

On a List Variant of the Multiplicative 1-2-3 Conjecture

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Abstract

The 1-2-3 Conjecture asks whether almost all graphs can be (edge-)labelled with 1, 2, 3 so that no two adjacent vertices are incident to the same sum of labels. In the last decades, several aspects of this problem have been studied in literature, including more general versions and slight variations. Notable such variations include the List 1-2-3 Conjecture variant, in which edges must be assigned labels from dedicated lists of three labels, and the Multiplicative 1-2-3 Conjecture variant, in which labels 1, 2, 3 must be assigned to the edges so that adjacent vertices are incident to different products of labels. Several results obtained towards these two variants led to observe some behaviours that are distant from those of the original conjecture.

In this work, we consider the list version of the Multiplicative 1-2-3 Conjecture, proposing the first study dedicated to this very problem. In particular, given any graph G , we wonder about the minimum k such that G can be labelled as desired when its edges must be assigned labels from dedicated lists of size k . Exploiting a relationship between our problem and the List 1-2-3 Conjecture, we provide upper bounds on k when G belongs to particular classes of graphs. We further improve some of these bounds through dedicated arguments.

Keywords: proper labelling; 1-2-3 Conjecture; product; list.

1 Introduction

Let G be a graph and ℓ be a k -labelling of G , i.e., an assignment $\ell : E(G) \rightarrow \{1, \dots, k\}$ of labels $1, \dots, k$ to the edges of G . For every vertex v of G , one can compute, as a colour, the *sum* $\sigma_\ell(v)$ of labels assigned by ℓ to the edges incident to v , that is

$$\sigma_\ell(v) = \sum_{w \in N(v)} \ell(vw).$$

We say that ℓ is *s-proper* if σ_ℓ is a proper vertex-colouring of G , i.e., if, for every edge uv of G , we have $\sigma_\ell(u) \neq \sigma_\ell(v)$. We denote by $\chi_\Sigma(G)$ the smallest $k \geq 1$, if any, such that G admits s-proper k -labellings. It turns out that $\chi_\Sigma(G)$ is defined, i.e., that G admits s-proper labellings, if and only if G has no connected component isomorphic to K_2 . For this reason, when investigating s-proper labellings, we generally focus on so-called *nice graphs*, which are those graphs with no connected component isomorphic to K_2 , i.e., having their parameter χ_Σ being properly defined.

The **1-2-3 Conjecture**, introduced in [11] by Karoński, Łuczak and Thomason in 2004, presumes that the maximum value of $\chi_\Sigma(G)$ for a nice graph G should never exceed 3; that is:

1-2-3 Conjecture. *If G is a nice graph, then $\chi_\Sigma(G) \leq 3$.*

Several aspects towards this conjecture have been investigated to date. For an in-depth review of most of our knowledge on the problem, we refer the reader to the survey [17] by Seamone. Let us mention, as notable evidence towards the 1-2-3 Conjecture, that it is known to hold for 3-colourable graphs [11], and that $\chi_\Sigma(G) \leq 5$ holds for every nice graph G [10].

Our investigations in this work are primarily related to two variants of the 1-2-3 Conjecture, being its **Multiplicative** and **List variants**, which we recall in what follows.

As the name suggests, the Multiplicative variant is related to products of labels rather than to sums of labels. To the best of our knowledge, considering products of labels was first considered by Anholcer in a more global version of the problem [2]. The terminology is as follows. Let G be a graph and ℓ be a labelling of G . This time, for every vertex v of G , we compute, as a colour, the *product* $\pi_\ell(v)$ of labels incident to v by ℓ , that is

$$\pi_\ell(v) = \prod_{w \in N(v)} \ell(vw).$$

We say that ℓ is *p-proper* if π_ℓ is a proper vertex-colouring of G , i.e., if, for every edge uv of G , we have $\pi_\ell(u) \neq \pi_\ell(v)$. Assuming G is nice, we denote by $\chi_\Pi(G)$ the smallest k such that G admits p-proper k -labellings.

