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Viscosity Solutions of system of PDEs with Interconnected Obstacles and nonlinear Neumann Boundary Conditions

Brahim BOUFOUSSI * Saïd HAMADENE † Manal JAKANI ‡

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Abstract

This paper investigates the Hamilton-Jacobi-Bellman system of equations associated with the m -states optimal switching problem in finite horizon when the state process is constrained to live in a connected bounded closed domain. We show existence and uniqueness of the solution in viscosity sense of the system. The main tool is the notion of systems of generalized reflected backward stochastic differential equations with oblique reflection and the Feynman-Kac representation of their solutions in the Markovian framework.

Keywords: Generalized reflected backward stochastic differential equations; Viscosity solution of PDEs; Variational inequalities; Nonlinear Neumann boundary conditions; Optimal switching.

1 Introduction

The purpose of the present paper is to provide existence and uniqueness of continuous viscosity solution to the following system of m -variational inequalities with interconnected obstacles and nonlinear Neumann boundary conditions: $\forall i = 1, \dots, m, \forall t \in [0, T)$,

$$\left\{ \begin{array}{l} \min\{u^i(t, x) - \max_{j \neq i} (u^j(t, x) - g_{ij}(t, x)); -\partial_t u^i(t, x) - \mathcal{L}u^i(t, x) \\ \quad - f_i(t, x, (u^k(t, x))_{k=1, \dots, m}, \sigma^\top(x) D_x u^i(t, x))\} = 0, \quad x \in D; \\ \frac{\partial u^i}{\partial l}(t, x) + \psi_i(t, x, u^i(t, x)) = 0, \quad x \in \partial D; \\ u^i(T, x) = h_i(x), \quad x \in \bar{D}, \end{array} \right. \quad (1.1)$$

where the operator \mathcal{L} is given by

$$\mathcal{L} = \frac{1}{2} Tr\{(\sigma \sigma^\top) D_{xx}^2 \cdot\} + b^\top D_x \cdot;$$

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and at a point $x \in \partial D$,

$$\frac{\partial}{\partial l} = \langle \nabla \phi(x), D_x \cdot \rangle = \sum_{i=1}^d \frac{\partial \phi}{\partial x_i}(x) \frac{\partial}{\partial x_i}.$$

The system (1.1) is of type Hamilton-Jacobi-Bellman (HJB for short) associated with the optimal switching problem when the state process X , which is a diffusion with generator \mathcal{L} , is constrained to live in a bounded connected domain $D := \{\phi > 0\}$.

Optimal switching models often arise in the analysis of industrial projects related to investment in electricity and valuation of energy storage (see e.g., [2, 3, 4, 6, 7, 9, 10] etc.). In a standard optimal switching problem, a decision maker controls a dynamical system over time by choosing successively its working modes from a discrete set. Therefore a switching strategy is given by $\alpha := (\tau_n, \theta_n)_{n \geq 0}$ where τ_n are stopping times such that $\tau_n \leq \tau_{n+1}$ and θ_n is a random variable with values in $J := \{1, \dots, m\}$. At time τ_n , the decision maker switches the system from its current mode θ_{n-1} to the next mode θ_n . When the system is in mode i at time s and works a short period of time ds , it provides a profit $f_i(s, \omega)ds$. On the other hand, switching the system from mode i to mode $j \neq i$ at time s costs $g_{ij}(s, \omega)$. Thus, the problem of the decision maker is to look for a strategy α^* which maximizes the performance $\Gamma_0(\alpha)$ given by:

$$\Gamma_0(\alpha) := \mathbb{E} \left[\int_0^T f_\alpha(s) ds - G_T^\alpha + \xi^\alpha \right],$$

where

- (i) $f_\alpha(s)$ is the instantaneous profit;
- (ii) G_T^α is the total switching cost;
- (iii) ξ^α is the terminal profit.

It is well known that the performance $\sup_\alpha \Gamma_0(\alpha)$ is related to the following system of backward stochastic differential equations with oblique reflection: $\forall i = 1, \dots, m, \forall t \leq T$,

$$\begin{cases} Y_t^i = \xi^i + \int_t^T f_i(s) ds + K_T^i - K_t^i - \int_t^T Z_s^i dB_s; \\ Y_t^i \geq \max_{j \neq i} (Y_t^j - g_{ij}(t)); \\ \int_0^T \{Y_t^i - \max_{j \neq i} (Y_t^j - g_{ij}(t))\} dK_t^i = 0. \end{cases} \quad (1.2)$$

Both optimal performance and optimal strategy are provided by $\vec{Y} = (Y_1, \dots, Y_m)$ the unique solution of (1.2) (see e.g. [11, 12, 13, 14] etc).

In the aforementioned works, when randomness stems from a standard diffusion process $X^{t,x}$ ($(t, x) \in [0, T] \times \mathbb{R}^k$) solution of the following standard SDE:

$$dX_s^{t,x} = b(s, X_s^{t,x}) ds + \sigma(s, X_s^{t,x}) dB_s, s \in [t, T] \text{ and } X_t^{t,x} = x \quad (1.3)$$

(i.e. $f_i(s, \omega) = f_i(s, X_s^{t,x}(\omega))$, $g_{ij}(s, \omega) = g_{ij}(s, X_s^{t,x}(\omega))$ and the bequest $\xi^i(\omega) = h_i(X_T^{t,x}(\omega))$ for any i, j) the HJB system of equations associated with this switching problem takes the following form: $\forall i = 1, \dots, m, \forall (t, x) \in [0, T] \times \mathbb{R}^k$,

$$\begin{cases} \min\{u^i(t, x) - \max_{j \neq i} (u^j(t, x) - g_{ij}(t, x)); -\partial_t u^i(t, x) - \mathcal{L}u^i(t, x) - f_i(t, x)\} = 0; \\ u^i(T, x) = h_i(x). \end{cases} \quad (1.4)$$

The solution $(u^i)_{i=1, \dots, m}$ is nothing but the value function of the switching problem (see e.g., [6, 10]). The process $X^{t,x}$ of equation (1.3) is unconstrained and roughly speaking lives in \mathbb{R}^k . However in

real life, there are several situations where $X^{t,x}$ is constrained to stay in a given bounded domain D , i.e. has the following dynamic:

$$\begin{cases} dX_s^{t,x} = b(X_s^{t,x})ds + \sigma(X_s^{t,x})dB_s + \nabla\phi(X_s^{t,x})dA_s^{t,x}, & s \in [t, T]; \\ X_s^{t,x} \in \bar{D} \text{ and } A_s^{t,x} = \int_t^s \chi_{\{X_r^{t,x} \in \partial D\}} dA_r^{t,x}, & s \in [t, T]; \\ X_t^{t,x} = x, & \text{for } s \leq t; \end{cases} \quad (1.5)$$

where $A^{t,x}$ is an increasing process and a part of the solution. The gradient $\nabla\phi(x)$ coincides with the unit normal pointing towards the interior of $D := \{\phi > 0\}$. The assumptions on b , σ and ϕ will be specified later. The quantity $\nabla\phi(X_s^{t,x})dA_s^{t,x}$ stands for the actions or efforts made in such a way to satisfy the constraint $X^{t,x} \in \bar{D}$.

So let us give two examples which highlight our model. The first one is related to electricity production. Indeed, consider a hydro-power station with a dam and several working modes. Denote by X_s the level of water in the dam at time s , which is obviously a stochastic process. For safety reasons the water level should not exceed a specific level ℓ_1 . On the other hand, for operational reasons this level should not be below another level ℓ_0 . Thus, at each time s , one should have $X_s \in [\ell_0, \ell_1] (= \bar{D})$ by evacuating water when X_s is prone to overlap ℓ_1 and stopping production when X_s reaches ℓ_0 . As a consequence, the working mode of the station is chosen according to parameters which include the level of water X in the dam which is a constrained stochastic process. So if α is a strategy of switching or management of the hydro-power station, then its yield from t to T is given by:

$$\Gamma_t(\alpha) := \mathbb{E} \left[\int_t^T f_\alpha(s)ds + \int_t^T \psi_\alpha(s)dA_s^{t,x} - G_T^\alpha + \xi^\alpha \right]. \quad (1.6)$$

The quantity $\int_t^T \psi_\alpha(s)dA_s^{t,x}$ stands for the cost of maintaining the level of the dam in the appropriate zone.

Another example is related to the interest rates in economies. Actually, assume that the process $(X_s)_{s \leq T}$ stands for the evolution of the interest rates in several economies. The central bank of each economy has the ability to control the interest rate in such a way to keep it above a minimum threshold to avoid inflation and below a fixed ceiling to promote investments for the well being of the economy. Therefore $X^{t,x}$ is constrained to stay in a bounded domain and instead of satisfying (1.3), we would rather it satisfies equation (1.5). The quantity $\nabla\phi(X_s^{t,x})dA_s^{t,x}$ stands for the actions of the central banks to keep the interest rates in appropriate levels. Consider now an investor who invests his/her capital in the economy which provides the best yield amongst those economies. Therefore a strategy of investment is of switching type and when a strategy α is implemented the yield is given by the quantity (1.6). Once more $\int_t^T \psi_\alpha(s)dA_s^{t,x}$ is the effect of the interventions of the central banks on the return of the investment.

We believe that switching problems which look like the previous ones are interesting applications of the model we study in this article.

In this framework of randomness which comes from this constrained Markov process $X^{t,x}$ solution of equation (1.5), the HJB system associated with the switching problem is the system of PDEs with Neumann boundary conditions (1.1). According to our knowledge, this system has not been considered yet. Therefore the novelty of this paper is to deal with this system (1.1) and to complete the literature on this subject of switching problems and their associated HJB systems.

To deal with the switching problem when $X^{t,x}$ is a solution of (1.5), we are led to study the following system of reflected generalized BSDEs with oblique reflection: $\forall i = 1, \dots, m, \forall s \in [t, T]$,

$$\begin{cases} Y_s^{i,t,x} = h_i(X_T^{t,x}) + \int_s^T f_i(r, X_r^{t,x}, \bar{Y}_r^{t,x}, Z_r^{i,t,x})dr + \int_s^T \psi_i(r, X_r^{t,x}, Y_r^{i,t,x})dA_r^{t,x} + K_T^{i,t,x} - K_s^{i,t,x} \\ \quad - \int_s^T Z_r^{i,t,x}dB_r; \\ Y_s^{i,t,x} \geq \max_{j \neq i} (Y_s^{j,t,x} - g_{ij}(s, X_s^{t,x})); \\ \int_0^T \{Y_s^{i,t,x} - \max_{j \neq i} (Y_s^{j,t,x} - g_{ij}(s, X_s^{t,x}))\} dK_s^{i,t,x} = 0. \end{cases} \quad (1.7)$$

Note that without the term $\int_s^T \psi_i(r, X_r^{t,x}, Y_r^{i,t,x}) dA_r^{t,x}$, the system of reflected BSDEs with interconnected obstacles has been investigated in several papers ([11, 12, 14] etc). However, to our knowledge this generalized version has not been considered.

The main contribution of our work, is to provide existence and uniqueness of continuous viscosity solution $(u^i)_{i \in J}$ for system (1.1). As a by-product, we obtain that for any $i \in J$ and $s \in [t, T]$, $u^i(s, X_s^{t,x})$ is the optimal payoff when at time s the system is in working mode i .

The paper is organized as follows. In Section 2, we formulate our problem in a non Markovian setting and we consider the following system of reflected generalized BSDEs with oblique reflection which is more general than (1.7): $\forall i = 1, \dots, m, \forall t \leq T$,

$$\begin{cases} Y_t^i = \xi^i + \int_t^T f_i(s, \vec{Y}_s, Z_s^i) ds + \int_t^T \psi_i(s, Y_s^i) dA_s + K_T^i - K_t^i - \int_t^T Z_s^i dB_s; \\ Y_t^i \geq \max_{j \neq i} (Y_t^j - g_{ij}(t)); \\ \int_0^T \{Y_t^i - \max_{j \neq i} (Y_t^j - g_{ij}(t))\} dK_t^i = 0. \end{cases} \quad (1.8)$$

We first show the existence of a solution $(Y^i, Z^i, K^i)_{i \in J}$ to system (1.8) using a scheme obtained by Picard iterations. Then, we establish the link between the components Y^i , $i \in J$, of the solution and the value functions of the switching problem, from which follows the uniqueness of the solution of system (1.8). The last section is devoted to study the system of PDEs (1.1). We define the notion of a viscosity solution for that system, then we provide a comparison result between its sub-solution and super-solution. Finally, we show the existence of a solution $(u^i)_{i=1, \dots, m}$ of system (1.1). This is given by the Feynman Kac representation, that relates the PDEs system (1.1) to the generalized RBSDEs system (1.8) considered in the second section, namely for any $i \in J$, $Y_s^{i,t,x} = u^i(s, X_s^{t,x})$, $\forall s \in [t, T]$.

2 System of Generalized Reflected BSDEs

2.1 Assumptions and notations

Let $T > 0$ be a finite time horizon. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a fixed probability space on which is defined a d -dimensional Brownian motion $B = (B_t)_{0 \leq t \leq T}$, where $\mathcal{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ is the completed filtration of $(\sigma(B_s, 0 \leq s \leq t))_{t \leq T}$ with all \mathbb{P} -null sets of \mathcal{F}_0 . Let $(A_t)_{t \geq 0}$ be a continuous one-dimensional increasing \mathcal{F}_t -progressively measurable process such that $A_0 = 0$. We introduce the following spaces:

$$\begin{aligned} \mathcal{H}^2 &= \{(\psi_t)_{0 \leq t \leq T} \text{ } \mathcal{F}_t\text{-progressively measurable process s.t. } \mathbb{E}[\int_0^T |\psi_t|^2 dt] < \infty\}; \\ \mathcal{S}^2 &= \{(\psi_t)_{0 \leq t \leq T} \text{ } \mathcal{F}_t\text{-progressively measurable process s.t. } \mathbb{E}[\sup_{0 \leq t \leq T} |\psi_t|^2] < \infty\}; \\ \mathcal{A}^2 &= \{(K_t)_{0 \leq t \leq T} \text{ } \mathcal{F}_t\text{-adapted continuous increasing process s.t. } K_0 = 0, \mathbb{E}[K_T^2] < \infty\}. \end{aligned}$$

Let us introduce the following assumptions:

Assumptions (A1):

Let m be a positive integer, $i \in J$ and $\mu > 0$;

$$(H_1) \quad \xi^i \text{ is a random variable in } L^2(\Omega, \mathcal{F}_T, \mathbb{P}, \mathbb{R}), \text{ satisfying } \mathbb{E}(e^{\mu A_T} | \xi^i|^2) < \infty.$$

$$(H_2) \quad f_i : [0, T] \times \Omega \times \mathbb{R}^m \times \mathbb{R}^{1 \times d} \longrightarrow \mathbb{R} \text{ and } \psi_i : [0, T] \times \Omega \times \mathbb{R} \longrightarrow \mathbb{R} \text{ s.t.}$$

- (i) $f_i(\cdot, \cdot, \vec{y}, z)$ and $\psi_i(\cdot, \cdot, y)$ are progressively measurable;
- (ii) $\mathbb{E} \int_0^T e^{\mu A_t} |f_i(t, \vec{0}, 0)|^2 dt < +\infty$ and $\mathbb{E} \int_0^T e^{\mu A_t} |\psi_i(t, 0)|^2 dA_t < +\infty$;
- (iii) f_i and ψ_i are Lipschitz continuous with respect to (\vec{y}, z) and y respectively;
- (iv) $\exists \beta < 0$ such that $(y - y')(\psi_i(t, y) - \psi_i(t, y')) \leq \beta |y - y'|^2$;

(v) For any $j \neq i$, $f_i(t, \vec{y}, z)$ is non-decreasing w.r.t. y_j , whenever the other components are fixed.

(H_3) For any $i, j \in J$, we have:

- (i) $g_{ij} : [0, T] \times \Omega \rightarrow \mathbb{R}$ is continuous w.r.t. t , non negative such that $g_{ii} = 0$;
- (ii) $\xi^i \geq \max_{j \neq i}(\xi^j - g_{ij}(T))$;
- (iii) g_{ij} satisfies the non free loop property, i.e., $\forall i_1, \dots, i_k$ such that $i_1 \neq i_2, i_1 = i_k$ and $\text{card}\{i_1, \dots, i_k\} = k - 1$, we have:

$$g_{i_1 i_2} + \dots + g_{i_{k-1} i_k} > 0. \quad (2.1)$$

The assumptions (H_1), (H_2) (i) – (iv) are frequently encountered in the study of GBSDEs (see [16, 18]) then carried on in the reflected version by Ren et al. in [20, 21]. One of the difficulties usually faced in multidimensional Reflected BSDEs with interconnected obstacles is the dependence on the whole vector \vec{y} in the generators f_i . This can be overcome with the use of the monotonicity condition (A1) (H_2) (v) first introduced in [12]. The assumptions (H_3) on the costs g_{ij} considered in [6, 11, 12, 13] are actually reasonable. In fact, the first point (H_3)(i) is necessary so that additional costs will not be charged for staying in the same mode. Hypothesis (H_3)(iii) reminds that going from a mode i , and making successive switches all the way along back to the same mode is not free. Finally, the assumption on the terminal values (H_3)(ii) shows that there is no interest in making switches at time T .

2.2 Existence and uniqueness of the solution

This section outlines two fundamental results of our work, namely, the existence and uniqueness of a solution for the following system of generalized reflected BSDEs with oblique reflection: $\forall i = 1, \dots, m, \forall t \leq T$

$$\begin{cases} Y^i \in \mathcal{S}^2, Z^i \in \mathcal{H}^{2,d}, K^i \in \mathcal{A}^2; \\ Y_t^i = \xi^i + \int_t^T f_i(s, \vec{Y}_s, Z_s^i) ds + \int_t^T \psi_i(s, Y_s^i) dA_s + K_T^i - K_t^i - \int_t^T Z_s^i dB_s; \\ Y_t^i \geq \max_{j \neq i} (Y_t^j - g_{ij}(t)); \\ \int_0^T \{Y_t^i - \max_{j \neq i} (Y_t^j - g_{ij}(t))\} dK_t^i = 0. \end{cases} \quad (2.2)$$

We first state the existence result:

Theorem 2.1 *Assume that assumptions (A1) are fulfilled. Then the system of generalized reflected BSDEs (2.2) has a solution.*

Sketch of the Proof: The proof consists of several steps. First, we construct an increasing sequence of standard generalized reflected BSDEs using the results of [18, 21]. Then, we give a priori estimates for the solutions. In step 3, using Peng's monotonic limit Theorem to show that our iterative scheme converges to the solution that we are looking for. Finally, we show the minimal boundary condition.

One of the tools used in this technique is the comparison theorem for generalized reflected BSDEs that we give in the remark below:

Remark 2.1 *For $i = 1, 2$, let Y^i be the unique solution of a one-dimensional generalized reflected BSDEs with data $(\xi^i, f^i, \psi^i, S^i)$. Note that, according to [21], Y^i is obtained as an increasing limit of*

a sequence of solutions of standard generalized BSDEs. Then, it is not difficult to see that if $\xi^1 \leq \xi^2$, $f^1 \leq f^2$, $\psi^1 \leq \psi^2$ and $S^1 \leq S^2$, we get:

$$Y_t^1 \leq Y_t^2, \forall t \in [0, T], \mathbb{P}\text{-a.s.}$$

Indeed, it suffices to consider the associated approximations and to apply the comparison result for generalized BSDEs (see Theorem 1.4 [18]).

Proof of Theorem 2.1.

Step 1: Iterative scheme.

