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# Viscosity Solutions of system of PDEs with Interconnected Obstacles and nonlinear Neumann Boundary Conditions

Brahim BOUFOUSSI \*      Saïd HAMADENE †      Manal JAKANI ‡

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## Abstract

This paper investigates the Hamilton-Jacobi-Bellman system of equations associated with the  $m$ -states optimal switching problem in finite horizon when the state process lives in a connected bounded closed domain. We show existence and uniqueness of the solution in viscosity sense of the system. We use systems of generalized reflected backward stochastic differential equations with oblique reflection and the Feynman-Kac representation of their solutions in the Markovian framework.

**Keywords:** Generalized Reflected Backward stochastic differential equations, Viscosity solution of PDEs, Variational inequalities, Nonlinear Neumann Boundary conditions, Optimal switching.

## 1 Introduction

The purpose of the present paper is to provide existence and uniqueness of continuous viscosity solution to the following system of  $m$ -variational inequalities with interconnected obstacles and nonlinear Neumann boundary conditions:  $\forall i = 1, \dots, m, \forall t \in [0, T)$ ,

$$\left\{ \begin{array}{l} \min\{u^i(t, x) - \max_{j \neq i}(u^j(t, x) - g_{ij}(t, x)); -\partial_t u^i(t, x) - \mathcal{L}u^i(t, x) \\ \quad - f_i(t, x, (u^k(t, x))_{k=1, \dots, m}, \sigma^\top(x) D_x u^i(t, x))\} = 0, \quad x \in D; \\ \frac{\partial u^i}{\partial l}(t, x) + \psi_i(t, x, u^i(t, x)) = 0, \quad x \in \partial D; \\ u^i(T, x) = h_i(x), \quad x \in \bar{D}, \end{array} \right. \quad (1.1)$$

where the operator  $\mathcal{L}$  is given by

$$\mathcal{L} = \frac{1}{2} Tr\{(\sigma \sigma^\top) D_{xx}^2\} + b^\top D_x.$$

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and at a point  $x \in \partial D$ ,

$$\frac{\partial}{\partial l} = \langle \nabla \phi(x), D_x \cdot \rangle = \sum_{i=1}^d \frac{\partial \phi}{\partial x_i}(x) \frac{\partial}{\partial x_i}.$$

This system of equations is of Hamilton-Jacobi-Bellman (HJB for short) type associated with the optimal switching problem when the state process  $X$ , which is a diffusion with generator  $\mathcal{L}$ , is constrained to live in a bounded connected domain  $D := \{\phi < 0\}$ .

Optimal switching models often arise in the analysis of industrial projects related to investment in electricity and valuation of energy storage (see e.g. [3, 4, 5, 7, 11, 13, 14] etc). In a standard optimal switching problem, a decision maker controls a dynamical system over time by choosing successively its working modes from a discrete set. Therefore a switching strategy is given by  $\alpha := (\tau_n, \theta_n)_{n \geq 0}$  where  $\tau_n$  are stopping times such that  $\tau_n \leq \tau_{n+1}$  and  $\theta_n$  is a random variable with values in  $J := \{1, \dots, m\}$ . At time  $\tau_n$ , the decision maker switches the system from its current mode  $\theta_{n-1}$  to another one  $\theta_n$ . When the system is in mode  $i$  at time  $s$  and works a short period of time  $ds$ , it provides a profit  $f_i(s, \omega)ds$ . On the other hand, switching the system from mode  $i$  to mode  $j \neq i$  at  $s$  costs  $g_{ij}(s, \omega)$ . Therefore the problem of the decision maker is to look for a strategy  $\alpha^*$  which maximizes the performance  $\Gamma(\alpha)$  which is equal to the yield provided by the system when it runs under the strategy  $\alpha$ . This problem turns into the solvability of a system of backward stochastic differential equations with oblique reflection (see e.g. [15, 16, 17, 18] etc) of the following form:  $\forall i = 1, \dots, m, \forall t \leq T$ ,

$$\begin{cases} Y_t^i = \xi^i + \int_t^T f_i(s)ds + K_T^i - K_t^i - \int_t^T Z_s^i dB_s; \\ Y_t^i \geq \max_{j \neq i} (Y_t^j - g_{ij}(t)); \\ \int_0^T \{Y_t^i - \max_{j \neq i} (Y_t^j - g_{ij}(t))\} dK_t^i = 0. \end{cases}$$

Then, both optimal performance and optimal strategy are provided by  $\bar{Y} = (Y_1, \dots, Y_m)$ .

In the aforementioned works, when randomness stems from a standard diffusion process  $X^{t,x} ((t, x) \in [0, T] \times \mathbb{R}^k)$  solution of the following standard SDE:

$$dX_s^{t,x} = b(s, X_s^{t,x})ds + \sigma(s, X_s^{t,x})dB_s, s \in [t, T] \text{ and } X_t^{t,x} = x \quad (1.2)$$

(i.e.  $f_i(s, \omega) = f_i(s, X_s^{t,x}(\omega))$ ,  $g_{ij}(s, \omega) = g_{ij}(s, X_s^{t,x}(\omega))$  and the bequest  $\xi^i(\omega) = h_i(X_T^{t,x}(\omega))$  for any  $i, j$ ) the Hamilton-Jacobi-Bellman system of equations associated with the switching problem takes the following form:  $\forall i = 1, \dots, m$ ,

$$\begin{cases} \min\{u^i(t, x) - \max_{j \neq i} (u^j(t, x) - g_{ij}(t, x)); -\partial_t u^i(t, x) - \mathcal{L}u^i(t, x) - f_i(t, x)\} = 0; \\ u^i(T, x) = h_i(x). \end{cases} \quad (1.3)$$

The solution  $(u^i)_{i=1, \dots, m}$  is nothing but the value functions of the switching problem (see e.g. [7, 14]). The process  $X^{t,x}$  of (1.2) is unconstrained and roughly speaking lives in  $\mathbb{R}^k$ . However in real life, there are several situations where  $X^{t,x}$  is constrained to stay in a given bounded domain  $D$ . Let us give two examples.

The first one is related to the interest rates in economies. For instance, assume that the process  $(X_t)_{t \leq T}$  stands for the evolution of the interest rates in several economies. The central bank of each economy acts in such a way to keep the interest rate in some bounded domain for the well being of the economies. Now assume there is an investor who holds a capital which he/she invests in one of the economies and switches it, from one economy to

another, according to the return of the investment. This return depends also on the interest rates in these economies. Therefore the investor will make his investment strategy also according to  $X^{t,x}$  which is constrained to stay in a bounded domain.

Another example is related to electricity production. Indeed, consider a hydro-power station with a dam and several working modes. Let us denote by  $X_t$  the level of water in the dam at time  $t$ , which is obviously a stochastic process. For safety reasons the water level should not exceed a specific level  $\ell_1$ . On the other hand, for operational reasons this level should not be below another level  $\ell_0$ . Thus at each time  $t$  one should have  $X_t \in [\ell_0, \ell_1]$  by evacuating water when  $X_t$  is prone to overlap  $\ell_1$  and stopping production when  $X_t$  reaches  $\ell_0$ . As a consequence, the working mode of the station is chosen according to parameters which include the level of water  $X$  in the dam which is a constrained stochastic process.

To deal with switching problems which look like the previous ones is the main motivation of this article.

So assume that  $X^{t,x}$ , instead of satisfying (1.2), satisfies the following equation:

$$\begin{cases} dX_s^{t,x} = b(X_s^{t,x})ds + \sigma(X_s^{t,x})dB_s + \nabla\phi(X_s^{t,x})dA_s^{t,x}, & s \in [t, T]; \\ X_s^{t,x} \in \bar{D} \text{ and } A_s^{t,x} = \int_t^s \chi_{\{X_r^{t,x} \in \partial D\}} dA_r^{t,x}, & s \in [t, T]; \\ X_t^{t,x} = x, & \text{for } s \leq t; \end{cases} \quad (1.4)$$

$A^{t,x}$  is an increasing process and a part of the solution. The gradient  $\nabla\phi(x)$  coincides with the unit normal pointing towards the interior of  $D := \{\phi < 0\}$ . The assumptions on  $b$ ,  $\sigma$  and  $\phi$  will be specified later. The term  $\nabla\phi(X_s^{t,x})dA_s^{t,x}$  represents the damage faced once the state process  $X^{t,x}$  hits the boundary of  $D$ , otherwise, it is null as long as  $X^{t,x} \in D$ .

In the case when randomness comes from this constrained Markov process  $X^{t,x}$  solution of (1.4), as previously highlighted, the HJB system associated with the switching problem is the system of PDEs with Neumann boundary conditions (1.1).

To deal with the switching problem when  $X^{t,x}$  is a solution of (1.4), we are led to study the following system of reflected generalized BSDEs with oblique reflection:

$$\begin{cases} \forall i = 1, \dots, m, \forall s \in [t, T], \\ \begin{cases} Y_s^{i,t,x} = h_i(X_T^{t,x}) + \int_s^T f_i(r, X_r^{t,x}, \bar{Y}_r^{t,x}, \bar{Z}_r^{i,t,x})dr + \int_s^T \psi_i(r, X_r^{t,x}, Y_r^{i,t,x})dA_r^{t,x} + K_T^{i,t,x} - K_s^{i,t,x} \\ \quad - \int_s^T Z_r^{i,t,x}dB_r; \\ Y_s^{i,t,x} \geq \max_{j \neq i} (Y_s^{j,t,x} - g_{ij}(s, X_s^{t,x})); \\ \int_0^T \{Y_s^{i,t,x} - \max_{j \neq i} (Y_s^{j,t,x} - g_{ij}(s, X_s^{t,x}))\} dK_s^{i,t,x} = 0. \end{cases} \end{cases} \quad (1.5)$$

In the above system, the quantity  $\psi_i(r, X_r^{t,x}, Y_r^{i,t,x})dA_r^{t,x}$  stands for the payoff generated by the system when the constraint  $X^{t,x} \in D$  is not satisfied.

The main contribution of our work, is to provide existence and uniqueness of continuous viscosity solution  $(u^i)_{i \in J}$  for system (1.1). As a by-product, we obtain that for any  $i \in J$  and  $s \in [t, T]$ ,  $u^i(s, X_s^{t,x})$  is the optimal payoff when at time  $s$  the system is in working mode  $i$ .

The paper is organized as follows. In Section 2, we consider the following system of reflected generalized BSDEs which is more general than the one in (1.5) since we do not assume any particularity of randomness:  $\forall i = 1, \dots, m, \forall t \leq T$ ,

$$\begin{cases} Y_t^i = \xi^i + \int_t^T f_i(s, \bar{Y}_s, \bar{Z}_s^i)ds + \int_t^T \psi_i(s, Y_s^i)dA_s + K_T^i - K_t^i - \int_t^T Z_s^i dB_s; \\ Y_t^i \geq \max_{j \neq i} (Y_t^j - g_{ij}(t)); \\ \int_0^T \{Y_t^i - \max_{j \neq i} (Y_t^j - g_{ij}(t))\} dK_t^i = 0. \end{cases} \quad (1.6)$$

We first show the existence of a solution  $(Y^i, Z^i, K^i)_{i \in J}$  to system (1.6) using a scheme obtained by Picard iterations. Then, we establish the link between the components  $Y^i$ ,  $i \in J$ , of the solution and the value functions of the switching problem. This link will allow us to show uniqueness of the solution of system of RGBSDEs (1.6). The last section is devoted to study the system of PDEs (1.1). We define the notion of a viscosity solution for that system, then provide a comparison result between its sub-solution and super-solution. Finally, we show that there is a solution  $(u^i)_{i=1, \dots, m}$  of system (1.1), this solution is given by the Feynman Kac representation, that relates the PDEs system (1.1) to the generalized RBSDEs system (1.6) considered in the second section, namely for any  $i \in J$ ,  $Y_s^{i,t,x} = u^i(s, X_s^{t,x})$ ,  $\forall s \in [t, T]$ .

## 2 System of Generalized Reflected BSDEs

### 2.1 Assumptions and notations

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a fixed probability space on which is defined a standard Brownian motion  $B = (B_t)_{0 \leq t \leq T}$  for a fixed finite horizon  $T > 0$ , and  $\mathcal{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$  be the completed filtration of  $(\sigma(B_s, 0 \leq s \leq t))_{t \leq T}$  with all  $\mathbb{P}$ -null sets of  $\mathcal{F}_0$ . Let  $(A_t)_{t \geq 0}$  be a continuous one-dimensional increasing  $\mathcal{F}_t$ -progressively measurable process such that  $A_0 = 0$ . Let us introduce the following spaces:

$$\mathcal{H}^2 = \{(\psi_t)_{0 \leq t \leq T} \text{ } \mathcal{F}_t\text{-progressively measurable process such that } \mathbb{E}[\int_0^T |\psi_t|^2 dt] < \infty\};$$

$$\mathcal{S}^2 = \{(\psi_t)_{0 \leq t \leq T} \text{ } \mathcal{F}_t\text{-progressively measurable process s.t. } \mathbb{E}[\sup_{0 \leq t \leq T} |\psi_t|^2] < \infty\};$$

$$\mathcal{A}^2 = \{(K_t)_{0 \leq t \leq T} \text{ } \mathcal{F}_t\text{-adapted continuous increasing process s.t. } K_0 = 0, \mathbb{E}[K_T^2] < \infty\}.$$

In this section, we consider the problem of existence and uniqueness of a solution for system (2.1). For this objective, let us introduce the following assumptions:

#### Assumptions (A1):

Let  $m$  be a positive integer,  $i \in J$  and  $\mu > 0$ ;

(H<sub>1</sub>)  $\xi^i$  is a random variable in  $L^2(\Omega, \mathcal{F}_T, \mathbb{P}, \mathbb{R})$ , satisfying  $\mathbb{E}(e^{\mu A_T} |\xi^i|^2) < \infty$ .

(H<sub>2</sub>)  $f_i : [0, T] \times \Omega \times \mathbb{R}^m \times \mathbb{R}^{1 \times d} \longrightarrow \mathbb{R}$  and  $\psi_i : [0, T] \times \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$  s.t.

(i)  $f_i(\cdot, \cdot, \vec{y}, z)$  and  $\psi_i(\cdot, \cdot, y)$  are progressively measurable.

(ii)  $\mathbb{E} \int_0^T e^{\mu A_t} |f_i(t, \vec{0}, 0)|^2 dt < +\infty$  and  $\mathbb{E} \int_0^T e^{\mu A_t} |\psi_i(t, 0)|^2 dA_t < +\infty$ .

(iii)  $f_i$  and  $\psi_i$  are Lipschitz continuous with respect to  $(\vec{y}, z)$  and  $y$  respectively.

(iv)  $\exists \beta < 0$  such that  $(y - y')(\psi_i(t, y) - \psi_i(t, y')) \leq \beta |y - y'|^2$ .

(v) For any  $j \neq i$ ,  $f_i(t, \vec{y}, z)$  is non-decreasing in  $y_j$ , whenever the other components are fixed.

(H<sub>3</sub>) For any  $i, j \in J$ , we have:

i)  $g_{ij} : [0, T] \times \Omega \longrightarrow \mathbb{R}$  is continuous w.r.t.  $t$ , non negative such that  $g_{ii} = 0$ .

ii)  $\xi^i \geq \max_{j \neq i} (\xi^j - g_{ij}(T))$ .

iii)  $g_{ij}$  satisfies the non free loop property, i.e.,  $\forall i_1, \dots, i_k$  such that  $i_1 \neq i_2, i_1 = i_k$  and  $\text{card}\{i_1, \dots, i_k\} = k - 1$ , we have:

$$g_{i_1 i_2} + \dots + g_{i_{k-1} i_k} > 0.$$

The assumptions  $(H_1)$ ,  $(H_2)$  (i) – (iv) are frequently encountered in the study of GBSDEs [20, 22] then carried on in the reflected version [25, 26]. The difficulties usually faced in multidimensional Reflected BSDEs with interconnected obstacles, include the dependence on the whole vector  $\vec{y}$  in the generators  $f_i$ . This can be overcome with the use of the monotonicity condition (A1)  $(H_2)$  (v) first introduced in [16]. Also, hypotheses  $(H_3)$  were considered in [7, 15, 16, 17] etc. Those assumptions on the costs are actually reasonable. In fact, the first point is necessary so that additional costs will not be charged for staying in the same mode. (iii) reminds that if we go from a mode  $i$ , then making successive switches all the way along back to the same mode is not free. Finally, the assumption on the terminal values shows that there is no interest in making switches at time  $T$ .

## 2.2 Existence and uniqueness of the solution

The current section outlines two fundamental results of our work. It focuses on existence and uniqueness of a solution for system (2.1).

**Theorem 2.1** *Assume that assumptions (A1) are fulfilled. Then the system of generalized reflected BSDEs (2.1) below has a solution:  $\forall i = 1, \dots, m, \forall t \leq T$*

$$\begin{cases} Y^i \in \mathcal{S}^2, Z^i \in \mathcal{H}^{2,d}, K^i \in \mathcal{A}^2; \\ Y_t^i = \xi^i + \int_t^T f_i(s, \vec{Y}_s, Z_s^i) ds + \int_t^T \psi_i(s, Y_s^i) dA_s + K_T^i - K_t^i - \int_t^T Z_s^i dB_s; \\ Y_t^i \geq \max_{j \neq i} (Y_t^j - g_{ij}(t)); \\ \int_0^T \{Y_t^i - \max_{j \neq i} (Y_t^j - g_{ij}(t))\} dK_t^i = 0. \end{cases} \quad (2.1)$$

**Proof:** The proof extends loosely in the same direction as in [16] and requires various steps.

**Step 1 : Iterative scheme.**

Let us set

$$\begin{aligned} \bar{\xi} &= \max_{i=1, \dots, m} |\xi^i|, & \bar{f}(s, y, z) &= \max_{i=1, \dots, m} f_i(s, y, \dots, y, z), & \bar{\psi}(s, y) &= \max_{i=1, \dots, m} \psi_i(s, y); \\ \underline{\xi} &= \min_{i=1, \dots, m} |\xi^i|, & \underline{f}(s, y, z) &= \min_{i=1, \dots, m} f_i(s, y, \dots, y, z), & \underline{\psi}(s, y) &= \min_{i=1, \dots, m} \psi_i(s, y). \end{aligned}$$

Consider the following standard GBSDEs:  $\forall t \leq T$ ,

$$\begin{cases} \bar{Y} \in \mathcal{S}^2, \bar{Z} \in \mathcal{H}^{2,d}; \\ \bar{Y}_t = \bar{\xi} + \int_t^T \bar{f}(s, \bar{Y}_s, \bar{Z}_s) ds + \int_t^T \bar{\psi}(s, \bar{Y}_s) dA_s - \int_t^T \bar{Z}_s dB_s, \end{cases} \quad (2.2)$$

and

$$\begin{cases} \underline{Y} \in \mathcal{S}^2, \underline{Z} \in \mathcal{H}^{2,d}; \\ \underline{Y}_t = \underline{\xi} + \int_t^T \underline{f}(s, \underline{Y}_s, \underline{Z}_s) ds + \int_t^T \underline{\psi}(s, \underline{Y}_s) dA_s - \int_t^T \underline{Z}_s dB_s. \end{cases} \quad (2.3)$$

By **Theorem 1.6** [22], the generalized BSDEs (2.2) and (2.3) have unique solutions. Next, for  $i = 1, \dots, m$  and  $n \geq 1$ , we set  $Y^{i,0} = \underline{Y}$  and we define recursively  $Y^{i,n}$  via the following GRBSDEs whose solutions exist and are unique thanks to [26]:

$$\begin{cases} Y^{i,n} \in \mathcal{S}^2, Z^{i,n} \in \mathcal{H}^{2,d}, K^{i,n} \in \mathcal{A}^2; \\ Y_t^{i,n} = \xi^i + \int_t^T f_i(s, Y_s^{1,n-1}, \dots, Y_s^{i,n}, \dots, Y_s^{m,n-1}, Z_s^{i,n}) ds + \int_t^T \psi_i(s, Y_s^{i,n}) dA_s + K_T^{i,n} - K_t^{i,n} \\ \quad - \int_t^T Z_s^{i,n} dB_s, \forall t \leq T; \\ Y_t^{i,n} \geq \max_{j \neq i} (Y_t^{j,n-1} - g_{ij}(t)), \forall t \leq T; \\ \int_0^T \{Y_t^{i,n} - \max_{j \neq i} (Y_t^{j,n-1} - g_{ij}(t))\} dK_t^{i,n} = 0. \end{cases} \quad (2.4)$$

Note that  $(\bar{Y}, \bar{K} = 0, \bar{Z})$  is also a solution for the following GRBSDE:

$$\begin{cases} \bar{Y}_t = \bar{\xi} + \int_t^T \bar{f}(s, \bar{Y}_s, \bar{Z}_s) ds + \int_t^T \bar{\psi}(s, \bar{Y}_s) dA_s + \bar{K}_T - \bar{K}_t - \int_t^T \bar{Z}_s dB_s; \\ \bar{Y}_t \geq \max_{j \neq i} (\bar{Y}_t - g_{ij}(t)); \\ \int_0^T \{\bar{Y}_t - \max_{j \neq i} (\bar{Y}_t - g_{ij}(t))\} d\bar{K}_t = 0. \end{cases} \quad (2.5)$$

**Remark 2.1** For  $i = 1, 2$ , let  $Y^i$  be the unique solution of a one-dimensional generalized reflected BSDEs with data  $(\xi^i, f^i, \psi^i, S^i)$ . Note that, according to [26],  $Y^i$  is obtained as an increasing limit of a sequence of solutions of standard generalized BSDEs. Then, it is not difficult to see that if  $\xi^1 \leq \xi^2$ ,  $f^1 \leq f^2$ ,  $\psi^1 \leq \psi^2$  and  $S^1 \leq S^2$ , we get:

$$Y_t^1 \leq Y_t^2, \forall t \in [0, T], \mathbb{P}\text{-a.s.}$$

It suffices to consider the associated approximations and to apply the comparison result for generalized BSDEs (see **Theorem 1.4** [22]).

