 A plane graph G is *path-partitioned* if for $m \geq 1$, there exists a colouring $\chi : E(G) \rightarrow$
 182 $\{1, \dots, m\}$ such that for each $i \in \{1, \dots, m\}$, the edges in $\chi^{-1}(i)$ induce a path $P_i \subseteq G$ where
 183 any two distinct paths P_i and P_j do not tangentially intersect. Indeed, every underlying
 184 planar graph $G(\Sigma)$ of a set of simple strings Σ is path-partitioned. Moreover, every path-
 185 partitioned plane graph can be obtained by subdividing a planarization of an ordinary set of
 186 simple strings. To extend the previously introduced notation we refer to each P_i as a string.
 187 The concepts of rainbow and obstruction naturally extend to the context of path-partitioned
 188 plane graphs.

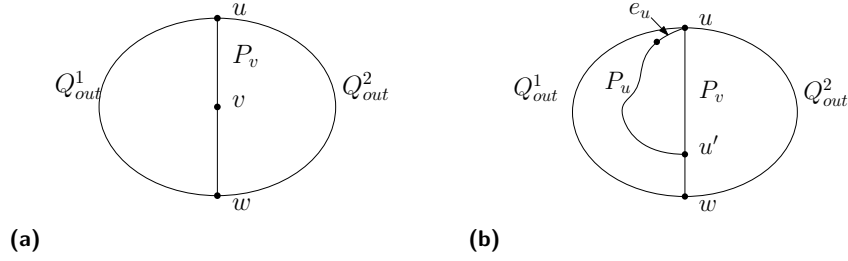
189 Suppose that G is a path-partitioned plane graph. Given $v \in V(G)$, a *near-obstruction at*
 190 v is a cycle C with at most three rainbows and such that v is a rainbow of C . Understanding
 191 how near-obstructions behave is the key ingredient needed in the proof of Theorem 2:

192 ▶ **Lemma 5.** *Let G be a path-partitioned plane graph and let $v \in V(G)$. Suppose that C_1
 193 and C_2 are two near-obstructions at v such that the union of the closed disks bounded by C_1
 194 and C_2 contains a small open ball centered at v . Suppose that one of the following two holds:*

- 195 1. *no obstruction of G contains v ; or*
- 196 2. *the two edges of C_1 incident with v are the same as the two edges of C_2 incident with v .*

197 *Then G has an obstruction not including v .*

198 Given a plane graph G , a cycle $C \subseteq G$ and a vertex $v \in V(C)$, *the edges at v inside C*
 199 *are the edges of G incident with v drawn inside C . Consider a homeomorphism from a small*
 200 *disc neighbourhood of v to the plane so that each edge segment incident with v is a straight*
 201 *ray from the origin (which is v). Since no two strings intersect tangentially at v , we may*
 202 *assume that the rotation at v has substrings of the same colour making an angle of π at v .*



■ **Figure 3** Auxiliary figures used in the proof of Lemma 5.

203 The angles between rays are the *angles at v* and we associate to them the set of edges at v
 204 drawn as rays inside them. From this geometric perspective, it is obvious that, if an angle α
 205 is less than π , then α is rainbow. This proves the second of the following facts.

206 ► **Useful Facts.** *Let G be a plane path-partitioned graph and let $v \in V(G)$. Then*

- 207 1. *if α, β are two angles at v with $\alpha \subseteq \beta$ and β is rainbow, then α is rainbow; and*
 208 2. *if α and β are two angles such $\bar{\alpha}$ is not rainbow and β is a proper subangle of the*
 209 *complement $\bar{\alpha}$ of α , then β is rainbow.*

210 **Proof of Lemma 5.** By way of contradiction, suppose that G has no obstruction not includ-
 211 ing v . The “small ball” hypothesis implies that v is not in the outer face of the subgraph
 212 $C_1 \cup C_2$.

213 We claim that $|V(C_1) \cap V(C_2)| \geq 3$. Suppose not. For $i = 1, 2$, let e_i and f_i be the edges
 214 of C_i at v and let Δ_i be the closed disk bounded by C_i . From the “small ball” hypothesis it
 215 follows that (i) Δ_1 contains the edges e_2 and f_2 ; and (ii) the points near v in the exterior of
 216 Δ_2 are contained in Δ_1 . These two properties imply that the path $C_2 - v$ intersects C_1 at
 217 least twice, and because $v \in V(C_1) \cap V(C_2)$, $|V(C_1) \cap V(C_2)| \geq 3$.

218 From the last paragraph we know that $C_1 \cup C_2$ is 2-connected, and hence the outer face
 219 of $C_1 \cup C_2$ is bounded by a cycle C_{out} . We will assume that

220 (*) the cycles C_1 and C_2 satisfying the hypothesis of Lemma 5 are chosen so that the number
 221 of vertices of G in the disk bounded by C_{out} is minimal.

222 Useful Fact 1 applied to the interior angles at vertices of C_{out} shows that every vertex
 223 that is a rainbow in C_{out} is also a rainbow in each of the cycles in $\{C_1, C_2\}$ containing it.
 224 We can assume that C_{out} is not an obstruction or else we are done. We may relabel C_1 and
 225 C_2 so that two of the rainbows of C_{out} , say p and q , are also rainbows in C_1 . Neither p nor q
 226 is v because $v \notin V(C_{out})$. Because C_1 is a near-obstruction, p, q and v are the only rainbows
 227 of C_1 .

228 Since $v \notin V(C_{out})$, by following C_1 in the two directions starting at v , we find a path
 229 $P_v \subseteq C_1$ containing v in which only the ends u and w of P_v are in C_{out} (note that $u \neq w$
 230 because $\{p, q\} \subseteq V(C_1) \cap V(C_{out})$). See Figure 3a.

231 As v is in the interior face of C_{out} , P_v is also in the interior of C_{out} . Let Q_{out}^1, Q_{out}^2 be
 232 the uw -paths of C_{out} . One of the two closed disks bounded by $P_v \cup Q_{out}^1$ and $P_v \cup Q_{out}^2$
 233 contains C_1 . By symmetry, we may assume that C_1 is contained in the first disk. Since
 234 $C_{out} \subseteq C_1 \cup C_2$, this implies that Q_{out}^2 is a subpath of C_2 .

235 Our desired contradiction will be to find three rainbows in C_2 distinct from v . We
 236 find the first: let $C_1 - (P_v)$ be the uw -path in C_1 distinct from P_v . The disk bounded by

237 $(C_1 - (P_v)) \cup Q_{out}^2$ contains the one bounded by C_1 . Useful Fact 1 applied to the interior
 238 angles at the vertices of $(C_1 - (P_v)) \cup Q_{out}^2$ implies that each vertex in $C_1 - (P_v)$ that is a
 239 rainbow in $(C_1 - (P_v)) \cup Q_{out}^2$ is also rainbow in C_1 . Since C_1 has at most two rainbows in
 240 $C_1 - (P_v)$, namely p and q , $(C_1 - (P_v)) \cup Q_{out}^2$ has a third rainbow r_1 in the interior of Q_{out}^2
 241 (else $(C_1 - (P_v)) \cup Q_{out}^2$ is an obstruction and we are done). Note that r_1 is also a rainbow
 242 for C_2 .