Let us set

$$\begin{aligned} \bar{\xi} &= \max_{i=1, \dots, m} |\xi^i|, & \bar{f}(s, y, z) &= \max_{i=1, \dots, m} f_i(s, y, \dots, y, z), & \bar{\psi}(s, y) &= \max_{i=1, \dots, m} \psi_i(s, y); \\ \underline{\xi} &= \min_{i=1, \dots, m} |\xi^i|, & \underline{f}(s, y, z) &= \min_{i=1, \dots, m} f_i(s, y, \dots, y, z), & \underline{\psi}(s, y) &= \min_{i=1, \dots, m} \psi_i(s, y). \end{aligned}$$

Consider the following standard GBSDEs: $\forall t \leq T$,

$$\begin{cases} \bar{Y} \in \mathcal{S}^2, \bar{Z} \in \mathcal{H}^{2,d}; \\ \bar{Y}_t = \bar{\xi} + \int_t^T \bar{f}(s, \bar{Y}_s, \bar{Z}_s) ds + \int_t^T \bar{\psi}(s, \bar{Y}_s) dA_s - \int_t^T \bar{Z}_s dB_s, \end{cases} \quad (2.3)$$

and

$$\begin{cases} \underline{Y} \in \mathcal{S}^2, \underline{Z} \in \mathcal{H}^{2,d}; \\ \underline{Y}_t = \underline{\xi} + \int_t^T \underline{f}(s, \underline{Y}_s, \underline{Z}_s) ds + \int_t^T \underline{\psi}(s, \underline{Y}_s) dA_s - \int_t^T \underline{Z}_s dB_s. \end{cases} \quad (2.4)$$

By Theorem 1.6 in [18], the generalized BSDEs (2.3) and (2.4) have unique solutions.

Next, for $i = 1, \dots, m$ and $n \geq 1$, we set $Y^{i,0} = \underline{Y}$ and we define recursively $Y^{i,n}$ via the following GRBSDEs whose solutions exist and are unique thanks to [21]:

$$\begin{cases} Y^{i,n} \in \mathcal{S}^2, Z^{i,n} \in \mathcal{H}^{2,d}, K^{i,n} \in \mathcal{S}^2; \\ Y_t^{i,n} = \xi^i + \int_t^T f_i(s, Y_s^{1,n-1}, \dots, Y_s^{i,n}, \dots, Y_s^{m,n-1}, Z_s^{i,n}) ds + \int_t^T \psi_i(s, Y_s^{i,n}) dA_s + K_T^{i,n} - K_t^{i,n} \\ \quad - \int_t^T Z_s^{i,n} dB_s, \forall t \leq T; \\ Y_t^{i,n} \geq \max_{j \neq i} (Y_t^{j,n-1} - g_{ij}(t)), \forall t \leq T; \\ \int_0^T \{Y_t^{i,n} - \max_{j \neq i} (Y_t^{j,n-1} - g_{ij}(t))\} dK_t^{i,n} = 0. \end{cases} \quad (2.5)$$

Note that $(\bar{Y}, \bar{K} = 0, \bar{Z})$ is also a solution for the following GRBSDE:

$$\begin{cases} \bar{Y}_t = \bar{\xi} + \int_t^T \bar{f}(s, \bar{Y}_s, \bar{Z}_s) ds + \int_t^T \bar{\psi}(s, \bar{Y}_s) dA_s + \bar{K}_T - \bar{K}_t - \int_t^T \bar{Z}_s dB_s; \\ \bar{Y}_t \geq \max_{j \neq i} (\bar{Y}_t - g_{ij}(t)); \\ \int_0^T \{\bar{Y}_t - \max_{j \neq i} (\bar{Y}_t - g_{ij}(t))\} d\bar{K}_t = 0. \end{cases} \quad (2.6)$$

Fix $i \in J$, by Remark 2.1 and the monotonicity condition on f_i , we can deduce by induction on n , that

$$\underline{Y} \leq Y^{i,n} \leq Y^{i,n+1} \leq \bar{Y}. \quad (2.7)$$

We conclude that the sequence $(Y^{i,n})_{n \geq 0}$ converges pointwise to Y^i .

Step 2 : A priori estimates.

Let $i = 1, \dots, m, t \leq T$ and $n \geq 1$, we see that

$$|Y_t^{i,n}| \leq \max\{|\underline{Y}_t|, |\bar{Y}_t|\},$$

then

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |Y_t^{i,n}|^2 \right] \leq \mathbb{E} \left[\max \left\{ \sup_{0 \leq t \leq T} |\underline{Y}_t|^2, \sup_{0 \leq t \leq T} |\bar{Y}_t|^2 \right\} \right]. \quad (2.8)$$

First, we should point out that both functions $\underline{\psi}$ and $\bar{\psi}$ satisfy the assumption (H_2) (iv), which is a crucial condition to apply Proposition 1.1 in [18], then the following estimates hold:

$$\begin{aligned} \mathbb{E} \left[\sup_{0 \leq t \leq T} |\bar{Y}_t|^2 + \int_0^T |\bar{Y}_t|^2 dA_t \right] &\leq C \mathbb{E} \left(|\bar{\xi}|^2 + \int_0^T |\bar{f}(t, 0, 0)|^2 dt + \int_0^T |\bar{\psi}(t, 0)|^2 dA_t \right); \\ \mathbb{E} \left[\sup_{0 \leq t \leq T} |\underline{Y}_t|^2 + \int_0^T |\underline{Y}_t|^2 dA_t \right] &\leq C \mathbb{E} \left(|\underline{\xi}|^2 + \int_0^T |\underline{f}(t, 0, 0)|^2 dt + \int_0^T |\underline{\psi}(t, 0)|^2 dA_t \right), \end{aligned} \quad (2.9)$$

which implies that

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |Y_t^{i,n}|^2 + \int_0^T |Y_t^{i,n}|^2 dA_t \right] \leq C. \quad (2.10)$$

We apply Itô's formula to $|Y_t^{i,n}|^2$, we get

$$\begin{aligned} |Y_t^{i,n}|^2 + \int_0^t \|Z_s^{i,n}\|^2 ds &= |\xi^i|^2 + 2 \int_t^T Y_s^{i,n} f_i(s, Y_s^{1,n-1}, \dots, Y_s^{i,n}, \dots, Y_s^{m,n-1}, Z_s^{i,n}) ds \\ &\quad + 2 \int_t^T Y_s^{i,n} \Psi_i(s, Y_s^{i,n}) dA_s + 2 \int_t^T Y_s^{i,n} dK_s^{i,n} - 2 \int_t^T Y_s^{i,n} Z_s^{i,n} dB_s. \end{aligned}$$

Taking the expectation in both sides yields:

$$\begin{aligned} \mathbb{E} \left(|Y_t^{i,n}|^2 \right) + \mathbb{E} \left(\int_t^T \|Z_s^{i,n}\|^2 ds \right) &= \mathbb{E} \left(|\xi^i|^2 \right) + 2 \mathbb{E} \left(\int_t^T Y_s^{i,n} f_i(s, Y_s^{1,n-1}, \dots, Y_s^{i,n}, \dots, Y_s^{m,n-1}, Z_s^{i,n}) ds \right) \\ &\quad + 2 \mathbb{E} \left(\int_t^T Y_s^{i,n} \Psi_i(s, Y_s^{i,n}) dA_s \right) + 2 \mathbb{E} \left(\int_t^T Y_s^{i,n} dK_s^{i,n} \right). \end{aligned}$$

Next, using assumptions (i), (ii) and (iv), we get for $C_1, C_2, C_3 > 0$:

$$\begin{aligned} \mathbb{E} \left(|Y_t^{i,n}|^2 \right) + \mathbb{E} \left(\int_t^T \|Z_s^{i,n}\|^2 ds \right) &\leq \mathbb{E} \left(|\xi^i|^2 \right) + (1 + C_1 + C_2) \mathbb{E} \left(\sup_{0 \leq t \leq T} |Y_t^{i,n}|^2 \right) \\ &\quad + 2 \mathbb{E} \left(\int_t^T M \left\| (Y_s^{1,n-1}, \dots, Y_s^{i-1,n-1}, Y_s^{i,n}, Y_s^{i+1,n-1}, \dots, Y_s^{m,n-1}) \right\| |Y_s^{i,n}| ds \right) \\ &\quad + C_1 \mathbb{E} \left(\int_t^T \|Z_s^{i,n}\|^2 ds \right) + 2 \mathbb{E} \left(\int_t^T \beta |Y_s^{i,n}|^2 dA_s \right) + C_3 \mathbb{E} \left(\int_t^T |Y_s^{i,n}|^2 dA_s \right) \\ &\quad + \mathbb{E} \left(\int_t^T |f_i(s, \vec{0}, 0)|^2 ds \right) + \frac{1}{C_3} \mathbb{E} \left(\int_t^T |\Psi_i(s, 0)|^2 dA_s \right) + \frac{1}{C_2} \mathbb{E} \left(|K_T^{i,n} - K_t^{i,n}|^2 \right). \end{aligned} \quad (2.11)$$

Now, from (2.5), we have

$$\begin{aligned} K_T^{i,n} - K_t^{i,n} &= Y_t^{i,n} - \xi^i - \int_t^T f_i(s, Y_s^{1,n-1}, \dots, Y_s^{i,n}, \dots, Y_s^{m,n-1}, Z_s^{i,n}) ds - \int_t^T \Psi_i(s, Y_s^{i,n}) dA_s \\ &\quad + \int_t^T Z_s^{i,n} dB_s. \end{aligned}$$

Then, by taking the expectation and using (2.10), we obtain by standard computations the following estimate for $K_T^{i,n} - K_t^{i,n}$:

$$\mathbb{E} \left(|K_T^{i,n} - K_t^{i,n}|^2 \right) \leq C_4 \left[1 + \mathbb{E} \left(\int_t^T \|Z_s^{i,n}\|^2 ds \right) \right],$$

where $C_4 > 0$. Back to (2.11), we use (2.10) to conclude that $\mathbb{E} \left(\int_t^T \|Z_s^{i,n}\|^2 ds \right) \leq C$, where C does not depend on n , then for any $i \in J$ and $n \geq 1$, we have:

$$\mathbb{E} \left(\sup_{0 \leq t \leq T} |Y_t^{i,n}|^2 + \int_0^T |Y_t^{i,n}|^2 dA_t + \int_0^T \|Z_t^{i,n}\|^2 dt \right) \leq C. \quad (2.12)$$

Step 3: Monotonic limit result.

Fix $i = 1, \dots, m, t \leq T$ and $n \geq 1$. Recall (2.7) and (2.9), then as $Y^{i,n}$ converges increasingly to Y^i , we have:

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |Y_t^i|^2 + \int_0^T |Y_t^i|^2 dA_t \right] \leq C. \quad (2.13)$$

By dominated convergence we get

$$\mathbb{E} \left(\int_0^T |Y_t^{i,n} - Y_t^i|^2 dt \right) \xrightarrow{n \rightarrow \infty} 0. \quad (2.14)$$

Also, we showed in the previous step that $(Z^{i,n})_{n \geq 1}$ are bounded in \mathcal{H}^2 and so are $(f_i(s, Y_s^{1,n-1}, \dots, Y_s^{i,n}, \dots, Y_s^{m,n-1}, Z_s^{i,n}))_{n \geq 1}$ in $L^2([0, T] \times \Omega)$. In order to apply Peng's monotonic limit theorem [19], let us note that the equation satisfied by $Y^{i,n}$ can be rewritten as follows:

$$\begin{aligned} Y_t^{i,n} + \int_0^t \psi_i(s, Y_s^i) dA_s &= Y_0^{i,n} - \int_0^t f_i(s, Y_s^{1,n-1}, \dots, Y_s^{i,n}, \dots, Y_s^{m,n-1}, Z_s^{i,n}) ds \\ &\quad - \int_0^t \{ \psi_i(s, Y_s^{i,n}) - \psi_i(s, Y_s^i) \} dA_s - K_t^{i,n} + \int_0^t Z_s^{i,n} dB_s. \end{aligned} \quad (2.15)$$

Since ψ_i is non-increasing w.r.t. y , the process $\int_0^t \{ \psi_i(s, Y_s^{i,n}) - \psi_i(s, Y_s^i) \} dA_s$ is increasing for each $n \geq 1$. Hence, using the estimate (2.13) together with the properties of both f_i and ψ_i , we can apply Theorem 2.1 in [19] to the process $Y^{i,n} + \int_0^t \psi_i(s, Y_s^i) dA_s$, which converges increasingly to $Y^i + \int_0^t \psi_i(s, Y_s^i) dA_s$, with Y^i is càdlàg. Therefore, the limit $Y^i + \int_0^t \psi_i(s, Y_s^i) dA_s$ has the following form:

$$Y_t^i + \int_0^t \psi_i(s, Y_s^i) dA_s = Y_0^i - \int_0^t f_i(s) ds - K_t^i + \int_0^t Z_s^i dB_s, \quad (2.16)$$

where K^i is càdlàg and denotes the weak limit of $\int_0^t \{ \psi_i(s, Y_s^{i,n}) - \psi_i(s, Y_s^i) \} dA_s + K^{i,n}$. Moreover, Z^i is the weak limit of $Z^{i,n}$ in \mathcal{H}^2 , which is also the strong limit in L^p for $p \in [1, 2)$. Then, we have:

$$\mathbb{E} \left(\int_0^T |Z_t^{i,n} - Z_t^i|^p dt \right) \xrightarrow{n \rightarrow \infty} 0, \forall p \in [1, 2).$$

We can use exactly the same techniques as in the proof of Theorem 2.4 in [19] to show that $\int_0^t f_i(s) ds = \int_0^t f_i(s, \vec{Y}_s, Z_s^i) ds$. Furthermore, from (2.7), (2.10) and (2.13), by dominated convergence theorem applied to the sequence $Y_t^{i,n}$ in $L^2([0, T] \times \Omega, dA_t \times d\mathbb{P})$ we get:

$$\mathbb{E} \left(\int_0^T |Y_t^{i,n} - Y_t^i|^2 dA_t \right) \xrightarrow{n \rightarrow \infty} 0. \quad (2.17)$$

The Lipschitz continuity of ψ_i implies

$$\mathbb{E} \left(\int_0^T | \psi_i(t, Y_t^{i,n}) - \psi_i(t, Y_t^i) |^2 dA_t \right) \leq M \mathbb{E} \left(\int_0^T |Y_t^{i,n} - Y_t^i|^2 dA_t \right) \xrightarrow{n \rightarrow \infty} 0.$$

This also means that K^i is the weak limit of $K^{i,n}$. In addition, K^i inherits the following properties from $K^{i,n}$:

(a) $K_0^i = 0$ and $\mathbb{E}(K_T^{i,2}) < +\infty$,

(b) K^i is increasing.

It follows that Y^i satisfies:

$$\begin{cases} Y_t^i = \xi^i + \int_t^T f_i(s, \vec{Y}_s, Z_s^i) ds + \int_t^T \psi_i(s, Y_s^i) dA_s + K_T^i - K_t^i - \int_t^T Z_s^i dB_s; \\ Y_t^i \geq \max_{j \neq i} (Y_t^j - g_{ij}(t)). \end{cases} \quad (2.18)$$

Step 4: The minimal boundary condition. Consider the following system:

$$\begin{cases} \tilde{Y}_t^i = \xi^i + \int_t^T f_i(s, Y_s^1, \dots, \tilde{Y}_s^i, \dots, Y_s^m, \tilde{Z}_s^i) ds + \int_t^T \psi_i(s, \tilde{Y}_s^i) dA_s + \tilde{K}_T^i - \tilde{K}_t^i - \int_t^T \tilde{Z}_s^i dB_s; \\ \tilde{Y}_t^i \geq \max_{j \neq i} (Y_t^j - g_{ij}(t)); \\ \int_0^T [\tilde{Y}_{t-}^i - \max_{j \neq i} (Y_{t-}^j - g_{ij}(t))] d\tilde{K}_t^i = 0. \end{cases} \quad (2.19)$$

Note that given $(Y^i, Z^i, K^i)_{i=1, \dots, m}$ solution of (2.18), the above system consists of m generalized reflected BSDEs where for each $i = 1, \dots, m$ only the i -th component of the generator f_i depends on the solution. Moreover, the barriers are known and càdlàg, this is a particular case of Ren-El Otmani [20], which gives the existence and uniqueness of the solution (2.19). Let $(\tilde{Y}^i, \tilde{Z}^i, \tilde{K}^i)_{i=1, \dots, m}$ be their unique solutions. By the comparison result [20], we have $Y_t^{i,n} \leq \tilde{Y}_t^i$ which implies that $Y_t^i \leq \tilde{Y}_t^i$. Next, we show that $\tilde{Y}_t^i \leq Y_t^i$. To this end, we apply Tanaka-Meyer's formula to the semi martingale $(\tilde{Y}_t^i - Y_t^i)_{0 \leq t \leq T}$ whose terminal value is equal to 0, we obtain:

$$\begin{aligned} (\tilde{Y}_t^i - Y_t^i)^+ &= - \int_t^T \chi_{\{\tilde{Y}_{s-}^i - Y_{s-}^i > 0\}} d(\tilde{Y}_s^i - Y_s^i) \\ &\quad - \sum_{t < s \leq T} [\chi_{\{\tilde{Y}_{s-}^i - Y_{s-}^i > 0\}} (\tilde{Y}_s^i - Y_s^i)^- + \chi_{\{\tilde{Y}_{s-}^i - Y_{s-}^i \leq 0\}} (\tilde{Y}_s^i - Y_s^i)^+] - \frac{1}{2} L_t^0. \end{aligned}$$

Here, the non-negative process $(L_t^0)_{t \leq T}$ stands for the local time of $\tilde{Y}^i - Y^i$ at 0. Then,

$$\begin{aligned} (\tilde{Y}_t^i - Y_t^i)^+ &\leq - \int_t^T \chi_{\{\tilde{Y}_{s-}^i - Y_{s-}^i > 0\}} d(\tilde{Y}_s^i - Y_s^i) \\ &= \int_t^T \chi_{\{\tilde{Y}_{s-}^i - Y_{s-}^i > 0\}} \{f_i(s, Y_s^1, \dots, \tilde{Y}_s^i, \dots, Y_s^m, \tilde{Z}_s^i) - f_i(s, Y_s^1, \dots, Y_s^i, \dots, Y_s^m, Z_s^i)\} ds \\ &\quad + \int_t^T \chi_{\{\tilde{Y}_{s-}^i - Y_{s-}^i > 0\}} \{\psi_i(s, \tilde{Y}_s^i) - \psi_i(s, Y_s^i)\} dA_s + \int_t^T \chi_{\{\tilde{Y}_{s-}^i - Y_{s-}^i > 0\}} d(\tilde{K}_s^i - K_s^i) \\ &\quad - \int_t^T \chi_{\{\tilde{Y}_{s-}^i - Y_{s-}^i > 0\}} (\tilde{Z}_s^i - Z_s^i) dB_s. \end{aligned}$$

The Skorokhod condition in system (2.19) shows that $\int_t^T \chi_{\{\tilde{Y}_{s-}^i - Y_{s-}^i > 0\}} d(\tilde{K}_s^i - K_s^i) \leq 0$.

In fact, we have:

$$\int_t^T \chi_{\{\tilde{Y}_{s-}^i - Y_{s-}^i > 0\}} d(\tilde{K}_s^i - K_s^i) = \int_t^T \chi_{\{\tilde{Y}_{s-}^i - Y_{s-}^i > 0\}} d\tilde{K}_s^i - \int_t^T \chi_{\{\tilde{Y}_{s-}^i - Y_{s-}^i > 0\}} dK_s^i.$$

The inequality in the system (2.18), implies that $\tilde{Y}_{s-}^i > \max_{j \neq i} (Y_{s-}^j - g_{ij}(s))$ whenever $\tilde{Y}_{s-}^i > Y_{s-}^i$, then $d\tilde{K}_s^i = 0$ on the set $\{\tilde{Y}_{s-}^i - Y_{s-}^i > 0\}$.