Fix  $i \in J$ , by Remark 2.1 and the monotonicity condition on  $f_i$ , we can deduce by induction on  $n$ , that

$$\underline{Y} \leq Y^{i,n} \leq Y^{i,n+1} \leq \bar{Y}. \quad (2.6)$$

Then we conclude that the sequence  $(Y^{i,n})_{n \geq 0}$  converges pointwisly to  $Y^i$ .

**Step 2 :** We show that for any  $i \in J$  and  $n \geq 1$ ,

$$\mathbb{E} \left( \sup_{0 \leq t \leq T} |Y_t^{i,n}|^2 + \int_0^T \|Z_s^{i,n}\|^2 ds \right) \leq C. \quad (2.7)$$

Let  $i = 1, \dots, m$ ,  $t \leq T$  and  $n \geq 1$ , we see that

$$|Y_t^{i,n}| \leq \max\{|\underline{Y}_t|, |\bar{Y}_t|\},$$

then

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y_t^{i,n}|^2 \right] \leq \mathbb{E} \left[ \max \left\{ \sup_{0 \leq t \leq T} |\underline{Y}_t|^2, \sup_{0 \leq t \leq T} |\bar{Y}_t|^2 \right\} \right]. \quad (2.8)$$

So as to identify the limit, we are supposed to establish some estimates with the use of the ones stated in **Proposition 1.1** [22]. We have in particular:

$$\begin{aligned}\mathbb{E} \left[ \sup_{0 \leq t \leq T} |\bar{Y}_t|^2 + \int_0^T |\bar{Y}_t|^2 dA_t \right] &\leq C \mathbb{E} \left( |\bar{\xi}|^2 + \int_0^T |\bar{f}(t, 0, 0)|^2 dt + \int_0^T |\bar{\psi}(t, 0)|^2 dA_t \right); \\ \mathbb{E} \left[ \sup_{0 \leq t \leq T} |\underline{Y}_t|^2 + \int_0^T |\underline{Y}_t|^2 dA_t \right] &\leq C \mathbb{E} \left( |\underline{\xi}|^2 + \int_0^T |\underline{f}(t, 0, 0)|^2 dt + \int_0^T |\underline{\psi}(t, 0)|^2 dA_t \right),\end{aligned}\tag{2.9}$$

which implies that

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y_t^{i,n}|^2 + \int_0^T |Y_t^{i,n}|^2 dA_t \right] \leq C.\tag{2.10}$$

We apply Itô's formula to  $|Y_t^{i,n}|^2$ , we get,

$$\begin{aligned}|Y_t^{i,n}|^2 + \int_0^t \|Z_s^{i,n}\|^2 ds &= |\xi^i|^2 + 2 \int_t^T Y_s^{i,n} f_i(s, Y_s^{1,n-1}, \dots, Y_s^{i,n}, \dots, Y_s^{m,n-1}, Z_s^{i,n}) ds \\ &\quad + 2 \int_t^T Y_s^{i,n} \Psi_i(s, Y_s^{i,n}) dA_s + 2 \int_t^T Y_s^{i,n} dK_s^{i,n} - 2 \int_t^T Y_s^{i,n} Z_s^{i,n} dB_s.\end{aligned}$$

Taking the expectation in both sides yields

$$\begin{aligned}\mathbb{E} \left( |Y_t^{i,n}|^2 \right) + \mathbb{E} \left( \int_t^T \|Z_s^{i,n}\|^2 ds \right) &= \mathbb{E} \left( |\xi^i|^2 \right) \\ &\quad + 2 \mathbb{E} \left( \int_t^T Y_s^{i,n} f_i(s, Y_s^{1,n-1}, \dots, Y_s^{i,n}, \dots, Y_s^{m,n-1}, Z_s^{i,n}) ds \right) \\ &\quad + 2 \mathbb{E} \left( \int_t^T Y_s^{i,n} \Psi_i(s, Y_s^{i,n}) dA_s \right) + 2 \mathbb{E} \left( \int_t^T Y_s^{i,n} dK_s^{i,n} \right).\end{aligned}$$

Using assumptions (i), (ii) and (iv), we get for  $C_1, C_2, C_3 > 0$ :

$$\begin{aligned}\mathbb{E} \left( |Y_t^{i,n}|^2 \right) + \mathbb{E} \left( \int_t^T \|Z_s^{i,n}\|^2 ds \right) &\leq \mathbb{E} \left( |\xi^i|^2 \right) + (1 + C_1 + C_2) \mathbb{E} \left( \sup_{0 \leq t \leq T} |Y_t^{i,n}|^2 \right) \\ &\quad + 2 \mathbb{E} \left( \int_t^T M \| (Y_s^{1,n-1}, \dots, Y_s^{i-1,n-1}, Y_s^{i,n}, Y_s^{i+1,n-1}, \dots, Y_s^{m,n-1}) \| |Y_s^{i,n}| ds \right) \\ &\quad + C_1 \mathbb{E} \left( \int_t^T \|Z_s^{i,n}\|^2 ds \right) + 2 \mathbb{E} \left( \int_t^T \beta |Y_s^{i,n}|^2 dA_s \right) + C_3 \mathbb{E} \left( \int_t^T |Y_s^{i,n}|^2 dA_s \right) \\ &\quad + \mathbb{E} \left( \int_t^T |f_i(s, \vec{0}, 0)|^2 ds \right) + \frac{1}{C_3} \mathbb{E} \left( \int_t^T |\Psi_i(s, 0)|^2 dA_s \right) + \frac{1}{C_2} \mathbb{E} \left( |K_T^{i,n} - K_t^{i,n}|^2 \right).\end{aligned}\tag{2.11}$$

From (2.4), we have:

$$\begin{aligned}K_T^{i,n} - K_t^{i,n} &= Y_t^{i,n} - \xi^i - \int_t^T f_i(s, Y_s^{1,n-1}, \dots, Y_s^{i,n}, \dots, Y_s^{m,n-1}, Z_s^{i,n}) ds - \int_t^T \Psi_i(s, Y_s^{i,n}) dA_s \\ &\quad + \int_t^T Z_s^{i,n} dB_s,\end{aligned}$$

taking the expectation and using (2.10), we obtain by standard computations the following estimate for  $K_T^{i,n} - K_t^{i,n}$ :

$$\mathbb{E} \left( |K_T^{i,n} - K_t^{i,n}|^2 \right) \leq C_4 \left[ 1 + \mathbb{E} \left( \int_t^T \|Z_s^{i,n}\|^2 ds \right) \right].$$



where  $C_4 > 0$ . Back to (2.11), we use (2.10) to conclude that  $\mathbb{E} \left( \int_t^T \|Z_s^{i,n}\|^2 ds \right) \leq C$ , where  $C$  does not depend on  $n$ , then the estimate (2.7) holds true.

**Step 3 : Monotonic limit result.**

Fix  $i = 1, \dots, m, t \leq T$  and  $n \geq 1$ . Recall (2.6) and (2.9), then as  $Y^{i,n}$  converges increasingly to  $Y^i$ , we have:

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y_t^i|^2 + \int_0^T |Y_t^i|^2 dA_t \right] \leq C. \quad (2.12)$$

By dominated convergence we get

$$\mathbb{E} \left( \int_0^T |Y_t^{i,n} - Y_t^i|^2 dt \right) \xrightarrow{n \rightarrow \infty} 0. \quad (2.13)$$

Also, we showed in the previous step that  $(Z^{i,n})_{n \geq 1}$  are bounded in  $\mathcal{H}^2$  and so are  $(f_i(s, Y_s^{1,n-1}, \dots, Y_s^{i,n}, \dots, Y_s^{m,n-1}, Z_s^{i,n}))_{n \geq 1}$  in  $L^2([0, T] \times \Omega)$ . In order to apply Peng's monotonic limit theorem [23], let us note that the equation satisfied by  $Y^{i,n}$  can be rewritten as follows:

$$\begin{aligned} Y_t^{i,n} - \int_0^t \psi_i(s, Y_s^i) dA_s &= Y_0^{i,n} - \int_0^t f_i(s, Y_s^{1,n-1}, \dots, Y_s^{i,n}, \dots, Y_s^{m,n-1}, Z_s^{i,n}) ds \\ &\quad - \int_0^t \{ \psi_i(s, Y_s^{i,n}) - \psi_i(s, Y_s^i) \} dA_s - K_t^{i,n} + \int_0^t Z_s^{i,n} dB_s. \end{aligned} \quad (2.14)$$

Since  $\psi_i$  is non-increasing w.r.t.  $y$ , the process  $\int_0^t \{ \psi_i(s, Y_s^{i,n}) - \psi_i(s, Y_s^i) \} dA_s$  is increasing for each  $n \geq 1$ . Hence, using the estimate (2.12) together with the properties of both  $f_i$  and  $\psi_i$ , we can apply **Theorem 2.1** in [23] to the process  $Y^{i,n} - \int_0^t \psi_i(s, Y_s^i) dA_s$ , which converges increasingly to  $Y^i - \int_0^t \psi_i(s, Y_s^i) dA_s$ , with  $Y^i$  is càdlàg. Therefore, the limit  $Y^i - \int_0^t \psi_i(s, Y_s^i) dA_s$  has the following form:

$$Y_t^i - \int_0^t \psi_i(s, Y_s^i) dA_s = Y_0^i - \int_0^t f_i(s) ds - K_t^i + \int_0^t Z_s^i dB_s, \quad (2.15)$$

where  $K^i$  is càdlàg and denotes the weak limit of  $\int_0^t \{ \psi_i(s, Y_s^{i,n}) - \psi_i(s, Y_s^i) \} dA_s + K_t^{i,n}$ . Moreover,  $Z^i$  is the weak limit of  $Z^{i,n}$  in  $\mathcal{H}^2$ , which also happens to be the strong limit in  $L^p$  for  $p \in [1, 2)$ . Then we have:

$$\mathbb{E} \left( \int_0^T |Z_t^{i,n} - Z_t^i|^p dt \right) \xrightarrow{n \rightarrow \infty} 0, \forall p \in [1, 2).$$

We can use exactly the same techniques as in the proof of **Theorem 2.4** in [23] to show that  $\int_0^t f_i(s) ds = \int_0^t f_i(s, \vec{Y}_s, Z_s^i) ds$ . Furthermore, from (2.6), (2.10) and (2.12), we get by dominated convergence theorem applied to the sequence  $Y_t^{i,n}$  in  $L^2([0, T] \times \Omega, dA_t \times d\mathbb{P})$

$$\mathbb{E} \left( \int_0^T |Y_t^{i,n} - Y_t^i|^2 dA_t \right) \xrightarrow{n \rightarrow \infty} 0. \quad (2.16)$$

Then the Lipschitz continuity of  $\psi_i$  implies

$$\mathbb{E} \left( \int_0^T | \psi_i(t, Y_t^{i,n}) - \psi_i(t, Y_t^i) |^2 dA_t \right) \leq M \mathbb{E} \left( \int_0^T |Y_t^{i,n} - Y_t^i|^2 dA_t \right) \xrightarrow{n \rightarrow \infty} 0.$$

This also means that  $K^i$  is the weak limit of  $K^{i,n}$ . In addition,  $K^i$  inherits the following properties from  $K^{i,n}$ :

- $K_0^i = 0$  and  $\mathbb{E}(K_T^i)^2 < +\infty$ ,
- $K^i$  is increasing.

It follows that  $Y^i$  satisfies:

$$\begin{cases} Y_t^i = \xi^i + \int_t^T f_i(s, \vec{Y}_s, \vec{Z}_s) ds + \int_t^T \psi_i(s, Y_s^i) dA_s + K_T^i - K_t^i - \int_t^T Z_s^i dB_s; \\ Y_t^i \geq \max_{j \neq i} (Y_t^j - g_{ij}(t)). \end{cases} \quad (2.17)$$

Now consider the following system:

$$\begin{cases} \tilde{Y}_t^i = \xi^i + \int_t^T f_i(s, Y_s^1, \dots, \tilde{Y}_s^i, \dots, Y_s^m, \tilde{Z}_s^i) ds + \int_t^T \psi_i(s, \tilde{Y}_s^i) dA_s + \tilde{K}_T^i - \tilde{K}_t^i - \int_t^T \tilde{Z}_s^i dB_s; \\ \tilde{Y}_t^i \geq \max_{j \neq i} (Y_t^j - g_{ij}(t)); \\ \int_0^T [\tilde{Y}_{t^-}^i - \max_{j \neq i} (Y_{t^-}^j - g_{ij}(t))] d\tilde{K}_t^i = 0. \end{cases} \quad (2.18)$$

Note that the barriers in the above Generalized Reflected BSDEs are càdlàg, this is a particular case of the work by Ren-El Otmani [25]. Let  $(\tilde{Y}^i, \tilde{Z}^i, \tilde{K}^i)_{i=1, \dots, m}$  be their unique solutions. By the comparison result [25], we have  $Y_t^{i,n} \leq \tilde{Y}_t^i$  which implies that  $Y_t^i \leq \tilde{Y}_t^i$ .

We next show that  $\tilde{Y}_t^i \leq Y_t^i$ . To this end, we apply Tanaka-Meyer's formula to the semi martingale  $(\tilde{Y}_t^i - Y_t^i)_{0 \leq t \leq T}$  whose terminal value is equal to 0, we obtain:

$$\begin{aligned} (\tilde{Y}_t^i - Y_t^i)^+ &= - \int_t^T \chi_{\{\tilde{Y}_{s^-}^i - Y_{s^-}^i > 0\}} d(\tilde{Y}_s^i - Y_s^i) \\ &\quad - \sum_{t < s \leq T} [\chi_{\{\tilde{Y}_{s^-}^i - Y_{s^-}^i > 0\}} (\tilde{Y}_s^i - Y_s^i)^- + \chi_{\{\tilde{Y}_{s^-}^i - Y_{s^-}^i \leq 0\}} (\tilde{Y}_s^i - Y_s^i)^+] - \frac{1}{2} L_t^0. \end{aligned}$$

Here, the non-negative process  $(L_t^0)_{t \leq T}$  stands for the local time of  $\tilde{Y}^i - Y^i$  at 0. Then,

$$\begin{aligned} (\tilde{Y}_t^i - Y_t^i)^+ &\leq - \int_t^T \chi_{\{\tilde{Y}_{s^-}^i - Y_{s^-}^i > 0\}} d(\tilde{Y}_s^i - Y_s^i) \\ &= \int_t^T \chi_{\{\tilde{Y}_{s^-}^i - Y_{s^-}^i > 0\}} \{f_i(s, Y_s^1, \dots, \tilde{Y}_s^i, \dots, Y_s^m, \tilde{Z}_s^i) - f_i(s, Y_s^1, \dots, Y_s^i, \dots, Y_s^m, \tilde{Z}_s^i)\} ds \\ &\quad + \int_t^T \chi_{\{\tilde{Y}_{s^-}^i - Y_{s^-}^i > 0\}} \{\psi_i(s, \tilde{Y}_s^i) - \psi_i(s, Y_s^i)\} dA_s + \int_t^T \chi_{\{\tilde{Y}_{s^-}^i - Y_{s^-}^i > 0\}} d(\tilde{K}_s^i - K_s^i) \\ &\quad - \int_t^T \chi_{\{\tilde{Y}_{s^-}^i - Y_{s^-}^i > 0\}} (\tilde{Z}_s^i - Z_s^i) dB_s. \end{aligned}$$

The Skorokhod condition in system (2.18) shows that  $\int_t^T \chi_{\{\tilde{Y}_{s^-}^i - Y_{s^-}^i > 0\}} d(\tilde{K}_s^i - K_s^i) \leq 0$ .

In fact,

$$\int_t^T \chi_{\{\tilde{Y}_{s^-}^i - Y_{s^-}^i > 0\}} d(\tilde{K}_s^i - K_s^i) = \int_t^T \chi_{\{\tilde{Y}_{s^-}^i - Y_{s^-}^i > 0\}} d\tilde{K}_s^i - \int_t^T \chi_{\{\tilde{Y}_{s^-}^i - Y_{s^-}^i > 0\}} dK_s^i.$$

The inequality in the system (2.17), implies that  $\tilde{Y}_{s^-}^i > \max_{j \neq i} (Y_{s^-}^j - g_{ij}(s))$  whenever  $\tilde{Y}_{s^-}^i > Y_{s^-}^i$ , then  $d\tilde{K}_s^i = 0$  on the set  $\{\tilde{Y}_{s^-}^i - Y_{s^-}^i > 0\}$ .

From assumption (A1) (H<sub>2</sub>) (iv), one can observe that  $\psi_i(s, \tilde{Y}_s^i) - \psi_i(s, Y_s^i)$  is negative since  $\tilde{Y}_s^i \geq Y_s^i$ , which implies that  $\int_t^T \chi_{\{\tilde{Y}_{s^-}^i - Y_{s^-}^i > 0\}} \{\psi_i(s, \tilde{Y}_s^i) - \psi_i(s, Y_s^i)\} dA_s \leq 0$ . Then

$$(\tilde{Y}_t^i - Y_t^i)^+ \leq \int_t^T \chi_{\{\tilde{Y}_{s^-}^i - Y_{s^-}^i > 0\}} \{f_i(s, Y_s^1, \dots, \tilde{Y}_s^i, \dots, Y_s^m, \tilde{Z}_s^i) - f_i(s, Y_s^1, \dots, Y_s^i, \dots, Y_s^m, \tilde{Z}_s^i)\}$$

$$\begin{aligned}
& + f_i(s, Y_s^1, \dots, Y_s^i, \dots, Y_s^m, \tilde{Z}_s^i) - f_i(s, Y_s^1, \dots, Y_s^i, \dots, Y_s^m, Z_s^i) \} ds \\
& - \int_t^T \mathcal{X}_{\{\tilde{Y}_s^i - Y_s^i > 0\}} (\tilde{Z}_s^i - Z_s^i) dB_s \\
& \leq \int_t^T M \mathcal{X}_{\{\tilde{Y}_s^i - Y_s^i > 0\}} (\tilde{Y}_s^i - Y_s^i)^+ ds \\
& + \int_t^T \mathcal{X}_{\{\tilde{Y}_s^i - Y_s^i > 0\}} \mathcal{X}_{\{\tilde{Z}_s^i \neq Z_s^i\}} [f_i(s, \tilde{Y}_s, \tilde{Z}_s^i) - f_i(s, \tilde{Y}_s, Z_s^i)] (\tilde{Z}_s^i - Z_s^i)^{-1} (\tilde{Z}_s^i - Z_s^i) ds \\
& - \int_t^T \mathcal{X}_{\{\tilde{Y}_s^i - Y_s^i > 0\}} (\tilde{Z}_s^i - Z_s^i) dB_s.
\end{aligned}$$

Set  $M_t = \int_0^t \mathcal{X}_{\{\tilde{Z}_s^i \neq Z_s^i\}} [f_i(s, \tilde{Y}_s, \tilde{Z}_s^i) - f_i(s, \tilde{Y}_s, Z_s^i)] (\tilde{Z}_s^i - Z_s^i)^{-1} dB_s$ ,  $t \leq T$ . By Girsanov's theorem, as  $f_i$  is Lipschitz continuous with respect to  $z$ , the process

$$\tilde{B}_t = B_t - \int_0^t \mathcal{X}_{\{\tilde{Z}_s^i \neq Z_s^i\}} [f_i(s, \tilde{Y}_s, \tilde{Z}_s^i) - f_i(s, \tilde{Y}_s, Z_s^i)] (\tilde{Z}_s^i - Z_s^i)^{-1} ds, t \leq T;$$

is a Brownian motion under  $\tilde{\mathbb{P}}$ , where  $d\tilde{\mathbb{P}} = \varepsilon(M)_T d\mathbb{P}$  and  $\varepsilon(M)_t = e^{M_t - \frac{1}{2}\langle M \rangle_t}$ . It follows that:

$$\begin{aligned}
\mathbb{E}^{\tilde{\mathbb{P}}}[(\tilde{Y}_t^i - Y_t^i)^+] & \leq \mathbb{E}^{\tilde{\mathbb{P}}}[\int_t^T M \mathcal{X}_{\{\tilde{Y}_s^i - Y_s^i > 0\}} (\tilde{Y}_s^i - Y_s^i)^+ ds] - \mathbb{E}^{\tilde{\mathbb{P}}}[\int_t^T \mathcal{X}_{\{\tilde{Y}_s^i - Y_s^i > 0\}} (\tilde{Z}_s^i - Z_s^i) d\tilde{B}_s], \\
& \leq \mathbb{E}^{\tilde{\mathbb{P}}}[\int_t^T M (\tilde{Y}_s^i - Y_s^i)^+ ds].
\end{aligned}$$

By Gronwall's Lemma, we have  $(\tilde{Y}_t^i - Y_t^i)^+ = 0$   $\tilde{\mathbb{P}}$ -a.s.. As  $\tilde{\mathbb{P}}$  and  $\mathbb{P}$  are equivalent, we get  $\mathbb{P}$ -a.s.  $\forall t \leq T$ ,  $\tilde{Y}_t^i \leq Y_t^i$  and then  $\tilde{Y}^i = Y^i$ .