243 To find another rainbow in C_2 , consider the edge e_u of C_2 incident to u and not in Q_{out}^2 .
 244 We claim that either u is a rainbow in C_2 or that e_u is not included in the closed disk
 245 bounded by $P_v \cup Q_{out}^2$. Seeking a contradiction, suppose that u is not a rainbow of C_2 and
 246 that e_u is included in the disk. Then Useful Fact 2 implies that u is a rainbow in C_1 . As p
 247 and q are the only rainbows of C_1 in C_{out} , u is one of p and q . Therefore u is a rainbow in
 248 C_{out} , and hence, a rainbow in C_2 , a contradiction.

249 If u is a rainbow in C_2 , then this is the desired second one. Otherwise, e_u is not in the
 250 closed disk bounded by $P_v \cup Q_{out}^2$. Let $P_u \subseteq C_2$ be the path starting at u , continuing on e_u
 251 and ending on the first vertex u' in P_v that we encounter. Let C_u be the cycle consisting of
 252 P_u and the uu' -subpath uP_vu' of P_v . See Figure 3b.

253 \triangleright **Claim 6.** If P_u does not have a rainbow of C_u in its interior, then either C_u is an
 254 obstruction not containing v or:

- 255 (a) C_u and C_2 are near-obstructions at v satisfying the same conditions as C_1 and C_2 in
 256 Lemma 5; and
- 257 (b) the closed disk bounded by the outer cycle of $C_u \cup C_2$ contains fewer vertices than the
 258 disk bounded by C_{out} .

259 **Proof.** Suppose that all the rainbows of C_u are located in uP_vu' . If z is a rainbow of C_u ,
 260 then $z \in \{u, v, u'\}$, as otherwise z is a rainbow of C_1 distinct from p , q and v , a contradiction.
 261 Thus, if $v \notin V(C_u)$, then C_u is the desired obstruction. We may assume that $v \in V(C_u)$.

262 If $u' = w$, then $C_2 = P_u \cup Q_{out}^2$, violating the assumption that $v \in V(C_2)$. Thus $u' \neq w$.
 263 If $u' = v$, then the rainbows of C_u are included in $\{u, u'\}$, and hence C_u is an obstruction.
 264 However, the existence of C_u shows that both alternatives (1) and (2) in Lemma 5 fail:
 265 condition (1) fails because C_u contains v and (2) fails because the edge of P_u incident with v
 266 is in $E(C_2) \setminus E(C_1)$. Thus $u' \neq v$.

267 The previous two paragraphs show that C_u is a near-obstruction at v with rainbows u ,
 268 v and u' . Since the interior of C_u near v is the same as the interior of C_1 near v , the pair
 269 (C_u, C_2) satisfies the ‘‘small ball’’ hypothesis. Thus, (a) holds.

270 Let C'_{out} be the outer cycle of $C_u \cup C_2$. From the fact that $C_u \cup C_2 \subseteq C_1 \cup C_2$ it follows
 271 that the disk bounded by C_{out} includes the disk bounded by C'_{out} .

272 Since $p, q \in V(C_{out})$, p and q are in the disk bounded by C_{out} . If both p and q are in
 273 C_2 , then p, q and r_1 are rainbows in C_2 , and also distinct from v , contradicting that C_2 is a
 274 near-obstruction for v . If, say $p \notin V(C_2)$, then p is not in the disk bounded by C'_{out} , which
 275 implies (b). \blacktriangleleft

276 From Claim 6(b) and assumption (*) either C_u is the desired obstruction or P_u contains
 277 a rainbow r_2 of C_2 in its interior. We assume the latter as otherwise we are done.

278 In the same way, the last rainbow r_3 comes by considering the edge of $C_2 - Q_{out}^2$ incident
 279 with w . It follows that v, r_1, r_2 and r_3 are four different rainbows in C_2 , contradicting the
 280 fact that C_2 is a near-obstruction. \blacktriangleleft

281 4 Proof of Theorem 2

282 In this section we prove that a set of strings with no obstructions can be extended to an
283 arrangement of pseudolines.

284 **Proof of Theorem 2.** It was shown in Observation 3 that the existence of obstructions
285 implies non-extendibility. For the converse, suppose that Σ is a set of strings for which $G(\Sigma)$
286 has no obstructions.

287 We start by reducing to the case where the point set $\bigcup \Sigma$ is connected: iteratively add a
288 new string in a face of $\bigcup \Sigma$ connecting two connected components of $\bigcup \Sigma$. No obstruction is
289 introduced at each step (obstructions are cycles), and, eventually, the obtained set $\bigcup \Sigma$ is
290 connected. An extension of the new set of strings contains an extension for the original set,
291 thus we may assume that $\bigcup \Sigma$ is connected.

292 Our proof is algorithmic, and consists of repeatedly applying one of the three steps
293 described below.

- 294 ■ **Disentangling Step.** If a string $\sigma \in \Sigma$ has an end a with degree at least 2 in $G(\Sigma)$,
295 then we slightly extend the a -end of σ into one of the faces incident with a .
- 296 ■ **Face-Escaping Step.** If a string $\sigma \in \Sigma$ has an end a with degree 1 in $G(\Sigma)$, and is
297 incident with an interior face, then we extend the a -end of σ until it intersects some point
298 in the boundary of this face.
- 299 ■ **Exterior-Meeting Step.** Assuming that all the strings in Σ have their two ends in
300 the outer face and these ends have degree 1 in $G(\Sigma)$, we extend the ends of two disjoint
301 strings so that they meet in the outer face.

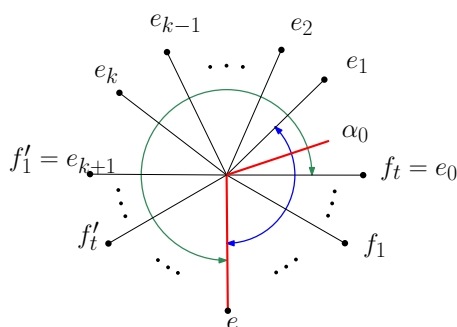
302 Each of these three steps either increases the number of pairs of strings that intersect, or
303 increase the number crossings (recall that a crossing between σ and σ' is a non-tangential
304 intersection point in $\sigma \cap \sigma'$ that is not an end of σ or σ'). Moreover, these steps can be
305 performed as long as one of the next two conditions holds: (1) at least one string does not
306 have an end incident with the outer face; and (2) there is a pair of strings that do not
307 cross. If none of (1) and (2) hold, then our set of strings is extendible into an arrangement
308 of pseudolines. Henceforth, we will show that, if performed correctly, none of these steps
309 introduces an obstruction. The proof for each step can be read independently.