Now, from assumption (A1) (H₂) (iv), one can observe that $\psi_i(s, \tilde{Y}_s^i) - \psi_i(s, Y_s^i)$ is negative since

$\tilde{Y}_s^i \geq Y_s^i$, which implies that $\int_t^T \mathcal{X}_{\{\tilde{Y}_{s^-}^i - Y_{s^-}^i > 0\}} \{\psi_i(s, \tilde{Y}_s^i) - \psi_i(s, Y_s^i)\} dA_s \leq 0$. Then

$$\begin{aligned} (\tilde{Y}_t^i - Y_t^i)^+ &\leq \int_t^T \mathcal{X}_{\{\tilde{Y}_{s^-}^i - Y_{s^-}^i > 0\}} \{f_i(s, Y_s^1, \dots, \tilde{Y}_s^i, \dots, Y_s^m, \tilde{Z}_s^i) - f_i(s, Y_s^1, \dots, Y_s^i, \dots, Y_s^m, \tilde{Z}_s^i) \\ &\quad + f_i(s, Y_s^1, \dots, Y_s^i, \dots, Y_s^m, \tilde{Z}_s^i) - f_i(s, Y_s^1, \dots, Y_s^i, \dots, Y_s^m, Z_s^i)\} ds \\ &\quad - \int_t^T \mathcal{X}_{\{\tilde{Y}_{s^-}^i - Y_{s^-}^i > 0\}} (\tilde{Z}_s^i - Z_s^i) dB_s \\ &\leq \int_t^T M \mathcal{X}_{\{\tilde{Y}_{s^-}^i - Y_{s^-}^i > 0\}} (\tilde{Y}_s^i - Y_s^i)^+ ds \\ &\quad + \int_t^T \mathcal{X}_{\{\tilde{Y}_{s^-}^i - Y_{s^-}^i > 0\}} \mathcal{X}_{\{\tilde{Z}_s^i \neq Z_s^i\}} [f_i(s, \tilde{Y}_s, \tilde{Z}_s^i) - f_i(s, \tilde{Y}_s, Z_s^i)] (\tilde{Z}_s^i - Z_s^i)^{-1} (\tilde{Z}_s^i - Z_s^i) ds \\ &\quad - \int_t^T \mathcal{X}_{\{\tilde{Y}_{s^-}^i - Y_{s^-}^i > 0\}} (\tilde{Z}_s^i - Z_s^i) dB_s. \end{aligned}$$

Set $M_t = \int_0^t \mathcal{X}_{\{\tilde{Z}_s^i \neq Z_s^i\}} [f_i(s, \tilde{Y}_s, \tilde{Z}_s^i) - f_i(s, \tilde{Y}_s, Z_s^i)] (\tilde{Z}_s^i - Z_s^i)^{-1} dB_s$, $t \leq T$. By Girsanov's theorem, as f_i is Lipschitz continuous with respect to z , the process

$$\tilde{B}_t = B_t - \int_0^t \mathcal{X}_{\{\tilde{Z}_s^i \neq Z_s^i\}} [f_i(s, \tilde{Y}_s, \tilde{Z}_s^i) - f_i(s, \tilde{Y}_s, Z_s^i)] (\tilde{Z}_s^i - Z_s^i)^{-1} ds, \quad t \leq T;$$

is a Brownian motion under $\tilde{\mathbb{P}}$, where $d\tilde{\mathbb{P}} = \varepsilon(M)_T d\mathbb{P}$ and $\varepsilon(M)_t = e^{M_t - \frac{1}{2}\langle M \rangle_t}$. It follows that:

$$\begin{aligned} \mathbb{E}^{\tilde{\mathbb{P}}}[(\tilde{Y}_t^i - Y_t^i)^+] &\leq \mathbb{E}^{\tilde{\mathbb{P}}}\left[\int_t^T M \mathcal{X}_{\{\tilde{Y}_{s^-}^i - Y_{s^-}^i > 0\}} (\tilde{Y}_s^i - Y_s^i)^+ ds\right] - \int_t^T \mathcal{X}_{\{\tilde{Y}_{s^-}^i - Y_{s^-}^i > 0\}} (\tilde{Z}_s^i - Z_s^i) d\tilde{B}_s, \\ &\leq \mathbb{E}^{\tilde{\mathbb{P}}}\left[\int_t^T M (\tilde{Y}_s^i - Y_s^i)^+ ds\right]. \end{aligned}$$

Finally, by Gronwall's Lemma, we have $(\tilde{Y}_t^i - Y_t^i)^+ = 0$ $\tilde{\mathbb{P}}$ -a.s.. As $\tilde{\mathbb{P}}$ and \mathbb{P} are equivalent, we get \mathbb{P} -a.s. $\forall t \leq T$, $\tilde{Y}_t^i \leq Y_t^i$ and then $\tilde{Y}^i = Y^i$.

Then, by classical arguments we show that $\tilde{K}^i = K^i$ \mathbb{P} -a.s. and $\tilde{Z}^i = Z^i$ $dt \times d\mathbb{P}$ -a.e., which means that (Y, K, Z) is a solution of (2.19).

To conclude, it suffices to show K^i and Y^i are continuous.

As $(K_t^i)_{t \geq 0}$ is a non-decreasing process, $\Delta K_t^i \geq 0$. Suppose there exists i_1 such that $\Delta K_t^{i_1} > 0$ then $\Delta Y_t^{i_1} < 0$. But

$$Y_t^{i_1} = \max_{k \neq i_1} (Y_t^k - g_{i_1 k}(t)).$$

Then there exists $i_2 \in J - \{i_1\}$ such that

$$Y_t^{i_1} = Y_t^{i_2} - g_{i_1 i_2}(t) > Y_t^{i_1} \geq \max_{k \neq i_1} (Y_t^k - g_{i_1 k}(t)).$$

This implies that

$$\Delta Y_t^{i_2} < 0 \text{ and } \Delta K_t^{i_2} > 0.$$

By repeating the same procedure in the finite set J , we can find a loop $i_1, \dots, i_p = i_1$ such that

$$Y_t^{i_1} = Y_t^{i_2} - g_{i_1 i_2}(t), Y_t^{i_2} = Y_t^{i_3} - g_{i_2 i_3}(t), \dots, Y_t^{i_{p-1}} = Y_t^{i_p} - g_{i_{p-1} i_p}(t).$$

By adding respective hand-sides in the above equations, we get $\sum_{k=1}^{p-1} g_{i_k i_{k+1}}(t) = 0$, which contradicts

the non-free loop property. Therefore, K^i is continuous and Y^i as well. Now as i is arbitrary then the processes K^i, Y^i , $i \in J$ are continuous.

As a result, we can rewrite our system (2.19) by deleting the limits. \square

In order to establish the uniqueness result, we construct a contraction with an appropriate norm. The idea is to recall the link existing between generalized reflected BSDEs and switching problems. More precisely, using the first existence result, we consider an initial system of generalized reflected BSDEs with interconnected obstacles in which the data do not depend on the solution. Our goal is to represent its solution in terms of the value functions of one dimensional switching problem, which gives the uniqueness of the latter system. As a by-product, we obtain the contraction property thanks to a comparison result for multidimensional generalized reflected BSDEs systems that we state in the following proposition:

Proposition 2.1 *Let $(Y^i)_{i \in J}$ and $(\tilde{Y}^i)_{i \in J}$ be respectively the unique solutions to the generalized reflected BSDEs system (2.2), with respective data $((\xi^i)_{i \in J}, (f_i)_{i \in J}, (\psi_i)_{i \in J}, (g_{ij})_{(i,j) \in J^2})$ and $((\tilde{\xi}^i)_{i \in J}, (\tilde{f}_i)_{i \in J}, (\tilde{\psi}_i)_{i \in J}, (\tilde{g}_{ij})_{(i,j) \in J^2})$. If for any $i \in J$, $\xi^i \leq \tilde{\xi}^i$, $f_i \leq \tilde{f}_i$, $\psi_i \leq \tilde{\psi}_i$ and $g_{ij} \geq \tilde{g}_{ij}$, then we have:*

$$Y_t^i \leq \tilde{Y}_t^i, \forall i \in J, \forall t \leq T, \mathbb{P}\text{-a.s.}$$

Proof. We apply Remark 2.1 to the increasing approximation schemes (2.5) which converge respectively to $(Y^i)_{i \in J}$ and $(\tilde{Y}^i)_{i \in J}$. We deduce the comparison by taking the limit. \square

Theorem 2.2 *Under assumptions (A1), the solution of system (2.2) is unique .*

Proof. For $\mu > 0$ and $\lambda > 0$, let $\mathcal{M}_{\mu, \lambda}^2(A)$ denote the set of progressively measurable processes $(\rho_t)_{0 \leq t \leq T}$ s.t.

$$\|\rho\|_{\mathcal{M}_{\mu, \lambda}^2(A)} := \mathbb{E} \left(\int_0^T e^{\mu t + \lambda A_t} |\rho_t|^2 dt + \int_0^T e^{\mu t + \lambda A_t} |\rho_t|^2 dA_t \right) < \infty.$$

Let $\bar{u} = (u^i)_{i=1, \dots, m}$ be such that $u^i \in \mathcal{M}_{\mu, \lambda}^2(A)$, $\forall i = 1, \dots, m$, and consider the following system: $\forall i = 1, \dots, m, \forall t \leq T$,

$$\begin{cases} Y^{u,i} \in \mathcal{S}^2, & Z^{u,i} \in \mathcal{H}^{2,d}, & K^{u,i} \in \mathcal{S}^2; \\ Y_t^{u,i} = \xi^i + \int_t^T f_i(s, \bar{u}_s^i, Z_s^{u,i}) ds + \int_t^T \psi_i(s, u_s^i) dA_s + K_T^{u,i} - K_t^{u,i} - \int_t^T Z_s^{u,i} dB_s, \\ Y_t^{u,i} \geq \max_{j \neq i} (Y_t^{u,j} - g_{ij}(t)), \\ \int_0^T [Y_t^{u,i} - \max_{j \neq i} (Y_t^{u,j} - g_{ij}(t))] dK_t^{u,i} = 0. \end{cases} \quad (2.20)$$

Let $(Y^{u,i}, Z^{u,i}, K^{u,i})_{i=1, \dots, m}$ be a solution of system (2.20) which exists thanks to Theorem 2.1. We shall stress that for fixed $\bar{u} \in \mathcal{M}_{\mu, \lambda}^{2,m}(A)$ the solution $Y^{u,i} \in \mathcal{M}_{\mu, \lambda}^2(A)$, $\forall i = 1, \dots, m$. In fact, by assumptions (H_1) , $(H_2)(ii)$ and $(H_2)(iii)$, we know that \underline{Y}^u and \bar{Y}^u the solutions obtained by Theorem 1.6 in [18] for the generalized BSDEs (2.3) and (2.4) with respective data $(\underline{\xi}, \underline{f}(\cdot, \bar{u}^i, \cdot), \underline{\psi}(\cdot, u^i))$ and $(\bar{\xi}, \bar{f}(\cdot, \bar{u}^i, \cdot), \bar{\psi}(\cdot, u^i))$ are both elements of $\mathcal{M}_{\mu, \lambda}^2(A)$, then using $\underline{Y}^u \leq Y^{u,i} \leq \bar{Y}^u$, the result follows.

Our first task, is to show that the solution of system (2.20) is unique. For this, we rely on the connection between generalized reflected BSDE and optimal switching. Let us first recall that a switching control α is a pair of subsequences $(\tau_n, \theta_n)_{n \geq 0}$, where τ_n are stopping times such that $\tau_n \leq \tau_{n+1}$ and θ_n is a random variable with values in $J := \{1, \dots, m\}$. If $\mathbb{P}[\tau_n^* < T, \forall n \geq 0] = 0$, we say that α is admissible.

Let $i \in J$ and $t \in [0, T]$, we define a class of admissible switching controls by:

$$\mathcal{D}_t^i = \{ \alpha = ((\tau_n)_{n \geq 0}, (\theta_n)_{n \geq 0}) \in \mathcal{D}, \theta_0 = i, \tau_0 = 0, \tau_1 \geq t \text{ and } \mathbb{E}(G_T^\alpha)^2 < \infty \},$$

where \mathcal{D} denotes the set of admissible switching controls and G_s^α is the cumulative switching costs up to time s , expressed by:

$$G_s^\alpha = \sum_{n \geq 1} g_{\theta_{n-1}, \theta_n}(\tau_n) \mathcal{X}_{[\tau_n \leq s]}, \quad s < T \text{ and } G_T^\alpha = \lim_{s \rightarrow T} G_s^\alpha = \sum_{n \geq 1} g_{\theta_{n-1}, \theta_n}(\tau_n) \mathcal{X}_{[\tau_n < T]}.$$

Let $\alpha = ((\tau_n)_{n \geq 0}, (\theta_n)_{n \geq 0}) \in \mathcal{D}_t^i$ and define:

$$\begin{aligned}\xi^\alpha &= \sum_{n \geq 0} \xi^{\theta_n} \chi_{[\tau_n \leq T < \tau_{n+1}]}, \\ f_\alpha(t, \vec{u}, z) &= \sum_{n \geq 0} f_{\theta_n}(t, \vec{u}, z) \chi_{[\tau_n \leq t < \tau_{n+1}]}, \\ \psi_\alpha(t, u^\alpha) &= \sum_{n \geq 0} \psi_{\theta_n}(t, u^{\theta_n}) \chi_{[\tau_n \leq t < \tau_{n+1}]}. \end{aligned} \quad (2.21)$$

Let us consider the following switching equation:

$$\begin{cases} P^\alpha, \mathbb{E}(\sup_{0 \leq t \leq T} |P_t^\alpha|^2) < \infty \text{ and } Q^\alpha \in \mathcal{H}^{2,d}; \\ P_t^\alpha = \xi^\alpha + \int_t^T f_\alpha(s, \vec{u}_s, Q_s^\alpha) ds + \int_t^T \psi_\alpha(s, u_s^\alpha) dA_s - \int_t^T Q_s^\alpha dB_s - (G_T^\alpha - G_t^\alpha), t \leq T. \end{cases} \quad (2.22)$$

In the above equation, ξ^α , f_α and ψ_α are respectively the reward received at time T , the running reward received on D and the additional reward once the limit of the domain is reached while adhering to the strategy α .

By setting $\bar{P}_t^\alpha := P_t^\alpha - G_t^\alpha$, we get a Generalized BSDE with standard generators and a terminal value satisfying $\mathbb{E}[(\xi^\alpha - G_T^\alpha)^2] < \infty$. All together with the adaptedness of G^α , there exists a unique solution (P^α, Q^α) for (2.22) thanks to [18].

We go back to system (2.20), to write the equation of $Y^{u,i}$ between t and τ_1 :

$$\begin{aligned} Y_t^{u,i} &= Y_{\tau_1}^{u,i} + \int_t^{\tau_1} f_i(s, \vec{u}_s, Z_s^{u,i}) ds + K_{\tau_1}^{u,i} - K_t^{u,i} + \int_t^{\tau_1} \psi_i(s, u_s^i) dA_s - \int_t^{\tau_1} Z_s^{u,i} dB_s \\ &= Y_{\tau_1}^{u,i} \chi_{[\tau_1=T]} + Y_{\tau_1}^{u,i} \chi_{[\tau_1 < T]} + \int_t^{\tau_1} f_\alpha(s, \vec{u}_s, Z_s^\alpha) ds + \int_t^{\tau_1} \psi_\alpha(t, u_s^\alpha) dA_s \\ &\quad + K_{\tau_1}^{u,i} - K_t^{u,i} - \int_t^{\tau_1} Z_s^\alpha dB_s, \end{aligned}$$

where we have used (2.21) together with $Z_s^\alpha = Z_s^{u, \theta_0} \chi_{[\tau_0 \leq s < \tau_1]} = Z_s^{u,i}$ on $[t, \tau_1]$. Then, as

$$Y_{\tau_1}^{u,i} \geq \max_{j \neq i} (Y_{\tau_1}^{u,j} - g_{ij}(\tau_1)) \geq Y_{\tau_1}^{u, \theta_1} - g_{i\theta_1}(\tau_1),$$

we get: $\forall i = 1, \dots, m, \forall s \in [t, \tau_1]$

$$\begin{aligned} Y_s^{u,i} &\geq (Y_{\tau_1}^{u, \theta_1} - g_{i, \theta_1}(\tau_1)) \chi_{[\tau_1 < T]} + \xi^{i=\theta_0} \chi_{[\tau_1=T]} + \int_t^{\tau_1} f_\alpha(s, \vec{u}_s, Z_s^\alpha) ds + \int_t^{\tau_1} \psi_\alpha(s, u_s^\alpha) dA_s \\ &\quad + K_{\tau_1}^{u,i} - K_t^{u,i} - \int_t^{\tau_1} Z_s^\alpha dB_s; \\ &\geq Y_{\tau_2}^{u, \theta_1} \chi_{[\tau_1 < T]} + \int_t^{\tau_2} f_\alpha(s, \vec{u}_s, Z_s^\alpha) ds + \int_t^{\tau_2} \psi_\alpha(s, u_s^\alpha) dA_s + (K_{\tau_2}^{u,i} - K_{\tau_1}^{u,i}) \\ &\quad - \int_t^{\tau_2} Z_s^\alpha dB_s - g_{i, \theta_1}(\tau_1) \chi_{[\tau_1 < T]} + K_{\tau_1}^{u,i} - K_t^{u,i} + \xi^{\theta_0} \chi_{[\tau_1=T]}, \end{aligned}$$

where we used the definition of Y^{u, θ_1} between $[\tau_1, \tau_2]$. Repeating the same procedure as many times as necessary, we obtain:

$$Y_t^{u,i} \geq \xi^\alpha + \int_t^T f(s, \vec{u}_s, Z_s^\alpha) ds + \int_t^T \psi_\alpha(s, u_s^\alpha) dA_s - G_T^\alpha + \tilde{K}_T^\alpha - \int_t^T Z_s^\alpha dB_s.$$

where $\tilde{K}_T^\alpha = K_{\tau_1}^{u,i} - K_t^{u,i} + \sum_{n \geq 1} (K_{\tau_{n+1}}^{u, \theta_n} - K_{\tau_n}^{u, \theta_n})$, which is non negative. Thus

$$Y_t^{u,i} \geq \xi^\alpha + \int_t^T f_\alpha(s, \vec{u}_s, Z_s^\alpha) ds + \int_t^T \psi_\alpha(s, u_s^\alpha) dA_s - G_T^\alpha - \int_t^T Z_s^\alpha dB_s.$$

Then, we use the equation satisfied by (P^α, Q^α) and we have:

$$Y_t^{u,i} - (P_t^\alpha - G_t^\alpha) \geq \int_t^T [f_\alpha(s, \vec{u}_s, Z_s^\alpha) - f_\alpha(s, \vec{u}_s, Q_s^\alpha)] ds - \int_t^T (Z_s^\alpha - Q_s^\alpha) dB_s.$$

Once again, thanks to Girsanov's Theorem there exists $\tilde{\mathbb{P}}$, s.t. $d\tilde{\mathbb{P}} = \varepsilon(M)_T d\mathbb{P}$, where $\varepsilon(M)_t = e^{M_t - \frac{1}{2}\langle M \rangle_t}$ and $M_t = \int_0^t \chi_{\{Z_s^\alpha \neq Q_s^\alpha\}} [f_\alpha(s, \vec{u}_s, Z_s^\alpha) - f_\alpha(s, \vec{u}_s, Q_s^\alpha)] (Z_s^\alpha - Q_s^\alpha)^{-1} dB_s$.

Then

$$Y_t^{u,i} - (P_t^\alpha - G_t^\alpha) \geq \int_t^T (Z_s^\alpha - Q_s^\alpha) d\tilde{B}_s,$$

where $\tilde{B}_t = B_t - \int_0^t \chi_{\{Z_s^\alpha \neq Q_s^\alpha\}} [f_\alpha(s, \vec{u}_s, Z_s^\alpha) - f_\alpha(s, \vec{u}_s, Q_s^\alpha)] (Z_s^\alpha - Q_s^\alpha)^{-1} ds$, $t \leq T$ is a Brownian motion under $\tilde{\mathbb{P}}$. It follows that

$$\mathbb{E}^{\tilde{\mathbb{P}}} \left(Y_t^{u,i} - (P_t^\alpha - G_t^\alpha) \mid \mathcal{F}_t \right) \geq 0.$$

Finally, as $\varepsilon(M)_t$ is a non-negative martingale, we obtain: $\forall \alpha \in \mathcal{D}_t^i$,

$$Y_t^{u,i} \geq P_t^\alpha - G_t^\alpha, \mathbb{P}\text{-a.s.}$$

$$Y_t^{u,i} = \operatorname{esssup}_{\alpha \in \mathcal{D}_t^i} (P_t^\alpha - G_t^\alpha).$$

Now, in order to represent $(Y_t^{u,i})_{i=1,\dots,m}$ as the value function of the associated switching problem, we consider the strategy $\alpha^* = (\tau_n^*, \theta_n^*)_{n \geq 0}$ defined by:

$$\tau_0^* = 0, \theta_0^* = i,$$

$$\tau_{n+1}^* = \inf\{s \geq \tau_n^*, Y_s^{u,\theta_n^*} = \max_{j \neq \theta_n^*} (Y_s^{u,j} - g_{\theta_n^*,j}(s))\} \wedge T, \theta_{n+1}^* = \operatorname{argmax}_{j \neq \theta_n^*} (Y_{\tau_{n+1}^*}^{u,j} - g_{\theta_n^*,j}(\tau_{n+1}^*)).$$

Next, we show that α^* is an admissible strategy, that is, we need to prove that:

1. $\mathbb{P}[\tau_n^* < T, \forall n \geq 0] = 0$,
2. $\mathbb{E}(G_T^{\alpha^*})^2 < \infty$.