Then we use classic arguments to show that we have also  $\tilde{K}^i = K^i$   $\mathbb{P}$ -a.s. and  $\tilde{Z}^i = Z^i$   $dt \times d\mathbb{P}$ -a.e., which means that  $(Y, K, Z)$  satisfies the following system:  $\forall i = 1, \dots, m, \forall t \leq T$ ,

$$\begin{cases} Y_t^i = \xi^i + \int_t^T f_i(s, \vec{Y}_s, Z_s^i) ds + \int_t^T \psi_i(s, Y_s^i) dA_s + K_T^i - K_t^i - \int_t^T Z_s^i dB_s; \\ Y_t^i \geq \max_{j \neq i} (Y_t^j - g_{ij}(t)); \\ \int_0^T [Y_t^i - \max_{j \neq i} (Y_t^j - g_{ij}(t))] dK_t^i = 0. \end{cases}$$

**Step 4 :**  $K^i$  and  $Y^i$  are continuous.

As  $(K_t^i)_{t \geq 0}$  is a non-decreasing process,  $\Delta K_t^i \geq 0$ . Suppose there exists  $i_1$  such that  $\Delta K_t^{i_1} > 0$  then  $\Delta Y_t^{i_1} < 0$ . But

$$Y_{t^-}^{i_1} = \max_{k \neq i_1} (Y_{t^-}^k - g_{i_1 k}(t)).$$

Then there exists  $i_2 \in J^{-\{i_1\}}$  such that

$$Y_{t^-}^{i_1} = Y_{t^-}^{i_2} - g_{i_1 i_2}(t) > Y_t^{i_1} \geq \max_{k \neq i_1} (Y_t^k - g_{i_1 k}(t)).$$

This implies that

$$\Delta Y_t^{i_2} < 0 \text{ and } \Delta K_t^{i_2} > 0.$$

By repeating the same procedure in the finite set  $J$ , we can find a loop  $i_1, \dots, i_p = i_1$  such that

$$Y_{t^-}^{i_1} = Y_{t^-}^{i_2} - g_{i_1 i_2}(t), Y_{t^-}^{i_2} = Y_{t^-}^{i_3} - g_{i_2 i_3}(t), \dots, Y_{t^-}^{i_{p-1}} = Y_{t^-}^{i_p = i_1} - g_{i_{p-1} i_1}(t).$$

By adding member to member in the above equations, we get  $\sum_{k=1}^{p-1} g_{ik_{k+1}}(t) = 0$ , which contradicts the non-free loop property. Therefore,  $K^i$  are continuous and  $Y^i$  as well. As a result, we can rewrite our system by deleting the limits in Skorokhod condition:  $\forall i = 1, \dots, m, \forall t \leq T$ ,

$$\begin{cases} Y_t^i = \xi^i + \int_t^T f_i(s, \vec{Y}_s, Z_s^i) ds + \int_t^T \psi_i(s, Y_s^i) dA_s + K_T^i - K_t^i - \int_t^T Z_s^i dB_s, \\ Y_t^i \geq \max_{j \neq i} (Y_t^j - g_{ij}(t)), \\ \int_0^T [Y_t^i - \max_{j \neq i} (Y_t^j - g_{ij}(t))] dK_t^i = 0. \end{cases} \quad (2.19)$$

We next prove uniqueness, by providing a representation for the solution of system (2.1).

**Theorem 2.2** *Under assumptions (A1), the solution of system (2.1) is unique .*

**Proof:**

For  $\mu > 0$  and  $\lambda > 0$ , let  $\mathcal{M}_{\mu, \lambda}^2(A)$  denote the set of progressively measurable processes  $(\rho_t)_{0 \leq t \leq T}$  s.t.

$$\|\rho\|_{\mathcal{M}_{\mu, \lambda}^2(A)} := \mathbb{E} \left( \int_0^T e^{\mu t + \lambda A_t} |\rho_t|^2 dt + \int_0^T e^{\mu t + \lambda A_t} |\rho_t|^2 dA_t \right) < \infty,$$

To begin with, we shall stress that  $Y^i \in \mathcal{M}_{\mu, \lambda}^2(A)$ ,  $\forall i = 1, \dots, m$ , where  $(Y^i)_{i \in J}$  solves the system (2.1). In fact, by assumptions  $(H_1)$ ,  $(H_2)(ii)$  and  $(H_2)(iii)$ , we know that  $\underline{Y}$  and  $\bar{Y}$  the solutions obtained by **Theorem 1.6** [22] for the generalized BSDEs (2.2) and (2.3) are both elements of  $\mathcal{M}_{\mu, \lambda}^2(A)$ , then we conclude using  $\underline{Y} \leq Y^i \leq \bar{Y}$ .

Now, let  $\vec{u} = (u^i)_{i=1, \dots, m}$  be an element of  $\mathcal{M}_{\mu, \lambda}^2(A)$  and consider the following system:  $\forall i = 1, \dots, m, \forall t \leq T$ ,

$$\begin{cases} Y^{u,i} \in \mathcal{S}^2, \quad Z^{u,i} \in \mathcal{H}^{2,d}, \quad K^{u,i} \in \mathcal{A}^2; \\ Y_t^{u,i} = \xi^i + \int_t^T f_i(s, \vec{u}_s, Z_s^{u,i}) ds + \int_t^T \psi_i(s, u_s^i) dA_s + K_T^{u,i} - K_t^{u,i} - \int_t^T Z_s^{u,i} dB_s, \\ Y_t^{u,i} \geq \max_{j \neq i} (Y_t^{u,j} - g_{ij}(t)), \\ \int_0^T [Y_t^{u,i} - \max_{j \neq i} (Y_t^{u,j} - g_{ij}(t))] dK_t^{u,i} = 0. \end{cases} \quad (2.20)$$

A switching control  $\alpha$  is a pair of subsequences  $(\tau_n, \theta_n)_{n \geq 0}$ , where  $\tau_n$  are stopping times such that  $\tau_n \leq \tau_{n+1}$  and  $\theta_n$  is a random variable with values in  $J := \{1, \dots, m\}$ . If  $\mathbb{P}[\tau_n^* < T, \forall n \geq 0] = 0$ , we say that  $\alpha$  is admissible.

Let  $i \in J$  and  $t \in [0, T]$ , we define a class of admissible switching controls by:

$$\mathcal{D}_t^i = \left\{ \alpha = ((\tau_n)_{n \geq 0}, (\theta_n)_{n \geq 0}) \in \mathcal{D}, \theta_0 = i, \tau_0 = 0, \tau_1 \geq t \text{ and } \mathbb{E}(G_T^\alpha)^2 < \infty \right\},$$

where  $\mathcal{D}$  denotes the set of admissible switching controls and  $G_s^\alpha$  is the cumulative switching costs up to time  $s$ , expressed by:

$$G_s^\alpha = \sum_{n \geq 1} g_{\theta_{n-1}, \theta_n}(\tau_n) \chi_{[\tau_n \leq s]}, \quad s < T \text{ and } G_T^\alpha = \lim_{s \rightarrow T} G_s^\alpha = \sum_{n \geq 1} g_{\theta_{n-1}, \theta_n}(\tau_n) \chi_{[\tau_n < T]}.$$

Let  $\alpha = ((\tau_n)_{n \geq 0}, (\theta_n)_{n \geq 0}) \in \mathcal{D}_t^i$  and define:

$$\begin{aligned}\xi^\alpha &= \sum_{n \geq 0} \xi^{\theta_n} \chi_{[\tau_n \leq T < \tau_{n+1}],} \\ f_\alpha(t, \vec{u}, z) &= \sum_{n \geq 0} f_{\theta_n}(t, \vec{u}, z) \chi_{[\tau_n \leq t < \tau_{n+1}],} \\ \psi_\alpha(t, u^\alpha) &= \sum_{n \geq 0} \psi_{\theta_n}(s, u^{\theta_n}) \chi_{[\tau_n \leq t < \tau_{n+1}].}\end{aligned}\tag{2.21}$$

Let us consider the following switching equation:

$$\begin{cases} P^\alpha, \mathbb{E}(\sup_{0 \leq t \leq T} |P_t^\alpha|^2) < \infty \text{ and } Q^\alpha \in \mathcal{H}^{2,d}; \\ P_t^\alpha = \xi^\alpha + \int_t^T f_\alpha(s, \vec{u}_s^\alpha, Q_s^\alpha) ds + \int_t^T \psi_\alpha(s, u_s^\alpha) dA_s - \int_t^T Q_s^\alpha dB_s - (G_T^\alpha - G_t^\alpha), t \leq T. \end{cases}\tag{2.22}$$

In the above equation,  $\xi^\alpha$ ,  $f_\alpha$  and  $\psi_\alpha$  are respectively the reward received at time  $T$ , the running reward received on  $D$  and the additional reward once the limit of the domain is reached while adhering to the strategy  $\alpha$ .

By setting up  $\bar{P}_t^\alpha := P_t^\alpha - G_t^\alpha$ , we get a Generalized BSDE with standard generators and a terminal value satisfying  $\mathbb{E}[(\xi^\alpha - G_T^\alpha)^2] < \infty$ . All together with the adaptedness of  $G^\alpha$ , there exists a unique solution  $(P^\alpha, Q^\alpha)$  for (2.22) thanks to [22].

We go back to System (2.20), to write the equation of  $Y^{u,i}$  between  $t$  and  $\tau_1$  :

$$\begin{aligned}Y_t^{u,i} &= Y_{\tau_1}^{u,i} + \int_t^{\tau_1} f_i(s, \vec{u}_s, Z_s^{u,i}) ds + K_{\tau_1}^{u,i} - K_t^{u,i} + \int_t^{\tau_1} \psi_i(s, u_s^i) dA_s - \int_t^{\tau_1} Z_s^{u,i} dB_s \\ &= Y_{\tau_1}^{u,i} \chi_{[\tau_1=T]} + Y_{\tau_1}^{u,i} \chi_{[\tau_1 < T]} + \int_t^{\tau_1} f_\alpha(s, \vec{u}_s, Z_s^\alpha) ds + \int_t^{\tau_1} \psi_\alpha(t, u_s^\alpha) dA_s + K_{\tau_1}^{u,i} - K_t^{u,i} \\ &\quad - \int_t^{\tau_1} Z_s^\alpha dB_s,\end{aligned}$$

where we have used (2.21) and by noticing that  $Z_s^\alpha = Z_s^{u, \theta_0} \chi_{[\tau_0 \leq s < \tau_1]} = Z_s^{u,i}$  on  $[t, \tau_1]$ . Then, as  $Y_{\tau_1}^{u,i} \geq \max_{j \neq i} (Y_{\tau_1}^{u,j} - g_{ij}(\tau_1)) \geq Y_{\tau_1}^{u, \theta_1} - g_{i\theta_1}(\tau_1)$ , we get

$$\begin{aligned}Y_s^{u,i} &\geq (Y_{\tau_1}^{u, \theta_1} - g_{i, \theta_1}(\tau_1)) \chi_{[\tau_1 < T]} + \xi^{i=\theta_0} \chi_{[\tau_1=T]} + \int_t^{\tau_1} f_\alpha(s, \vec{u}_s, Z_s^\alpha) ds + \int_t^{\tau_1} \psi_\alpha(s, u_s^\alpha) dA_s \\ &\quad + K_{\tau_1}^{u,i} - K_t^{u,i} - \int_t^{\tau_1} Z_s^\alpha dB_s; \\ &\geq Y_{\tau_2}^{u, \theta_1} \chi_{[\tau_1 < T]} + \int_t^{\tau_2} f_\alpha(s, \vec{u}_s, Z_s^\alpha) ds + \int_t^{\tau_2} \psi_\alpha(s, u_s^\alpha) dA_s + (K_{\tau_2}^{u,i} - K_{\tau_1}^{u,i}) \\ &\quad - \int_t^{\tau_2} Z_s^\alpha dB_s - g_{i, \theta_1}(\tau_1) \chi_{[\tau_1 < T]} + K_{\tau_1}^{u,i} - K_t^{u,i} + \xi^{\theta_0} \chi_{[\tau_1=T]},\end{aligned}$$

where we used the equation for  $Y^{u, \theta_1}$  between  $[\tau_1, \tau_2]$ . Repeating the same procedure as many times as necessary, we get

$$Y_t^{u,i} \geq \xi^\alpha + \int_t^T f(s, \vec{u}_s, Z_s^\alpha) ds + \int_t^T \psi_\alpha(s, u_s^\alpha) dA_s - G_T^\alpha + \tilde{K}_T^\alpha - \int_t^T Z_s^\alpha dB_s,$$

where  $\tilde{K}_T^\alpha = K_{\tau_1}^{u,i} - K_t^{u,i} + \sum_{n \geq 1} (K_{\tau_{n+1}}^{u, \theta_n} - K_{\tau_n}^{u, \theta_n})$ , which is non negative. Thus

$$Y_t^{u,i} \geq \xi^\alpha + \int_t^T f_\alpha(s, \vec{u}_s, Z_s^\alpha) ds + \int_t^T \psi_\alpha(s, u_s^\alpha) dA_s - G_T^\alpha - \int_t^T Z_s^\alpha dB_s.$$

We then use the equation satisfied by  $(P^\alpha, Q^\alpha)$  to get

$$Y_t^{u,i} - (P_t^\alpha - G_t^\alpha) \geq \int_t^T [f_\alpha(s, \vec{u}_s, Z_s^\alpha) - f_\alpha(s, \vec{u}_s, Q_s^\alpha)] ds - \int_t^T (Z_s^\alpha - Q_s^\alpha) dB_s.$$

Once again, thanks to Girsanov's Theorem there exists  $\tilde{\mathbb{P}}$ , s.t.  $d\tilde{\mathbb{P}} = \varepsilon(M)_T d\mathbb{P}$ , where  $\varepsilon(M)_t = e^{M_t - \frac{1}{2}\langle M \rangle_t}$  and  $M_t = \int_0^t \chi_{\{Z_s^\alpha \neq Q_s^\alpha\}} [f_\alpha(s, \vec{u}_s, Z_s^\alpha) - f_\alpha(s, \vec{u}_s, Q_s^\alpha)] (Z_s^\alpha - Q_s^\alpha)^{-1} dB_s$ . Then

$$Y_t^{u,i} - (P_t^\alpha - G_t^\alpha) \geq \int_t^T (Z_s^\alpha - Q_s^\alpha) d\tilde{B}_s,$$

where  $\tilde{B}_t = B_t - \int_0^t \chi_{\{Z_s^\alpha \neq Q_s^\alpha\}} [f_\alpha(s, \vec{u}_s, Z_s^\alpha) - f_\alpha(s, \vec{u}_s, Q_s^\alpha)] (Z_s^\alpha - Q_s^\alpha)^{-1} ds$ ,  $t \leq T$  is a Brownian motion under  $\tilde{\mathbb{P}}$ . It follows that

$$\mathbb{E}^{\tilde{\mathbb{P}}} \left( Y_t^{u,i} - (P_t^\alpha - G_t^\alpha) \mid \mathcal{F}_t \right) \geq 0.$$

Finally, as  $\varepsilon(M)_t$  is a non-negative martingale, we obtain for all  $\alpha \in \mathcal{D}_t^i$

$$Y_t^{u,i} \geq P_t^\alpha - G_t^\alpha, \quad \mathbb{P}\text{-a.s.}$$

Our purpose is to give a representation for  $Y_t^{u,i}$  as follows:

$$Y_t^{u,i} = \operatorname{esssup}_{\alpha \in \mathcal{D}_t^i} (P_t^\alpha - G_t^\alpha).$$

Consider the strategy  $\alpha^* = (\tau_n^*, \theta_n^*)_{n \geq 0}$  defined by:

$$\tau_0^* = 0, \theta_0^* = i,$$

$$\tau_{n+1}^* = \inf\{s \geq \tau_n^*, Y_s^{u,\theta_n^*} = \max_{j \neq \theta_n^*} (Y_s^{u,j} - g_{\theta_n^*,j}(s))\} \wedge T, \theta_{n+1}^* = \operatorname{argmax}_{j \neq \theta_n^*} (Y_{\tau_{n+1}^*}^{u,j} - g_{\theta_n^*,j}(\tau_{n+1}^*));$$

we need to show that  $\alpha^*$  is an admissible strategy under which  $Y^{u,i}$  is optimal. Let us first prove that:

- 1)  $\mathbb{P}[\tau_n^* < T, \forall n \geq 0] = 0$ ,
- 2)  $\mathbb{E}(G_T^{\alpha^*})^2 < \infty$ .

We proceed by contradiction assuming that  $\mathbb{P}[\tau_n^* < T, \forall n \geq 0] > 0$ . By definition of  $\alpha^*$ , this means that

$$\mathbb{P} \left[ Y_{\tau_{n+1}^*}^{u,\theta_n^*} = Y_{\tau_{n+1}^*}^{u,\theta_{n+1}^*} - g_{\theta_n^*,\theta_{n+1}^*}(\tau_{n+1}^*), \theta_n^* \neq \theta_{n+1}^*, \forall n \geq 1 \right] > 0.$$

Since  $J$  is finite, there exists a loop  $i_0, \dots, i_k, i_{k+1} = i_0$  such that  $i_1 \neq i_0$  and a subsequence  $(n_q)_{q \geq 1}$  such that

$$\mathbb{P} \left[ Y_{\tau_{n_q+1}^*}^{u,i_l} = Y_{\tau_{n_q+1}^*}^{u,i_{l+1}} - g_{i_l,i_{l+1}}(\tau_{n_q+1}^*), l = 0, \dots, k, \forall q \geq 1, i_{k+1} = i_0 \right] > 0.$$

Consider now  $\tau = \lim_n \nearrow \tau_n^*$ , this implies that

$$\mathbb{P} \left[ Y_\tau^{u,i_l} = Y_\tau^{u,i_{l+1}} - g_{i_l,i_{l+1}}(\tau), l = 0, \dots, k, i_{k+1} = i_0 \right] > 0,$$

which leads to

$$\mathbb{P}[g_{i_0 i_1}(\tau) + \dots + g_{i_k i_0}(\tau) = 0] > 0,$$

that contradicts the non free loop property, then  $\mathbb{P}[\tau_n^* < T, \forall n \geq 0] = 0$  holds.