310 ► **Lemma 7 (Disentangling Step).** *Suppose that $\sigma \in \Sigma$ has an end a with degree at least 2 in*
311 *$G(\Sigma)$. Then we can extend the a -end of σ into one of the faces incident to a without creating*
312 *an obstruction.*

313 **Proof.** A pair of different edges f and f' in $G(\Sigma)$ incident with a are *twins* if they belong to
314 the same string in Σ . The edge $e \subseteq \sigma$ incident with a has no twin.

315 The fact that no pair of strings tangentially intersect at a tells us that if (f_1, f'_1) and
316 (f_2, f'_2) are pairs of twins, then f_1, f_2, f'_1, f'_2 occur in this cyclic order for either the clockwise
317 or counterclockwise rotation at a . Thus, we may assume that the counterclockwise rotation
318 at a restricted to the twins and e is $e, f_1, \dots, f_t, f'_1, \dots, f'_t$, where (f_i, f'_i) is a twin pair for
319 $i = 1, \dots, t$.

320 To avoid tangential intersections, the extension of σ at a must be in the angle between f_t
321 and f'_1 not containing e . Let e_1, \dots, e_k be the counterclockwise ordered list of non-twin edges
322 at a having an end in this angle (as depicted in Figure 4). We label $e_0 = f_t$ and $e_{k+1} = f'_1$.
323 If there are no twins, then let $e_0 = e_{k+1} = e$.



■ **Figure 4** Substrings included in the disk bounded by C_0 .

324 Let us consider all the possible extensions: for $i \in \{0, \dots, k\}$, let Σ_i be the set of strings
 325 obtained from Σ by slightly extending the a -end of σ into the face containing the angle
 326 between e_i and e_{i+1} . Let α_i be the new edge at a extending σ in Σ_i (see α_0 in Figure 4).

327 Seeking a contradiction, suppose that, for each $i \in \{0, \dots, k\}$, $G(\Sigma_i)$ contains an obstruction
 328 C_i . Since α_i contains a degree-1 vertex, α_i is not in C_i . Hence C_i is a cycle of $G(\Sigma)$. Thus,
 329 C_i is not an obstruction in $G(\Sigma)$ and becomes an obstruction in $G(\Sigma_i)$. This conversion has
 330 a simple explanation: in $G(\Sigma)$, C_i has exactly three rainbows, and one of them is a . After
 331 α_i is added, a is not a rainbow in C_i (witnessed by the edges e and α_i included in the new
 332 version of σ).

333 Recall from Section 3 that a *near-obstruction* at a is a cycle with exactly three rainbows,
 334 and one of them is a . Each of C_0, C_1, \dots, C_k is a near-obstruction at a in $G(\Sigma)$.

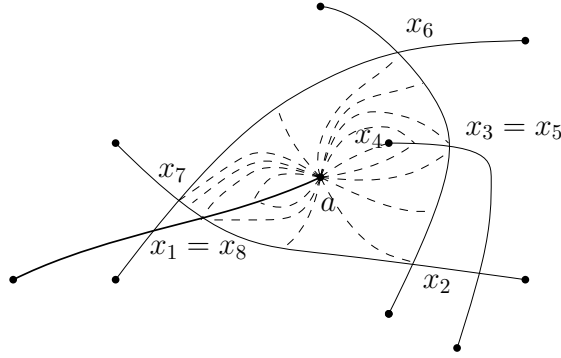
335 For a cycle $C \subseteq G$, let $\Delta(C)$ denote the closed disk bounded by C . Both e and α_0 are in
 336 $\Delta(C_0)$. Thus, either $\Delta(C_0) \supseteq \{e, f_1, f_2, \dots, f_t, e_1\}$ (blue bidirectional arrow in Figure 4) or
 337 $\Delta(C_0) \supseteq \{f_t, e_1, \dots, e_k, f'_1, f'_2, \dots, f'_t, e\}$ (green bidirectional arrow). We rule out the latter
 338 situation as the second list contains f_t and f'_t , and this would imply that a is not a rainbow
 339 for C_0 in $G(\Sigma)$.

340 We just showed that $\{e, e_0, e_1\} \subseteq \Delta(C_0)$. By symmetry, $\{e_k, e_{k+1}, e\} \subseteq \Delta(C_k)$. Consider
 341 the largest index $i \in \{0, 1, \dots, k-1\}$ for which $\{e, e_0, \dots, e_{i+1}\} \subseteq \Delta(C_i)$. By the choice of i ,
 342 and because $\{e, \alpha_{i+1}\} \subseteq \Delta(C_{i+1})$, $\{e, f'_t, \dots, f'_1, e_k, \dots, e_i\} \subseteq \Delta(C_{i+1})$. Apply Lemma 5 to
 343 the pair C_i and C_{i+1} , where C_i, C_{i+1} and a play the roles of C_1, C_2 and v . Condition 1 of
 344 Lemma 5 holds, and hence we obtain that $G(\Sigma)$ has an obstruction, a contradiction. ◀

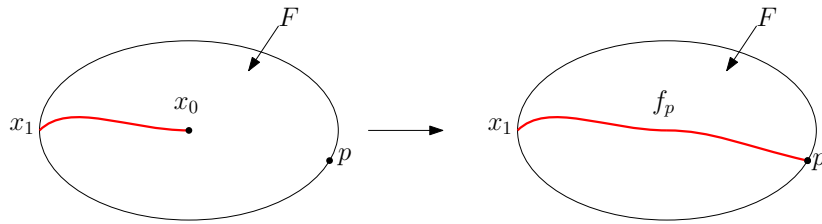
345 ▶ **Lemma 8 (Face-Escaping Step).** *Suppose that there is a string σ that has an end a with*
 346 *degree 1 in $G(\Sigma)$, and a is incident to an interior face F . Then there is an extension σ' of*
 347 *σ from its a -end to a point in the boundary of F such that the set $(\Sigma \setminus \{\sigma\}) \cup \{\sigma'\}$ has no*
 348 *obstruction.*

349 **Proof.** Let W be the closed boundary walk $(x_0, e_1, x_1, e_2, \dots, e_n, x_n)$ of F such that $x_0 =$
 350 $x_n = a$ and F is to the left as we traverse W (see Figure 5 for an illustration with $n = 9$).
 351 For $i = 1, \dots, n$ we let m_i be a point in the relative interior of e_i , and let P be the list of
 352 non-necessarily distinct points $m_1, x_1, m_2, x_2, \dots, m_n$, which are the potential ends for all
 353 the different extensions. For each $p \in P$, let Σ_p be the set of strings obtained from Σ by
 354 extending the a -end of σ by adding an arc α_p connecting a to p in F (see Figure 5). We
 355 assume that every two distinct arcs α_p and $\alpha_{p'}$ are internally disjoint.