For the first equality, We proceed by contradiction assuming that $\mathbb{P}[\tau_n^* < T, \forall n \geq 0] > 0$. By definition of α^* , this means that

$$\mathbb{P} \left[Y_{\tau_{n+1}^*}^{u,\theta_n^*} = Y_{\tau_{n+1}^*}^{u,\theta_{n+1}^*} - g_{\theta_n^*,\theta_{n+1}^*}(\tau_{n+1}^*), \theta_n^* \neq \theta_{n+1}^*, \forall n \geq 1 \right] > 0.$$

Since J is finite, there exists a loop $i_0, \dots, i_k, i_{k+1} = i_0$ such that $i_1 \neq i_0$ and a subsequence $(n_q)_{q \geq 1}$ such that

$$\mathbb{P} \left[Y_{\tau_{n_q+1}^*}^{u,i_l} = Y_{\tau_{n_q+1}^*}^{u,i_{l+1}} - g_{i_l,i_{l+1}}(\tau_{n_q+1}^*), l = 0, \dots, k, \forall q \geq 1, i_{k+1} = i_0 \right] > 0.$$

Consider now $\tau = \lim_n \nearrow \tau_n^*$, this implies that

$$\mathbb{P} \left[Y_\tau^{u,i_l} = Y_\tau^{u,i_{l+1}} - g_{i_l,i_{l+1}}(\tau), l = 0, \dots, k, i_{k+1} = i_0 \right] > 0,$$

which leads to

$$\mathbb{P} [g_{i_0 i_1}(\tau) + \dots + g_{i_k i_0}(\tau) = 0] > 0,$$

that contradicts the non free loop property, then assertion one holds.

To show the second property, we rewrite the equation for $Y^{u,i}$ under the strategy α^* and Skorhod Condition, which means that $K_s^{u,\theta_n^*} - K_{\tau_n^*}^{u,\theta_n^*} = 0$ for $\tau_n^* \leq s \leq \tau_{n+1}^*$, we get:

$$Y_t^{u,i} = \xi^{\alpha^*} + \int_t^{\tau_n^*} f_{\alpha^*}(s, \bar{u}_s, Z_s^\alpha) ds + \int_t^{\tau_n^*} \psi_{\alpha^*}(s, u_s^{\alpha^*}) dA_s - \int_t^{\tau_n^*} Z_s^{\alpha^*} dB_s - G_{\tau_n^*}^{\alpha^*}, \forall n \geq 1.$$

Next, by taking the limit w.r.t., we have: $\forall t \leq T$,

$$Y_t^{u,i} = \xi^{\alpha^*} + \int_t^T f_{\alpha^*}(s, \bar{u}_s, Z_s^\alpha) ds + \int_t^T \psi_{\alpha^*}(s, u_s^{\alpha^*}) dA_s - \int_t^T Z_s^{\alpha^*} dB_s - G_T^{\alpha^*}.$$

This implies that $\mathbb{E}(G_T^{\alpha^*})^2 < \infty$ since $(\bar{u}, Y^{u,i}, Z^{\alpha^*}) \in \mathcal{M}_{\mu,\lambda}^2(A) \times \mathcal{S}^2 \times \mathcal{H}^{2,d}$. It follows that $\alpha^* \in \mathcal{D}_t^i$ and $Y_t^{u,i} = P_t^{\alpha^*} - G_t^{\alpha^*}$. As a conclusion, we have:

$$Y_t^{u,i} = \operatorname{esssup}_{\alpha \in \mathcal{D}_t^i} (P_t^\alpha - G_t^\alpha), \quad (2.23)$$

which gives the uniqueness of the solution for system (2.20).

Now, let us introduce the following mapping:

$$\Phi(\bar{u}) = (Y^{u,i})_{i=1,\dots,m},$$

where $\bar{u} \in \mathcal{M}_{\mu,\lambda}^2(A)$, we show that Φ is a strict contraction on the space $\mathcal{M}_{\mu,\lambda}^2(A)$ equipped with the norm $\|\cdot\|_{\mathcal{M}_{\mu,\lambda}^2(A)}$.

Let us set, for \bar{u} and \bar{v} in $\mathcal{M}_{\mu,\lambda}^2(A)$:

$$\begin{aligned} F_i(s, z) &= f_i(s, \bar{u}_s, z) \vee f_i(s, \bar{v}_s, z), \\ \Psi_i(s) &= \psi_i(s, u_s^i) \vee \psi_i(s, v_s^i); \end{aligned}$$

and let $(\tilde{Y}, \tilde{K}, \tilde{Z})$ be the unique solution of the system below: $\forall i = 1, \dots, m, \forall t \leq T$,

$$\begin{cases} \tilde{Y}_t^i = \xi^i + \int_t^T F_i(s, \tilde{Z}_s^i) ds + \int_t^T \Psi_i(s) dA_s + \tilde{K}_T^i - \tilde{K}_t^i - \int_t^T \tilde{Z}_s^i dB_s, \\ \tilde{Y}_t^i \geq \max_{j \neq i} (\tilde{Y}_t^j - g_{ij}(t)), \\ \int_0^T \{ \tilde{Y}_t^i - \max_{j \neq i} (\tilde{Y}_t^j - g_{ij}(t)) \} d\tilde{K}_t^i = 0. \end{cases} \quad (2.24)$$

Moreover, as in (2.23), we have the following representation: $\forall t \leq T$,

$$\tilde{Y}_t^i = \operatorname{esssup}_{\alpha \in \mathcal{D}_t^i} (\tilde{P}_t^\alpha - G_t^\alpha) = \tilde{P}_t^{\alpha^*} - G_t^{\alpha^*}, \quad (2.25)$$

where $(\tilde{P}^\alpha, \tilde{Q}^\alpha)$ is the unique solution of the generalized BSDE (2.22) with data (F_α, Ψ_α) and α^* is the optimal strategy.

Now, by proposition 2.1, we can get the comparison between $Y^{u,i}$ and \tilde{Y}^i , and the same for $Y^{v,i}$ and \tilde{Y}^i . Then we get:

$$Y^{u,i} \leq \tilde{Y}^i \text{ and } Y^{v,i} \leq \tilde{Y}^i.$$

This combined with (2.23) and (2.25), implies that,

$$P^{\alpha^*} - G^{\alpha^*} \leq Y^{u,i} \leq \tilde{Y}^i = \tilde{P}^{\alpha^*} - G^{\alpha^*} \text{ and } P^{\alpha^*} - G^{\alpha^*} \leq Y^{v,i} \leq \tilde{Y}^i = \tilde{P}^{\alpha^*} - G^{\alpha^*},$$

where $(P^{\alpha^*}, Q^{\alpha^*})$ is the unique solution for (2.22) with data $(f_\alpha(\cdot, \bar{v}, \cdot), \psi_\alpha(\cdot, v^i))$. It follows that:

$$|Y^{u,i} - Y^{v,i}| \leq |\tilde{P}^{\alpha^*} - P^{\alpha^*}| + |\tilde{P}^{\alpha^*} - P^{\alpha^*}|. \quad (2.26)$$

Since both terms in the right hand side are similar, we give only the estimates for the first one. To start with, note that $(\tilde{P}^{\alpha^*} - P^{\alpha^*}, \tilde{Q}^{\alpha^*} - Q^{\alpha^*})$ is the unique solution for the following generalized BSDE.

$$\begin{aligned} \tilde{P}_t^{\alpha^*} - P_t^{\alpha^*} &= \int_t^T [F_{\alpha^*}(s, \tilde{Q}_s^{\alpha^*}) - f_{\alpha^*}(s, \bar{u}_s, Q_s^{\alpha^*})] ds + \int_t^T [\Psi_{\alpha^*}(s) - \psi_{\alpha^*}(s, u_s^{\alpha^*})] dA_s \\ &\quad - \int_t^T (\tilde{Q}_s^{\alpha^*} - Q_s^{\alpha^*}) dB_s, \forall t \leq T. \end{aligned}$$

Under the assumptions $(H_2)(ii)$ and $(H_2)(iii)$, and by Theorem 1.6 in [18] and the remarks therein, one can see that $(\int_0^t e^{\mu s + \lambda A_s} (\tilde{P}_s^{\alpha^*} - P_s^{\alpha^*})(\tilde{Q}_s^{\alpha^*} - Q_s^{\alpha^*}) dB_s)_{0 \leq t \leq T}$ is a uniformly integrable martingale. By Itô's formula applied with $e^{\mu t + \lambda A_t} |\tilde{P}_t^{\alpha^*} - P_t^{\alpha^*}|^2$, we have: $\forall t \leq T$,

$$\begin{aligned} e^{\mu t + \lambda A_t} |\tilde{P}_t^{\alpha^*} - P_t^{\alpha^*}|^2 &+ \int_t^T e^{\mu s + \lambda A_s} \|\tilde{Q}_s^{\alpha^*} - Q_s^{\alpha^*}\|^2 ds \\ &= 2 \int_t^T e^{\mu s + \lambda A_s} (\tilde{P}_s^{\alpha^*} - P_s^{\alpha^*}) [F_{\alpha^*}(s, \tilde{Q}_s^{\alpha^*}) - f_{\alpha^*}(s, \bar{u}_s, Q_s^{\alpha^*})] ds \\ &\quad - \mu \int_t^T e^{\mu s + \lambda A_s} |\tilde{P}_s^{\alpha^*} - P_s^{\alpha^*}|^2 ds + 2 \int_t^T e^{\mu s + \lambda A_s} (\tilde{P}_s^{\alpha^*} - P_s^{\alpha^*}) [\Psi_{\alpha^*}(s) - \psi_{\alpha^*}(s, u_s^{\alpha^*})] dA_s \\ &\quad - \lambda \int_t^T e^{\mu s + \lambda A_s} |\tilde{P}_s^{\alpha^*} - P_s^{\alpha^*}|^2 dA_s - 2 \int_t^T e^{\mu s + \lambda A_s} (\tilde{P}_s^{\alpha^*} - P_s^{\alpha^*})(\tilde{Q}_s^{\alpha^*} - Q_s^{\alpha^*}) dB_s. \end{aligned}$$

By taking the expectation and using the fact that $|x \vee y - y| \leq |x - y|$ for $x, y \in \mathbb{R}$, we obtain: $\forall t \leq T$,

$$\begin{aligned} \mathbb{E}[e^{\mu t + \lambda A_t} |\tilde{P}_t^{\alpha^*} - P_t^{\alpha^*}|^2] &+ \mathbb{E}[\int_t^T e^{\mu s + \lambda A_s} \|\tilde{Q}_s^{\alpha^*} - Q_s^{\alpha^*}\|^2 ds] \\ &\quad + \mu \mathbb{E}[\int_t^T e^{\mu s + \lambda A_s} |\tilde{P}_s^{\alpha^*} - P_s^{\alpha^*}|^2 ds] + \lambda \mathbb{E}[\int_t^T e^{\mu s + \lambda A_s} |\tilde{P}_s^{\alpha^*} - P_s^{\alpha^*}|^2 dA_s] \\ &\leq 2M \mathbb{E}[\int_t^T e^{\mu s + \lambda A_s} (\tilde{P}_s^{\alpha^*} - P_s^{\alpha^*}) \{ \|\bar{u} - \bar{v}\| + \|\tilde{Q}_s^{\alpha^*} - Q_s^{\alpha^*}\| \} ds] \\ &\quad + 2M \mathbb{E}[\int_t^T e^{\mu s + \lambda A_s} |\tilde{P}_s^{\alpha^*} - P_s^{\alpha^*}| \|\bar{u} - \bar{v}\| dA_s]. \end{aligned}$$

Next, using the inequality $2|ab| \leq \frac{1}{q}|a|^2 + q|b|^2$, $q > 0$, it follows that: $\forall t \leq T$,

$$\begin{aligned} \mathbb{E}[e^{\mu t + \lambda A_t} |\tilde{P}_t^{\alpha^*} - P_t^{\alpha^*}|^2] &+ \mathbb{E}[\int_t^T e^{\mu s + \lambda A_s} \|\tilde{Q}_s^{\alpha^*} - Q_s^{\alpha^*}\|^2 ds] \\ &\quad + \mu \mathbb{E}[\int_t^T e^{\mu s + \lambda A_s} |\tilde{P}_s^{\alpha^*} - P_s^{\alpha^*}|^2 ds] + \lambda \mathbb{E}[\int_t^T e^{\mu s + \lambda A_s} |\tilde{P}_s^{\alpha^*} - P_s^{\alpha^*}|^2 dA_s] \\ &\leq \mathbb{E}[\int_t^T e^{\mu s + \lambda A_s} |\tilde{P}_s^{\alpha^*} - P_s^{\alpha^*}|^2 ((qM^2 + 2M^2) ds + qM^2 dA_s)] \\ &\quad + \frac{1}{q} \mathbb{E}[\int_t^T e^{\mu s + \lambda A_s} \|\bar{u} - \bar{v}\|^2 (ds + dA_s)] + \frac{1}{2} \mathbb{E}[\int_t^T e^{\mu s + \lambda A_s} \|\tilde{Q}_s^{\alpha^*} - Q_s^{\alpha^*}\|^2 ds]. \end{aligned}$$

As $\mathbb{E}[e^{\mu t + \lambda A_t} |\tilde{P}_t^{\alpha^*} - P_t^{\alpha^*}|^2]$ and $\mathbb{E}[\int_t^T e^{\mu s + \lambda A_s} \|\tilde{Q}_s^{\alpha^*} - Q_s^{\alpha^*}\|^2 ds]$ are non-negative, and by choosing $\mu = 1 + qM^2 + 2M^2$ and $\lambda = 1 + qM^2$, where $q > 4$ we obtain: $\forall t \leq T$,

$$\begin{aligned} \mathbb{E}[\int_t^T e^{\mu s + \lambda A_s} |\tilde{P}_s^{\alpha^*} - P_s^{\alpha^*}|^2 ds + \int_t^T e^{\mu s + \lambda A_s} |\tilde{P}_s^{\alpha^*} - P_s^{\alpha^*}|^2 dA_s] \\ \leq \frac{1}{q} \mathbb{E}[\int_t^T e^{\mu s + \lambda A_s} \|\bar{u} - \bar{v}\|^2 ds + \int_t^T e^{\mu s + \lambda A_s} \|\bar{u} - \bar{v}\|^2 dA_s]. \end{aligned}$$

The same can be done with for the inequality with $\tilde{P}^{\alpha^*} - P^{\alpha^*}$. Hence, from (2.26), we obtain:

$\forall t \leq T$,

$$\begin{aligned} \mathbb{E} \left[\int_t^T e^{\mu s + \lambda A_s} |Y_s^{u,i} - Y_s^{v,i}|^2 ds + \int_t^T e^{\mu s + \lambda A_s} |Y_s^{u,i} - Y_s^{v,i}|^2 dA_s \right] \\ \leq \frac{4}{q} \mathbb{E} \left[\int_t^T e^{\mu s + \lambda A_s} \|\vec{u} - \vec{v}\|^2 ds + \int_t^T e^{\mu s + \lambda A_s} \|\vec{u} - \vec{v}\|^2 dA_s \right]. \end{aligned}$$

Then, we deduce that Φ is a strict contraction on $\mathcal{M}_{\mu, \lambda}^2(A)$ provided $\mu \geq 1 + qM^2 + 2M^2$ and $\lambda \geq 1 + qM^2$. Hence, it has a unique fixed point which gives the uniqueness of the solution for the system of generalized reflected BSDEs (2.2). \square

3 System of PDEs with Interconnected Obstacles and nonlinear Neumann Boundary Conditions

3.1 Preliminaries

Let D be an open bounded subset of \mathbb{R}^d , such that $D = \{\phi > 0\}$ and $\partial D = \{\phi = 0\}$. The function ϕ is in $C_b^2(\mathbb{R}^d)$ and satisfies $|\nabla \phi(x)| = 1$ whenever $x \in \partial D$, and $\nabla \phi(x)$ coincides with the unit normal pointing towards the interior of D . Then the interior sphere condition holds (see [18] and the references therein) *i.e.* there exists $r > 0$ such that for any $x \in \partial D$ and $y \in \bar{D}$ we have:

$$|y - x|^2 + r \langle \nabla \phi(x), y - x \rangle \geq 0. \quad (3.1)$$

Let (t, x) be in $[0, T] \times \bar{D}$ and $(X_s^{t,x}, A_s^{t,x})_{t \leq s \leq T}$ the solution of the reflected SDE below:

$$\begin{cases} dX_s^{t,x} = b(X_s^{t,x})ds + \sigma(X_s^{t,x})dB_s + \nabla \phi(X_s^{t,x})dA_s^{t,x}, & s \in [t, T]; \\ A_s^{t,x} = \int_t^s \mathcal{X}_{\{X_r^{t,x} \in \partial D\}} dA_r^{t,x}, & s \in [t, T]; \\ X_s^{t,x} = x, & s \leq t; \end{cases} \quad (3.2)$$

where $A_s^{t,x}$ is increasing, the functions $b: \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\sigma: \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ are Lipschitz. Note that $(X_s^{t,x}, A_s^{t,x})_{t \leq s \leq T}$ is valued in $\bar{D} \times \mathbb{R}_+$.

The following proposition is borrowed from Pardoux-Zhang [18]:

Proposition 3.1 *For each $t \in [0, T]$, there exists a constant C such that for all $x, x' \in \bar{D}$,*

$$\begin{aligned} \mathbb{E} \left[\sup_{t \leq s \leq T} |X_s^{t,x} - X_s^{t,x'}|^4 \right] &\leq C |x - x'|^4, \\ \mathbb{E} \left[\sup_{t \leq s \leq T} |A_s^{t,x} - A_s^{t,x'}|^4 \right] &\leq C |x - x'|^4. \end{aligned}$$

Moreover, for each $\mu > 0$, $s \in [t, T]$, there exists $C(\mu, s)$ such that for all $x \in \bar{D}$,

$$\mathbb{E}(e^{\mu A_s^{t,x}}) \leq C(\mu, s).$$

Suppose now that the data $((\xi_i)_{i \in J}, (f_i)_{i \in J}, (\psi_i)_{i \in J}, (g_{ij})_{(i,j) \in J^2})$ of the system of GRBSDEs (2.2) take the following form:

$$\begin{aligned} \xi_i(\omega) &= h_i(X_T^{t,x}(\omega)); \\ f_i(\omega, s, x, \vec{y}, z) &= f_i(s, X_r^{t,x}(\omega), \vec{y}, z); \\ \psi_i(\omega, s, x, y) &= \psi_i(s, X_s^{t,x}(\omega), y); \\ g_{ij}(\omega, s, x) &= g_{ij}(s, X_s^{t,x}(\omega)). \end{aligned} \quad (3.3)$$

Next, we introduce the following assumptions which are an adaptation of the ones introduced in the first section once one is in the Markovian framework:

Assumptions (A2):

Let $i \in \{1, \dots, m\}$ and $\mu > 0$,

(H₁) Let $f_i : [0, T] \times \bar{D} \times \mathbb{R}^m \times \mathbb{R}^{1 \times d} \rightarrow \mathbb{R}$ and $\psi_i : [0, T] \times \bar{D} \times \mathbb{R} \rightarrow \mathbb{R}$ be such that

- (i) $(t, x) \rightarrow f_i(t, x, \bar{y}, z)$ and $(t, x) \rightarrow \psi_i(t, x, y)$ are continuous uniformly in (\bar{y}, z) and y respectively;
- (ii) f_i and ψ_i are Lipschitz continuous w.r.t. (\bar{y}, z) and y respectively;
- (iii) $\exists \beta < 0$ such that $(y - y')(\psi_i(t, x, y) - \psi_i(t, x, y')) \leq \beta |y - y'|^2$;
- (iv) $f_i(t, x, \bar{y}, z)$ is non-decreasing w.r.t. y_j for $j \neq i$, whenever the other components are fixed.