We turn to rewrite the equation for  $Y^{u,i}$  under the strategy  $\alpha^*$  and Skorokhod Condition which means that  $K_s^{u,\theta_n^*} - K_{\tau_n^*}^{u,\theta_n^*} = 0$  for  $\tau_n^* \leq s \leq \tau_{n+1}^*$ , resulting in

$$Y_t^{u,i} = \xi^{\alpha^*} + \int_t^{\tau_n^*} f_{\alpha^*}(s, \vec{u}_s, Z_s^\alpha) ds + \int_t^{\tau_n^*} \Psi_{\alpha^*}(s, u_s^{\alpha^*}) dA_s - \int_t^{\tau_n^*} Z_s^{\alpha^*} dB_s - G_{\tau_n^*}^{\alpha^*}, \forall n \geq 1.$$

By taking the limit, we get

$$Y_t^{u,i} = \xi^{\alpha^*} + \int_t^T f_{\alpha^*}(s, \vec{u}_s, Z_s^\alpha) ds + \int_t^T \Psi_{\alpha^*}(s, u_s^{\alpha^*}) dA_s - \int_t^T Z_s^{\alpha^*} dB_s - G_T^{\alpha^*}.$$

This implies that  $\mathbb{E}(G_T^{\alpha^*})^2 < \infty$  since we have  $(\vec{u}, Y^{u,i}, Z^{\alpha^*}) \in \mathcal{M}_{\mu,\lambda}^2(A) \times \mathcal{S}^2 \times \mathcal{H}^{2,d}$ . It follows that  $\alpha^* \in \mathcal{D}_t^i$  and  $Y_t^{u,i} = P_t^{\alpha^*} - G_t^{\alpha^*}$ . As a conclusion, we have

$$Y_t^{u,i} = \operatorname{esssup}_{\alpha \in \mathcal{D}_t^i} (P_t^\alpha - G_t^\alpha),$$

that gives the uniqueness of the solution for system (2.20).

Now, let us introduce the following mapping:

$$\Phi(\vec{u}) = (Y^{u,i})_{i=1,\dots,m},$$

where  $\vec{u} \in \mathcal{M}_{\mu,\lambda}^2(A)$ , the objective is to show that  $\Phi$  is a strict contraction on  $\mathcal{M}_{\mu,\lambda}^2(A)$  equipped with the norm  $\|\cdot\|_{\mathcal{M}_{\mu,\lambda}^2(A)}$ .

Let us set, for  $\vec{u}$  and  $\vec{v}$  in  $\mathcal{M}_{\mu,\lambda}^2(A)$ :

$$\begin{aligned} F_i(s, z) &= f_i(s, \vec{u}_s, z) \vee f_i(s, \vec{v}_s, z), \\ \Psi_i(s) &= \psi_i(s, u_s^i) \vee \psi_i(s, v_s^i); \end{aligned}$$

and let  $(\tilde{Y}, \tilde{K}, \tilde{Z})$  be the unique solution of the system below:  $\forall i = 1, \dots, m, \forall t \leq T$ ,

$$\begin{cases} \tilde{Y}_t^i = \xi^i + \int_t^T F_i(s, \tilde{Z}_s^i) ds + \int_t^T \Psi_i(s) dA_s + \tilde{K}_T^i - \tilde{K}_t^i - \int_t^T \tilde{Z}_s^i dB_s, \\ \tilde{Y}_t^i \geq \max_{j \neq i} (\tilde{Y}_t^j - g_{ij}(t)), \\ \int_0^T \{\tilde{Y}_t^i - \max_{j \neq i} (\tilde{Y}_t^j - g_{ij}(t))\} d\tilde{K}_t^i = 0. \end{cases} \quad (2.23)$$

We define now the following equation of switching:

$$\tilde{P}_t^\alpha = \xi^\alpha + \int_t^T F_\alpha(s, \tilde{Q}_s^\alpha) ds + \int_t^T \Psi_\alpha(s) dA_s - \int_t^T \tilde{Q}_s^\alpha dB_s - (G_T^\alpha - G_t^\alpha).$$

We know that  $\tilde{Y}_t^i = \operatorname{esssup}_{\alpha \in \mathcal{D}_t^i} (\tilde{P}_t^\alpha - G_t^\alpha) = \tilde{P}_t^{\alpha^*} - G_t^{\alpha^*}$ , where  $\alpha^*$  is the optimal strategy. Next,

we use Proposition 2.1 to prove that  $Y^{u,i} \leq \tilde{Y}^i$  and  $Y^{v,i} \leq \tilde{Y}^i$ .

For  $\alpha \in \mathcal{D}^i$ , let  $(P^\alpha, Q^\alpha)$  be the unique solution for (2.22) and  $(P'^\alpha, Q'^\alpha)$  the solution to the same generalized BSDE with data  $(f_\alpha(\cdot, \vec{v}, \cdot), \psi_\alpha(\cdot, v^i))$ . Then we have

$$P^{\alpha^*} - G^{\alpha^*} \leq Y^{u,i} \leq \tilde{Y}^i = \tilde{P}^{\alpha^*} - G^{\alpha^*} \text{ and } P'^{\alpha^*} - G^{\alpha^*} \leq Y^{v,i} \leq \tilde{Y}^i = \tilde{P}^{\alpha^*} - G^{\alpha^*}.$$

It yields

$$|Y^{u,i} - Y^{v,i}| \leq |\tilde{P}^{\alpha^*} - P^{\alpha^*}| + |\tilde{P}^{\alpha^*} - P'^{\alpha^*}|. \quad (2.24)$$

We want to express  $\|Y^{u,i} - Y^{v,i}\|_{\mathcal{M}_{\mu,\lambda}^2(A)}$  in terms of  $\|\vec{u} - \vec{v}\|_{\mathcal{M}_{\mu,\lambda}^2(A)}$ . To start with, note that  $(\tilde{P}^{\alpha^*} - P^{\alpha^*}, \tilde{Q}^{\alpha^*} - Q^{\alpha^*})$  is the unique solution for the following generalized BSDE.

$$\begin{aligned} \tilde{P}_t^{\alpha^*} - P_t^{\alpha^*} &= \int_t^T [F_{\alpha^*}(s, \tilde{Q}_s^{\alpha^*}) - f_{\alpha^*}(s, \vec{u}_s, Q_s^{\alpha^*})] ds + \int_t^T [\Psi_{\alpha^*}(s) - \psi_{\alpha^*}(s, u_s^{\alpha^*})] dA_s \\ &\quad - \int_t^T (\tilde{Q}_s^{\alpha^*} - Q_s^{\alpha^*}) dB_s, \forall t \leq T. \end{aligned}$$

Under the assumptions  $(H_2)(ii)$  and  $(H_2)(iii)$ , by **Theorem 1.6** [22] and the remarks therein,  $(\int_0^t e^{\mu s + \lambda A_s} (\tilde{P}_s^{\alpha^*} - P_s^{\alpha^*})(\tilde{Q}_s^{\alpha^*} - Q_s^{\alpha^*}) dB_s)_{0 \leq t \leq T}$  is a uniformly integrable martingale, a fact that will be used in the sequel.

By Itô's formula applied to  $e^{\mu t + \lambda A_t} |\tilde{P}_t^{\alpha^*} - P_t^{\alpha^*}|^2$ , we have:  $\forall t \leq T$ ,

$$\begin{aligned} e^{\mu t + \lambda A_t} |\tilde{P}_t^{\alpha^*} - P_t^{\alpha^*}|^2 &+ \int_t^T e^{\mu s + \lambda A_s} \|\tilde{Q}_s^{\alpha^*} - Q_s^{\alpha^*}\|^2 ds \\ &= 2 \int_t^T e^{\mu s + \lambda A_s} (\tilde{P}_s^{\alpha^*} - P_s^{\alpha^*}) [F_{\alpha^*}(s, \tilde{Q}_s^{\alpha^*}) - f_{\alpha^*}(s, \vec{u}_s, Q_s^{\alpha^*})] ds \\ &\quad - \mu \int_t^T e^{\mu s + \lambda A_s} |\tilde{P}_s^{\alpha^*} - P_s^{\alpha^*}|^2 ds + 2 \int_t^T e^{\mu s + \lambda A_s} (\tilde{P}_s^{\alpha^*} - P_s^{\alpha^*}) [\Psi_{\alpha^*}(s) - \psi_{\alpha^*}(s, u_s^{\alpha^*})] dA_s \\ &\quad - \lambda \int_t^T e^{\mu s + \lambda A_s} |\tilde{P}_s^{\alpha^*} - P_s^{\alpha^*}|^2 dA_s - 2 \int_t^T e^{\mu s + \lambda A_s} (\tilde{P}_s^{\alpha^*} - P_s^{\alpha^*})(\tilde{Q}_s^{\alpha^*} - Q_s^{\alpha^*}) dB_s. \end{aligned}$$

Then by taking the expectation and using the fact that  $|x \vee y - y| \leq |x - y|$ , we obtain:  $\forall t \leq T$ ,

$$\begin{aligned} &\mathbb{E}[e^{\mu t + \lambda A_t} |\tilde{P}_t^{\alpha^*} - P_t^{\alpha^*}|^2] + \mathbb{E}[\int_t^T e^{\mu s + \lambda A_s} \|\tilde{Q}_s^{\alpha^*} - Q_s^{\alpha^*}\|^2 ds] \\ &\quad + \mu \mathbb{E}[\int_t^T e^{\mu s + \lambda A_s} |\tilde{P}_s^{\alpha^*} - P_s^{\alpha^*}|^2 ds] + \lambda \mathbb{E}[\int_t^T e^{\mu s + \lambda A_s} |\tilde{P}_s^{\alpha^*} - P_s^{\alpha^*}|^2 dA_s] \\ &\leq 2M \mathbb{E}[\int_t^T e^{\mu s + \lambda A_s} (\tilde{P}_s^{\alpha^*} - P_s^{\alpha^*}) \{ \|\vec{u} - \vec{v}\| + \|\tilde{Q}_s^{\alpha^*} - Q_s^{\alpha^*}\| \} ds] \\ &\quad + 2M \mathbb{E}[\int_t^T e^{\mu s + \lambda A_s} |\tilde{P}_s^{\alpha^*} - P_s^{\alpha^*}| \|\vec{u} - \vec{v}\| dA_s]. \end{aligned}$$

Hence,  $\forall t \leq T$ ,

$$\begin{aligned} &\mathbb{E}[e^{\mu t + \lambda A_t} |\tilde{P}_t^{\alpha^*} - P_t^{\alpha^*}|^2] + \mathbb{E}[\int_t^T e^{\mu s + \lambda A_s} \|\tilde{Q}_s^{\alpha^*} - Q_s^{\alpha^*}\|^2 ds] \\ &\quad + \mu \mathbb{E}[\int_t^T e^{\mu s + \lambda A_s} |\tilde{P}_s^{\alpha^*} - P_s^{\alpha^*}|^2 ds] + \lambda \mathbb{E}[\int_t^T e^{\mu s + \lambda A_s} |\tilde{P}_s^{\alpha^*} - P_s^{\alpha^*}|^2 dA_s] \\ &\leq \mathbb{E}[\int_t^T e^{\mu s + \lambda A_s} |\tilde{P}_s^{\alpha^*} - P_s^{\alpha^*}|^2 ((qM^2 + 2M^2) ds + qM^2 dA_s)] \\ &\quad + \frac{1}{q} \mathbb{E}[\int_t^T e^{\mu s + \lambda A_s} \|\vec{u} - \vec{v}\|^2 (ds + dA_s)] + \frac{1}{2} \mathbb{E}[\int_t^T e^{\mu s + \lambda A_s} \|\tilde{Q}_s^{\alpha^*} - Q_s^{\alpha^*}\|^2 ds]. \end{aligned}$$

As  $\mathbb{E}[e^{\mu t + \lambda A_t} |\tilde{P}_t^{\alpha^*} - P_t^{\alpha^*}|^2]$  and  $\mathbb{E}[\int_t^T e^{\mu s + \lambda A_s} \|\tilde{Q}_s^{\alpha^*} - Q_s^{\alpha^*}\|^2 ds]$  are non-negative, and by choosing  $\mu = 1 + qM^2 + 2M^2$  and  $\lambda = 1 + qM^2$ , where  $q > 4$  we obtain:  $\forall t \leq T$ ,

$$\mathbb{E}[\int_t^T e^{\mu s + \lambda A_s} |\tilde{P}_s^{\alpha^*} - P_s^{\alpha^*}|^2 ds + \int_t^T e^{\mu s + \lambda A_s} |\tilde{P}_s^{\alpha^*} - P_s^{\alpha^*}|^2 dA_s]$$



$$\leq \frac{1}{q} \mathbb{E}[\int_t^T e^{\mu s + \lambda A_s} \|\bar{u} - \bar{v}\|^2 ds + \int_t^T e^{\mu s + \lambda A_s} \|\bar{u} - \bar{v}\|^2 dA_s].$$

We proceed likewise to obtain the inequality with  $\tilde{P}^{\alpha^*} - P^{\alpha^*}$ . Finally, we go back to (2.24) and we obtain:  $\forall t \leq T$ ,

$$\begin{aligned} & \mathbb{E}[\int_t^T e^{\mu s + \lambda A_s} |Y_s^{u,i} - Y_s^{v,i}|^2 ds + \int_t^T e^{\mu s + \lambda A_s} |Y_s^{u,i} - Y_s^{v,i}|^2 dA_s] \\ & \leq \frac{4}{q} \mathbb{E}[\int_t^T e^{\mu s + \lambda A_s} \|\bar{u} - \bar{v}\|^2 ds + \int_t^T e^{\mu s + \lambda A_s} \|\bar{u} - \bar{v}\|^2 dA_s]. \end{aligned}$$

We deduce that  $\Phi$  is a strict contraction on  $\mathcal{M}_{\mu, \lambda}^2(A)$  provided  $\mu \geq 1 + qM^2 + 2M^2$  and  $\lambda \geq 1 + qM^2$ . Hence, it has a unique fixed point which gives the uniqueness of the solution for system of generalized reflected BSDEs (2.1).

We end this section by giving a comparison result for generalized reflected BSDEs systems:

**Proposition 2.1** *Let  $(Y^i)_{i \in J}$  and  $(\tilde{Y}^i)_{i \in J}$  be respectively the unique solutions to the generalized reflected BSDEs system (2.1), with respective data  $((\xi^i)_{i \in J}, (f_i)_{i \in J}, (\psi_i)_{i \in J}, (g_{ij})_{(i,j) \in J^2})$  and  $((\tilde{\xi}^i)_{i \in J}, (\tilde{f}_i)_{i \in J}, (\tilde{\psi}_i)_{i \in J}, (\tilde{g}_{ij})_{(i,j) \in J^2})$ . If for any  $i \in J$ ,  $\xi^i \leq \tilde{\xi}^i$ ,  $f_i \leq \tilde{f}_i$ ,  $\psi_i \leq \tilde{\psi}_i$  and  $g_{ij} \geq \tilde{g}_{ij}$ , then we have:*

$$Y_t^i \leq \tilde{Y}_t^i, \forall i \in J, \forall t \leq T, \mathbb{P} - a.s..$$

**Proof:** We apply Remark 2.1 to the increasing approximation schemes (2.4) which converge respectively to  $(Y^i)_{i \in J}$  and  $(\tilde{Y}^i)_{i \in J}$ . Then we deduce the comparison by taking the limit.

### 3 System of PDEs with Interconnected Obstacles and nonlinear Neumann Boundary Conditions

This section is devoted to the existence and uniqueness of solution in viscosity sense for system of PDEs with interconnected obstacles and conditions on the boundary. To begin with, let  $D$  be an open bounded subset of  $\mathbb{R}^d$ , such that  $D = \{\phi > 0\}$  and  $\partial D = \{\phi = 0\}$ . The function  $\phi$  is in  $C_b^2(\mathbb{R}^d)$  and satisfies  $|\nabla \phi(x)| = 1$  whenever  $x \in \partial D$ , and  $\nabla \phi(x)$  coincides with the unit normal pointing towards the interior of  $D$ . Then the interior sphere condition holds (see [22] and the references therein) *i.e.* there exists  $r > 0$  such that for any  $x \in \partial D$  and  $y \in \bar{D}$  we have:

$$|y - x|^2 + r \langle \nabla \phi(x), y - x \rangle \geq 0. \quad (3.1)$$

Let  $(t, x)$  be in  $[0, T] \times \bar{D}$  and  $(X_s^{t,x}, A_s^{t,x})_{t \leq s \leq T}$  the solution of the reflected SDE below:

$$\begin{cases} dX_s^{t,x} = b(X_s^{t,x})ds + \sigma(X_s^{t,x})dB_s + \nabla \phi(X_s^{t,x})dA_s^{t,x}, & s \in [t, T]; \\ A_s^{t,x} = \int_t^s \mathcal{X}_{\{X_r^{t,x} \in \partial D\}} dA_r^{t,x}, & s \in [t, T]; \\ X_s^{t,x} = x, & s \leq t; \end{cases} \quad (3.2)$$

where  $A^{t,x}$  is increasing, the functions  $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$  and  $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$  are Lipschitz. Note that  $(X_s^{t,x}, A_s^{t,x})_{t \leq s \leq T}$  is valued in  $\bar{D} \times \mathbb{R}_+$ .

The following proposition is borrowed from Pardoux-Zhang [22]:

**Proposition 3.1** For each  $t \in [0, T]$ , there exists a constant  $C$  such that for all  $x, x' \in \bar{D}$ ,

$$\mathbb{E} \left[ \sup_{t \leq s \leq T} |X_s^{t,x} - X_s^{t,x'}|^4 \right] \leq C |x - x'|^4,$$

$$\mathbb{E} \left[ \sup_{t \leq s \leq T} |A_s^{t,x} - A_s^{t,x'}|^4 \right] \leq C |x - x'|^4.$$

Moreover, for all  $p \geq 1$ , there exists a constant  $C_p$  such that for all  $(s, x) \in [t, T] \times \bar{D}$ ,

$$\mathbb{E}(|A_s^{t,x}|^p) \leq C_p(1 + t^p),$$

and for each  $\mu > 0$ ,  $s \in [t, T]$ , there exists  $C(\mu, s)$  such that for all  $x \in \bar{D}$ ,

$$\mathbb{E}(e^{\mu A_s^{t,x}}) \leq C(\mu, s).$$

Suppose now that the data  $((\xi_i)_{i \in J}, (f_i)_{i \in J}, (\psi_i)_{i \in J}, (g_{ij})_{(i,j) \in J^2})$  of the system of GRBS-DEs (2.1) take the following form:

$$\begin{aligned} \xi_i(\omega) &= h_i(X_T^{t,x}(\omega)); \\ f_i(\omega, s, x, \vec{y}, z) &= f_i(s, X_r^{t,x}(\omega), \vec{y}, z); \\ \psi_i(\omega, s, x, y) &= \psi_i(s, X_s^{t,x}(\omega), y); \\ g_{ij}(\omega, s, x) &= g_{ij}(s, X_s^{t,x}(\omega)). \end{aligned} \tag{3.3}$$

The following assumptions are an adaptation of the ones introduced in the first section once one is in the Markovian framework:

**Assumptions (A2):**

Let  $i \in \{1, \dots, m\}$  and  $\mu > 0$ ,

(H<sub>1</sub>) Let  $f_i : [0, T] \times \bar{D} \times \mathbb{R}^m \times \mathbb{R}^{1 \times d} \rightarrow \mathbb{R}$  and  $\psi_i : [0, T] \times \bar{D} \times \mathbb{R} \rightarrow \mathbb{R}$  be such that

(i)  $f_i(t, x, \vec{0}, 0)$  and  $\psi_i(t, x, 0)$  are jointly continuous.

(ii)  $(t, x) \rightarrow f_i(t, x, \vec{y}, z)$  and  $(t, x) \rightarrow \psi_i(t, x, y)$  are uniformly continuous w.r.t.  $(\vec{y}, z)$  and  $y$  respectively.

(iii)  $f_i$  and  $\psi_i$  are Lipschitz continuous w.r.t.  $(\vec{y}, z)$  and  $y$  respectively.

(iv)  $\exists \beta < 0$  such that  $(y - y')(\psi_i(t, x, y) - \psi_i(t, x, y')) \leq \beta |y - y'|^2$ .

(v)  $f_i(t, x, \vec{y}, z)$  is non-decreasing w.r.t.  $y_j$  for  $j \neq i$ , whenever the other components are fixed.

(H<sub>2</sub>) For any  $j \in \{1, \dots, m\}$ , we have:

- $g_{ij} : [0, T] \times \bar{D} \rightarrow \mathbb{R}$  is continuous, non-negative such that  $g_{ii} = 0$ .