356 Let f_p be the edge $e_1 \cup \alpha_p$ in $G(\Sigma_p)$; f_p has ends x_1 and p . Also, let $\sigma^p = \sigma \cup \alpha_p$. See
 357 Figure 6. Seeking a contradiction, suppose that each $G(\Sigma_p)$ has an obstruction.



■ **Figure 5** All possible extensions in the Face-Escaping Step.



■ **Figure 6** Transforming Σ into Σ_p .

358 \triangleright **Claim 9.** Let $p \in P$. Then there exists an obstruction C_p in $G(\Sigma_p)$ including f_p . Moreover,

- 359 (1) if $p \in \sigma$, then C_p can be chosen so that all its edges are included in σ^p ; and
 360 (2) if $p \notin \sigma$, then every obstruction includes f_p .

361 **Proof.** First, if $p \in \sigma$, then the string σ^p self-intersects at p ; thus σ^p has a simple close curve
 362 including f_p . In this case let C_p be the cycle in $G(\Sigma_p)$ representing this simple closed curve
 363 without rainbows, and thus (1) holds.

364 Second, assume that $p \notin \sigma$ and let C_p be any obstruction of $G(\Sigma_p)$. For (2), we will show
 365 that $f_p \in E(C_p)$.

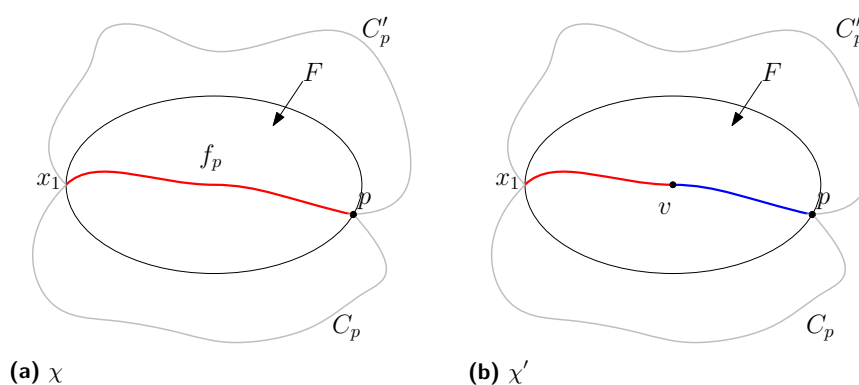
366 Seeking a contradiction, suppose that $f_p \notin E(C_p)$.

367 If $p = m_i$ for $i \in \{1, \dots, n\}$, since m_i is the only vertex whose rotation in $G(\Sigma)$ differs
 368 from its rotation in $G(\Sigma_{m_i})$, $m_i \in V(C_p)$. Consider the cycle C of $G(\Sigma)$ obtained from C_p
 369 by replacing the subpath (x_{i-1}, m_i, x_i) by the edge $x_{i-1}x_i$. For each vertex $v \in V(C)$ the
 370 colors of the edges of $G(\Sigma)$ at v included in the disk bounded by C are the same as in $G(\Sigma_p)$
 371 for the disk bounded by $V(C_p)$. Thus, C is an obstruction for $G(\Sigma)$, a contradiction.

372 Suppose now that p is one of x_1, \dots, x_{n-1} . The only vertex in $G(\Sigma)$ whose rotation is
 373 different in $G(\Sigma_p)$ is p . Therefore, p is a point that is a rainbow for C_p in $G(\Sigma)$, but not a
 374 rainbow in $G(\Sigma_p)$, as witnessed by the two edges of σ^p that are incident with p and inside
 375 C_p . This contradicts the assumption that $p \notin \sigma$. Hence $f_p \in E(C_p)$. \blacktriangleleft

376 Henceforth we assume that, for $p \in P$, C_p is an obstruction in $G(\Sigma_p)$ as in Claim 9.

377 More can be said about the obstructions in $G(\Sigma_p)$, but for this we need some terminology.
 378 If we orient an edge e in a plane graph, then the *sides* of e are either the points near e that
 379 are to the right of e , or the points near e to the left of e . For any cycle C of G through e ,
 380 exactly one side of e lies inside C . This is the side of e *covered* by C . For the next claim
 381 and in the rest of the proof we will assume that for $p \in P$, f_p is oriented from x_1 to p .



■ **Figure 7** The two edge colorings χ and χ' discussed in the proof of Claim 10.

382 ▷ **Claim 10.** For $p \in P$ with $p \notin \sigma$, every obstruction in $G(\Sigma_p)$ covers the same side of f_p .

383 **Proof.** Suppose that for $p \in P$ there are obstructions C_p and C'_p covering both sides of f_p .
 384 Let G' be the plane graph obtained from $G(\Sigma_p)$ by subdividing f_p , and let v be the new
 385 degree-2 vertex inside f_p .

386 We consider the edge-colouring χ induced by the strings in Σ_p . Let χ' be a new colouring
 387 obtained from χ by replacing the colour of the edge vp by a new colour not used in χ
 388 (see Figure 7). It is immediate that (i) χ' induces a path-partition in G' ; and in the next
 389 paragraph we show that (ii) C_p and C'_p are near-obstructions for v with respect to χ' .

390 Consider the set of edges in the rotation at p inside the disk bounded by C_p and assume
 391 they are colored by χ . No edge from this set (except f_p) can have the same color as f_p or else
 392 $p \in \sigma$, contradicting our hypothesis. Therefore, p is a rainbow in C_p in χ if and only if p is a
 393 rainbow in C_p in χ' . Thus, when we switch from χ to χ' , v is the only vertex of C_p switching
 394 identity (where the identity is to be or not to be a rainbow). As C_p is an obstruction for χ ,
 395 then C'_p is a near obstruction at v for χ' . Likewise, C_p is a near obstruction for χ' .