(H₂) For any $j \in \{1, \dots, m\}$, we have:

- (i) $g_{ij} : [0, T] \times \bar{D} \rightarrow \mathbb{R}$ is continuous, non-negative such that $g_{ii} = 0$;
- (ii) g_{ij} satisfies the non-free loop property, i.e., $\forall (t, x) \in [0, T] \times \bar{D}$, $\forall i_1, \dots, i_k$ such that $i_1 \neq i_2, i_1 = i_k$ and $\text{card}\{i_1, \dots, i_k\} = k - 1$, we have:

$$g_{i_1 i_2}(t, x) + \dots + g_{i_{k-1} i_k}(t, x) > 0.$$

(H₃) $h_i : \bar{D} \rightarrow \mathbb{R}$, is continuous s.t. $h_i(x) \geq \max_{j \neq i} (h_j(x) - g_{ij}(T, x))$, $\forall x \in \bar{D}$.

These assumptions, in combination with the properties of the solution of equation (3.2) (see [17]), show that **Assumptions (A1)** are well satisfied. In this context, we consider the following Markovian GBSDEs system with oblique reflection: $\forall (t, x) \in [0, T] \times \bar{D}$, $\forall i \in \{1, \dots, m\}$,

$$\left\{ \begin{array}{l} Y_s^{i,t,x} = h_i(X_T^{t,x}) + \int_s^T f_i(r, X_r^{t,x}, \bar{Y}_r^{t,x}, Z_r^{i,t,x}) dr + \int_s^T \psi_i(r, X_r^{t,x}, Y_r^{i,t,x}) dA_r^{t,x} + K_T^{i,t,x} - K_s^{i,t,x} \\ \quad - \int_s^T Z_r^{i,t,x} dB_r, \\ Y_s^{i,t,x} \geq \max_{j \neq i} (Y_s^{j,t,x} - g_{ij}(s, X_s^{t,x})), \\ \int_0^T \{Y_s^{i,t,x} - \max_{j \neq i} (Y_s^{j,t,x} - g_{ij}(s, X_s^{t,x}))\} dK_s^{i,t,x} = 0. \end{array} \right. \quad (3.4)$$

Note that thanks to Theorem 2.1 and Theorem 2.2, the solution of this system exists and is unique. On the basis of the studies conducted on generalized reflected BSDEs [21], and several works on multi-switching problems, amongst them [11], the HJB system associated with our switching problem is the following: $\forall i = 1, \dots, m$,

$$\left\{ \begin{array}{l} \min \{ u^i(t, x) - \max_{j \neq i} (u^j(t, x) - g_{ij}(t, x)); -\partial_t u^i(t, x) - \mathcal{L} u^i(t, x) \\ \quad - f_i(t, x, (u^k(t, x))_{k=1, \dots, m}, (\sigma^\top D_x u^i)(t, x)) \} = 0, \quad (t, x) \in [0, T] \times D; \\ \frac{\partial u^i}{\partial l}(t, x) + \psi_i(t, x, u^i(t, x)) = 0, \quad (t, x) \in [0, T] \times \partial D; \\ u^i(T, x) = h_i(x), \quad x \in \bar{D}, \end{array} \right. \quad (3.5)$$

where the operator \mathcal{L} is defined by $\mathcal{L} = \frac{1}{2} \text{Tr}(\sigma \sigma^\top) D_{xx}^2 + b^\top D_x \cdot$, and at a point $x \in \partial D$, $\frac{\partial}{\partial l} = \langle \nabla \phi(x), D_x \cdot \rangle = \sum_{i=1}^d \frac{\partial \phi}{\partial x_i}(x) \frac{\partial}{\partial x_i}$.

Since we are interested in finding a function (u_1, \dots, u_m) which solves (3.5) in viscosity sense, we are led to recall some definitions introduced in [5].

Definition 3.1 For a locally bounded function $u : [0, T] \times \bar{D} \rightarrow \mathbb{R}$, we define its lower semicontinuous envelope u_* and its upper semicontinuous envelope u^* as follows:

$$u_*(t, x) = \liminf_{\substack{(t', x') \rightarrow (t, x) \\ t' < T}} u(t', x') \text{ and } u^*(t, x) = \limsup_{\substack{(t', x') \rightarrow (t, x) \\ t' < T}} u(t', x').$$

Definition 3.2 (Subjets and Superjets)

(i) For a lower semi continuous (lsc) (resp. upper semi continuous (usc)) function $u : [0, T] \times \bar{D} \rightarrow \mathbb{R}$, we define the parabolic subjet $J^{2-}u(t, x)$ (resp. superjet $J^{2+}u(t, x)$) of u at $(t, x) \in [0, T] \times \bar{D}$, the set of triples $(p, q, M) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$ s.t. for any $(t', x') \in (0, T) \times \bar{D}$, we have:

$$u(t', x') \geq u(t, x) + p(t' - t) + \langle q, x' - x \rangle + \frac{1}{2} \langle x' - x, M(x' - x) \rangle + o(|t' - t| + |x' - x|^2)$$

(resp.

$$u(t', x') \leq u(t, x) + p(t' - t) + \langle q, x' - x \rangle + \frac{1}{2} \langle x' - x, M(x' - x) \rangle + o(|t' - t| + |x' - x|^2)).$$

(ii) For $(t, x) \in [0, T] \times \bar{D}$, we define the parabolic limiting subjet $\bar{J}^{2-}u(t, x)$ (resp. the parabolic limiting superjet $\bar{J}^{2+}u(t, x)$) as follows:

$$\begin{aligned} \bar{J}^{2-}u(t, x) &= \{(p, q, M) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d : \exists (t_n, x_n, p_n, q_n, M_n) \in [0, T] \times \bar{D} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d \\ &\text{s.t. } (p_n, q_n, M_n) \in J^{2-}u(t_n, x_n) \text{ and } (t_n, x_n, p_n, q_n, M_n) \rightarrow (t, x, p, q, M) \text{ as } n \rightarrow \infty\}, \\ \text{(resp.} \end{aligned}$$

$$\begin{aligned} \bar{J}^{2+}u(t, x) &= \{(p, q, M) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d : \exists (t_n, x_n, p_n, q_n, M_n) \in [0, T] \times \bar{D} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d \\ &\text{s.t. } (p_n, q_n, M_n) \in J^{2+}u(t_n, x_n) \text{ and } (t_n, x_n, p_n, q_n, M_n) \rightarrow (t, x, p, q, M) \text{ as } n \rightarrow \infty\}, \end{aligned}$$

where \mathbb{S}^d is the set of symmetric real matrices of dimension d .

We are ready to give the definition of viscosity subsolution and supersolution that we will adopt throughout this section.

Definition 3.3 (Viscosity solution)

(i) A function $(u_1, \dots, u_m) : [0, T] \times \bar{D} \rightarrow \mathbb{R}^m$ such that for any $i \in J$, u_i is lsc (resp. usc), is called a viscosity supersolution (resp. subsolution) to (3.5), if for any $i \in J$, we have:

$$\left\{ \begin{array}{l} \min_{j \neq i} \{u_i(t, x) - \max_{j \neq i} (u_j(t, x) - g_{ij}(t, x)); \\ -p - b(x)^\top q - \frac{1}{2} \text{Tr}[\sigma \sigma^\top(x) M] - f_i(t, x, (u_k(t, x))_{k=1, \dots, m}, \sigma^\top(x) q)\} \geq 0, \\ (t, x) \in [0, T] \times D, (p, q, M) \in \bar{J}^{2-}u_i(t, x); \\ \min_{j \neq i} \{u_i(t, x) - \max_{j \neq i} (u_j(t, x) - g_{ij}(t, x)); -p - b(x)^\top q - \frac{1}{2} \text{Tr}[\sigma \sigma^\top(x) M] \\ -f_i(t, x, (u_k(t, x))_{k=1, \dots, m}, \sigma^\top(x) q)\} \vee \{-\langle \nabla \phi(x), q \rangle - \psi_i(t, x, u_i(t, x))\} \geq 0, \\ (t, x) \in [0, T] \times \partial D, (p, q, M) \in \bar{J}^{2-}u_i(t, x); \\ u_i(T, x) \geq h_i(x), \quad x \in \bar{D}. \end{array} \right. \quad (3.6)$$

(resp.

$$\left\{ \begin{array}{l} \min\{u_i(t,x) - \max_{j \neq i}(u_j(t,x) - g_{ij}(t,x)); \\ -p - b(x)^\top q - \frac{1}{2} \text{Tr}[\sigma \sigma^\top(x)M] - f_i(t,x, (u_k(t,x))_{k=1,\dots,m}, \sigma^\top(x)q)\} \leq 0, \\ (t,x) \in [0,T] \times D, (p,q,M) \in \mathcal{J}^{2+} u_i(t,x); \\ \min\{u_i(t,x) - \max_{j \neq i}(u_j(t,x) - g_{ij}(t,x)); -p - b(x)^\top q - \frac{1}{2} \text{Tr}[\sigma \sigma^\top(x)M] \\ - f_i(t,x, (u_k(t,x))_{k=1,\dots,m}, \sigma^\top(x)q)\} \wedge \{-\langle \nabla \phi(x), q \rangle - \psi_i(t,x, u_i(t,x))\} \leq 0, \\ (t,x) \in [0,T] \times \partial D, (p,q,M) \in \mathcal{J}^{2+} u_i(t,x); \\ u_i(T,x) \leq h_i(x), \quad x \in \bar{D}. \end{array} \right. \quad (3.7)$$

(ii) A locally bounded function $(u_1, \dots, u_m) : [0, T] \times \bar{D} \rightarrow \mathbb{R}$ is called a viscosity solution if (u_{1*}, \dots, u_{m*}) (resp. (u_1^*, \dots, u_m^*)) is a viscosity supersolution (resp. subsolution).

3.2 Uniqueness of the viscosity solution

In this paragraph, we deal with the issue of uniqueness of the solution for system (3.5). To do so, we establish the comparison property between subsolutions and supersolutions of system (3.5).

Theorem 3.1 Assume $(u_i)_{i \in J}$ and $(v^i)_{i \in J}$ are respectively an u.s.c subsolution and l.s.c. supersolution of system (3.5). Then we have: $\forall i \in J$

$$u_i \leq v^i, \text{ on } [0, T] \times \bar{D}.$$

Proof. The proof will be obtained by contradiction and is based on the classical maximum principle. The main idea is to assume the existence of (\bar{t}, \bar{x}) such that for some $i \in J$, we have $(u_i - v^i)(\bar{t}, \bar{x}) := \max_{(t,x) \in [0,T] \times \bar{D}} (u_i - v^i)(t,x) > 0$, this maximum exists since $u_i - v^i$ is usc and $[0, T] \times \bar{D}$ is compact. Then we use the doubling variable technique and Crandall-Ishii-Lions's Lemma in order to find elements of the subjet and the superjet of u_i and v^i respectively, for which the subsolution property and the supersolution property hold. The key ingredient to finding a contradiction is the specific monotonicity condition that will be assumed in the first step of our proof. In the second step, we generalize our result via a transformation of the system (3.5).

Step 1: We first assume that there exists a constant $\lambda < -m \cdot (\max_{j \in J} C_f^j)$, (C_f^j being the Lipschitz constant of f_j for $j \in J$) and verifying:

$\forall i \in J, \forall t, x, y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_m, y, \bar{y}, z$, if $y \geq \bar{y}$ then

$$f_i(t, x, y_1, \dots, y_{i-1}, y, y_{i+1}, \dots, y_m, z) - f_i(t, x, y_1, \dots, y_{i-1}, \bar{y}, y_{i+1}, \dots, y_m, z) \leq \lambda(y - \bar{y}). \quad (3.8)$$

Let $(u_i)_{i \in J}$ and $(v^i)_{i \in J}$ be respectively subsolution and supersolution to the system (3.5). The difficulty of this part is that our system of PDEs involves two inequalities. In order to avoid the boundary conditions, we produce two appropriate approximations $(u_i^\varepsilon)_{i \in J}$ and $(v_\varepsilon^i)_{i \in J}$ such that for each $i \in J$, $u_i^\varepsilon \rightarrow u_i$ and $v_\varepsilon^i \rightarrow v^i$ uniformly as $\varepsilon \rightarrow 0$. Next, we are going to show that they are respectively subsolution and supersolution for two PDEs systems that will be given explicitly. Finally, we show that $u_i^\varepsilon \leq v_\varepsilon^i$, which allows to conclude that $u_i \leq v^i$.

In order to construct these approximations, we apply Lemma 7.6 in [5] with $-\nabla \phi$, then there exists $\varphi \in C^2(\bar{D})$ that satisfies $\langle -\nabla \phi(x), D\varphi(x) \rangle \geq 1, \forall x \in \partial D$ and φ is positive on \bar{D} . Then put:

$$u_i^\varepsilon(t, x) = u_i(t, x) - \varepsilon \varphi(x) - C \quad \text{and} \quad v_\varepsilon^i(t, x) = v^i(t, x) + \frac{\varepsilon}{t} + \varepsilon \varphi(x) + C,$$

where, C is a positive constant that will be chosen later on.

Let us show that u_i^ε is a subsolution to a specific PDEs system. First, note that for any $x \in \bar{D}$, we have:

$$u_i^\varepsilon(T, x) \leq u_i(T, x) \leq h_i(x).$$

Now, let $(t, x) \in [0, T] \times \bar{D}$ and $(p^\varepsilon, q^\varepsilon, M^\varepsilon) \in \bar{J}^{2+} u_i^\varepsilon(t, x)$. If we set $p = p^\varepsilon$, $q = q^\varepsilon + \varepsilon D\varphi(x)$ and $M = M^\varepsilon + \varepsilon D^2\varphi(x)$, we can show easily that $(p, q, M) \in \bar{J}^{2+} u_i(t, x)$ and we have:

$$\begin{aligned} & \min\{u_i^\varepsilon(t, x) - \max_{j \neq i}(u_j^\varepsilon(t, x) - g_{ij}(t, x)); \\ & -p^\varepsilon - b(x)^\top q^\varepsilon - \frac{1}{2} \text{Tr}[\sigma \sigma^\top(x) M^\varepsilon] - f_i(t, x, \bar{u}^\varepsilon(t, x), \sigma^\top(x) q^\varepsilon)\} \\ & = \min\{u_i(t, x) - \max_{j \neq i}(u_j(t, x) - g_{ij}(t, x)); \\ & -p - b(x)^\top q - \frac{1}{2} \text{Tr}[\sigma \sigma^\top(x) M] - f_i(t, x, \bar{u}(t, x), \sigma^\top(x) q) + b(x)^\top (\varepsilon D\varphi(x)) \\ & + \frac{1}{2} \text{Tr}[\sigma \sigma^\top(x) (\varepsilon D^2\varphi(x))] + f_i(t, x, \bar{u}(t, x), \sigma^\top(x) q) - f_i(t, x, \bar{u}^\varepsilon(t, x), \sigma^\top(x) q^\varepsilon)\}. \end{aligned}$$

Since b, σ are continuous on \bar{D} and $\varphi \in C^2(\bar{D})$, there exists $\kappa_1 > 0$ satisfying:

$$b(x)^\top (\varepsilon D\varphi(x)) + \frac{1}{2} \text{Tr}[\sigma \sigma^\top(x) (\varepsilon D^2\varphi(x))] \leq \varepsilon \kappa_1.$$

Next with $f_i(t, x, \bar{u}(t, x), \sigma^\top(x) q) - f_i(t, x, \bar{u}^\varepsilon(t, x), \sigma^\top(x) q^\varepsilon)$, we use a linearization procedure and both assumptions $(H_1)(ii)$ and (3.8) to write:

$$\begin{aligned} & f_i(t, x, \bar{u}(t, x), \sigma^\top(x) q) - f_i(t, x, \bar{u}^\varepsilon(t, x), \sigma^\top(x) q^\varepsilon) \\ & \leq \max_{j \in J} C_f^j (m-1) (\varepsilon \varphi(x) + C) + \lambda (\varepsilon \varphi(x) + C) + \max_{j \in J} C_f^j | \sigma^\top(x) (\varepsilon D^2\varphi(x)) |, \\ & \leq (m \max_{j \in J} C_f^j + \lambda) (\varepsilon \varphi(x) + C) - \max_{j \in J} C_f^j (\varepsilon \varphi(x) + C) + \kappa_2 \varepsilon, \text{ for some } \kappa_2 > 0. \end{aligned}$$

Recall that $m \max_{j \in J} C_f^j + \lambda < 0$ and $\varphi \geq 0$, then we deduce that

$$\begin{aligned} & b(x)^\top (\varepsilon D\varphi(x)) + \frac{1}{2} \text{Tr}[\sigma \sigma^\top(x) (\varepsilon D^2\varphi(x))] + f_i(t, x, \bar{u}(t, x), \sigma^\top(x) q) \\ & - f_i(t, x, \bar{u}^\varepsilon(t, x), \sigma^\top(x) q^\varepsilon) \leq \varepsilon \kappa_3 - C \max_{j \in J} C_f^j, \end{aligned} \quad (3.9)$$

where κ_3 is positive. By choosing $C = \varepsilon \kappa_3 / \max_{j \in J} C_f^j$, we obtain that the right hand side of (3.9) is negative and then,

$$\begin{aligned} & \min\{u_i^\varepsilon(t, x) - \max_{j \neq i}(u_j^\varepsilon(t, x) - g_{ij}(t, x)); \\ & -p^\varepsilon - b(x)^\top q^\varepsilon - \frac{1}{2} \text{Tr}[\sigma \sigma^\top(x) M^\varepsilon] - f_i(t, x, \bar{u}^\varepsilon(t, x), \sigma^\top(x) q^\varepsilon)\} \\ & \leq \min\{u_i(t, x) - \max_{j \neq i}(u_j(t, x) - g_{ij}(t, x)); \\ & -p - b(x)^\top q - \frac{1}{2} \text{Tr}[\sigma \sigma^\top(x) M] - f_i(t, x, \bar{u}(t, x), \sigma^\top(x) q)\}. \end{aligned} \quad (3.10)$$

On the other hand, we have:

$$\begin{aligned} & -\langle \nabla \phi(x), q^\varepsilon \rangle - \psi_i(t, x, u_i^\varepsilon(t, x)) \\ & = -\langle \nabla \phi(x), q \rangle - \psi_i(t, x, u^i(t, x) - \varepsilon \varphi(x) - C) - \varepsilon \langle -\nabla \phi(x), D\varphi(x) \rangle. \end{aligned}$$

As ψ_i is non-increasing and using Lemma 3.1, we get for $x \in \partial D$:

$$-\langle \nabla \phi(x), q^\varepsilon \rangle - \psi_i(t, x, u_i^\varepsilon(t, x)) + \varepsilon \leq -\langle \nabla \phi(x), q \rangle - \psi_i(t, x, u_i(t, x)). \quad (3.11)$$

Moreover, u^i satisfies (3.7), we deduce from (3.10): $\forall x \in D$,

$$\begin{aligned} & \min\{u_i^\varepsilon(t, x) - \max_{j \neq i}(u_j^\varepsilon(t, x) - g_{ij}(t, x)); \\ & -p^\varepsilon - b(x)^\top q^\varepsilon - \frac{1}{2} \text{Tr}[\sigma \sigma^\top(x) M^\varepsilon] - f_i(t, x, \bar{u}^\varepsilon(t, x), \sigma^\top(x) q^\varepsilon)\} \leq 0, \end{aligned}$$

whereas, when $x \in \partial D$, it follows from (3.10) and (3.11) that:

$$\begin{aligned}
& \min\{u_i^\varepsilon(t, x) - \max_{j \neq i}(u_j^\varepsilon(t, x) - g_{ij}(t, x)); -p^\varepsilon - b(x)^\top q^\varepsilon - \frac{1}{2} \text{Tr}[\sigma \sigma^\top(x) M^\varepsilon] \\
& \quad - f_i(t, x, \bar{u}^\varepsilon(t, x), \sigma^\top(x) q^\varepsilon)\} \wedge \{-\langle \nabla \phi(x), q^\varepsilon \rangle - \psi_i(t, x, u_i^\varepsilon(t, x)) + \varepsilon\} \\
& \leq \min\{u_i(t, x) - \max_{j \neq i}(u_j(t, x) - g_{ij}(t, x)); -p - b(x)^\top q - \frac{1}{2} \text{Tr}[\sigma \sigma^\top(x) M] \\
& \quad - f_i(t, x, (u_k(t, x))_{k=1, \dots, m}, \sigma^\top(x) q)\} \wedge \{-\langle \nabla \phi(x), q \rangle - \psi_i(t, x, u_i(t, x))\} \\
& \leq 0.
\end{aligned}$$