- $g_{ij}$  satisfies the non-free loop property, i.e.,  $\forall (t, x) \in [0, T] \times \bar{D}$ ,  $\forall i_1, \dots, i_k$  such that  $i_1 \neq i_2$ ,  $i_1 = i_k$  and  $\text{card}\{i_1, \dots, i_k\} = k - 1$ , we have:

$$g_{i_1 i_2}(t, x) + \dots + g_{i_{k-1} i_k}(t, x) > 0.$$

(H<sub>3</sub>)  $h_i : x \rightarrow h_i(x)$ , is continuous s.t.  $h_i(x) \geq \max_{j \neq i} (h_j(x) - g_{ij}(T, x))$ ,  $\forall x \in \bar{D}$ .

These assumptions, in combination with the properties of the solution of (3.2) (see [21]), show that **Assumptions (A1)** are well satisfied. Then by Theorem 2.1 and Theorem 2.2, for any  $(t, x) \in [0, T] \times \bar{D}$  and  $i \in J$ , there exists a unique solution  $(Y_s^{i,t,x}, K_s^{i,t,x}, Z_s^{i,t,x})_{t \leq s \leq T}$  to the following system:

$$\begin{cases} Y_s^{i,t,x} = h_i(X_T^{t,x}) + \int_s^T f_i(r, X_r^{t,x}, \bar{Y}_r^{t,x}, Z_r^{i,t,x}) dr + \int_s^T \psi_i(r, X_r^{t,x}, Y_r^{i,t,x}) dA_r^{t,x} + K_T^{i,t,x} - K_s^{i,t,x} \\ \quad - \int_s^T Z_r^{i,t,x} dB_r, \\ Y_s^{i,t,x} \geq \max_{j \neq i} (Y_s^{j,t,x} - g_{ij}(s, X_s^{t,x})), \\ \int_0^T \{Y_s^{i,t,x} - \max_{j \neq i} (Y_s^{j,t,x} - g_{ij}(s, X_s^{t,x}))\} dK_s^{i,t,x} = 0. \end{cases}$$

On the basis of the studies conducted on generalized reflected BSDEs [26], and several works on multi-switching problems, amongst them [15], the HJB system associated with our switching problem is the following:  $\forall i = 1, \dots, m$ ,

$$\begin{cases} \min\{u^i(t, x) - \max_{j \neq i} (u^j(t, x) - g_{ij}(t, x)); -\partial_t u^i(t, x) - \mathcal{L}u^i(t, x) \\ \quad - f_i(t, x, (u^k(t, x))_{k=1, \dots, m}, (\sigma^\top D_x u^i)(t, x))\} = 0, (t, x) \in [0, T] \times D; \\ \frac{\partial u^i}{\partial l}(t, x) + \psi_i(t, x, u^i(t, x)) = 0, (t, x) \in [0, T] \times \partial D; \\ u^i(T, x) = h_i(x), x \in \bar{D}, \end{cases} \quad (3.4)$$

where the operator  $\mathcal{L}$  is defined by  $\mathcal{L} = \frac{1}{2} \text{Tr}(\sigma \sigma^\top) D_{xx}^2 + b^\top D_x \cdot$ , and at a point  $x \in \partial D$ ,

$$\frac{\partial}{\partial l} = \langle \nabla \phi(x), D_x \cdot \rangle = \sum_{i=1}^d \frac{\partial \phi}{\partial x_i}(x) \frac{\partial}{\partial x_i}.$$

Since we are interested in finding a function  $(u_1, \dots, u_m)$  which solves (3.4) in viscosity sense, we are led to recall some definitions introduced in [6].

**Definition 3.1** For a locally bounded function  $u : [0, T] \times \bar{D} \rightarrow \mathbb{R}$ , we define its lower semicontinuous envelope  $u_*$  and its upper semicontinuous envelope  $u^*$  as follows:

$$u_*(t, x) = \liminf_{\substack{(t', x') \rightarrow (t, x) \\ t' < T}} u(t', x') \quad \text{and} \quad u^*(t, x) = \overline{\lim}_{\substack{(t', x') \rightarrow (t, x) \\ t' < T}} u(t', x').$$

**Definition 3.2** (Subjets and Superjets)

(i) For a lower semi continuous (lsc) (resp. upper semi continuous (usc)) function  $u : [0, T] \times \bar{D} \rightarrow \mathbb{R}$ , we define the parabolic subjet  $\mathcal{J}^{2-}u(t, x)$  (resp. superjet  $\mathcal{J}^{2+}u(t, x)$ ) of  $u$  at  $(t, x) \in [0, T] \times \bar{D}$ , the set of triples  $(p, q, M) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$  s.t. for any  $(t', x') \in (0, T) \times \bar{D}$ , we have:

$$u(t', x') \geq u(t, x) + p(t' - t) + \langle q, x' - x \rangle + \frac{1}{2} \langle x' - x, M(x' - x) \rangle + o(|t' - t| + |x' - x|^2)$$

(resp.  $u(t', x') \leq u(t, x) + p(t' - t) + \langle q, x' - x \rangle + \frac{1}{2} \langle x' - x, M(x' - x) \rangle + o(|t' - t| + |x' - x|^2)$ ).

(ii) For  $(t, x) \in [0, T] \times \bar{D}$ , we define the parabolic limiting subjet  $\bar{\mathcal{J}}^{2-}u(t, x)$  (resp. the parabolic limiting superjet  $\bar{\mathcal{J}}^{2+}u(t, x)$ ) as follows:

$$\bar{J}^{2-}u(t,x) = \{(p,q,M) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d : \exists(t_n,x_n,p_n,q_n,M_n) \in [0,T] \times \bar{D} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$$

$$s.t. (p_n,q_n,M_n) \in J^{2-}u(t_n,x_n) \text{ and } (t_n,x_n,p_n,q_n,M_n) \longrightarrow (t,x,p,q,M) \text{ as } n \rightarrow \infty\},$$

$$(\text{resp. } \bar{J}^{2+}u(t,x) = \{(p,q,M) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d : \exists(t_n,x_n,p_n,q_n,M_n) \in [0,T] \times \bar{D} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$$

$$(p_n,q_n,M_n) \in J^{2+}u(t_n,x_n) \text{ and } (t_n,x_n,p_n,q_n,M_n) \longrightarrow (t,x,p,q,M) \text{ as } n \rightarrow \infty\},$$

where  $\mathbb{S}^d$  is the set of symmetric real matrices of dimension  $d$ .

We are ready to give the definition of viscosity subsolution and supersolution that we will adopt throughout this section.

**Definition 3.3** (Viscosity solution)

(i) A function  $(u_1, \dots, u_m) : [0, T] \times \bar{D} \longrightarrow \mathbb{R}^m$  such that for any  $i \in J$ ,  $u_i$  is lsc (resp. usc), is called a viscosity supersolution (resp. subsolution) to (3.4), if for any  $i \in J$ , we have:

$$\left\{ \begin{array}{l} \min\{u_i(t,x) - \max_{j \neq i}(u_j(t,x) - g_{ij}(t,x)); \\ -p - b(x)^\top q - \frac{1}{2}Tr[\sigma\sigma^\top(x)M] - f_i(t,x, (u_k(t,x))_{k=1,\dots,m}, \sigma^\top(x)q)\} \geq 0, \\ (t,x) \in [0,T] \times D, (p,q,M) \in \bar{J}^{2-}u_i(t,x); \\ \\ \min\{u_i(t,x) - \max_{j \neq i}(u_j(t,x) - g_{ij}(t,x)); -p - b(x)^\top q - \frac{1}{2}Tr[\sigma\sigma^\top(x)M] \\ - f_i(t,x, (u_k(t,x))_{k=1,\dots,m}, \sigma^\top(x)q)\} \vee \{-\langle \nabla\phi(x), q \rangle - \psi_i(t,x, u_i(t,x))\} \geq 0, \\ (t,x) \in [0,T] \times \partial D, (p,q,M) \in \bar{J}^{2-}u_i(t,x); \\ \\ u_i(T,x) \geq h_i(x), \quad x \in \bar{D}. \end{array} \right. \quad (3.5)$$

(resp.

$$\left\{ \begin{array}{l} \min\{u_i(t,x) - \max_{j \neq i}(u_j(t,x) - g_{ij}(t,x)); \\ -p - b(x)^\top q - \frac{1}{2}Tr[\sigma\sigma^\top(x)M] - f_i(t,x, (u_k(t,x))_{k=1,\dots,m}, \sigma^\top(x)q)\} \leq 0, \\ (t,x) \in [0,T] \times D, (p,q,M) \in \bar{J}^{2+}u_i(t,x); \\ \\ \min\{u_i(t,x) - \max_{j \neq i}(u_j(t,x) - g_{ij}(t,x)); -p - b(x)^\top q - \frac{1}{2}Tr[\sigma\sigma^\top(x)M] \\ - f_i(t,x, (u_k(t,x))_{k=1,\dots,m}, \sigma^\top(x)q)\} \wedge \{-\langle \nabla\phi(x), q \rangle - \psi_i(t,x, u_i(t,x))\} \leq 0, \\ (t,x) \in [0,T] \times \partial D, (p,q,M) \in \bar{J}^{2+}u_i(t,x); \\ \\ u_i(T,x) \leq h_i(x), \quad x \in \bar{D}. \end{array} \right. \quad (3.6)$$

(ii) A locally bounded function  $(u_1, \dots, u_m) : [0, T] \times \bar{D} \longrightarrow \mathbb{R}$  is called a viscosity solution if  $(u_{1*}, \dots, u_{m*})$  (resp.  $(u_1^*, \dots, u_m^*)$ ) is a viscosity supersolution (resp. subsolution).

### 3.1 Uniqueness of the viscosity solution

In this paragraph, we deal with the uniqueness of the solution to system (3.4). It will be obtained by the comparison between subsolutions and supersolutions of the PDEs system (3.4):

**Theorem 3.1** *If  $(u_i)_{i \in J}$  and  $(v^i)_{i \in J}$  are respectively subsolution and supersolution of (3.1), then for any  $i \in J$*

$$u_i \leq v^i, \text{ on } [0, T] \times \bar{D}.$$

**Proof:** The proof will be obtained in two steps.

**Step 1 :** We first assume that there exists a constant  $\lambda$  such that  $\lambda < -m \cdot (\max_{j \in J} C_f^j)$ , ( $C_f^j$  being the Lipschitz constant of  $f_j$  for  $j \in J$ ) and verifying:  
 $\forall i \in J, \forall t, x, y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_m, y, \bar{y}, z$ , if  $y \geq \bar{y}$  then

$$f_i(t, x, y_1, \dots, y_{i-1}, y, y_{i+1}, \dots, y_m, z) - f_i(t, x, y_1, \dots, y_{i-1}, \bar{y}, y_{i+1}, \dots, y_m, z) \leq \lambda(y - \bar{y}). \quad (3.7)$$

Let  $(u_i)_{i \in J}$  and  $(v^i)_{i \in J}$  be respectively subsolution and supersolution to the system (3.4). We first construct an approximation  $u_i^\varepsilon$  and  $v_\varepsilon^i$  such that  $u_i^\varepsilon \rightarrow u_i$  and  $v_\varepsilon^i \rightarrow v^i$  uniformly as  $\varepsilon \rightarrow 0$ , then we are going to show that they are respectively subsolution and supersolution for two PDEs systems that will be given explicitly. Next, we show that  $u_i^\varepsilon \leq v_\varepsilon^i$ , which would permit to conclude that  $u_i \leq v^i$ . We recall the following lemma, from [6]:

**Lemma 3.1** *Let  $v$  be in  $C(\partial D, \mathbb{R}^d)$  satisfying  $-\langle \nabla \phi(x), v(x) \rangle > 0, \forall x \in \partial D$ , then there exists  $\varphi$  in  $C^2(\bar{D})$  such that*

$$\langle v(x), D\varphi(x) \rangle \geq 1, \forall x \in \partial D \text{ and } \varphi \geq 0 \text{ on } \bar{D}. \quad \square \quad (3.8)$$

We apply the previous lemma for  $v = -\nabla \phi$ , then there exists  $\varphi \in C^2(\bar{D})$ , that satisfies  $\langle -\nabla \phi(x), D\varphi(x) \rangle \geq 1, \forall x \in \partial D$  and  $\varphi$  is positive on  $\bar{D}$ . Then put:

$$u_i^\varepsilon(t, x) = u_i(t, x) - \varepsilon \varphi(x) - C \quad \text{and} \quad v_\varepsilon^i(t, x) = v^i(t, x) + \frac{\varepsilon}{t} + \varepsilon \varphi(x) + C,$$

where,  $C$  is a positive constant that will be chosen later on.

Let us show that  $u_i^\varepsilon$  is a subsolution to a specific PDEs system. First, note that for any  $x \in \bar{D}$ , we have:  $u_i^\varepsilon(T, x) \leq u_i(T, x) \leq h_i(x)$ .

Now, let  $(t, x)$  be in  $[0, T] \times \bar{D}$  and  $(p^\varepsilon, q^\varepsilon, M^\varepsilon) \in \bar{J}^{2+} u_i^\varepsilon(t, x)$ . If we set  $p = p^\varepsilon, q = q^\varepsilon + \varepsilon D\varphi(x)$  and  $M = M^\varepsilon + \varepsilon D^2\varphi(x)$ , we can show easily that  $(p, q, M) \in \bar{J}^{2+} u_i(t, x)$  and we have:

$$\begin{aligned} & \min\{u_i^\varepsilon(t, x) - \max_{j \neq i} (u_j^\varepsilon(t, x) - g_{ij}(t, x)); -p^\varepsilon - b(x)^\top q^\varepsilon - \frac{1}{2} Tr[\sigma \sigma^\top(x) M^\varepsilon] \\ & - f_i(t, x, \bar{u}^\varepsilon(t, x), \sigma^\top(x) q^\varepsilon)\} = \min\{u_i(t, x) - \max_{j \neq i} (u_j(t, x) - g_{ij}(t, x)); \\ & -p - b(x)^\top q - \frac{1}{2} Tr[\sigma \sigma^\top(x) M] - f_i(t, x, \bar{u}(t, x), \sigma^\top(x) q) + b(x)^\top (\varepsilon D\varphi(x)) \\ & + \frac{1}{2} Tr[\sigma \sigma^\top(x) (\varepsilon D^2\varphi(x))] + f_i(t, x, \bar{u}(t, x), \sigma^\top(x) q) - f_i(t, x, \bar{u}^\varepsilon(t, x), \sigma^\top(x) q^\varepsilon)\}. \end{aligned}$$

Since  $b, \sigma$  are continuous on  $\bar{D}$  and  $\varphi \in C^2(\bar{D})$ , there exists  $\kappa_1 > 0$  satisfying:

$$b(x)^\top (\varepsilon D\varphi(x)) + \frac{1}{2} Tr[\sigma \sigma^\top(x) (\varepsilon D^2\varphi(x))] \leq \varepsilon \kappa_1.$$

Next with  $f_i(t, x, \bar{u}(t, x), \sigma^\top(x)q) - f_i(t, x, \bar{u}^\varepsilon(t, x), \sigma^\top(x)q^\varepsilon)$ , we use a linearization procedure and both assumptions  $(H_1)(iii)$  and (3.7) to write:

$$\begin{aligned} & f_i(t, x, \bar{u}(t, x), \sigma^\top(x)q) - f_i(t, x, \bar{u}^\varepsilon(t, x), \sigma^\top(x)q^\varepsilon) \\ & \leq \max_{j \in J} C_f^j (m-1)(\varepsilon\varphi(x) + C) + \lambda(\varepsilon\varphi(x) + C) + \kappa_2 \|\sigma^\top(x)(\varepsilon D^2\varphi(x))\|; \\ & \leq (m.\max_{j \in J} C_f^j + \lambda)(\varepsilon\varphi(x) + C) - \max_{j \in J} C_f^j (\varepsilon\varphi(x) + C) + \kappa_3 \varepsilon. \end{aligned}$$

Recall that  $m.\max_{j \in J} C_f^j + \lambda < 0$  and  $\varphi \geq 0$ , then we deduce that

$$\begin{aligned} & b(x)^\top(\varepsilon D\varphi(x)) + \frac{1}{2}Tr[\sigma\sigma^\top(x)(\varepsilon D^2\varphi(x))] \\ & + f_i(t, x, \bar{u}(t, x), \sigma^\top(x)q) - f_i(t, x, \bar{u}^\varepsilon(t, x), \sigma^\top(x)q^\varepsilon) \leq \varepsilon\kappa_4 - C.\max_{j \in J} C_f^j, \quad (3.9) \end{aligned}$$

where  $\kappa_4$  is positive. By choosing  $C = \varepsilon\kappa_4 / \max_{j \in J} C_f^j$ , we obtain that the right hand side of (3.9) is negative and then,

$$\begin{aligned} & \min\{u_i^\varepsilon(t, x) - \max_{j \neq i} (u_j^\varepsilon(t, x) - g_{ij}(t, x)); -p^\varepsilon - b(x)^\top q^\varepsilon - \frac{1}{2}Tr[\sigma\sigma^\top(x)M^\varepsilon] \\ & \quad - f_i(t, x, \bar{u}^\varepsilon(t, x), \sigma^\top(x)q^\varepsilon)\} \leq \min\{u_i(t, x) - \max_{j \neq i} (u_j(t, x) - g_{ij}(t, x)); \\ & \quad -p - b(x)^\top q - \frac{1}{2}Tr[\sigma\sigma^\top(x)M] - f_i(t, x, \bar{u}(t, x), \sigma^\top(x)q)\}. \quad (3.10) \end{aligned}$$

On the other hand, we have:

$$\begin{aligned} & -\langle \nabla\phi(x), q^\varepsilon \rangle - \psi_i(t, x, u_i^\varepsilon(t, x)) \\ & = -\langle \nabla\phi(x), q \rangle - \psi_i(t, x, u^i(t, x) - \varepsilon\varphi(x) - C) - \varepsilon\langle -\nabla\phi(x), D\varphi(x) \rangle. \end{aligned}$$

As  $\psi_i$  is non-increasing and using Lemma 3.1, we get for  $x \in \partial D$ :

$$-\langle \nabla\phi(x), q^\varepsilon \rangle - \psi_i(t, x, u_i^\varepsilon(t, x)) + \varepsilon \leq -\langle \nabla\phi(x), q \rangle - \psi_i(t, x, u_i(t, x)). \quad (3.11)$$

Moreover,  $u^i$  satisfies (3.6), we deduce from (3.10):  $\forall x \in D$ ,

$$\begin{aligned} & \min\{u_i^\varepsilon(t, x) - \max_{j \neq i} (u_j^\varepsilon(t, x) - g_{ij}(t, x)); \\ & \quad -p^\varepsilon - b(x)^\top q^\varepsilon - \frac{1}{2}Tr[\sigma\sigma^\top(x)M^\varepsilon] - f_i(t, x, \bar{u}^\varepsilon(t, x), \sigma^\top(x)q^\varepsilon)\} \leq 0, \end{aligned}$$

whereas, when  $x \in \partial D$ , it follows from (3.10) and (3.11) that:

$$\begin{aligned} & \min\{u_i^\varepsilon(t, x) - \max_{j \neq i} (u_j^\varepsilon(t, x) - g_{ij}(t, x)); -p^\varepsilon - b(x)^\top q^\varepsilon - \frac{1}{2}Tr[\sigma\sigma^\top(x)M^\varepsilon] \\ & \quad - f_i(t, x, \bar{u}^\varepsilon(t, x), \sigma^\top(x)q^\varepsilon)\} \wedge \{-\langle \nabla\phi(x), q^\varepsilon \rangle - \psi_i(t, x, u_i^\varepsilon(t, x)) + \varepsilon\}; \\ & \leq \min\{u_i(t, x) - \max_{j \neq i} (u_j(t, x) - g_{ij}(t, x)); -p - b(x)^\top q - \frac{1}{2}Tr[\sigma\sigma^\top(x)M] \end{aligned}$$

$$-f_i(t, x, (u_k(t, x))_{k=1, \dots, m}, \sigma^\top(x)q) \wedge \{-\langle \nabla \phi(x), q \rangle - \psi_i(t, x, u_i(t, x))\} \leq 0.$$