396 As Condition 2 of Lemma 5 holds for $C_1 = C_p$, $C_2 = C'_p$ and $v = v$ with respect to
 397 χ' , G' has an obstruction not containing v in χ' . However, this implies the existence of an
 398 obstruction in $G(\Sigma)$ with respect to χ , a contradiction. ◀

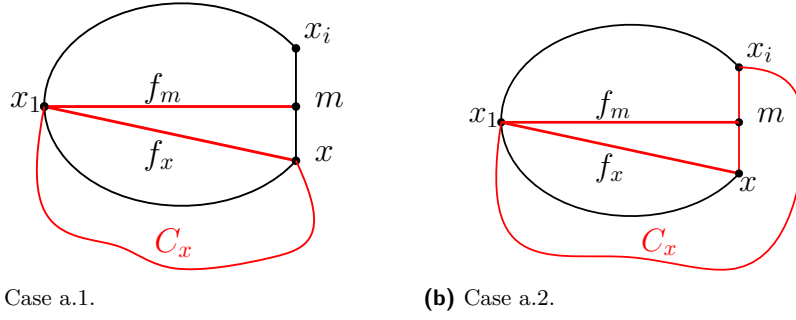
399 Recall that the boundary walk of F is $W = (x_0, e_1, \dots, e_n, x_n)$, with $x_0 = x_n = a$. Since
 400 x_1 and x_{n-1} are in σ , the extreme obstructions C_{x_1} and $C_{x_{n-1}}$ cover the right of f_{x_1} and the
 401 left of $f_{x_{n-1}}$, respectively. Thus, there are two consecutive vertices x_{i-1}, x_i in $W - a$, such
 402 that the interior of $C_{x_{i-1}}$ covers the right of $f_{x_{i-1}}$ and the interior of C_{x_i} covers the left of
 403 f_{x_i} . Moreover, we may assume that the interior of C_{m_i} includes the left of f_{m_i} (otherwise
 404 we reflect our drawing).

405 The next claim is the last ingredient to obtain a final contradiction. To make the notation
 406 simpler, we let $x = x_{i-1}$ and $m = m_i$.

407 ▷ **Claim 11.** Exactly one of the following holds:

- 408 (a) $x \in \sigma$, $m \notin \sigma$ and $G(\Sigma_m)$ has an obstruction covering the side of f_m not covered by C_m ;
 409 or
 410 (b) $x \notin \sigma$ and $G(\Sigma_x)$ has an obstruction covering the side of f_x not covered by C_x .

411 **Proof.** By redrawing the arcs representing f_x and f_m , we will assume that they only intersect
 412 at x_1 . In particular this redrawing creates two copies of the edge e_1 .



■ **Figure 8** Illustrations for Claim 11.a.

413 First, suppose that $x \in \sigma$. For (a) we have two cases depending on whether xx_i is an
 414 edge in C_x .

415 **Case a.1** $xx_i \notin E(C_x)$. See Figure 8a.

416 Let C'_m be the cycle obtained from C_x by replacing the edge f_x by the path $P = (x_1,$
 417 $f_m, m, mx, x)$. Since $x \in \sigma$, by the choice of C_x (Claim 9), all the edges in C_x are in σ^x .
 418 Therefore, by Claim 9.1, all the edges in C'_m , with the possible exception of mx , are in σ^m .
 419 Thus C'_m is an obstruction in $G(\Sigma_m)$.

420 Now we show that C'_m covers the right side of f_m . The disk bounded by $P \cup f_x$ is to the
 421 right of f_m as this side of $P \cup f_x$ is included in the bounded face F . Since the interior of C_x
 422 is to the right of f_x , the interior of C'_m covers the right side of f_m .

423 Finally, note that $m \notin \sigma$, or else, $C'_m \subseteq \sigma^m$ and hence by the choice of C_m , and Claim
 424 10, $C'_m = C_m$. However, this contradicts that C_m covers the left side of f_m . Thus, (a) holds.

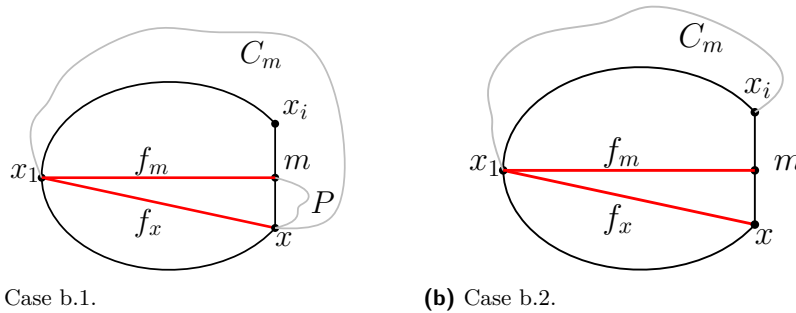
425 **Case a.2.** $xx_i \in E(C_x)$. See Figure 8b.

426 Let C'_m be the cycle obtained from C_x by replacing the path (x_1, f_x, x, xx_i, x_i) by $(x_1, f_m, m,$
 427 $mx_i, x_i)$. Since $x \in \sigma$, by the choice of C_x (Claim 9), all the edges in C_x are in σ^x . Therefore
 428 all the edges in C'_m are in σ^m . Thus C'_m is an obstruction in $G(\Sigma_m)$.

429 Now we show that C'_m covers the right side of f_m . The disk bounded by $f_x \cup f_m \cup xm$ is
 430 to the right of f_m as this side of $f_x \cup f_m \cup xm$ is included in the bounded face F . Since the
 431 interior of C_x is to the right of f_x , the interior of C'_m covers the right side of f_m .

432 Finally, as $C'_m \subseteq \sigma^m$ and by the choice of C_m , $C'_m = C_m$. However, this contradicts the
 433 assumption that C_m covers the left side of f_m . Thus, (a) holds.

434 Turning to (b), let us suppose that $x \notin \sigma$.



■ **Figure 9** Illustrations for Claim 11.b.

435 **Case b.1.** $x \in V(C_m)$. See Figure 9a.

436 Let T be the triangle bounded by f_x , f_m and xm . The interior face of T is to the left of
 437 f_x and to the right of f_m . Let P be the mx -path of $C_m - f_m$ and let P' be the xx_1 -path of
 438 $C_m - m$. Since the interior face of T is a subset of F , P and P' are drawn in the closure of
 439 the exterior of T (possibly $P = (m, mx, x)$).

440 Let C be the simple closed curve bounded by $P \cup f_x \cup f_m$ (in other words, C is obtained
 441 from T by replacing xm by P). Seeking a contradiction, suppose that xm is in the closed
 442 exterior of C . Then, P' is included inside the cycle $C' = P + xm$. Since $V(C') \subseteq V(C_m)$
 443 and C_m is included in the disk bounded by C' , the number of rainbows in C' is at most
 444 the number of rainbows in C_m . Then C' is an obstruction in $G(\Sigma_m)$ not containing f_m ,
 445 contradicting Claim 9.2. Thus, xm is inside C .

446 Our last observation implies that P' is an arc connecting x_1 and x in the exterior of C .
 447 Since the interior of C_m covers the left of f_m , the interior of $C'_x = P' + f_x$ covers the left of
 448 f_x . The cycle C'_x is an obstruction because $V(C'_x) \subseteq V(C_m)$ and C_m is included inside C'_x .

449 **Case b.2.** $x \notin V(C_m)$. See Figure 9b.