We conclude that, $\forall i \in J$, u_i^ε satisfies the following system:

$$\left\{ \begin{array}{l} \min\{u_i^\varepsilon(t, x) - \max_{j \neq i}(u_j^\varepsilon(t, x) - g_{ij}(t, x)); \\ -p^\varepsilon - b(x)^\top q^\varepsilon - \frac{1}{2} \text{Tr}[\sigma \sigma^\top(x) M^\varepsilon] - f_i(t, x, \bar{u}^\varepsilon(t, x), \sigma^\top(x) q^\varepsilon)\} \leq 0, \\ (t, x) \in [0, T] \times D, (p^\varepsilon, q^\varepsilon, M^\varepsilon) \in \bar{J}^{2+} u_i^\varepsilon(t, x); \\ \min\{u_i^\varepsilon(t, x) - \max_{j \neq i}(u_j^\varepsilon(t, x) - g_{ij}(t, x)); -p^\varepsilon - b(x)^\top q^\varepsilon - \frac{1}{2} \text{Tr}[\sigma \sigma^\top(x) M^\varepsilon] \\ - f_i(t, x, \bar{u}^\varepsilon(t, x), \sigma^\top(x) q^\varepsilon)\} \wedge \{-\langle \nabla \phi(x), q^\varepsilon \rangle - \psi_i(t, x, u_i^\varepsilon(t, x)) + \varepsilon\} \leq 0, \\ (t, x) \in [0, T] \times \partial D, (p^\varepsilon, q^\varepsilon, M^\varepsilon) \in \bar{J}^{2+} u_i^\varepsilon(t, x); \\ u_i^\varepsilon(T, x) \leq h_i(x), \quad x \in \bar{D}. \end{array} \right. \quad (3.12)$$

Similarly, recall that the family $(v^i)_{i \in J}$ satisfies (3.6). Then, with a slight modification and taking into account the assumption (3.8) and the fact that the mapping $\psi_i(t, x, \cdot)$ is non-increasing, we show that $\forall i \in J$, v_ε^i satisfies the following system of inequalities:

$$\left\{ \begin{array}{l} \min\{v_\varepsilon^i(t, x) - \max_{j \neq i}(v_\varepsilon^j(t, x) - g_{ij}(t, x)); \\ -p_\varepsilon - b(x)^\top q_\varepsilon - \frac{1}{2} \text{Tr}[\sigma \sigma^\top(x) M_\varepsilon] - f_i(t, x, \bar{v}_\varepsilon(t, x), \sigma^\top(x) q_\varepsilon) - \frac{\varepsilon}{\bar{\rho}^2}\} \geq 0, \\ (t, x) \in [0, T] \times D, (p_\varepsilon, q_\varepsilon, M_\varepsilon) \in \bar{J}^{2-} v_\varepsilon^i(t, x); \\ \min\{v_\varepsilon^i(t, x) - \max_{j \neq i}(v_\varepsilon^j(t, x) - g_{ij}(t, x)); -p_\varepsilon - b(x)^\top q_\varepsilon - \frac{1}{2} \text{Tr}[\sigma \sigma^\top(x) M_\varepsilon] \\ - f_i(t, x, \bar{v}_\varepsilon(t, x), \sigma^\top(x) q_\varepsilon) - \frac{\varepsilon}{\bar{\rho}^2}\} \vee \{-\langle \nabla \phi(x), q_\varepsilon \rangle - \psi_i(t, x, v_\varepsilon^i(t, x)) - \varepsilon\} \geq 0, \\ (t, x) \in [0, T] \times \partial D, (p_\varepsilon, q_\varepsilon, M_\varepsilon) \in \bar{J}^{2-} v_\varepsilon^i(t, x); \\ v_\varepsilon^i(T, x) \geq h_i(x), \quad x \in \bar{D}. \end{array} \right. \quad (3.13)$$

Now, let us show by contradiction that $u_i^\varepsilon \leq v_\varepsilon^i$, $\forall i \in J$.

Assume that $\max_{[0, T] \times \bar{D}} \max_{i \in J} (u_i^\varepsilon - v_\varepsilon^i) > 0$, then there exists $(\bar{t}, \bar{x}) \in [0, T] \times \bar{D}$ such that

$$\max_{i \in J} (u_i^\varepsilon(\bar{t}, \bar{x}) - v_\varepsilon^i(\bar{t}, \bar{x})) > 0.$$

As in [11], there exists $k \in \tilde{J} = \{j \in J, u_j^\varepsilon(\bar{t}, \bar{x}) - v_\varepsilon^j(\bar{t}, \bar{x}) = \max_{k \in J} (u_k^\varepsilon(\bar{t}, \bar{x}) - v_\varepsilon^k(\bar{t}, \bar{x}))\}$ such that

$$u_k^\varepsilon(\bar{t}, \bar{x}) > \max_{j \neq k} (u_j^\varepsilon(\bar{t}, \bar{x}) - g_{kj}(\bar{t}, \bar{x})). \quad (3.14)$$

This is obtained mainly due to the non-free loop property.

Let us fix $j \in \tilde{J}$ that satisfies (3.14) and suppose first that $\bar{x} \in \partial D$. For (t, x, y) in $[0, T] \times \bar{D}^2$, we

define the following function: $\forall \alpha > 0$,

$$\Phi_\alpha^j(t, x, y) = u_j^\varepsilon(t, x) - v_\varepsilon^j(t, y) - \Psi_j^\alpha(t, x, y),$$

where $\Psi_j^\alpha(t, x, y) = \frac{\alpha}{2} |x - y|^2 - \psi_j(\bar{t}, \bar{x}, u_j^\varepsilon(\bar{t}, \bar{x})) \langle \nabla \phi(\bar{x}), x - y \rangle + |x - \bar{x}|^4 + |t - \bar{t}|^2$.

Let $(t_\alpha, x_\alpha, y_\alpha)$ be the maximum point of Φ_α^j over $[0, T] \times \bar{D}^2$, which exists due to the upper semi-continuity of $u_j^\varepsilon - v_\varepsilon^j$ and the compactness of \bar{D} , and let M_α be defined as follows:

$$M_\alpha = \max_{(t, x, y) \in [0, T] \times \bar{D} \times \bar{D}} \Phi_\alpha^j(t, x, y).$$

It is clear that this supremum is achieved only if t_α is in $(0, T)$.

Next, we can see that

$$M_\alpha \geq \max_{(t, x, y) \in [0, T] \times \bar{D}/x=y} u_j^\varepsilon(t, x) - v_\varepsilon^j(t, x) - |x - \bar{x}|^4 - |t - \bar{t}|^2 = u_j^\varepsilon(\bar{t}, \bar{x}) - v_\varepsilon^j(\bar{t}, \bar{x}).$$

Note that M_α is non-increasing w.r.t. α . On the other hand, we have $\alpha |x_\alpha - y_\alpha|^2 \xrightarrow{\alpha \rightarrow \infty} 0$ (see Lemma 3.1 [5]). It follows that $(x_\alpha - y_\alpha) \xrightarrow{\alpha \rightarrow \infty} 0$, then we get $(t_\alpha, x_\alpha, y_\alpha) \xrightarrow{\alpha \rightarrow \infty} (\hat{t}, \hat{x}, \hat{x})$, which implies that

$$u_j^\varepsilon(\bar{t}, \bar{x}) - v_\varepsilon^j(\bar{t}, \bar{x}) \leq u_j^\varepsilon(\hat{t}, \hat{x}) - v_\varepsilon^j(\hat{t}, \hat{x}).$$

As (\bar{t}, \bar{x}) is the maximum point of $u_j^\varepsilon - v_\varepsilon^j$, we necessarily have:

$$u_j^\varepsilon(\bar{t}, \bar{x}) - v_\varepsilon^j(\bar{t}, \bar{x}) = u_j^\varepsilon(\hat{t}, \hat{x}) - v_\varepsilon^j(\hat{t}, \hat{x}).$$

Thanks to the semi-continuity of the functions u_j^ε and v_ε^j , we have:

$$\begin{aligned} u_j^\varepsilon(\bar{t}, \bar{x}) - v_\varepsilon^j(\bar{t}, \bar{x}) &\leq \liminf_{\alpha \rightarrow \infty} (u_j^\varepsilon(t_\alpha, x_\alpha) - v_\varepsilon^j(t_\alpha, y_\alpha)) \\ &\leq \overline{\lim}_{\alpha \rightarrow \infty} (u_j^\varepsilon(t_\alpha, x_\alpha) - v_\varepsilon^j(t_\alpha, y_\alpha)) \\ &\leq u_j^\varepsilon(\bar{t}, \bar{x}) - v_\varepsilon^j(\bar{t}, \bar{x}). \end{aligned}$$

Then $\lim_{\alpha \rightarrow \infty} u_j^\varepsilon(t_\alpha, x_\alpha) - v_\varepsilon^j(t_\alpha, y_\alpha) = u_j^\varepsilon(\bar{t}, \bar{x}) - v_\varepsilon^j(\bar{t}, \bar{x})$, thus $(t_\alpha, x_\alpha) \xrightarrow{\alpha \rightarrow \infty} (\bar{t}, \bar{x})$.

We deduce from the results above that

$$\liminf_{\alpha \rightarrow \infty} u_j^\varepsilon(t_\alpha, x_\alpha) \geq u_j^\varepsilon(\bar{t}, \bar{x}) - v_\varepsilon^j(\bar{t}, \bar{x}) + \liminf_{\alpha \rightarrow \infty} v_\varepsilon^j(t_\alpha, y_\alpha) \geq u_j^\varepsilon(\bar{t}, \bar{x}) \geq \overline{\lim}_{\alpha \rightarrow \infty} u_j^\varepsilon(t_\alpha, x_\alpha).$$

It follows that $(u_j^\varepsilon(t_\alpha, x_\alpha), v_\varepsilon^j(t_\alpha, y_\alpha)) \xrightarrow{\alpha \rightarrow \infty} (u_j^\varepsilon(\bar{t}, \bar{x}), v_\varepsilon^j(\bar{t}, \bar{x}))$. Then as $(u_j^\varepsilon)_{j \in J}$ are usc and $(g_{ij})_{(i, j) \in J^2}$ are continuous, by (3.14) we get:

$$u_j^\varepsilon(t_\alpha, x_\alpha) > \max_{k \neq j} (u_k^\varepsilon(t_\alpha, x_\alpha) - g_{jk}(t_\alpha, x_\alpha)). \quad (3.15)$$

Now, going back to Ψ_j^α and note that

$$D_x \Psi_j^\alpha(t, x, y) = \alpha(x - y) - \psi_j(\bar{t}, \bar{x}, u_j^\varepsilon(\bar{t}, \bar{x})) \nabla \phi(\bar{x}) + 4|x - \bar{x}|^2(x - \bar{x}),$$

$$D_y \Psi_j^\alpha(t, x, y) = -\alpha(x - y) + \psi_j(\bar{t}, \bar{x}, u_j^\varepsilon(\bar{t}, \bar{x})) \nabla \phi(\bar{x}) \text{ and } \partial_t \Psi_j^\alpha(t, x, y) = 2(t - \bar{t}).$$

We recall the interior sphere condition (3.1) and we distinguish two cases. If $x_\alpha \in \partial D$, we have:

$$\begin{aligned} &-\langle \nabla \phi(x_\alpha), D_x \Psi_j^\alpha(t_\alpha, x_\alpha, y_\alpha) \rangle - \psi_j(t_\alpha, x_\alpha, u_j^\varepsilon(t_\alpha, x_\alpha)) \\ &= -\alpha \langle \nabla \phi(x_\alpha), x_\alpha - y_\alpha \rangle - 4|x_\alpha - \bar{x}|^2 \langle \nabla \phi(x_\alpha), x_\alpha - \bar{x} \rangle - \psi_j(t_\alpha, x_\alpha, u_j^\varepsilon(t_\alpha, x_\alpha)) \\ &\quad + \psi_j(\bar{t}, \bar{x}, u_j^\varepsilon(\bar{t}, \bar{x})) \langle \nabla \phi(\bar{x}), \nabla \phi(x_\alpha) \rangle, \\ &\geq -\frac{\alpha}{r} |x_\alpha - y_\alpha|^2 - \frac{4}{r} |x_\alpha - \bar{x}|^4 + \psi_j(\bar{t}, \bar{x}, u_j^\varepsilon(\bar{t}, \bar{x})) \langle \nabla \phi(\bar{x}), \nabla \phi(x_\alpha) \rangle \\ &\quad - \psi_j(t_\alpha, x_\alpha, u_j^\varepsilon(t_\alpha, x_\alpha)). \end{aligned}$$

In view of the convergences above, we know that the right hand side tends to 0 as $\alpha \rightarrow \infty$. Then for α large enough we deduce that

$$-\langle \nabla \phi(x_\alpha), D_x \Psi_j^\alpha(t_\alpha, x_\alpha, y_\alpha) \rangle - \psi_j(t_\alpha, x_\alpha, u_j^\varepsilon(t_\alpha, x_\alpha)) > -\varepsilon. \quad (3.16)$$

Similarly, if $y_\alpha \in \partial D$, we get:

$$\begin{aligned} & -\langle \nabla \phi(y_\alpha), -D_y \Psi_j^\alpha(t_\alpha, x_\alpha, y_\alpha) \rangle - \psi_j(t_\alpha, y_\alpha, v_\varepsilon^j(t_\alpha, y_\alpha)) \\ &= \alpha \langle \nabla \phi(y_\alpha), y_\alpha - x_\alpha \rangle + \psi_j(\bar{t}, \bar{x}, u_j^\varepsilon(\bar{t}, \bar{x})) \langle \nabla \phi(\bar{x}), \nabla \phi(y_\alpha) \rangle - \psi_j(t_\alpha, x_\alpha, v_\varepsilon^j(t_\alpha, y_\alpha)), \\ &\leq \alpha |x_\alpha - y_\alpha|^2 + \psi_j(\bar{t}, \bar{x}, u_j^\varepsilon(\bar{t}, \bar{x})) \langle \nabla \phi(\bar{x}), \nabla \phi(y_\alpha) \rangle - \psi_j(t_\alpha, x_\alpha, v_\varepsilon^j(t_\alpha, y_\alpha)), \\ &\leq \alpha |x_\alpha - y_\alpha|^2 + \psi_j(\bar{t}, \bar{x}, u_j^\varepsilon(\bar{t}, \bar{x})) \langle \nabla \phi(\bar{x}), \nabla \phi(y_\alpha) \rangle - \psi_j(\bar{t}, \bar{x}, u_j^\varepsilon(\bar{t}, \bar{x})) \langle \nabla \phi(\bar{x}), \nabla \phi(\bar{x}) \rangle \\ &\quad + \psi_j(\bar{t}, \bar{x}, v_\varepsilon^j(\bar{t}, \bar{x})) - \psi_j(t_\alpha, x_\alpha, v_\varepsilon^j(t_\alpha, y_\alpha)) + \psi_j(\bar{t}, \bar{x}, u_j^\varepsilon(\bar{t}, \bar{x})) - \psi_j(\bar{t}, \bar{x}, v_\varepsilon^j(\bar{t}, \bar{x})). \end{aligned}$$

We know that

$$\begin{aligned} & \alpha |x_\alpha - y_\alpha|^2 + \psi_j(\bar{t}, \bar{x}, u_j^\varepsilon(\bar{t}, \bar{x})) \langle \nabla \phi(\bar{x}), \nabla \phi(y_\alpha) \rangle \\ & - \psi_j(\bar{t}, \bar{x}, u_j^\varepsilon(\bar{t}, \bar{x})) \langle \nabla \phi(\bar{x}), \nabla \phi(\bar{x}) \rangle + \psi_j(\bar{t}, \bar{x}, v_\varepsilon^j(\bar{t}, \bar{x})) - \psi_j(t_\alpha, x_\alpha, v_\varepsilon^j(t_\alpha, y_\alpha)) \rightarrow 0, \end{aligned}$$

then it can be strictly dominated by ε for α large. In addition, since ψ_i is non-increasing, we have $\psi_j(\bar{t}, \bar{x}, u_j^\varepsilon(\bar{t}, \bar{x})) - \psi_j(\bar{t}, \bar{x}, v_\varepsilon^j(\bar{t}, \bar{x})) \leq 0$. It follows that:

$$-\langle \nabla \phi(y_\alpha), -D_y \Psi_j^\alpha(t_\alpha, x_\alpha, y_\alpha) \rangle - \psi_j(t_\alpha, y_\alpha, v_\varepsilon^j(t_\alpha, y_\alpha)) - \varepsilon < 0. \quad (3.17)$$

Remember that u_j^ε is a subsolution for system (3.12) and v_ε^j is a supersolution for (3.13). Then (3.16) and (3.17) lead to the following inequalities: $\forall (\bar{t}, \bar{x}) \in [0, T] \times \partial D, \forall \alpha$ large enough,

$$\begin{aligned} & -\partial_t \Psi_j^\alpha(t_\alpha, x_\alpha, y_\alpha) - b(x_\alpha)^\top D_x \Psi_j^\alpha(t_\alpha, x_\alpha, y_\alpha) - \frac{1}{2} \text{Tr}[\sigma \sigma^\top(x_\alpha) M^\varepsilon] \\ & \quad - f_i(t_\alpha, x_\alpha, \bar{u}^\varepsilon(t_\alpha, x_\alpha), \sigma^\top(x_\alpha) D_x \Psi_j^\alpha(t_\alpha, x_\alpha, y_\alpha)) \leq 0, \\ & \quad \text{for } (\partial_t \Psi_j^\alpha(t_\alpha, x_\alpha, y_\alpha), D_x \Psi_j^\alpha(t_\alpha, x_\alpha, y_\alpha), M^\varepsilon) \in \bar{J}^{2+} u_j^\varepsilon(t_\alpha, x_\alpha); \\ & -\partial_t \Psi_j^\alpha(t_\alpha, x_\alpha, y_\alpha) - b(y_\alpha)^\top (-D_y \Psi_j^\alpha(t_\alpha, x_\alpha, y_\alpha)) - \frac{1}{2} \text{Tr}[\sigma \sigma^\top(y_\alpha) M_\varepsilon] \\ & \quad - f_i(t_\alpha, y_\alpha, (v_\varepsilon^k(t_\alpha, y_\alpha))_{k=1, \dots, m}, \sigma^\top(y_\alpha) (-D_y \Psi_j^\alpha(t_\alpha, x_\alpha, y_\alpha))) \geq 0, \\ & \quad \text{for } (\partial_t \Psi_j^\alpha(t_\alpha, x_\alpha, y_\alpha), -D_y \Psi_j^\alpha(t_\alpha, x_\alpha, y_\alpha), M_\varepsilon) \in \bar{J}^{2-} v_\varepsilon^j(t_\alpha, y_\alpha). \end{aligned}$$

Before we proceed, let us emphasize on the fact that if $\bar{x} \in D$, the subsequence (x_α, y_α) is in $D \times D$ for α large enough, then the inequalities above hold true. Therefore, we can apply Crandall-Ishii-Lions's Lemma (Theorem 3.2 in [5]) with u_j^ε and v_ε^j on $(0, T) \times \bar{D}$ and $\varepsilon = \frac{1}{\alpha}$ to find $(p_\alpha^\mu, q_\alpha^\mu, M_\alpha^\mu) \in \bar{J}^{2+} u_j^\varepsilon(t_\alpha, x_\alpha)$ and $(p_\alpha^v, q_\alpha^v, M_\alpha^v) \in \bar{J}^{2-} v_\varepsilon^j(t_\alpha, x_\alpha)$ such that

$$\begin{aligned} p_\alpha^\mu - p_\alpha^v &= \partial_t \Psi_j^\alpha(t_\alpha, x_\alpha, y_\alpha) = 2(t_\alpha - \bar{t}), \\ q_\alpha^\mu - q_\alpha^v &= -(D_x \Psi_j^\alpha(t_\alpha, x_\alpha, y_\alpha) - D_y \Psi_j^\alpha(t_\alpha, x_\alpha, y_\alpha)) \end{aligned}$$

and

$$-(\alpha + \|A\|) \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} M_\alpha^\mu & 0 \\ 0 & M_\alpha^v \end{pmatrix} \leq A + \frac{1}{\alpha} A^2, \text{ where } A = D_{(x,y)}^2 \Psi_j^\alpha(t_\alpha, x_\alpha, y_\alpha).$$