We conclude that  $\forall i \in J$ ,  $u_i^\varepsilon$  satisfies the following system:

$$\left\{ \begin{array}{l} \min\{u_i^\varepsilon(t, x) - \max_{j \neq i}(u_j^\varepsilon(t, x) - g_{ij}(t, x)); \\ -p^\varepsilon - b(x)^\top q^\varepsilon - \frac{1}{2}Tr[\sigma \sigma^\top(x)M^\varepsilon] - f_i(t, x, \bar{u}^\varepsilon(t, x), \sigma^\top(x)q^\varepsilon)\} \leq 0, \\ (t, x) \in [0, T] \times D, (p^\varepsilon, q^\varepsilon, M^\varepsilon) \in \bar{J}^{2+} u_i^\varepsilon(t, x); \\ \\ \min\{u_i^\varepsilon(t, x) - \max_{j \neq i}(u_j^\varepsilon(t, x) - g_{ij}(t, x)); -p^\varepsilon - b(x)^\top q^\varepsilon - \frac{1}{2}Tr[\sigma \sigma^\top(x)M^\varepsilon] \\ -f_i(t, x, \bar{u}^\varepsilon(t, x), \sigma^\top(x)q^\varepsilon)\} \wedge \{-\langle \nabla \phi(x), q^\varepsilon \rangle - \psi_i(t, x, u_i^\varepsilon(t, x)) + \varepsilon\} \leq 0, \\ (t, x) \in [0, T] \times \partial D, (p^\varepsilon, q^\varepsilon, M^\varepsilon) \in \bar{J}^{2+} u_i^\varepsilon(t, x); \\ \\ u_i^\varepsilon(T, x) \leq h_i(x), \quad x \in \bar{D}. \end{array} \right. \quad (3.12)$$

As for  $v_\varepsilon$ , recall that  $v^i$  satisfies (3.5). Then, with a slight modification and taking into account of assumptions (3.7), (H2)-(v) and the fact that the mapping  $\psi_i(t, x, \cdot)$  is non-increasing, we show that  $\forall i \in J$ ,  $v_\varepsilon^i$  satisfies the following system of inequalities:

$$\left\{ \begin{array}{l} \min\{v_\varepsilon^i(t, x) - \max_{j \neq i}(v_\varepsilon^j(t, x) - g_{ij}(t, x)); \\ -p_\varepsilon - b(x)^\top q_\varepsilon - \frac{1}{2}Tr[\sigma \sigma^\top(x)M_\varepsilon] - f_i(t, x, \bar{v}_\varepsilon(t, x), \sigma^\top(x)q_\varepsilon) - \frac{\varepsilon}{l^2}\} \geq 0, \\ (t, x) \in [0, T] \times D, (p_\varepsilon, q_\varepsilon, M_\varepsilon) \in \bar{J}^{2-} v_\varepsilon^i(t, x); \\ \\ \min\{v_\varepsilon^i(t, x) - \max_{j \neq i}(v_\varepsilon^j(t, x) - g_{ij}(t, x)); -p_\varepsilon - b(x)^\top q_\varepsilon - \frac{1}{2}Tr[\sigma \sigma^\top(x)M_\varepsilon] \\ -f_i(t, x, \bar{v}_\varepsilon(t, x), \sigma^\top(x)q_\varepsilon) - \frac{\varepsilon}{l^2}\} \vee \{-\langle \nabla \phi(x), q_\varepsilon \rangle - \psi_i(t, x, v_\varepsilon^i(t, x)) - \varepsilon\} \geq 0, \\ (t, x) \in [0, T] \times \partial D, (p_\varepsilon, q_\varepsilon, M_\varepsilon) \in \bar{J}^{2-} v_\varepsilon^i(t, x); \\ \\ v_\varepsilon^i(T, x) \geq h_i(x), \quad x \in \bar{D}. \end{array} \right. \quad (3.13)$$

Now, let us show by contradiction that  $u_i^\varepsilon \leq v_\varepsilon^i, \forall i \in J$ . Assume that  $\max_{[0, T] \times \bar{D}} \max_{i \in J} (u_i^\varepsilon - v_\varepsilon^i) > 0$ .

Then there exists  $(\bar{t}, \bar{x}) \in [0, T] \times \bar{D}$  such that

$$\max_{i \in J} (u_i^\varepsilon(\bar{t}, \bar{x}) - v_\varepsilon^i(\bar{t}, \bar{x})) > 0.$$

As in [15], there exists  $k \in \tilde{J} = \{j \in J, u_j^\varepsilon(\bar{t}, \bar{x}) - v_\varepsilon^j(\bar{t}, \bar{x}) = \max_{k \in J} (u_k^\varepsilon(\bar{t}, \bar{x}) - v_\varepsilon^k(\bar{t}, \bar{x}))\}$  such that

$$u_k^\varepsilon(\bar{t}, \bar{x}) > \max_{j \neq k} (u_j^\varepsilon(\bar{t}, \bar{x}) - g_{kj}(\bar{t}, \bar{x})), \quad (3.14)$$

This is obtained mainly due to the non-free loop property.

Let us fix  $j \in \tilde{J}$  that satisfies (3.14) and suppose first that  $\bar{x} \in \partial D$ . For  $(t, x, y)$  in  $[0, T] \times \bar{D}^2$ , we define the following function:

$$\Phi_\alpha^j(t, x, y) = u_j^\varepsilon(t, x) - v_\varepsilon^j(t, y) - \Psi_j^\alpha(t, x, y),$$

where  $\Psi_j^\alpha(t, x, y) = \frac{\alpha}{2} |x - y|^2 - \psi_j(\bar{t}, \bar{x}, u_j^\varepsilon(\bar{t}, \bar{x})) \langle \nabla \phi(\bar{x}), x - y \rangle + |x - \bar{x}|^4 + |t - \bar{t}|^2$ .

Let  $(t_\alpha, x_\alpha, y_\alpha)$  be the maximum point of  $\Phi_\alpha^j$  over  $[0, T] \times \bar{D}^2$ , which exists due to the upper semicontinuity of  $u_j^\varepsilon - v_\varepsilon^j$  and the compactness of  $\bar{D}$ , and let  $M_\alpha$  be defined as follows:

$$M_\alpha = \max_{(t, x, y) \in [0, T] \times \bar{D} \times \bar{D}} \Phi_\alpha^j(t, x, y).$$

It is clear that this supremum is achieved only if  $t_\alpha$  is in  $(0, T)$ .

Next, we can see that

$$M_\alpha \geq \max_{(t, x, y) \in [0, T] \times \bar{D} / x=y} u_j^\varepsilon(t, x) - v_\varepsilon^j(t, x) - |x - \bar{x}|^4 - |t - \bar{t}|^2 = u_j^\varepsilon(\bar{t}, \bar{x}) - v_\varepsilon^j(\bar{t}, \bar{x}).$$

Note that  $M_\alpha$  is non-increasing w.r.t.  $\alpha$ . On the other hand, we have  $\alpha |x_\alpha - y_\alpha|^2 \xrightarrow{\alpha \rightarrow \infty} 0$  (see **Lemma 3.1** [6]). It follows that  $(x_\alpha - y_\alpha) \xrightarrow{\alpha \rightarrow \infty} 0$ , then we get  $(t_\alpha, x_\alpha, y_\alpha) \xrightarrow{\alpha \rightarrow \infty} (\hat{t}, \hat{x}, \hat{x})$ , which implies that

$$u_j^\varepsilon(\bar{t}, \bar{x}) - v_\varepsilon^j(\bar{t}, \bar{x}) \leq u_j^\varepsilon(\hat{t}, \hat{x}) - v_\varepsilon^j(\hat{t}, \hat{x}),$$

as  $(\bar{t}, \bar{x})$  is the maximum point of  $u_j^\varepsilon - v_\varepsilon^j$ , we necessarily have

$$u_j^\varepsilon(\bar{t}, \bar{x}) - v_\varepsilon^j(\bar{t}, \bar{x}) = u_j^\varepsilon(\hat{t}, \hat{x}) - v_\varepsilon^j(\hat{t}, \hat{x}).$$

Thanks to the semi-continuity of the functions  $u_j^\varepsilon$  and  $v_\varepsilon^j$ , we have

$$u_j^\varepsilon(\bar{t}, \bar{x}) - v_\varepsilon^j(\bar{t}, \bar{x})$$

$$\leq \liminf_{\alpha \rightarrow \infty} (u_j^\varepsilon(t_\alpha, x_\alpha) - v_\varepsilon^j(t_\alpha, y_\alpha)) \leq \overline{\lim}_{\alpha \rightarrow \infty} (u_j^\varepsilon(t_\alpha, x_\alpha) - v_\varepsilon^j(t_\alpha, y_\alpha)) \leq u_j^\varepsilon(\bar{t}, \bar{x}) - v_\varepsilon^j(\bar{t}, \bar{x}).$$

Then  $\lim_{\alpha \rightarrow \infty} u_j^\varepsilon(t_\alpha, x_\alpha) - v_\varepsilon^j(t_\alpha, y_\alpha) = u_j^\varepsilon(\bar{t}, \bar{x}) - v_\varepsilon^j(\bar{t}, \bar{x})$ , thus  $(t_\alpha, x_\alpha) \xrightarrow{\alpha \rightarrow \infty} (\bar{t}, \bar{x})$ .

Also, we deduce from the results above that

$$\liminf_{\alpha \rightarrow \infty} u_j^\varepsilon(t_\alpha, x_\alpha) = u_j^\varepsilon(\bar{t}, \bar{x}) - v_\varepsilon^j(\bar{t}, \bar{x}) + \liminf_{\alpha \rightarrow \infty} v_\varepsilon^j(t_\alpha, y_\alpha) \geq u_j^\varepsilon(\bar{t}, \bar{x}) \geq \overline{\lim}_{\alpha \rightarrow \infty} u_j^\varepsilon(t_\alpha, x_\alpha).$$

It follows that  $(u_j^\varepsilon(t_\alpha, x_\alpha), v_\varepsilon^j(t_\alpha, y_\alpha)) \xrightarrow{\alpha \rightarrow \infty} (u_j^\varepsilon(\bar{t}, \bar{x}), v_\varepsilon^j(\bar{t}, \bar{x}))$ , then as  $(u_j^\varepsilon)_{j \in J}$  are usc and  $(g_{ij})_{(i, j) \in J^2}$  are continuous, by (3.14) we get:

$$u_j^\varepsilon(t_\alpha, x_\alpha) > \max_{k \neq j} (u_k^\varepsilon(t_\alpha, x_\alpha) - g_{jk}(t_\alpha, x_\alpha)). \quad (3.15)$$

Go back now to  $\Psi_j^\alpha$ , we note that

$$D_x \Psi_j^\alpha(t, x, y) = \alpha(x - y) - \psi_j(\bar{t}, \bar{x}, u_j^\varepsilon(\bar{t}, \bar{x})) \nabla \phi(\bar{x}) + 4|x - \bar{x}|^2(x - \bar{x}),$$

$$D_y \Psi_j^\alpha(t, x, y) = -\alpha(x - y) + \psi_j(\bar{t}, \bar{x}, u_j^\varepsilon(\bar{t}, \bar{x})) \nabla \phi(\bar{x}) \text{ and } \partial_t \Psi_j^\alpha(t, x, y) = 2(t - \bar{t}).$$

Now, recall the interior sphere condition (3.1) and we distinguish two cases. If  $x_\alpha \in \partial D$ , we have:

$$-\langle \nabla \phi(x_\alpha), D_x \Psi_j^\alpha(t_\alpha, x_\alpha, y_\alpha) \rangle - \psi_j(t_\alpha, x_\alpha, u_j^\varepsilon(t_\alpha, x_\alpha)) = -\alpha \langle \nabla \phi(x_\alpha), x_\alpha - y_\alpha \rangle$$



$$\begin{aligned}
& -4 |x_\alpha - \bar{x}|^2 \langle \nabla \phi(x_\alpha), x_\alpha - \bar{x} \rangle + \psi_j(\bar{t}, \bar{x}, u_j^\varepsilon(\bar{t}, \bar{x})) \langle \nabla \phi(\bar{x}), \nabla \phi(x_\alpha) \rangle - \psi_j(t_\alpha, x_\alpha, u_j^\varepsilon(t_\alpha, x_\alpha)), \\
& \geq -\frac{\alpha}{r} |x_\alpha - y_\alpha|^2 - \frac{4}{r} |x_\alpha - \bar{x}|^4 + \psi_j(\bar{t}, \bar{x}, u_j^\varepsilon(\bar{t}, \bar{x})) \langle \nabla \phi(\bar{x}), \nabla \phi(x_\alpha) \rangle - \psi_j(t_\alpha, x_\alpha, u_j^\varepsilon(t_\alpha, x_\alpha)).
\end{aligned}$$

In view of the convergences above, we know that the right hand side tends to 0 as  $\alpha \rightarrow \infty$ . Then for  $\alpha$  large enough we deduce that

$$-\langle \nabla \phi(x_\alpha), D_x \Psi_j^\alpha(t_\alpha, x_\alpha, y_\alpha) \rangle - \psi_j(t_\alpha, x_\alpha, u_j^\varepsilon(t_\alpha, x_\alpha)) > -\varepsilon. \quad (3.16)$$

Similarly, if  $y_\alpha \in \partial D$ , we get

$$\begin{aligned}
& -\langle \nabla \phi(y_\alpha), -D_y \Psi_j^\alpha(t_\alpha, x_\alpha, y_\alpha) \rangle - \psi_j(t_\alpha, y_\alpha, v_\varepsilon^j(t_\alpha, y_\alpha)) \\
& = \alpha \langle \nabla \phi(y_\alpha), y_\alpha - x_\alpha \rangle + \psi_j(\bar{t}, \bar{x}, u_j^\varepsilon(\bar{t}, \bar{x})) \langle \nabla \phi(\bar{x}), \nabla \phi(y_\alpha) \rangle - \psi_j(t_\alpha, x_\alpha, v_\varepsilon^j(t_\alpha, y_\alpha)), \\
& \leq \alpha |x_\alpha - y_\alpha|^2 + \psi_j(\bar{t}, \bar{x}, u_j^\varepsilon(\bar{t}, \bar{x})) \langle \nabla \phi(\bar{x}), \nabla \phi(y_\alpha) \rangle - \psi_j(t_\alpha, x_\alpha, v_\varepsilon^j(t_\alpha, y_\alpha)), \\
& \leq \alpha |x_\alpha - y_\alpha|^2 + \psi_j(\bar{t}, \bar{x}, u_j^\varepsilon(\bar{t}, \bar{x})) \langle \nabla \phi(\bar{x}), \nabla \phi(y_\alpha) \rangle - \psi_j(\bar{t}, \bar{x}, u_j^\varepsilon(\bar{t}, \bar{x})) \langle \nabla \phi(\bar{x}), \nabla \phi(\bar{x}) \rangle \\
& \quad + \psi_j(\bar{t}, \bar{x}, v_\varepsilon^j(\bar{t}, \bar{x})) - \psi_j(t_\alpha, x_\alpha, v_\varepsilon^j(t_\alpha, y_\alpha)) + \psi_j(\bar{t}, \bar{x}, u_j^\varepsilon(\bar{t}, \bar{x})) - \psi_j(\bar{t}, \bar{x}, v_\varepsilon^j(\bar{t}, \bar{x})).
\end{aligned}$$

We know that

$$\begin{aligned}
& \alpha |x_\alpha - y_\alpha|^2 + \psi_j(\bar{t}, \bar{x}, u_j^\varepsilon(\bar{t}, \bar{x})) \langle \nabla \phi(\bar{x}), \nabla \phi(y_\alpha) \rangle \\
& \quad - \psi_j(\bar{t}, \bar{x}, u_j^\varepsilon(\bar{t}, \bar{x})) \langle \nabla \phi(\bar{x}), \nabla \phi(\bar{x}) \rangle + \psi_j(\bar{t}, \bar{x}, v_\varepsilon^j(\bar{t}, \bar{x})) - \psi_j(t_\alpha, x_\alpha, v_\varepsilon^j(t_\alpha, y_\alpha)) \rightarrow 0,
\end{aligned}$$

then it can be strictly dominated by  $\varepsilon$  for  $\alpha$  large. In addition, since  $\psi_i$  is non-increasing we have  $\psi_j(\bar{t}, \bar{x}, u_j^\varepsilon(\bar{t}, \bar{x})) - \psi_j(\bar{t}, \bar{x}, v_\varepsilon^j(\bar{t}, \bar{x})) \leq 0$ . It follows that:

$$-\langle \nabla \phi(y_\alpha), -D_y \Psi_j^\alpha(t_\alpha, x_\alpha, y_\alpha) \rangle - \psi_j(t_\alpha, y_\alpha, v_\varepsilon^j(t_\alpha, y_\alpha)) - \varepsilon < 0. \quad (3.17)$$

Remember that  $u_j^\varepsilon$  is a subsolution for system (3.12) and  $v_\varepsilon^j$  is a supersolution for (3.13). Then (3.16) and (3.17) lead to the following inequalities:  $\forall (\bar{t}, \bar{x}) \in [0, T] \times \partial D$ ,  $\forall \alpha$  large enough,

$$\begin{aligned}
& -\partial_t \Psi_j^\alpha(t_\alpha, x_\alpha, y_\alpha) - b(x_\alpha)^\top D_x \Psi_j^\alpha(t_\alpha, x_\alpha, y_\alpha) - \frac{1}{2} \text{Tr}[\sigma \sigma^\top(x_\alpha) M^\varepsilon] \\
& \quad - f_i(t_\alpha, x_\alpha, \bar{u}^\varepsilon(t_\alpha, x_\alpha), \sigma^\top(x_\alpha) D_x \Psi_j^\alpha(t_\alpha, x_\alpha, y_\alpha)) \leq 0, \\
& \quad \text{for } (\partial_t \Psi_j^\alpha(t_\alpha, x_\alpha, y_\alpha), D_x \Psi_j^\alpha(t_\alpha, x_\alpha, y_\alpha), M^\varepsilon) \in \bar{J}^{2+} u_j^\varepsilon(t_\alpha, x_\alpha); \\
& -\partial_t \Psi_j^\alpha(t_\alpha, x_\alpha, y_\alpha) - b(y_\alpha)^\top (-D_y \Psi_j^\alpha(t_\alpha, x_\alpha, y_\alpha)) - \frac{1}{2} \text{Tr}[\sigma \sigma^\top(y_\alpha) M_\varepsilon] \\
& \quad - f_i(t_\alpha, y_\alpha, (v_\varepsilon^k(t_\alpha, y_\alpha))_{k=1, \dots, m}, \sigma^\top(y_\alpha) (-D_y \Psi_j^\alpha(t_\alpha, x_\alpha, y_\alpha))) \geq 0, \\
& \quad \text{for } (\partial_t \Psi_j^\alpha(t_\alpha, x_\alpha, y_\alpha), -D_y \Psi_j^\alpha(t_\alpha, x_\alpha, y_\alpha), M_\varepsilon) \in \bar{J}^{2-} v_\varepsilon^j(t_\alpha, y_\alpha);
\end{aligned}$$

Before we proceed, let us emphasize on the fact that if  $\bar{x} \in D$ , the subsequence  $(x_\alpha, y_\alpha)$  is in  $D \times D$  for  $\alpha$  large enough, then the inequalities above hold true. Therefore, we can

apply Crandall-Ishii-Lions's Lemma (**Theorem 3.2** [6]) with  $u_j^\varepsilon$  and  $v_\varepsilon^j$  on  $(0, T) \times \bar{D}$  and  $\varepsilon = \frac{1}{\alpha}$  to find  $(p_\alpha^u, q_\alpha^u, M_\alpha^u) \in \bar{J}^{2+} u_j^\varepsilon(t_\alpha, x_\alpha)$  and  $(p_\alpha^v, q_\alpha^v, M_\alpha^v) \in \bar{J}^{2-} v_\varepsilon^j(t_\alpha, x_\alpha)$  such that

$$p_\alpha^u - p_\alpha^v = \partial_t \Psi_j^\alpha(t_\alpha, x_\alpha, y_\alpha) = 2(t_\alpha - \bar{t}),$$

$$q_\alpha^u - q_\alpha^v = -(D_x \Psi_j^\alpha(t_\alpha, x_\alpha, y_\alpha) - D_y \Psi_j^\alpha(t_\alpha, x_\alpha, y_\alpha)),$$

and  $-(\alpha + \|A\|) \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} M_\alpha^u & 0 \\ 0 & M_\alpha^v \end{pmatrix} \leq A + \frac{1}{\alpha} A^2$ , where  $A = D_{(x,y)}^2 \Psi_j^\alpha(t_\alpha, x_\alpha, y_\alpha)$ .