450 In this case we let C'_x be the cycle obtained by replacing the path (x_1, f_m, m, mx_i, x_i) in
 451 C_m by the path $P = (x_1, f_x, x, xx_i, x_i)$ in $G(\Sigma_x)$. Since C_m covers the left of f_m and F is
 452 bounded, C'_x covers the left of f_x .

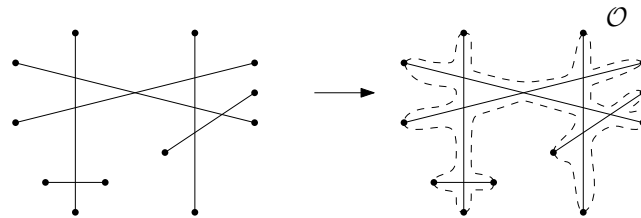
453 To show that C'_x is an obstruction, note that C_m is inside C'_x and that $V(C'_x) \setminus \{x\} \subseteq$
 454 $V(C_m)$. Thus, all the rainbows of C'_x in $V(C'_x) \setminus \{x\}$ are also rainbows in C_m . Since $x \notin \sigma$,
 455 we see that x is a rainbow in C'_x , but is not a vertex of C_m . To compensate, we note
 456 that m is a rainbow in C_m that is not in $V(C'_x)$: if m is not rainbow, both f_m and xx_i are
 457 included in σ , implying that $x \in \sigma$. This shows that C'_x has at most as many rainbows as
 458 C_m . Therefore C'_x is the desired obstruction. ◀

459 Claims 10 and 11 contradict each other, so, for some $p \in P$, $G(\Sigma_p)$ has no obstructions.

460

461 ▶ **Lemma 12** (Exterior-Meeting Step). *If all the strings in Σ have their ends on the outer*
 462 *face of $G(\Sigma)$ and the ends have degree 1 in $G(\Sigma)$, then we can extend a pair disjoint strings*
 463 *so that they intersect without creating an obstruction.*

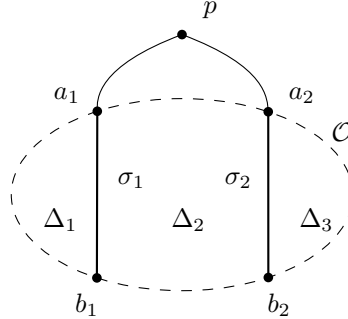
464 **Proof.** First, consider a simple closed curve in the outerface of $\bigcup \Sigma$ closely following the
 465 outerboundary of $\bigcup \Sigma$. Then, by slightly modifying this curve, we obtain a simple closed
 466 curve \mathcal{O} containing all the ends of the strings in Σ , but otherwise disjoint from $\bigcup \Sigma$. See
 467 Figure 10.



468 ■ **Figure 10** Construction of the curve \mathcal{O} .

468 Suppose σ_1, σ_2 are two disjoint strings in Σ . For $i = 1, 2$, let a_i, b_i be the ends of σ_i ;
 469 since σ_1 and σ_2 do not cross, we may assume that these ends occur in the cyclic order $a_1, b_1,$

470 b_2, a_2 . We extend the a_i -ends of σ_1 and σ_2 so that they meet in a point p in the outer face,
 471 and so that all the ends of σ_1 and σ_2 remain incident with the outer face (Figure 11). Let Σ'
 472 be the obtained set of strings.



■ **Figure 11** Exterior-Meeting Step.

473 Seeking a contradiction, suppose that $G(\Sigma')$ has an obstruction C . Since $G(\Sigma)$ has no
 474 obstruction, $p \in V(C)$. Our contradiction will be to find three rainbows in C . Note that
 475 p is a rainbow. To obtain a second rainbow, traverse C starting from p towards a_1 . Let
 476 d_1 be the first vertex during our traversal that is not in the extended σ_1 , and let c_1 be its
 477 neighbour in σ_1 , one step before we reach d_1 . Since b_1 has degree one, $c_1 \neq b_1$.

478 The strings σ_1 and σ_2 divide the disk bounded by \mathcal{O} into three closed regions $\Delta_1, \Delta_2,$
 479 Δ_3 such that $\Delta_1 \cap \Delta_2 = \sigma_1, \Delta_2 \cap \Delta_3 = \sigma_2$ and $\Delta_1 \cap \Delta_3 = \emptyset$ (see Figure 11).

480 \triangleright **Claim 13.** The cycle C has a rainbow included in Δ_1 .

481 **Proof.** First, suppose that $d_1 \notin \Delta_1$. In this case, c_1 is a rainbow because otherwise there
 482 would be a string σ that tangentially intersects σ_1 at c_1 . Thus, if $d_1 \notin \Delta_1$, then c_1 is the
 483 desired rainbow.

484 Second, suppose that $d_1 \in \Delta_1$. Let P_1 be the path of C starting at c_1 , continuing on the
 485 edge $c_1 d_1$, and ending at the first vertex we encounter in σ_1 . Let C' be the cycle enclosed by
 486 $P_1 \cup \sigma_1$. Since C' is not an obstruction, there is one rainbow of C' that is an interior vertex
 487 of P_1 ; this is the desired rainbow of C . This concludes the proof of Claim 13. ◀

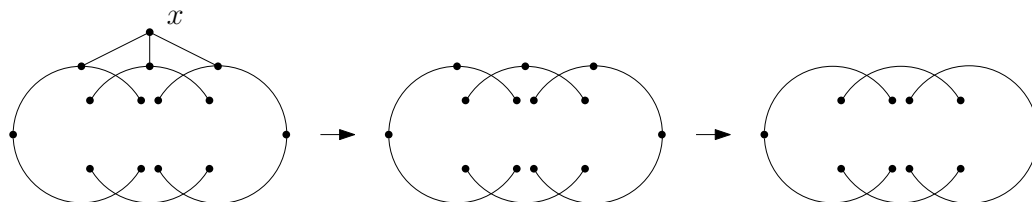
488 Considering σ_2 instead of σ_1 , Claim 13 yields a third rainbow in C inside the region Δ_3
 489 analogous to Δ_1 , contradicting that C is an obstruction. Hence Lemma 12 holds. ◀

490 We iteratively apply the Disentangling Step, Face-Escaping Step or Exterior-Meeting Step
 491 without creating obstructions. Each step increases the number of pairwise intersecting strings
 492 in Σ until we reach a stage where the strings are pairwise intersecting and all of them have
 493 their two ends in the unbounded face. From this we extend them into an arrangement of
 494 pseudolines. This concludes the proof of Theorem 2. ◀

495 **5 Finding obstructions and extending strings in polynomial time**

496 We start this section by describing an algorithm to detect obstructions. Henceforth, we
 497 assume that the input to the problem is the planarization $G(\Sigma)$ of an ordinary set of s strings
 498 Σ . For the running-time analysis, we assume that n and m are the number of vertices and
 499 edges in $G(\Sigma)$, respectively. Since $G(\Sigma)$ is planar, $m = O(n)$. Moreover, if Σ is pseudolinear,
 500 then $n \leq \binom{s}{2} + 2s = \binom{s+2}{2} - 1$. At the end of this section we explain how to extend Σ (if
 501 possible) in polynomial time.