By replacing the latter in the above inequations, we obtain:

$$\begin{aligned} -p_\alpha^\mu - b(x_\alpha)^\top q_\alpha^\mu - \frac{1}{2} \text{Tr}[\sigma \sigma^\top(x_\alpha) M_\alpha^\mu] - f_j(t_\alpha, x_\alpha, \bar{u}^\varepsilon(t_\alpha, x_\alpha), \sigma^\top(x_\alpha) q_\alpha^\mu) &\leq 0, \\ -p_\alpha^\nu - b(y_\alpha)^\top q_\alpha^\nu - \frac{1}{2} \text{Tr}[\sigma \sigma^\top(y_\alpha) M_\alpha^\nu] - f_j(t_\alpha, y_\alpha, \bar{v}_\varepsilon(t_\alpha, y_\alpha), \sigma^\top(y_\alpha) q_\alpha^\nu) &\geq 0. \end{aligned}$$

Combining these two inequalities, gives:

$$\begin{aligned} -(p_\alpha^\mu - p_\alpha^\nu) - (b(x_\alpha)^\top q_\alpha^\mu - b(y_\alpha)^\top q_\alpha^\nu) - \frac{1}{2} \text{Tr}[\sigma \sigma^\top(x_\alpha) M_\alpha^\mu - \sigma \sigma^\top(y_\alpha) M_\alpha^\nu] \\ - \{f_j(t_\alpha, x_\alpha, \bar{u}^\varepsilon(t_\alpha, x_\alpha), \sigma^\top(x_\alpha) q_\alpha^\mu) - f_j(t_\alpha, y_\alpha, \bar{v}_\varepsilon(t_\alpha, y_\alpha), \sigma^\top(y_\alpha) q_\alpha^\nu)\} \leq 0. \end{aligned}$$

Thanks to the uniform continuity of f_j , the Lipschitz assumption on b and σ and the above convergences, we can find some Σ_α such that $\lim_{\alpha \rightarrow \infty} \Sigma_\alpha \leq 0$ and

$$-\{f_j(t_\alpha, x_\alpha, \bar{u}^\varepsilon(t_\alpha, x_\alpha), \sigma^\top(x_\alpha) q_\alpha^\mu) - f_j(t_\alpha, x_\alpha, \bar{v}_\varepsilon(t_\alpha, y_\alpha), \sigma^\top(x_\alpha) q_\alpha^\mu)\} \leq \Sigma_\alpha.$$

Yet again, f_j is Lipschitz w.r.t. to \bar{y} , and as long as it verifies (3.8), we obtain:

$$-\lambda(u_j^\varepsilon(t_\alpha, x_\alpha) - v_\varepsilon^j(t_\alpha, y_\alpha)) - \sum_{k \neq j} \Theta_\alpha^{jk}(u_k^\varepsilon(t_\alpha, x_\alpha) - v_\varepsilon^k(t_\alpha, y_\alpha)) \leq \Sigma_\alpha,$$

where Θ_α^{jk} stands for the increment rate of f_j with respect to y_k for $k \neq j$. Observe that Θ_α^{jk} is nonnegative and bounded by C_f^j the Lipschitz constant, thanks to the monotonicity condition on f_j , then we have:

$$\begin{aligned} -\lambda(u_j^\varepsilon(t_\alpha, x_\alpha) - v_\varepsilon^j(t_\alpha, y_\alpha)) &\leq \sum_{k \neq j} \Theta_\alpha^{jk}(u_k^\varepsilon(t_\alpha, x_\alpha) - v_\varepsilon^k(t_\alpha, y_\alpha))^+ + \Sigma_\alpha, \\ &\leq C_f^j \sum_{k \neq j} (u_k^\varepsilon(t_\alpha, x_\alpha) - v_\varepsilon^k(t_\alpha, y_\alpha))^+ + \Sigma_\alpha. \end{aligned}$$

Taking the superior limit in both sides as $\alpha \rightarrow \infty$, with the semicontinuity of u_k^ε and v_ε^k , we obtain: $\forall j \in \bar{J}$

$$\begin{aligned} -\lambda(u_j^\varepsilon(\bar{t}, \bar{x}) - v_\varepsilon^j(\bar{t}, \bar{x})) &\leq C_f^j \sum_{k \neq j} (u_k^\varepsilon(\bar{t}, \bar{x}) - v_\varepsilon^k(\bar{t}, \bar{x}))^+, \\ &\leq (m-1)C_f^j(u_j^\varepsilon(\bar{t}, \bar{x}) - v_\varepsilon^j(\bar{t}, \bar{x})), \end{aligned}$$

which is contradictory since $u_j^\varepsilon(\bar{t}, \bar{x}) - v_\varepsilon^j(\bar{t}, \bar{x}) > 0$ and $-\lambda > mC_f^j$.

Thus $\forall i \in J$, $u_i^\varepsilon \leq v_\varepsilon^i$ on $[0, T] \times \bar{D}$. By taking the limit as $\varepsilon \rightarrow 0$, we obtain the desired result.

Step 2: Now, we deal with the general case, i.e. without assuming the assumption (3.8). Let λ be arbitrary in \mathbb{R} and consider the following system: $\forall i = 1, \dots, m$,

$$\left\{ \begin{array}{l} \min_{j \neq i} \{ \bar{u}^j(t, x) - \max_{j \neq i} (\bar{u}^j(t, x) - e^{\lambda t} g_{ij}(t, x)); -\partial_t \bar{u}^i(t, x) - \mathcal{L} \bar{u}^i(t, x) + \lambda \bar{u}^i(t, x) \\ \quad - e^{\lambda t} f_i(t, x, (e^{-\lambda t} \bar{u}^k(t, x))_{k=1, \dots, m}, e^{-\lambda t} \sigma^\top(x) D_x \bar{u}^i(t, x)) \} = 0, \quad (t, x) \in [0, T] \times D; \\ \frac{\partial \bar{u}^i}{\partial t} + e^{\lambda t} \psi_i(t, x, e^{-\lambda t} \bar{u}^i(t, x)) = 0, \quad (t, x) \in [0, T] \times \partial D; \\ \bar{u}^i(T, x) = e^{\lambda T} h_i(x), \quad x \in \bar{D}. \end{array} \right. \quad (3.18)$$

It turns out that $(\bar{u}_i(t, x) = e^{\lambda t} u_i(t, x))_{i \in J}$ and $(\bar{v}^i(t, x) = e^{\lambda t} v^i(t, x))_{i \in J}$ are respectively subsolution and supersolution to system (3.18), provided $(u_i)_{i \in J}$ a subsolution and $(v^i)_{i \in J}$ a supersolution to

system (3.5).

By choosing λ small enough, the functions F_i defined by: $\forall i \in J$

$$F_i(t, x, \vec{y}, z) = -\lambda y_i + f_i(t, x, e^{-\lambda t} \vec{y}, e^{-\lambda t} z),$$

satisfy assumption (3.8). As a result of the first step, we obtain $\tilde{u}_i \leq \tilde{v}^i$, then $u_i \leq v^i$, $\forall i \in J$. \square

The following corollary is an immediate conclusion of the comparison between the subsolution and supersolution of system (3.5):

Corollary 3.1 *If the solution of the system of PDEs (3.5) exists, it is unique and continuous.*

3.3 Existence of the viscosity solution

Now, it only remains to prove the existence of the solution for the PDEs system (3.5). To this end, we rely on the connection existing between the system of PDEs (3.5) and the system of generalized RBSDEs (2.2).

Corollary 3.2 *Let $(Y^{i,t,x}, K^{i,t,x}, Z^{i,t,x})_{i=1,\dots,m}$ be the unique solution of the Markovian system (3.4). Then, there exists a family of deterministic lower semicontinuous functions $(v^i(t,x))_{i \in J}$ such that*

$$\forall (t,x) \in [0, T] \times \bar{D}, \forall s \in [t, T], Y_s^{i,t,x} = v^i(s, X_s^{t,x}). \quad (3.19)$$

Proof. It has been shown in the previous section that

$$\underline{Y}^{t,x} \leq Y^{i,t,x,n} \leq Y^{i,t,x,n+1} \leq \bar{Y}^{t,x}, \forall n \geq 1, \quad (3.20)$$

where $(Y^{i,t,x,n})_{i \in J}$ is the solution of the system (2.5) in the Markovian case. Then, according to [21], there exists a family of deterministic continuous functions $(v^{i,n})_{i \in J}$ such that:

$$Y_s^{i,t,x,n} = v^{i,n}(s, X_s^{t,x}), \forall n \geq 1.$$

Once more, since the setting is Markovian, there exist \underline{v} and \bar{v} two continuous solutions of two PDEs associated with the GBSDEs (2.3) and (2.4) whose solutions are respectively $\underline{Y}^{t,x}$ and $\bar{Y}^{t,x}$ (see [18] for more details). Then from (3.20) we obtain:

$$\underline{v} \leq v^{i,n} \leq v^{i,n+1} \leq \bar{v}, \forall n \geq 1.$$

Thus $v^{i,n}$ converges increasingly to v^i and the continuity of $v^{i,n}$ ensures the lower semicontinuity of v^i . But $Y^{i,n}$ converges increasingly to $Y^{i,t,x}$ and $Y_s^{i,t,x} = v^i(s, X_s^{t,x}), \forall s \in [t, T]$. \square

Next, we show that $(v^i)_{i \in J}$ is solution of the PDEs system (3.5).

Theorem 3.2 *The function $(v^i)_{i \in J}$ is the unique continuous solution of the system of variational inequalities with inter-connected obstacles and non linear Neumann boundaries (3.5).*

Proof.

Part 1: Supersolution property.

First, note that $\forall i \in J$, v^i is lsc, i.e. $v^i = v^i_*$. By construction $v^i = \lim_{n \rightarrow \infty} v^{i,n}$, $i \in J$, where $v^{i,n}$ is a viscosity solution of the following PDE:

$$\left\{ \begin{array}{l} \min\{v^{i,n}(t,x) - \max_{j \neq i} (v^{j,n-1}(t,x) - g_{ij}(t,x)); -\partial_t v^{i,n}(t,x) - \mathcal{L} v^{i,n}(t,x) \\ -f_i(t,x, (v^{1,n-1}, \dots, v^{i-1,n-1}, v^{i,n}, v^{i+1,n-1}, \dots, v^{m,n-1})(t,x), \sigma^\top(x) D_x v^{i,n}(t,x))\} = 0, \\ (t,x) \in [0, T] \times D; \\ \frac{\partial v^{i,n}}{\partial t}(t,x) + \Psi_i(t,x, v^{i,n}(t,x)) = 0, (t,x) \in [0, T] \times \partial D; \\ v^{i,n}(T,x) = h_i(x), \quad x \in \bar{D}. \end{array} \right. \quad (3.21)$$

Let us fix $i \in J$, $(t, x) \in [0, T) \times \partial D$ and $(p, q, M) \in \bar{J}^{2-} v^i(t, x)$. By Lemma 6.1 in [5], there exist $n_j \rightarrow \infty$ and $x_j \in \bar{D}$ such that

$$(t_j, x_j, v^{i, n_j}(t_j, x_j), p_j, q_j, M_j) \xrightarrow{j \rightarrow \infty} (t, x, v^i(t, x), p, q, M).$$

Since x_j is in \bar{D} , we can extract a subsequence x_{j_l} which is either in ∂D or D while preserving the above convergence. We are going to show that in both cases, v^i solves (3.5).

If $x_{j_l} \in D$, we recall that $(t_{j_l}, x_{j_l}, v^{i, n_{j_l}}(t_{j_l}, x_{j_l}), p_{j_l}, q_{j_l}, M_{j_l}) \xrightarrow{l \rightarrow \infty} (t, x, v^i(t, x), p, q, M)$. Next from the viscosity supersolution property for $v^{i, n_{j_l}}$, we get:

$$\begin{aligned} & -p_{j_l} - b(x_{j_l})^\top q_{j_l} - \frac{1}{2} \text{Tr}[\sigma \sigma^\top(x_{j_l}) M_{j_l}] \\ & \geq f_i(t_{j_l}, x_{j_l}, (v^{1, n_{j_l}-1}, \dots, v^{i-1, n_{j_l}-1}, v^{i, n_{j_l}}, v^{i+1, n_{j_l}-1}, \dots, v^{m, n_{j_l}-1})(t_{j_l}, x_{j_l}), \sigma^\top(x_{j_l}) q_{j_l}). \end{aligned}$$

On the other hand, D is a bounded subset, and for any $i \in J$, $v^{i, n}$ is continuous, then there exists a subsequence $(l_k)_{k \geq 0}$ such that $(v^{i, n_{l_k}-1}(t_{l_k}, x_{l_k}))_{k \geq 0}$ is convergent.

As $(v^{i, n})_{n \geq 1}$ is increasing w.r.t. n and uniformly dominated, we get from ([1] page 91):

$$v^i(t, x) = v_*^i(t, x) = \lim_{\substack{(t', x') \rightarrow (t, x) \\ l \rightarrow \infty}} v^{i, l}(t', x').$$

It follows immediately that

$$\begin{aligned} & \min\{v^i(t, x) - \max_{j \neq i} (v^j(t, x) - g_{ij}(t, x)); \\ & -p - b(x)^\top q - \frac{1}{2} \text{Tr}[\sigma \sigma^\top(x) M] - f_i(t, x, (v^i(t, x))_{i=1, \dots, m}, \sigma^\top(x) q)\} \geq 0. \end{aligned}$$

Otherwise, $(x_{j_l})_{l \geq 0} \in \partial D$ and $(t_{j_l}, x_{j_l}, v^{i, n_{j_l}}(t_{j_l}, x_{j_l}), p_{j_l}, q_{j_l}, M_{j_l}) \xrightarrow{l \rightarrow \infty} (t, x, v^i(t, x), p, q, M)$. As $v^{i, n_{j_l}}$ is a viscosity supersolution of system (3.21), by considering the boundary condition, we get:

$$\begin{aligned} & \{-p_{j_l} - b(x_{j_l})^\top q_{j_l} - \frac{1}{2} \text{Tr}[\sigma \sigma^\top(x_{j_l}) M_{j_l}] \\ & - f_i(t_{j_l}, x_{j_l}, (v^{1, n_{j_l}-1}, \dots, v^{i-1, n_{j_l}-1}, v^{i, n_{j_l}}, v^{i+1, n_{j_l}-1}, \dots, v^{m, n_{j_l}-1})(t_{j_l}, x_{j_l}), \sigma^\top(x_{j_l}) q_{j_l})\} \\ & \vee \{-\langle \nabla \phi(x_{j_l}), q_{j_l} \rangle - \psi_i(t_{j_l}, x_{j_l}, v^{i, n_{j_l}}(t_{j_l}, x_{j_l}))\} \geq 0. \end{aligned}$$

Suppose now that,

$$\begin{aligned} & -p_{j_l} - b(x_{j_l})^\top q_{j_l} - \frac{1}{2} \text{Tr}[\sigma \sigma^\top(x_{j_l}) M_{j_l}] \\ & \geq f_i(t_{j_l}, x_{j_l}, (v^{1, n_{j_l}-1}, \dots, v^{i-1, n_{j_l}-1}, v^{i, n_{j_l}}, v^{i+1, n_{j_l}-1}, \dots, v^{m, n_{j_l}-1})(t_{j_l}, x_{j_l}), \sigma^\top(x_{j_l}) q_{j_l}). \end{aligned}$$

Then as previously, we obtain:

$$-p - b(x)^\top q - \frac{1}{2} \text{Tr}[\sigma \sigma^\top(x) M] - f_i(t, x, (v^i(t, x))_{i=1, \dots, m}, \sigma^\top(x) q) \geq 0.$$

If not, we would have

$$-\langle \nabla \phi(x_{j_l}), q_{j_l} \rangle - \psi_i(t_{j_l}, x_{j_l}, v^{i, n_{j_l}}(t_{j_l}, x_{j_l})) \geq 0.$$

By taking the limit, we get

$$-\langle \nabla \phi(x), q \rangle - \psi_i(t, x, v^i(t, x)) \geq 0.$$

Therefore,

$$\min\{v^i(t,x) - \max_{j \neq i}(v^j(t,x) - g_{ij}(t,x)); -p - b(x)^\top q - \frac{1}{2} \text{Tr}[\sigma \sigma^\top(x)M] \\ - f_i(t,x, (v^i(t,x))_{i=1,\dots,m}, \sigma^\top(x)q)\} \vee \{-\langle \nabla \phi(x), q \rangle - \psi_i(t,x, v^i(t,x))\} \geq 0.$$

Finally, for the terminal value, we know that $\forall x \in \bar{D}$, $v^{i,n}(T,x) = h_i(x)$, then we simply take the limit as $n \rightarrow \infty$. Thus, v^i is a viscosity supersolution for the following PDE

$$\begin{cases} \min\{v^i(t,x) - \max_{j \neq i}(v^j(t,x) - g_{ij}(t,x)); -\partial_t v^i(t,x) - \mathcal{L}v^i(t,x) \\ - f_i(t,x, (v^1, \dots, v^m)(t,x), \sigma^\top(x)D_x v^i(t,x))\} = 0, \quad (t,x) \in [0,T) \times D; \\ \frac{\partial v^i}{\partial l}(t,x) + \psi_i(t,x, v^i(t,x)) = 0, \quad (t,x) \in [0,T) \times \partial D; \\ v^i(T,x) = h_i(x), \quad x \in \bar{D}. \end{cases}$$

Since i is arbitrary in J we deduce that the m -tuple (v^1, \dots, v^m) is a viscosity supersolution of the system (3.5).

Part 2: Subsolution property.

We will now show that $(v^{i*})_{i \in J}$ is a subsolution for system (3.5), namely, we should check if the following inequalities are satisfied:

$$\begin{cases} \min\{v^{i*}(t,x) - \max_{j \neq i}(v^{j*}(t,x) - g_{ij}(t,x)); -p - b(x)^\top q - \frac{1}{2} \text{Tr}[\sigma \sigma^\top(x)M] \\ - f_i(t,x, (v^{i*}(t,x))_{i=1,\dots,m}, \sigma^\top(x)q)\} \leq 0, \quad (t,x) \in [0,T) \times D, \quad (p,q,M) \in \bar{J}^{2+} v^{i*}(t,x); \\ \min\{v^{i*}(t,x) - \max_{j \neq i}(v^{j*}(t,x) - g_{ij}(t,x)); -p - b(x)^\top q - \frac{1}{2} \text{Tr}[\sigma \sigma^\top(x)M] \\ - f_i(t,x, (v^{i*}(t,x))_{i=1,\dots,m}, \sigma^\top(x)q)\} \wedge \{-\langle \nabla \phi(x), q \rangle - \psi_i(t,x, v^{i*}(t,x))\} \leq 0, \\ (t,x) \in [0,T) \times \partial D, \quad (p,q,M) \in \bar{J}^{2+} v^{i*}(t,x); \\ v^{i*}(T,x) = h_i(x), \quad x \in \bar{D}. \end{cases}$$

Step 1: To begin with, we need to show that $v^{i*}(T,x) = h_i(x)$, $\forall i \in J$, $\forall x \in \bar{D}$. For this purpose we are going to show that

$$\min\{v^{i*}(T,x) - h_i(x); v^{i*}(T,x) - \max_{j \neq i}(v^{j*}(T,x) - g_{ij}(T,x))\} = 0, \quad \forall i \in J, \quad \forall x \in \bar{D}.$$

Let $i \in J$, $n \geq 1$ and $x \in \bar{D}$. We know that

$$v^{i*}(T,x) = \overline{\lim}_{\substack{(t',x') \rightarrow (T,x) \\ t' < T, x' \in \bar{D}}} v^i(t',x') \geq \overline{\lim}_{\substack{(t',x') \rightarrow (T,x) \\ t' < T, x' \in \bar{D}}} v^{i,n}(t',x'),$$

then as $v^{i,n}$ is continuous, we get $v^{i*}(T,x) \geq v^{i,n}(T,x) = h_i(x)$.