By replacing the latter in the above inequations, we obtain

$$-p_\alpha^u - b(x_\alpha)^\top q_\alpha^u - \frac{1}{2} \text{Tr}[\sigma \sigma^\top(x_\alpha) M_\alpha^u] - f_j(t_\alpha, x_\alpha, \bar{u}^\varepsilon(t_\alpha, x_\alpha), \sigma^\top(x_\alpha) q_\alpha^u) \leq 0,$$

$$-p_\alpha^v - b(y_\alpha)^\top q_\alpha^v - \frac{1}{2} \text{Tr}[\sigma \sigma^\top(y_\alpha) M_\alpha^v] - f_j(t_\alpha, y_\alpha, \bar{v}^\varepsilon(t_\alpha, y_\alpha), \sigma^\top(y_\alpha) q_\alpha^v) \geq 0.$$

Combining the two inequalities, gives

$$\begin{aligned} & -(p_\alpha^u - p_\alpha^v) - (b(x_\alpha)^\top q_\alpha^u - b(y_\alpha)^\top q_\alpha^v) - \frac{1}{2} \text{Tr}[\sigma \sigma^\top(x_\alpha) M_\alpha^u - \sigma \sigma^\top(y_\alpha) M_\alpha^v] \\ & - \{f_j(t_\alpha, x_\alpha, \bar{u}^\varepsilon(t_\alpha, x_\alpha), \sigma^\top(x_\alpha) q_\alpha^u) - f_j(t_\alpha, y_\alpha, \bar{v}^\varepsilon(t_\alpha, y_\alpha), \sigma^\top(y_\alpha) q_\alpha^v)\} \leq 0. \end{aligned}$$

Thanks to the uniform continuity of  $f_j$ , the Lipschitz assumption on  $b$  and  $\sigma$  and the above convergences, we can find some  $\Sigma_\alpha$  such that  $\overline{\lim}_{\alpha \rightarrow \infty} \Sigma_\alpha \leq 0$  and

$$-\{f_j(t_\alpha, x_\alpha, \bar{u}^\varepsilon(t_\alpha, x_\alpha), \sigma^\top(x_\alpha) q_\alpha^u) - f_j(t_\alpha, x_\alpha, \bar{v}^\varepsilon(t_\alpha, y_\alpha), \sigma^\top(x_\alpha) q_\alpha^u)\} \leq \Sigma_\alpha.$$

Yet again,  $f_j$  is Lipschitz w.r. to  $\bar{y}$ , and as long as it verifies (3.7), we obtain

$$-\lambda(u_j^\varepsilon(t_\alpha, x_\alpha) - v_\varepsilon^j(t_\alpha, y_\alpha)) - \sum_{k \neq j} \Theta_\alpha^{jk}(u_k^\varepsilon(t_\alpha, x_\alpha) - v_\varepsilon^k(t_\alpha, y_\alpha)) \leq \Sigma_\alpha,$$

where  $\Theta_\alpha^{jk}$  stands for the increment rate of  $f_j$  with respect to  $y_k$  for  $k \neq j$ . Observe that  $\Theta_\alpha^{jk}$  is nonnegative and bounded by  $C_f^j$  the Lipschitz constant, thanks to the monotonicity condition on  $f_j$ , then we have:

$$\begin{aligned} -\lambda(u_j^\varepsilon(t_\alpha, x_\alpha) - v_\varepsilon^j(t_\alpha, y_\alpha)) & \leq \sum_{k \neq j} \Theta_\alpha^{jk}(u_k^\varepsilon(t_\alpha, x_\alpha) - v_\varepsilon^k(t_\alpha, y_\alpha))^+ + \Sigma_\alpha, \\ & \leq C_f^j \sum_{k \neq j} (u_k^\varepsilon(t_\alpha, x_\alpha) - v_\varepsilon^k(t_\alpha, y_\alpha))^+ + \Sigma_\alpha. \end{aligned}$$

Taking the superior limit in both sides as  $\alpha \rightarrow \infty$ , with the semicontinuity of  $u_k^\varepsilon$  and  $v_\varepsilon^k$ , we obtain:  $\forall j \in \bar{J}$

$$\begin{aligned} -\lambda(u_j^\varepsilon(\bar{t}, \bar{x}) - v_\varepsilon^j(\bar{t}, \bar{x})) & \leq C_f^j \sum_{k \neq j} (u_k^\varepsilon(\bar{t}, \bar{x}) - v_\varepsilon^k(\bar{t}, \bar{x}))^+, \\ & \leq (m-1) C_f^j (u_j^\varepsilon(\bar{t}, \bar{x}) - v_\varepsilon^j(\bar{t}, \bar{x})), \end{aligned}$$

which is contradictory since  $u_j^\varepsilon(\bar{t}, \bar{x}) - v_\varepsilon^j(\bar{t}, \bar{x}) > 0$  and  $-\lambda > m C_f^j$ .

Thus  $\forall i \in J$ ,  $u_i^\varepsilon \leq v_\varepsilon^i$  on  $[0, T] \times \bar{D}$ . To conclude, it suffices to take the limit as  $\varepsilon \rightarrow 0$ .

**Step 2 : General case.**

To handle the general case in which the generator  $f_j$  is no longer claimed to satisfy the assumption (3.7), we consider the following system for  $\lambda$  arbitrary in  $\mathbb{R}$  and  $t \in [0, T]$ :

$$\begin{cases} \min\{\tilde{u}^i(t, x) - \max_{j \neq i}(\tilde{u}^j(t, x) - e^{\lambda t} g_{ij}(t, x)); -\partial_t \tilde{u}^i(t, x) - \mathcal{L} \tilde{u}^i(t, x) + \lambda \tilde{u}^i(t, x) \\ - e^{\lambda t} f_i(t, x, (e^{-\lambda t} \tilde{u}^k(t, x))_{k=1, \dots, m}, e^{-\lambda t} \sigma^\top(x) D_x \tilde{u}^i(t, x))\} = 0, (t, x) \in [0, T] \times D; \\ \frac{\partial \tilde{u}^i}{\partial t} + e^{\lambda t} \psi_i(t, x, e^{-\lambda t} \tilde{u}^i(t, x)) = 0, (t, x) \in [0, T] \times \partial D; \\ \tilde{u}^i(T, x) = e^{\lambda T} h_i(x), x \in \bar{D}. \end{cases} \quad (3.18)$$

It turns out that  $(\tilde{u}_i(t, x) = e^{\lambda t} u_i(t, x))_{i \in J}$  and  $(\tilde{v}^i(t, x) = e^{\lambda t} v^i(t, x))_{i \in J}$  are respectively sub-solution and supersolution to system (3.18), provided  $(u_i)_{i \in J}$  a subsolution and  $(v^i)_{i \in J}$  a supersolution to (3.4).

By choosing  $\lambda$  small enough, the function  $F_i$  defined by:

$$F_i(t, x, \vec{y}, z) = -\lambda y_i + f_i(t, x, e^{-\lambda t} \vec{y}, e^{-\lambda t} z), \forall i \in J;$$

satisfies (3.7) which shows thanks to the first part that  $\tilde{u}_i \leq \tilde{v}^i$ , then  $u_i \leq v^i$ ,  $\forall i \in J$ .

The following corollary is an immediate conclusion of the comparison between the subsolution and supersolution of system (3.4):

**Corollary 3.1** *If the solution of the system of PDEs (3.4) exists, it is unique and continuous.*

### 3.2 Existence of the viscosity solution

The only point remaining to complete our study is related to the problem of existence of the solution for the PDEs system (3.4). It is at this stage that we are often led to recall the connection between the system of PDEs (3.4) and the system of generalized RBSDEs (2.1).

**Corollary 3.2** *There exist deterministic lower semicontinuous functions  $(v^i(t, x))_{i \in J}$  such that*

$$\forall (t, x) \in [0, T] \times \bar{D}, \forall s \in [t, T], Y_s^i = v^i(s, X_s^{t, x}). \quad (3.19)$$

**Proof:** Back to system (2.4) whose solutions are  $(Y^{i, n})_{i \in J}$ , it has been shown in the previous section that

$$\underline{Y} \leq Y^{i, n} \leq Y^{i, n+1} \leq \bar{Y}. \quad (3.20)$$

We know that  $(Y^{i, n})_{i \in J}$  provide the solutions for the PDEs associated to the generalized reflected BSDEs (2.4), hence we get the following constructions provided by [26]:

$$Y_s^{i, n} = v^{i, n}(s, X_s^{t, x}).$$

Then from (3.20) we obtain:

$$\underline{v} \leq v^{i, n} \leq v^{i, n+1} \leq \bar{v},$$

where  $\underline{v}$  and  $\bar{v}$  are both continuous solutions for two PDEs associated with GBSDEs whose solutions are respectively  $\underline{Y}$  and  $\bar{Y}$  (see [22] for more details).

Thus  $v^{i,n}$  converges increasingly to  $v^i$  and the continuity of  $v^{i,n}$  ensures the lower semicontinuity of  $v^i$ . But  $Y_s^{i,n}$  converges increasingly to  $Y^i$ , therefore  $Y_s^i = v^i(s, X_s^{t,x}), \forall s \in [t, T]$ .

In what follows, we study the solvability in the viscosity sense of system of PDEs (3.4). We show that  $(v^i)_{i \in J}$  is actually a solution of this latter system.

**Theorem 3.2** *The function  $(v^i)_{i \in J}$  is the unique continuous solution of the system of variational inequalities with inter-connected obstacles and non linear Neumann boundaries (3.4).*

**Proof:**

Part 1: Supersolution property.

First, note that  $\forall i \in J$ ,  $v^i$  is lsc, i.e.  $v^i = v_*^i$ . By construction of  $(v^i)_{i \in J}$ , for any  $i \in J$ ,  $v^i = \lim_{n \rightarrow \infty} v^{i,n}$ , where  $v^{i,n}$  is a viscosity solution of the following PDE:

$$\begin{cases} \min\{v^{i,n}(t,x) - \max_{j \neq i}(v^{j,n-1}(t,x) - g_{ij}(t,x)); -\partial_t v^{i,n}(t,x) - \mathcal{L}v^{i,n}(t,x) \\ -f_i(t,x, (v^{1,n-1}, \dots, v^{i-1,n-1}, v^{i,n}, v^{i+1,n-1}, \dots, v^{m,n-1})(t,x), \sigma^\top(x)D_x v^{i,n}(t,x))\} = 0, \\ (t,x) \in [0, T) \times D; \\ \frac{\partial v^{i,n}}{\partial t}(t,x) + \psi_i(t,x, v^{i,n}(t,x)) = 0, (t,x) \in [0, T) \times \partial D; \\ v^{i,n}(T,x) = h_i(x), \quad x \in \bar{D}. \end{cases} \quad (3.21)$$

Let us fix  $i \in J$ ,  $(t,x) \in [0, T) \times \partial D$  and  $(p,q,M) \in \bar{J}^{2-} v^i(t,x)$ . By Lemma 6.1 [6], there exist  $n_j \rightarrow \infty$  and  $x_j \in \bar{D}$  such that

$$(t_j, x_j, v^{i,n_j}(t_j, x_j), p_j, q_j, M_j) \xrightarrow{j \rightarrow \infty} (t, x, v^i(t, x), p, q, M).$$

Since  $x_j$  is in  $\bar{D}$ , we can extract a subsequence  $x_{j_l}$  which is either in  $\partial D$  or  $D$  while preserving the above convergence. We are going to show that in both cases,  $v^i$  solves (3.4).

If  $x_{j_l} \in D$ , we recall that  $(t_{j_l}, x_{j_l}, v^{i,n_{j_l}}(t_{j_l}, x_{j_l}), p_{j_l}, q_{j_l}, M_{j_l}) \xrightarrow{l \rightarrow \infty} (t, x, v^i(t, x), p, q, M)$ . Next from the viscosity supersolution property for  $v^{i,n_{j_l}}$ , we get:

$$\begin{aligned} -p_{j_l} - b(x_{j_l})^\top q_{j_l} - \frac{1}{2} \text{Tr}[\sigma \sigma^\top(x_{j_l}) M_{j_l}] \\ \geq f_i(t_{j_l}, x_{j_l}, (v^{1,n_{j_l}-1}, \dots, v^{i-1,n_{j_l}-1}, v^{i,n_{j_l}}, v^{i+1,n_{j_l}-1}, \dots, v^{m,n_{j_l}-1})(t_{j_l}, x_{j_l}), \sigma^\top(x_{j_l}) q_{j_l}). \end{aligned}$$

On the other hand,  $D$  is a bounded subset, and for any  $i \in J$ ,  $v^{i,n}$  is continuous, then there exists a subsequence  $(l_k)_{k \geq 0}$  such that  $(v^{i,n_{l_k}-1}(t_{l_k}, x_{l_k}))_{k \geq 0}$  is convergent.

As  $(v^{i,n})_{n \geq 1}$  is increasing w.r.t.  $n$  and uniformly dominated, we get from ([2] page 91):

$$v^i(t,x) = v_*^i(t,x) = \lim_{\substack{(t',x') \rightarrow (t,x) \\ l \rightarrow \infty}} v^{i,l}(t',x').$$

It follows immediately that

$$\min\{v^i(t,x) - \max_{j \neq i}(v^j(t,x) - g_{ij}(t,x));$$

$$-p - b(x)^\top q - \frac{1}{2} \text{Tr}[\sigma \sigma^\top(x) M] - f_i(t, x, (v^i(t, x))_{i=1, \dots, m}, \sigma^\top(x) q) \geq 0.$$

Otherwise,  $(x_{j_l})_{l \geq 0} \in \partial D$  and  $(t_{j_l}, x_{j_l}, v^{i, n_{j_l}}(t_{j_l}, x_{j_l}), p_{j_l}, q_{j_l}, M_{j_l}) \xrightarrow{l \rightarrow \infty} (t, x, v^i(t, x), p, q, M)$ . As  $v^{i, n_{j_l}}$  is a viscosity supersolution to (3.21), considering now the boundary condition, we get:

$$\begin{aligned} & \{-p_{j_l} - b(x_{j_l})^\top q_{j_l} - \frac{1}{2} \text{Tr}[\sigma \sigma^\top(x_{j_l}) M_{j_l}] \\ & \quad - f_i(t_{j_l}, x_{j_l}, (v^{1, n_{j_l}-1}, \dots, v^{i-1, n_{j_l}-1}, v^{i, n_{j_l}}, v^{i+1, n_{j_l}-1}, \dots, v^{m, n_{j_l}-1})(t_{j_l}, x_{j_l}), \sigma^\top(x_{j_l}) q_{j_l})\} \\ & \quad \vee \{-\langle \nabla \phi(x_{j_l}), q_{j_l} \rangle - \psi_i(t_{j_l}, x_{j_l}, v^{i, n_{j_l}}(t_{j_l}, x_{j_l}))\} \geq 0. \end{aligned}$$

Suppose now that,

$$\begin{aligned} & -p_{j_l} - b(x_{j_l})^\top q_{j_l} - \frac{1}{2} \text{Tr}[\sigma \sigma^\top(x_{j_l}) M_{j_l}] \\ & \quad \geq f_i(t_{j_l}, x_{j_l}, (v^{1, n_{j_l}-1}, \dots, v^{i-1, n_{j_l}-1}, v^{i, n_{j_l}}, v^{i+1, n_{j_l}-1}, \dots, v^{m, n_{j_l}-1})(t_{j_l}, x_{j_l}), \sigma^\top(x_{j_l}) q_{j_l}). \end{aligned}$$

Then as previously, we obtain:

$$-p - b(x)^\top q - \frac{1}{2} \text{Tr}[\sigma \sigma^\top(x) M] - f_i(t, x, (v^i(t, x))_{i=1, \dots, m}, \sigma^\top(x) q) \geq 0.$$

If not, we would have

$$-\langle \nabla \phi(x_{j_l}), q_{j_l} \rangle - \psi_i(t_{j_l}, x_{j_l}, v^{i, n_{j_l}}(t_{j_l}, x_{j_l})) \geq 0.$$

By taking the limit, we get

$$-\langle \nabla \phi(x), q \rangle - \psi_i(t, x, v^i(t, x)) \geq 0.$$

Therefore,

$$\begin{aligned} & \min\{v^i(t, x) - \max_{j \neq i}(v^j(t, x) - g_{ij}(t, x)); -p - b(x)^\top q - \frac{1}{2} \text{Tr}[\sigma \sigma^\top(x) M] \\ & \quad - f_i(t, x, (v^i(t, x))_{i=1, \dots, m}, \sigma^\top(x) q)\} \vee \{-\langle \nabla \phi(x), q \rangle - \psi_i(t, x, v^i(t, x))\} \geq 0. \end{aligned}$$

Finally, for the terminal value, we know that  $\forall x \in \bar{D}$ ,  $v^{i, n}(T, x) = h_i(x)$ , then we simply take the limit as  $n \rightarrow \infty$ . Therefore,  $v^i$  is a viscosity supersolution for the following PDE

$$\begin{cases} \min\{v^i(t, x) - \max_{j \neq i}(v^j(t, x) - g_{ij}(t, x)); -\partial_t v^i(t, x) - \mathcal{L} v^i(t, x) \\ \quad - f_i(t, x, (v^1, \dots, v^m)(t, x), \sigma^\top(x) D_x v^i(t, x))\} = 0, (t, x) \in [0, T) \times D; \\ \frac{\partial v^i}{\partial t}(t, x) + \psi_i(t, x, v^i(t, x)) = 0, (t, x) \in [0, T) \times \partial D; \\ v^i(T, x) = h_i(x), x \in \bar{D}. \end{cases}$$

Since  $i$  is arbitrary in  $J$  we deduce that the  $m$ -tuple  $(v^1, \dots, v^m)$  is a viscosity supersolution for the system (3.4).

Part 2: Subsolution property.

We will now show that  $(v^{i*})_{i \in J}$  is a subsolution for system (3.4), namely, we should check if the following inequalities are satisfied:

$$\left\{ \begin{array}{l} \min\{v^{i*}(t,x) - \max_{j \neq i}(v^{j*}(t,x) - g_{ij}(t,x)); -p - b(x)^\top q - \frac{1}{2}Tr[\sigma \sigma^\top(x)M] \\ \quad - f_i(t,x, (v^{i*}(t,x))_{i=1,\dots,m}, \sigma^\top(x)q)\} \leq 0, \quad (t,x) \in [0,T] \times D, \quad (p,q,M) \in \bar{J}^{2+} v^{i*}(t,x); \\ \min\{v^{i*}(t,x) - \max_{j \neq i}(v^{j*}(t,x) - g_{ij}(t,x)); -p - b(x)^\top q - \frac{1}{2}Tr[\sigma \sigma^\top(x)M] \\ \quad - f_i(t,x, (v^{i*}(t,x))_{i=1,\dots,m}, \sigma^\top(x)q)\} \wedge \{-\langle \nabla \phi(x), q \rangle - \Psi_i(t,x, v^{i*}(t,x))\} \leq 0, \\ \quad (t,x) \in [0,T] \times \partial D, \quad (p,q,M) \in \bar{J}^{2+} v^{i*}(t,x); \\ v^{i*}(T,x) = h_i(x), \quad x \in \bar{D}. \end{array} \right.$$

**Step 1:** To begin with, we need to show that  $v^{i*}(T,x) = h_i(x)$ ,  $\forall i \in J, \forall x \in \bar{D}$ . For this purpose we are going to show

$$\min\{v^{i*}(T,x) - h_i(x); v^{i*}(T,x) - \max_{j \neq i}(v^{j*}(T,x) - g_{ij}(T,x))\} = 0, \quad \forall i \in J, \forall x \in \bar{D}.$$

Let  $i \in J, n \geq 1$  and  $x \in \bar{D}$ . We know that

$$v^{i*}(T,x) = \overline{\lim}_{\substack{(t',x') \rightarrow (T,x) \\ t' < T, x' \in \bar{D}}} v^i(t',x') \geq \overline{\lim}_{\substack{(t',x') \rightarrow (T,x) \\ t' < T, x' \in \bar{D}}} v^{i,n}(t',x'),$$

then as  $v^{i,n}$  is continuous, we get  $v^{i*}(T,x) \geq v^{i,n}(T,x) = h_i(x)$ .