502 Recall that each string in Σ receives a different colour; this induces an edge-colouring on
 503 $G(\Sigma)$ where each string is a monochromatic path. An *outer-rainbow* is a vertex $x \in V(G(\Sigma))$
 504 incident with the outer face and for which the edges incident with x have different colours.
 Next we describe the basic operation in our obstruction-detecting algorithm.



■ **Figure 12** From Σ to $\Sigma - x$.

505
 506 **Outer-rainbow deletion.** Given an outer-rainbow $x \in V(G(\Sigma))$, the instance $G(\Sigma - x)$ is
 507 defined by: first, removing x and the edges incident to x ; second, suppressing the degree-2
 508 vertices incident with edges of the same colour; and third, removing remaining degree-0
 509 vertices (Figure 12 illustrates this process). Edge colours are preserved.

510 It is easy to verify that $G(\Sigma - x)$ is the planarization of an arrangement of strings. The
 511 colours removed by this operation are those belonging to strings that are paths of length 1 in
 512 $G(\Sigma)$ incident with x . Our obstruction-detecting algorithm relies on the following property:

513 (**) if x is an outer-rainbow of $G(\Sigma)$, then there is an obstruction in $G(\Sigma)$ not including x if
 514 and only if there is an obstruction in $G(\Sigma - x)$.

515 This property holds because cycles in $G(\Sigma) - x$ and in $G(\Sigma - x)$ are in 1-1 correspon-
 516 dence: two cycles correspond to each other if they are the same simple closed curve. This
 517 correspondence is obstruction-preserving.

518 Let us now describe the two subroutines in our algorithm. For this, we remark that an
 519 outer-rainbow of $G(\Sigma)$ is a rainbow for any cycle containing it.

520 **Subroutine 1.** *Detecting an obstruction through two outer-rainbows x and y .*

- 521 (1) Find a cycle C through x and y whose edges are incident with the outer face of $G(\Sigma)$. If
 522 C exists, then this cycle is unique and can be described as the outer boundary of the
 523 block containing x and y . If no such C exists, then output *No obstruction through x and*
 524 *y* . Else, go to Step 2.
 525 (2) Find whether there is a third outer-rainbow $z \in V(C) \setminus \{x, y\}$. If such z exists, update
 526 $G(\Sigma) \leftarrow G(\Sigma - z)$ and go to Step 1. If no such z exists, output C .

527 *Correctness and running-time of Subroutine 1:* If an obstruction through x and y exists, then
 528 x and y are in the same block (some authors use the term ‘biconnected component’). Since
 529 x and y are incident with the outer face, the outer boundary of the block containing x and y
 530 is the cycle C from Step 1. This C can be found by considering outer boundary walk W of
 531 $G(\Sigma)$ and then by finding whether x and y belong to the same non-edge block of W . Finding
 532 W is $O(m)$ and computing the blocks of W via a DFS takes $O(m)$ time.

533 In Step 2, if there is a third outer rainbow z in C , then no obstruction through x and y
 534 contains z . Property (**) justifies the update that takes $O(m)$ time.

535 A full run from Step 1 to Step 2 takes $O(m)$. Moving from Step 2 to Step 1 occurs $O(n)$
 536 times. Thus, the total time for Subroutine 1 is $O(mn) = O(n^2)$.

537 **Subroutine 2.** *Detecting an obstruction through a single outer-rainbow x .*

- 538 (1) Find a cycle C through x whose edges are incident with the outer face of $G(\Sigma)$. If no
 539 such C exists, output *No obstruction through x* . Else, go to Step 2.
 540 (2) Find whether there is an outer-rainbow y in $V(C) \setminus \{x\}$. If no such y exists, output C .
 541 Else, apply Subroutine 1 to x and y ; if there is an obstruction C' through x and y , then
 542 output C' . Else, update $G(\Sigma) \leftarrow G(\Sigma - y)$ and go to Step 1.

543 *Correctness and running-time of Subroutine 2:* If $G(\Sigma)$ has an obstruction through x , then
 544 there is a non-edge block in $G(\Sigma)$ containing x . The outer boundary of this block is a cycle
 545 C through x having all edges incident with the outer face. As in Subroutine 1, Step 1 takes
 546 $O(m)$ time.

547 Detecting the existence of y in Step 2 is $O(m)$ because to detect rainbows in C , each edge
 548 incident with a vertex in $V(C)$ is verified at most twice. The update in Step 2 is justified
 549 by Property (**). Since Step 2 may use Subroutine 1, Step 2 takes $O(n^2)$ time. As moving
 550 from Step 2 to Step 1 occurs $O(n)$ times, the total running-time for Subroutine 2 is $O(n^3)$.

551 We are now ready for the algorithm to detect obstructions.

552 **Algorithm 1:** *Detecting obstructions in $G(\Sigma)$.*

- 553 (1) Find a cycle C having all edges incident with the outer face. If no such C exists, output
 554 *No obstruction*. Else, go to step 2.
 555 (2) Find whether there is an outer rainbow $x \in V(C)$. If not, output C . Else apply Subroutine
 556 2 to x ; if there is an obstruction C' through x , output C' . Else, update $G(\Sigma) \leftarrow G(\Sigma - x)$
 557 and go to Step 1.

558 *Correctness and running-time of Algorithm 1:* If $G(\Sigma)$ has an obstruction, then it has a
 559 non-trivial block whose outer boundary is a cycle C as in Step 1. As before, C and x as in
 560 Step 2 can be found in $O(m)$ steps. If C has not outer rainbow x , then C is an obstruction;
 561 Property (**) justifies the update in Step 2.

562 Since Step 2 may use Subroutine 2, a full run of Steps 1 and 2 takes $O(n^3)$ time. Since
 563 Step 2 goes to Step 1 $O(n)$ times, the running-time of Algorithm 1 is $O(n^4)$.

564 Algorithm 1 and the constructive proof of Theorem 2 imply the following result.

565 ► **Theorem 14.** *There is a polynomial-time algorithm to recognize and extend an ordinary
 566 set of strings that are extendible to an arrangement of pseudolines.*

567 **Proof.** Let Σ be an ordinary set of s strings. First, note that if $n = |V(G(\Sigma))|$, $m =$
 568 $|E(G(\Sigma))|$, and Σ is extendible, then $n \leq \binom{s}{2} + 2s$. Hence $n, m = O(s^2)$.