Besides

$$v^i(t,x) \geq \max_{j \neq i}(v^j(t,x) - g_{ij}(t,x)),$$

after passing to the limit, we get

$$v^{i*}(T,x) \geq \max_{j \neq i}(v^{j*}(T,x) - g_{ij}(T,x)),$$

then we obtain

$$\min\{v^{i*}(T,x) - h_i(x); v^{i*}(T,x) - \max_{j \neq i}(v^{j*}(T,x) - g_{ij}(T,x))\} \geq 0.$$

We now show that the left hand side cannot be strictly positive. To do so, we suppose to the contrary, that for some $x_0 \in \bar{D}$, there is $\varepsilon > 0$ such that

$$\min\{v^{i*}(T, x_0) - h_i(x_0); v^{i*}(T, x_0) - \max_{j \neq i}(v^{j*}(T, x_0) - g_{ij}(T, x_0))\} = 2\varepsilon.$$

Let $(t_k, x_k)_{k \geq 1}$ be a sequence in $[0, T] \times \bar{D}$ satisfying $(t_k, x_k) \xrightarrow[k \rightarrow \infty]{} (T, x_0)$ and $v^i(t_k, x_k) \xrightarrow[k \rightarrow \infty]{} v^{i*}(T, x_0)$, which exists thanks to Lemma 6.1 in [5]. As v^{i*} is usc on $[0, T] \times \bar{D}$ and $v^{i,n}$ converges increasingly to v^i , we can find $(Q^n)_{n \geq 0}$ a sequence of functions in $C^{1,2}([0, T] \times \bar{D})$ such that $Q^n \xrightarrow[n \rightarrow \infty]{} v^{i*}$. In addition, we suppose that on some neighborhood B_n of (T, x_0) , we have:

$$\min\{Q^n(t, x) - h_i(x); Q^n(t, x) - \max_{j \neq i}(v^{j*}(t, x) - g_{ij}(t, x))\} \geq \varepsilon, \forall (t, x) \in B_n. \quad (3.22)$$

We can assume that (3.22) holds on $B_k^n = [t_k, T] \times B(x_k, \delta_k^n) \cap D$, for some $\delta_k^n \in (0, 1)$ such that $B_k^n \subset B_n$. In case $x_k \in \partial D$, this means that we work only on open neighborhoods of D near ∂D . Since v^{i*} is usc on a bounded space, there exists $c > 0$ such that $|v^{i*}| \leq c$ on B_n . Then, we can assume that $Q^n \geq -2c$. Next define

$$V_k^n(t, x) = Q^n(t, x) + \frac{4c|x - x_k|^2}{\delta_k^{n2}} + \sqrt{T - t}.$$

Note that $V_k^n(t, x) \geq Q^n(t, x)$ and $(v^{i*} - V_k^n)(t, x) \leq -c, \forall (t, x) \in [t_k, T] \times \partial B(x_k, \delta_k^n)$.

On the other hand we have,

$$\begin{aligned} -\{\partial_t V_k^n(t, x) + \mathcal{L}V_k^n(t, x)\} &= -\{\partial_t Q^n(t, x) + \partial_t \sqrt{T - t} + D_x b(x)\{D_x Q^n(t, x) + \frac{8c(x - x_k)}{(\delta_k^{n2})^2}\} \\ &\quad + \frac{1}{2}\sigma \sigma^\top(x)\{D_{xx} Q^n(t, x) + \frac{8c}{(\delta_k^{n2})^2}\}\}. \end{aligned}$$

It is clear that $\frac{4c|x - x_k|^2}{\delta_k^{n2}} \in C_b^2$ and $Q^n \in C_b^{1,2}$, which implies that the derivatives are bounded. Then, as $\partial_t(\sqrt{T - t}) \xrightarrow[t \rightarrow T]{} -\infty$, we can choose k large enough so that:

$$-\{\partial_t V_k^n(t, x) + \mathcal{L}V_k^n(t, x)\} \geq 0, \forall (t, x) \in B_k^n.$$

Recall the SDE (3.2) and consider the following stopping times:

$$\begin{aligned} \theta_n^k &= \inf\{s \geq t_k; (s, X_s^{t_k, x_k}) \in (B_k^n)^c\} \wedge T, \\ \theta_k &= \inf\{s \geq t_k; v^i(s, X_s^{t_k, x_k}) = \max_{j \neq i}(v^j(s, X_s^{t_k, x_k}) - g_{ij}(s, X_s^{t_k, x_k}))\} \wedge T. \end{aligned}$$

Note that on $[t_k, \theta_n^k \wedge \theta_k]$, $\nabla \phi(X_s^{t, x}) dA_s^{t, x}$ vanishes in the SDE (3.2) since the support of $A^{t, x}$ is ∂D . Then, by Itô's formula, we obtain:

$$\begin{aligned} V_k^n(t_k, x_k) &= V_k^n(\theta_n^k \wedge \theta_k, X_{\theta_n^k \wedge \theta_k}^{t_k, x_k}) - \int_{t_k}^{\theta_n^k \wedge \theta_k} \{\partial_t V_k^n(r, X_r^{t_k, x_k}) + \mathcal{L}V_k^n(r, X_r^{t_k, x_k})\} dr \\ &\quad - \int_{t_k}^{\theta_n^k \wedge \theta_k} \sigma(X_r^{t_k, x_k}) D_x V_k^n(r, X_r^{t_k, x_k}) dB_r. \end{aligned}$$

Taking the expectation, we get

$$\begin{aligned}
V_k^n(t_k, x_k) &= \mathbb{E}[V_k^n(\theta_n^k \wedge \theta_k, X_{\theta_n^k \wedge \theta_k}^{t_k, x_k}) - \int_{t_k}^{\theta_n^k \wedge \theta_k} \{\partial_t V_k^n(r, X_r^{t_k, x_k}) + \mathcal{L}V_k^n(r, X_r^{t_k, x_k})\} dr] \\
&\geq \mathbb{E}[V_k^n(\theta_n^k, X_{\theta_n^k}^{t_k, x_k}) \mathcal{X}_{[\theta_n^k \leq \theta_k]} + V_k^n(\theta_k, X_{\theta_k}^{t_k, x_k}) \mathcal{X}_{[\theta_n^k > \theta_k]}] \\
&= \mathbb{E}[(V_k^n(\theta_n^k, X_{\theta_n^k}^{t_k, x_k}) \mathcal{X}_{[\theta_n^k < T]} + V_k^n(T, X_T^{t_k, x_k}) \mathcal{X}_{[\theta_n^k = T]}) \mathcal{X}_{[\theta_n^k \leq \theta_k]} + V_k^n(\theta_k, X_{\theta_k}^{t_k, x_k}) \mathcal{X}_{[\theta_n^k > \theta_k]}] \\
&\geq \mathbb{E}[\{(v^{i*}(\theta_n^k, X_{\theta_n^k}^{t_k, x_k}) + c) \mathcal{X}_{[\theta_n^k < T]} + (\varepsilon + h_i(X_T^{t_k, x_k})) \mathcal{X}_{[\theta_n^k = T]}\} \mathcal{X}_{[\theta_n^k \leq \theta_k]} \\
&\quad + (\varepsilon + \max_{j \neq i}(v^{j*}(\theta_k, X_{\theta_k}^{t_k, x_k}) - g_{ij}(\theta_k, X_{\theta_k}^{t_k, x_k}))) \mathcal{X}_{[\theta_n^k > \theta_k]}] \\
&\geq \mathbb{E}[\{(v^i(\theta_n^k, X_{\theta_n^k}^{t_k, x_k}) + c) \mathcal{X}_{[\theta_n^k < T]} + (\varepsilon + h_i(X_T^{t_k, x_k})) \mathcal{X}_{[\theta_n^k = T]}\} \mathcal{X}_{[\theta_n^k \leq \theta_k]} \\
&\quad + (\varepsilon + \max_{j \neq i}(v^j(\theta_k, X_{\theta_k}^{t_k, x_k}) - g_{ij}(\theta_k, X_{\theta_k}^{t_k, x_k}))) \mathcal{X}_{[\theta_n^k > \theta_k]}] \\
&\geq \mathbb{E}[v^i(\theta_n^k \wedge \theta_k, X_{\theta_n^k \wedge \theta_k}^{t_k, x_k})] + c \wedge \varepsilon.
\end{aligned}$$

Besides, recall that the process $Y^i = v^i(\cdot, X_\cdot)$ stopped at time $\theta_n^k \wedge \theta_k$ solves an explicit GRBSDE, given by:

$$\begin{aligned}
v^i(\theta_n^k \wedge \theta_k, X_{\theta_n^k \wedge \theta_k}^{t_k, x_k}) &= v^i(t_k, x_k) - \int_{t_k}^{\theta_n^k \wedge \theta_k} f_i(r, X_r^{t_k, x_k}, (v^i(r, X_r^{t_k, x_k}))_{i \in J}, Z_r^{i, t_k, x_k}) dr \\
&\quad - \int_{t_k}^{\theta_n^k \wedge \theta_k} \psi_i(r, X_r^{t_k, x_k}, v^i(r, X_r^{t_k, x_k})) dA_r^{t, x} - (K_{\theta_n^k \wedge \theta_k}^{i, t, x} - K_{t_k}^{i, t, x}) + \int_{t_k}^{\theta_n^k \wedge \theta_k} Z_r^{i, t_k, x_k} dB_r.
\end{aligned}$$

Once again, $dA^{t_k, x_k} = 0$ on $[t_k, \theta_k^n]$, also $dK^{i, t_k, x_k} = 0$ on $[t_k, \theta_k]$, then by taking the expectation we get:

$$\mathbb{E}\left(v^i(\theta_n^k \wedge \theta_k, X_{\theta_n^k \wedge \theta_k}^{t_k, x_k})\right) = \mathbb{E}\left(v^i(t_k, x_k) - \int_{t_k}^{\theta_n^k \wedge \theta_k} f_i(r, X_r^{t_k, x_k}, (v^i(r, X_r^{t_k, x_k}))_{i \in J}, Z_r^{i, t_k, x_k}) dr\right).$$

Then, as \bar{D} is bounded and using the properties of the solution $(X_t^{t_k, x_k})_{t \leq T}$, the estimates (2.12) and the properties of f_i we can show that:

$$\lim_{k \rightarrow \infty} \mathbb{E}\left(\int_{t_k}^{\theta_n^k \wedge \theta_k} f_i(r, X_r^{t_k, x_k}, v^i(r, X_r^{t_k, x_k}))_{i \in J}, Z_r^{i, t_k, x_k}) dr\right) = 0.$$

Hence, $\lim_{k \rightarrow \infty} V_k^n(t_k, x_k) \geq \lim_{k \rightarrow \infty} V_k^n(t_k, x_k) = v^i(T, x_0) + c \wedge \varepsilon$ where $c \wedge \varepsilon > 0$, however from the definition of $V_k^n(t, x)$ we have $\lim_{k \rightarrow \infty} V_k^n(t_k, x_k) = Q^n(T, x_0)$, which is contradictory since $Q^n \xrightarrow[k \rightarrow \infty]{} v^{i*}$. It follows that for any $x \in \bar{D}$

$$\min\{v^{i*}(T, x) - h_i(x); v^{i*}(T, x) - \max_{j \neq i}(v^{j*}(T, x) - g_{ij}(T, x))\} = 0.$$

Finally, we can use the non-free loop property of g_{ij} following the same method as in [11] to obtain the desired result.

Step 2: Let us show that $(v^{i*})_{i \in J}$ is a subsolution. First, we point out that $(v^{i, n})_{i \in J}$ are continuous and $v^i = \lim_{n \rightarrow \infty} v^{i, n}$, then by ([1] page 91)

$$v^{i*}(t, x) = \overline{\lim}_{n \rightarrow \infty} v^{i, n}(t, x) = \overline{\lim}_{(t', x') \rightarrow (t, x)} v^{i, n}(t', x'), \quad \forall i \in J, \forall (t, x) \in [0, T] \times \bar{D}.$$

From the construction of $v^{i,n}$, we have for any $i \in J$ and $n \geq 1$:

$$v^{i,n}(t,x) \geq \max_{j \neq i} (v^{j,n-1}(t,x) - g_{ij}(t,x)), \forall (t,x) \in [0,T] \times \bar{D};$$

taking the limit as $n \rightarrow \infty$, we get for any $i \in J$ and $(t,x) \in [0,T] \times \bar{D}$,

$$v^{i*}(t,x) \geq \max_{j \neq i} (v^{j*}(t,x) - g_{ij}(t,x)).$$

Let $i \in J$ and $(t,x) \in [0,T] \times \partial D$ be such that $v^{i*}(t,x) - \max_{j \neq i} (v^{j*}(t,x) - g_{ij}(t,x)) > 0$, and fix $(p,q,M) \in \bar{J}^{2+} v^{i*}(t,x)$. By Lemma 6.1 in [5], there exist $n_k \rightarrow \infty$, $x_k \in \bar{D}$ and $(p_k, q_k, M_k) \in \bar{J}^{2+} v^{i,n_k}(t_k, x_k)$ such that

$$(t_k, x_k, v^{i,n_k}(t_k, x_k), p_k, q_k, M_k) \xrightarrow[k \rightarrow \infty]{} (t, x, v^{i*}(t, x), p, q, M).$$

In the spirit of the proof of the first part, since x_k is in \bar{D} , we can extract a subsequence $(x_{k_l})_{l \geq 1}$ which is either in ∂D or D . It suffices to check out both possibilities. If $x_{k_l} \in D$, we recall that $(t_{k_l}, x_{k_l}, v^{i,n_{k_l}}(t_{k_l}, x_{k_l}), p_{k_l}, q_{k_l}, M_{k_l}) \xrightarrow[l \rightarrow \infty]{} (t, x, v^{i*}(t, x), p, q, M)$. Next, we obtain from the viscosity subsolution property for $v^{i,n_{k_l}}$:

$$\begin{aligned} & \min\{v^{i,n_{k_l}}(t_{k_l}, x_{k_l}) - \max_{j \neq i} (v^{j,n_{k_l}-1}(t_{k_l}, x_{k_l}) - g_{ij}(t_{k_l}, x_{k_l})); -p_{k_l} - b(x_{k_l})^\top q_{k_l} \\ & - \frac{1}{2} \text{Tr}[\sigma \sigma^\top(x_{k_l}) M_{k_l}] - f_i(t_{k_l}, x_{k_l}, (v^{1,n_{k_l}-1}, \dots, v^{i,n_{k_l}}, \dots, v^{m,n_{k_l}-1})(t_{k_l}, x_{k_l}), \sigma^\top(x_{k_l}) q_{k_l})\} \leq 0. \end{aligned}$$

We know that $v^{i*}(t,x) \geq \overline{\lim}_{l \rightarrow \infty} v^{i,n_{k_l}}(t_{k_l}, x_{k_l})$, then there exists $l_0 > 0$ s.t. $\forall l \geq l_0$, we have:

$$\begin{aligned} v^{i,n_{k_l}}(t_{k_l}, x_{k_l}) & \geq \max_{j \neq i} (v^{j,n_{k_l}}(t_{k_l}, x_{k_l}) - g_{ij}(t_{k_l}, x_{k_l})) \\ & \geq \max_{j \neq i} (v^{j,n_{k_l}-1}(t_{k_l}, x_{k_l}) - g_{ij}(t_{k_l}, x_{k_l})), \end{aligned}$$

since $v^{i*}(t,x) \geq \max_{j \neq i} (v^{j*}(t,x) - g_{ij}(t,x))$. It follows that for $l \geq l_0$,

$$\begin{aligned} & -p - b(x)^\top q - \frac{1}{2} \text{Tr}[\sigma \sigma^\top(x) M] \\ & \leq \overline{\lim}_{l \rightarrow \infty} f_i(t_{k_l}, x_{k_l}, (v^{1,n_{k_l}-1}, \dots, v^{i-1,n_{k_l}-1}, v^{i,n_{k_l}}, v^{i+1,n_{k_l}-1}, \dots, v^{m,n_{k_l}-1})(t_{k_l}, x_{k_l}), \sigma^\top(x_{k_l}) q_{k_l}). \end{aligned}$$

In the same manner as in Part 1, we show that for any $(t,x) \in [0,T] \times \partial D$ and $(p,q,M) \in \bar{J}^{2+} v^{i*}(t,x)$, we have:

$$\begin{aligned} & \min\{v^{i*}(t,x) - \max_{j \neq i} (v^{j*}(t,x) - g_{ij}(t,x)); \\ & -p - b(x)^\top q - \frac{1}{2} \text{Tr}[\sigma \sigma^\top(x) M] - f_i(t,x, (v^{i*}(t,x))_{i=1,\dots,m}, \sigma^\top(x) q)\} \leq 0. \end{aligned}$$

If not we can extract a subsequence x_{k_l} of x_k in ∂D , such that

$$(t_{k_l}, x_{k_l}, v^{i,n_{k_l}}(t_{k_l}, x_{k_l}), p_{k_l}, q_{k_l}, M_{k_l}) \xrightarrow[l \rightarrow \infty]{} (t, x, v^{i*}(t, x), p, q, M).$$

Then, there exists $l_0 > 0$ s.t. $\forall l \geq l_0$ and from the viscosity subsolution property of $v^{i,n_{k_l}}$ at $(t_{k_l}, x_{k_l}) \in [0,T] \times \partial D$, we have:

$$\begin{aligned} & \{-p_{k_l} - b(x_{k_l})^\top q_{k_l} - \frac{1}{2} \text{Tr}[\sigma \sigma^\top(x_{k_l}) M_{k_l}] \\ & - f_i(t_{k_l}, x_{k_l}, (v^{1,n_{k_l}-1}, \dots, v^{i-1,n_{k_l}-1}, v^{i,n_{k_l}}, v^{i+1,n_{k_l}-1}, \dots, v^{m,n_{k_l}-1})(t_{k_l}, x_{k_l}), \sigma^\top(x_{k_l}) q_{k_l})\} \\ & \wedge \{-\langle \nabla \phi(x_{k_l}), q_{k_l} \rangle - \psi_i(t_{k_l}, x_{k_l}, v^{i,n_{k_l}}(t_{k_l}, x_{k_l}))\} \leq 0. \end{aligned}$$

If

$$-p_{k_l} - b(x_{k_l})^\top q_{k_l} - \frac{1}{2} \text{Tr}[\sigma \sigma^\top(x_{k_l}) M_{k_l}] \\ \leq f_i(t_{k_l}, x_{k_l}, (v^{1, n_{k_l}-1}, \dots, v^{i-1, n_{k_l}-1}, v^{i, n_{k_l}}, v^{i+1, n_{k_l}-1}, \dots, v^{m, n_{k_l}-1})(t_{k_l}, x_{k_l}), \sigma^\top(x_{k_l}) q_{k_l})$$

similar computations, yield

$$-p - b(x)^\top q - \frac{1}{2} \text{Tr}[\sigma \sigma^\top(x) M] - f_i(t, x, (v^{i*}(t, x))_{i=1, \dots, m}, \sigma^\top(x) q) \leq 0.$$

Otherwise, the other inequality holds

$$-\langle \nabla \phi(x_{k_l}), q_{k_l} \rangle - \psi_i(t_{k_l}, x_{k_l}, v^{i, n_{k_l}}(t_{k_l}, x_{k_l})) \leq 0,$$

by taking the limit as $l \rightarrow \infty$, we get

$$-\langle \nabla \phi(x), q \rangle - \psi_i(t, x, v^{i*}(t, x)) \leq 0.$$

Before we finish the proof, we shall stress out that the inequalities in D are not difficult and can be handled likewise. As a consequence, v^i is a viscosity subsolution for the following PDE:

$$\begin{cases} \min\{v^i(t, x) - \max_{j \neq i} (v^j(t, x) - g_{ij}(t, x)); -\partial_t v^i(t, x) - \mathcal{L}v^i(t, x) \\ \quad - f_i(t, x, (v^1, \dots, v^m)(t, x), \sigma^\top(x) D_x v^i(t, x))\} = 0, (t, x) \in [0, T) \times D; \\ \frac{\partial v^i}{\partial l}(t, x) + \psi_i(t, x, v^i(t, x)) = 0, (t, x) \in [0, T) \times \partial D; \\ v^i(T, x) = h_i(x), x \in \bar{D}. \end{cases} \quad (3.23)$$

The fact that i is arbitrary in J , shows that the m -tuple (v^1, \dots, v^m) is the unique viscosity solution for the system (3.5). \square

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