Besides

$$v^i(t,x) \geq \max_{j \neq i}(v^j(t,x) - g_{ij}(t,x)),$$

after passing to the limit, we get

$$v^{i*}(T,x) \geq \max_{j \neq i}(v^{j*}(T,x) - g_{ij}(T,x)),$$

then we obtain

$$\min\{v^{i*}(T,x) - h_i(x); v^{i*}(T,x) - \max_{j \neq i}(v^{j*}(T,x) - g_{ij}(T,x))\} \geq 0.$$

We now show that the left hand side cannot be strictly positive. To do so, we suppose to the contrary, that for some  $x_0 \in \bar{D}$ , there is  $\varepsilon > 0$  such that

$$\min\{v^{i*}(T,x_0) - h_i(x_0); v^{i*}(T,x_0) - \max_{j \neq i}(v^{j*}(T,x_0) - g_{ij}(T,x_0))\} = 2\varepsilon.$$

Let  $(t_k, x_k)_{k \geq 1}$  be a sequence in  $[0, T] \times \bar{D}$  satisfying  $(t_k, x_k) \xrightarrow[k \rightarrow \infty]{} (T, x_0)$  and  $v^i(t_k, x_k) \xrightarrow[k \rightarrow \infty]{} v^{i*}(T, x_0)$ , which exists thanks to **Lemma 6.1** [6].

As  $v^{i*}$  is usc on  $[0, T] \times \bar{D}$  and  $v^{i,n}$  converges increasingly to  $v^i$ , we can find  $(Q^n)_{n \geq 0}$  a sequence of functions in  $C^{1,2}([0, T] \times \bar{D})$  such that  $Q^n \xrightarrow[n \rightarrow \infty]{} v^{i*}$ . In addition, we suppose that on some neighborhood  $B_n$  of  $(T, x_0)$ , we have:

$$\min\{Q^n(t,x) - h_i(x); Q^n(t,x) - \max_{j \neq i}(v^{j*}(t,x) - g_{ij}(t,x))\} \geq \varepsilon, \quad \forall (t,x) \in B_n. \quad (3.22)$$

We can assume that (3.22) holds on  $B_k^n = [t_k, T] \times B(x_k, \delta_k^n) \cap D$ , for some  $\delta_k^n \in (0, 1)$  such that  $B_k^n \subset B_n$ . In case  $x_k \in \partial D$ , this means that we work only on open neighborhoods of  $D$  near  $\partial D$ . Since  $v^{i*}$  is usc on a bounded space, there exists  $c > 0$  such that  $|v^{i*}| \leq c$  on  $B_n$ . Then, we can assume that  $Q^n \geq -2c$ . Next define

$$V_k^n(t, x) = Q^n(t, x) + \frac{4c|x-x_k|^2}{\delta_k^{n^2}} + \sqrt{T-t}.$$

Note that  $V_k^n(t, x) \geq Q^n(t, x)$  and  $(v^{i*} - V_k^n)(t, x) \leq -c, \forall (t, x) \in [t_k, T] \times \partial B(x_k, \delta_k^n)$ . On the other hand we have,

$$\begin{aligned} & -\{\partial_t V_k^n(t, x) + \mathcal{L}V_k^n(t, x)\} \\ &= -\{\partial_t Q^n(t, x) + \partial_t \sqrt{T-t} + D_x b(x) \{D_x Q^n(t, x) + \frac{8c(x-x_k)}{(\delta_k^n)^2}\} \\ & \quad + \frac{1}{2} \sigma \sigma^\top(x) \{D_{xx} Q^n(t, x) + \frac{8c}{(\delta_k^n)^2}\}\}. \end{aligned}$$

It is clear that  $\frac{4c|x-x_k|^2}{\delta_k^{n^2}} \in C_b^2$  and  $Q^n \in C_b^{1,2}$ , then the derivatives are bounded, together with  $\partial_t(\sqrt{T-t}) \xrightarrow{t \rightarrow T} -\infty$ . Hence, we can choose  $k$  large enough so that:

$$-\{\partial_t V_k^n(t, x) + \mathcal{L}V_k^n(t, x)\} \geq 0, \forall (t, x) \in B_k^n.$$

Recall the SDE (3.2) and consider the following stopping times:

$$\begin{aligned} \theta_n^k &= \inf\{s \geq t_k; (s, X_s^{t_k, x_k}) \in (B_k^n)^c\} \wedge T, \\ \theta_k &= \inf\{s \geq t_k; v^i(s, X_s^{t_k, x_k}) = \max_{j \neq i} (v^j(s, X_s^{t_k, x_k}) - g_{ij}(s, X_s^{t_k, x_k}))\} \wedge T. \end{aligned}$$

Note that on  $[t_k, \theta_n^k \wedge \theta_k]$ ,  $\nabla \phi(X_s^{t, x}) dA_s^{t, x}$  vanishes in the SDE (3.2) since the support of  $A^{t, x}$  is  $\partial D$ . Then, by Itô's formula, we obtain:

$$\begin{aligned} V_k^n(t_k, x_k) &= V_k^n(\theta_n^k \wedge \theta_k, X_{\theta_n^k \wedge \theta_k}^{t_k, x_k}) - \int_{t_k}^{\theta_n^k \wedge \theta_k} \{\partial_t V_k^n(r, X_r^{t_k, x_k}) + \mathcal{L}V_k^n(r, X_r^{t_k, x_k})\} dr \\ & \quad - \int_{t_k}^{\theta_n^k \wedge \theta_k} \sigma(X_r^{t_k, x_k}) D_x V_k^n(r, X_r^{t_k, x_k}) dB_r. \end{aligned}$$

Taking the expectation, we get

$$\begin{aligned} V_k^n(t_k, x_k) &= \mathbb{E}[V_k^n(\theta_n^k \wedge \theta_k, X_{\theta_n^k \wedge \theta_k}^{t_k, x_k}) - \int_{t_k}^{\theta_n^k \wedge \theta_k} \{\partial_t V_k^n(r, X_r^{t_k, x_k}) + \mathcal{L}V_k^n(r, X_r^{t_k, x_k})\} dr] \\ &\geq \mathbb{E}[V_k^n(\theta_n^k, X_{\theta_n^k}^{t_k, x_k}) \mathcal{X}_{[\theta_n^k \leq \theta_k]} + V_k^n(\theta_k, X_{\theta_k}^{t_k, x_k}) \mathcal{X}_{[\theta_n^k > \theta_k]}] \\ &= \mathbb{E}[\{V_k^n(\theta_n^k, X_{\theta_n^k}^{t_k, x_k}) \mathcal{X}_{[\theta_n^k < T]} + V_k^n(T, X_T^{t_k, x_k}) \mathcal{X}_{[\theta_n^k = T]}\} \mathcal{X}_{[\theta_n^k \leq \theta_k]} + V_k^n(\theta_k, X_{\theta_k}^{t_k, x_k}) \mathcal{X}_{[\theta_n^k > \theta_k]}] \\ &\geq \mathbb{E}[\{(v^{i*}(\theta_n^k, X_{\theta_n^k}^{t_k, x_k}) + c) \mathcal{X}_{[\theta_n^k < T]} + (\varepsilon + h_i(X_T^{t_k, x_k})) \mathcal{X}_{[\theta_n^k = T]}\} \mathcal{X}_{[\theta_n^k \leq \theta_k]} \\ & \quad + (\varepsilon + \max_{j \neq i} (v^{j*}(\theta_k, X_{\theta_k}^{t_k, x_k}) - g_{ij}(\theta_k, X_{\theta_k}^{t_k, x_k}))) \mathcal{X}_{[\theta_n^k > \theta_k]}] \\ &\geq \mathbb{E}[\{(v^i(\theta_n^k, X_{\theta_n^k}^{t_k, x_k}) + c) \mathcal{X}_{[\theta_n^k < T]} + (\varepsilon + h_i(X_T^{t_k, x_k})) \mathcal{X}_{[\theta_n^k = T]}\} \mathcal{X}_{[\theta_n^k \leq \theta_k]} \\ & \quad + (\varepsilon + \max_{j \neq i} (v^j(\theta_k, X_{\theta_k}^{t_k, x_k}) - g_{ij}(\theta_k, X_{\theta_k}^{t_k, x_k}))) \mathcal{X}_{[\theta_n^k > \theta_k]}] \end{aligned}$$

$$\geq \mathbb{E}[v^i(\theta_n^k \wedge \theta_k, X_{\theta_n^k \wedge \theta_k}^{t_k, x_k})] + c \wedge \varepsilon.$$

Besides, recall that the process  $Y^i = v^i(\cdot, X)$  stopped at time  $\theta_n^k \wedge \theta_k$  solves an explicit GRBSDE, given by:

$$\begin{aligned} v^i(\theta_n^k \wedge \theta_k, X_{\theta_n^k \wedge \theta_k}^{t_k, x_k}) &= v^i(t_k, x_k) - \int_{t_k}^{\theta_n^k \wedge \theta_k} f_i(r, X_r^{t_k, x_k}, (v^i(r, X_r^{t_k, x_k}))_{i \in J}, Z_r^{i, t_k, x_k}) dr \\ &\quad - \int_{t_k}^{\theta_n^k \wedge \theta_k} \psi_i(r, X_r^{t_k, x_k}, v^i(r, X_r^{t_k, x_k})) dA_r^{t, x} - (K_{\theta_n^k \wedge \theta_k}^{i, t, x} - K_{t_k}^{i, t, x}) + \int_{t_k}^{\theta_n^k \wedge \theta_k} Z_r^{i, t_k, x_k} dB_r. \end{aligned}$$

Once again,  $dA^{t_k, x_k} = 0$  on  $[t_k, \theta_n^k]$ , also  $dK^{i, t_k, x_k} = 0$  on  $[t_k, \theta_k]$ , then by taking the expectation we get:

$$\mathbb{E}\left(v^i(\theta_n^k \wedge \theta_k, X_{\theta_n^k \wedge \theta_k}^{t_k, x_k})\right) = \mathbb{E}\left(v^i(t_k, x_k) - \int_{t_k}^{\theta_n^k \wedge \theta_k} f_i(r, X_r^{t_k, x_k}, (v^i(r, X_r^{t_k, x_k}))_{i \in J}, Z_r^{i, t_k, x_k}) dr\right).$$

Then, as  $\bar{D}$  is bounded, we can show using the properties of the solution  $(X_t^{t_k, x_k})_{t \leq T}$  and the properties of  $f_i$  that

$$\lim_{k \rightarrow \infty} \mathbb{E}\left(\int_{t_k}^{\theta_n^k \wedge \theta_k} f_i(r, X_r^{t_k, x_k}, v^i(r, X_r^{t_k, x_k}))_{i \in J}, Z_r^{i, t_k, x_k}) dr\right) = 0.$$

Hence,  $\lim_{k \rightarrow \infty} V_k^n(t_k, x_k) \geq \lim_{k \rightarrow \infty} V_k^n(t_k, x_k) = v^i(T, x_0) + c \wedge \varepsilon$  where  $c \wedge \varepsilon > 0$ , however from the definition of  $V_k^n(t, x)$  we have  $\lim_{k \rightarrow \infty} V_k^n(t_k, x_k) = Q^n(T, x_0)$ , which is contradictory since  $Q^n \xrightarrow[k \rightarrow \infty]{} v^{i*}$ . It follows that for any  $x \in \bar{D}$

$$\min\{v^{i*}(T, x) - h_i(x); v^{i*}(T, x) - \max_{j \neq i}(v^{j*}(T, x) - g_{ij}(T, x))\} = 0.$$

Finally, we can use the non-free loop property of  $g_{ij}$  following the same method as in [15] to obtain the desired result.

**Step 2:** Let us show that  $(v^{i*})_{i \in J}$  is a subsolution. First, we point out that  $(v^{i, n})_{i \in J}$  are continuous and  $v^i = \lim_{n \rightarrow \infty} v^{i, n}$ , then by ([2] page 91)

$$v^{i*}(t, x) = \overline{\lim}_{n \rightarrow \infty} v^{i, n}(t, x) = \overline{\lim}_{\substack{(t', x') \rightarrow (t, x) \\ n \rightarrow \infty}} v^{i, n}(t', x'), \quad \forall i \in J, \forall (t, x) \in [0, T] \times \bar{D}.$$

From the construction of  $v^{i, n}$ , we have for any  $i \in J$  and  $n \geq 1$ :

$$v^{i, n}(t, x) \geq \max_{j \neq i}(v^{j, n-1}(t, x) - g_{ij}(t, x)), \quad \forall (t, x) \in [0, T] \times \bar{D};$$

taking the limit as  $n \rightarrow \infty$ , we get for any  $i \in J$  and  $(t, x) \in [0, T] \times \bar{D}$ ,

$$v^{i*}(t, x) \geq \max_{j \neq i}(v^{j*}(t, x) - g_{ij}(t, x)).$$

Let  $i \in J$  and  $(t, x) \in [0, T] \times \partial D$  be such that  $v^{i*}(t, x) - \max_{j \neq i}(v^{j*}(t, x) - g_{ij}(t, x)) > 0$ , and fix  $(p, q, M) \in \bar{J}^{2+} v^{i*}(t, x)$ . By Lemma 6.1 [6], there exist  $n_k \rightarrow \infty$ ,  $x_k \in \bar{D}$  and  $(p_k, q_k, M_k) \in \bar{J}^{2+} v^{i, n_k}(t_k, x_k)$  such that

$$(t_k, x_k, v^{i, n_k}(t_k, x_k), p_k, q_k, M_k) \xrightarrow[k \rightarrow \infty]{} (t, x, v^{i*}(t, x), p, q, M).$$



In the spirit of the proof of the first part, since  $x_k$  is in  $\bar{D}$ , we can extract a subsequence  $(x_{k_l})_{l \geq 1}$  which is either in  $\partial D$  or  $D$ . It suffices to check out both possibilities. If  $x_{k_l} \in D$ , we recall that  $(t_{k_l}, x_{k_l}, v^{i, n_{k_l}}(t_{k_l}, x_{k_l}), p_{k_l}, q_{k_l}, M_{k_l}) \xrightarrow{l \rightarrow \infty} (t, x, v^{i*}(t, x), p, q, M)$ . Next, we obtain from the viscosity subsolution property for  $v^{i, n_{k_l}}$ :

$$\begin{aligned} & \min\{v^{i, n_{k_l}}(t_{k_l}, x_{k_l}) - \max_{j \neq i}(v^{j, n_{k_l}-1}(t_{k_l}, x_{k_l}) - g_{ij}(t_{k_l}, x_{k_l})); -p_{k_l} - b(x_{k_l})^\top q_{k_l} - \frac{1}{2} \text{Tr}[\sigma \sigma^\top(x_{k_l}) M_{k_l}] \\ & - f_i(t_{k_l}, x_{k_l}, (v^{1, n_{k_l}-1}, \dots, v^{i-1, n_{k_l}-1}, v^{i, n_{k_l}}, v^{i+1, n_{k_l}-1}, \dots, v^{m, n_{k_l}-1})(t_{k_l}, x_{k_l}), \sigma^\top(x_{k_l}) q_{k_l})\} \leq 0. \end{aligned}$$

We know that  $v^{i*}(t, x) \geq \overline{\lim}_{l \rightarrow \infty} v^{i, n_{k_l}}(t_{k_l}, x_{k_l})$ , then there exists  $l_0 > 0$  s.t.  $\forall l \geq l_0$ , we have:

$$\begin{aligned} v^{i, n_{k_l}}(t_{k_l}, x_{k_l}) & \geq \max_{j \neq i}(v^{j, n_{k_l}}(t_{k_l}, x_{k_l}) - g_{ij}(t_{k_l}, x_{k_l})) \\ & \geq \max_{j \neq i}(v^{j, n_{k_l}-1}(t_{k_l}, x_{k_l}) - g_{ij}(t_{k_l}, x_{k_l})). \end{aligned}$$

since  $v^{i*}(t, x) \geq \max_{j \neq i}(v^{j*}(t, x) - g_{ij}(t, x))$ . It follows that for  $l \geq l_0$ ,

$$\begin{aligned} & -p - b(x)^\top q - \frac{1}{2} \text{Tr}[\sigma \sigma^\top(x) M] \\ & \leq \overline{\lim}_{l \rightarrow \infty} f_i(t_{k_l}, x_{k_l}, (v^{1, n_{k_l}-1}, \dots, v^{i-1, n_{k_l}-1}, v^{i, n_{k_l}}, v^{i+1, n_{k_l}-1}, \dots, v^{m, n_{k_l}-1})(t_{k_l}, x_{k_l}), \sigma^\top(x_{k_l}) q_{k_l}). \end{aligned}$$

In the same manner as in Part 1, we show that for any  $(t, x) \in [0, T] \times \partial D$  and  $(p, q, M) \in \bar{\mathcal{J}}^{2+} v^{i*}(t, x)$ , we have:

$$\begin{aligned} & \min\{v^{i*}(t, x) - \max_{j \neq i}(v^{j*}(t, x) - g_{ij}(t, x)); \\ & -p - b(x)^\top q - \frac{1}{2} \text{Tr}[\sigma \sigma^\top(x) M] - f_i(t, x, (v^{i*}(t, x))_{i=1, \dots, m}, \sigma^\top(x) q)\} \leq 0. \end{aligned}$$

If not we can extract a subsequence  $x_{k_l}$  of  $x_k$  in  $\partial D$ , such that

$$(t_{k_l}, x_{k_l}, v^{i, n_{k_l}}(t_{k_l}, x_{k_l}), p_{k_l}, q_{k_l}, M_{k_l}) \xrightarrow{l \rightarrow \infty} (t, x, v^{i*}(t, x), p, q, M).$$

Then, there exists  $l_0 > 0$  s.t.  $\forall l \geq l_0$  and from the viscosity subsolution property of  $v^{i, n_{k_l}}$  at  $(t_{k_l}, x_{k_l}) \in [0, T] \times \partial D$ , we have:

$$\begin{aligned} & \{-p_{k_l} - b(x_{k_l})^\top q_{k_l} - \frac{1}{2} \text{Tr}[\sigma \sigma^\top(x_{k_l}) M_{k_l}] \\ & - f_i(t_{k_l}, x_{k_l}, (v^{1, n_{k_l}-1}, \dots, v^{i-1, n_{k_l}-1}, v^{i, n_{k_l}}, v^{i+1, n_{k_l}-1}, \dots, v^{m, n_{k_l}-1})(t_{k_l}, x_{k_l}), \sigma^\top(x_{k_l}) q_{k_l})\} \\ & \wedge \{-\langle \nabla \phi(x_{k_l}), q_{k_l} \rangle - \psi_i(t_{k_l}, x_{k_l}, v^{i, n_{k_l}}(t_{k_l}, x_{k_l}))\} \leq 0. \end{aligned}$$

$$\begin{aligned} & \text{If } -p_{k_l} - b(x_{k_l})^\top q_{k_l} - \frac{1}{2} \text{Tr}[\sigma \sigma^\top(x_{k_l}) M_{k_l}] \\ & \leq f_i(t_{k_l}, x_{k_l}, (v^{1, n_{k_l}-1}, \dots, v^{i-1, n_{k_l}-1}, v^{i, n_{k_l}}, v^{i+1, n_{k_l}-1}, \dots, v^{m, n_{k_l}-1})(t_{k_l}, x_{k_l}), \sigma^\top(x_{k_l}) q_{k_l}), \end{aligned}$$

similar computations, yield

$$-p - b(x)^\top q - \frac{1}{2} \text{Tr}[\sigma \sigma^\top(x) M] \leq f_i(t, x, (v^{i*}(t, x))_{i=1, \dots, m}, \sigma^\top(x) q).$$

Otherwise, the other inequality holds

$$-\langle \nabla \phi(x_{k_l}), q_{k_l} \rangle - \psi_i(t_{k_l}, x_{k_l}, v^{i, n_{k_l}}(t_{k_l}, x_{k_l})) \leq 0,$$

by taking to the limit as  $l \rightarrow \infty$ , we get

$$-\langle \nabla \phi(x), q \rangle - \psi_i(t, x, v^{i*}(t, x)) \leq 0.$$

Before we finish the proof, we shall stress out that the inequalities in  $D$  are not difficult and can be handled likewise. As a consequence,  $v^i$  is a viscosity subsolution for the following PDE:

$$\begin{cases} \min\{v^i(t, x) - \max_{j \neq i}(v^j(t, x) - g_{ij}(t, x)); -\partial_t v^i(t, x) - \mathcal{L}v^i(t, x) \\ - f_i(t, x, (v^1, \dots, v^m)(t, x), \sigma^\top(x)D_x v^i(t, x))\} = 0, (t, x) \in [0, T) \times D; \\ \frac{\partial v^i}{\partial t}(t, x) + \psi_i(t, x, v^i(t, x)) = 0, (t, x) \in [0, T) \times \partial D; \\ v^i(T, x) = h_i(x), x \in \bar{D}. \end{cases} \quad (3.23)$$

The fact that  $i$  is arbitrary in  $J$ , shows that the  $m$ -tuple  $(v^1, \dots, v^m)$  is the unique viscosity solution for the system (3.4).

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