569 Assume that $G(\Sigma)$ has not obstructions, by first verifying that $n \leq \binom{s}{2} + 2s$ and then
 570 running Algorithm 1. For each end in each string in Σ , we keep track of whether one of the
 571 Disentangling, Face-Escaping or Exterior-Meeting Steps apply.

572 The Disentangling and Face-Escaping Steps consist on extending one end a of a fixed
 573 string $\sigma \in \Sigma$ in different ways to find an obstruction-free set of strings. For the Disentangling
 574 Step, the number of possible extensions is bounded by the maximum degree of $G(\Sigma)$; for the
 575 Face-Escaping Step, the number of possible extensions is bounded by twice the length of the
 576 face containing the end that we are extending. Thus, each step lead to $O(m)$ possibilities, and
 577 testing obstructions in each of them is $O(n^4)$. Thus, the Disentangling and the Face-Escaping
 578 Steps take $O(n^5)$ time.

579 The Exterior-Meeting Step is $O(m^2)$ because for this step we just need to record the
 580 number of the pairwise disjoint strings in Σ and the set of strings that have ends incident
 581 with the outer face; if all the strings have their ends in the outer boundary, the extension is
 582 performed as in the proof of Lemma 12.

583 As there is a total of $O(s^2)$ extending steps, extending Σ is $O(s^2(n^5 + m^2)) = O(s^{12})$. ◀

6 Concluding remarks

In this work we characterized in Theorem 2 sets of strings that can be extended into arrangements of pseudolines. Moreover, we showed that the obstructions to pseudolinearity can be detected in $O(n^4)$ time, where n is the number of vertices in the planarization of the set of strings.

An easy consequence of Theorem 2 is the following (presented before as Theorem 1).

► **Theorem 15.** *Let D be a non-pseudolinear good drawing of a graph H . Then there is a subset S of edge-arcs in $\{D[e] : e \in E(H)\}$, such that each $\sigma \in S$ has a substring $\sigma' \subseteq \sigma$ for which $\bigcup_{\sigma \in S} \sigma'$ is one of the drawings represented in Figure 2.*

Proof. Take C an obstruction of the planarization associated to D . Let $\delta(C) \subseteq V(C)$ be the vertices that in C are incident with two different strings in $\Sigma = \{D[e] : e \in E(H)\}$. We choose our obstruction C so that $|\delta(C)|$ is as small as possible.

Decompose C into a cyclic sequence of paths P_0, \dots, P_m , where P_i connects two points in $\delta(C)$ and it is otherwise disjoint from $\delta(C)$. Using Lemma 4, one can show that P_0, \dots, P_m belong to distinct edge-arcs $\sigma_0, \dots, \sigma_m \in \Sigma$, respectively. For each P_i , we consider the string σ'_i , obtained by slightly extending the ends of P_i that are not rainbows in C ; we extend them along σ_i .

Let $x \in \delta(C)$ be an end shared by P_{i-1} and P_i . If x is not a rainbow for C , then x is a crossing between σ_{i-1} and σ_i . Moreover, the arcs added to P_{i-1} and P_i at x to obtain σ'_{i-1} and σ'_i are inside C . If x is a rainbow in C , then P_i and P_{i-1} are not extended at x , and x acts as one of the degree-2 vertices in Figure 2. The rest of the points in $\delta(C)$ are crossings in $\bigcup_{i=0}^m \sigma'_i$ facing the interior of C . Since C has at most two rainbows, $\bigcup_{i=0}^m \sigma'_i$ is one of the drawings depicted in Figure 2. ◀

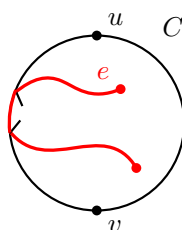
Theorem 2 can also be applied to show that a drawing of K_n is pseudolinear if and only if does not contain the B -configuration (Theorem 2.5.1 in [3]). We sketch the proof of a specific case of this theorem in the next two paragraphs and comment on the general case afterwards.

Suppose that $G(\Sigma)$ is the planarization of a non-pseudolinear drawing D of K_n for which we would like to show that D contains a B -configuration. Consider an obstruction C of $G(\Sigma)$ minimizing $|\delta(C)|$, where $\delta(C)$ are vertices of C incident with edges in C having different colours. For illustrative purposes, let us assume that C contains two vertices from $V(K_n)$. Since C is an obstruction, u and v are the only rainbows of C .

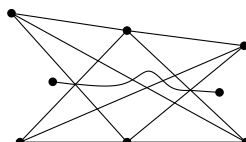
An edge e of K_n is *involved* in C if C contains a subarc of $D[e]$ (see Figure 13). By using Lemma 4 is not hard to show that every edge involved in C is drawn inside C . Consider all the vertices incident with an edge involved in C and let D' be the drawing of the complete graph induced by these vertices. Then, D' has at most two vertices in its outer boundary, namely u and v . Thus, the outer boundary of D' is incident with at least one crossing. The K_4 containing this crossing is drawn as in Figure 1 with its crossing incident with the outer face. This K_4 contains a B -configuration.

The proof for the general case, where C does not necessarily contains two vertices of K_n , is considered in full detail in [3], and uses the complete subgraph induced by the edges involved in C combined with the fact that $|\delta(C)|$ is minimal.

A drawing is *stretchable* if it is homeomorphic to a rectilinear drawing. There are pseudolinear drawings that are not stretchable. For instance, consider the Non-Pappus configuration in Figure 14. Nevertheless, the following is an immediate consequence of Thomassen's main result in [19].



■ **Figure 13** An edge e involved in the obstruction C .



■ **Figure 14** Non-Pappus configuration.

630 ▶ **Corollary 16.** *A 1-planar drawing of a graph is stretchable if and only if it is pseudolinear.*

631 **Proof.** If a drawing D is stretchable then clearly it is pseudolinear. To show the converse,
 632 suppose that D is pseudolinear. Then D does not contain any obstruction, and in particular,
 633 neither of the B - and W -configurations in Figure 2 occurs in D . This condition was shown
 634 in [19] to be equivalent to being stretchable. ◀

635 One can construct more general examples of pseudolinear drawings that are not stretchable
 636 by considering non-stretchable arrangements of pseudolines. However, such examples seem to
 637 inevitably have some edge with multiple crossings. This leads to a natural question.

638 ▷ **Question 17.** Is it true that if D is a pseudolinear drawing in which every edge is crossed
 639 at most twice, then D is stretchable?

640 We believe that there are other instances where pseudolinearity characterizes stretchability
 641 of drawings. A drawing is *near planar* if the removal of one edge produces a planar graph.
 642 One instance, is the following result by Eades et al. that can be translated to the language
 643 of pseudolines:

644 ▶ **Theorem 18.** [9] *A drawing of a near-planar graph is stretchable if and only if the drawing*
 645 *induced by the crossed edges is pseudolinear.*

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