Introduced in [18] by Skowronek-Kaziów in 2012, the **Multiplicative 1-2-3 Conjecture** stands as the straight product counterpart to the original 1-2-3 Conjecture, and, as mentioned earlier, as a local counterpart of the problem considered by Anholcer in [2]:

Multiplicative 1-2-3 Conjecture. *If G is a nice graph, then $\chi_\Pi(G) \leq 3$.*

The Multiplicative 1-2-3 Conjecture has received, to date, much less attention than the original 1-2-3 Conjecture. Towards it, the main results we know of are that the conjecture holds for 4-colourable graphs [4], and that $\chi_\Pi(G) \leq 4$ holds for every nice graph G [18].

The List variant of the 1-2-3 Conjecture is a generalisation where edges must be assigned labels from fixed-size lists that might be different from $\{1, 2, 3\}$. This is all defined accordingly to the following definitions. Let G be a graph and L be a *k-list assignment* (to the edges) of G , i.e., an assignment of sets of k real numbers to the edges. An *L-labelling* ℓ of G is a labelling where each edge is assigned a label from its list, i.e., $\ell(e) \in L(e)$ for every $e \in E(G)$. Note that the notion of s-proper labellings can be extended naturally to L -labellings. Now, for a nice graph G , we define $\text{ch}_\Sigma(G)$ as the smallest k such that G admits an s-proper L -labelling for every k -list assignment L .

The **List 1-2-3 Conjecture**, introduced in 2009 by Bartnicki, Grytczuk and Niwczyk in [3], is the straight analogue of the 1-2-3 Conjecture to the previous notions:

List 1-2-3 Conjecture. *If G is a nice graph, then $\text{ch}_\Sigma(G) \leq 3$.*

The List 1-2-3 Conjecture is of course much stronger than the original conjecture, and, as a matter of fact, there is still no known general constant upper bound on ch_Σ . To date, the best bound we know of, is that $\text{ch}_\Sigma(G) \leq \Delta(G) + 1$ holds for every nice graph G [7] (where $\Delta(G)$ denotes the maximum degree of G). Constant upper bounds were established for some classes of graphs; see later Section 2 for more details. For now, let us just mention that most of these results were established through an approach imagined by Bartnicki, Grytczuk and Niwczyk in [3], which is reminiscent of the studies on choosability of graphs, which relies on attacking the problem from an algebraic point of view. This will also be described further in a later section (Subsection 2.3), as this is an important point behind our investigations.

This work is dedicated to investigating a new problem inspired from the previous ones above, holding, essentially, as a **List Multiplicative 1-2-3 Conjecture**. Note that, for an L -labelling of a nice graph G , the notion of p-properness adapts naturally, and from this we can define $\text{ch}_\Pi(G)$

as the smallest k such that G admits a p-proper L -labelling for every k -list assignment L to its edges. This parameter ch_Π is precisely the one we study throughout this work. To the best of our knowledge, this parameter was, to date, only discussed briefly by Seamone in his survey [17], in which he suggests a few of its properties. This parameter is also somewhat close to other studied parameters, such as the notions of *product irregularity strength* [2] (related to labellings for which all vertices, not only the adjacent ones, must be incident to distinct products of labels) and *neighbour-product-distinguishing index* [12] (related to labellings for which the labels assigned to the edges must form a proper edge-colouring).

The current paper consists of two main sections. Section 2 stands as a preliminary section in which we raise first observations on the parameter ch_Π . In that section, we also explore the connections between our problem and the List 1-2-3 Conjecture, from which we get first systematic upper bounds on ch_Π . We also get to describing the algebraic approach through which we improve some of these first bounds. These improved bounds are gathered in Section 3, and are about both general graphs (Subsection 3.1) and particular classes of graphs, such as trees, planar graphs with large girth, and subcubic graphs (Subsection 3.2).

2 Preliminaries, tools, and connections with the sum variant

We here introduce all tools and preliminary materials needed to establish the results in later Section 3. In Subsection 2.1, we first state a few easy observations on the parameter ch_Π . In Subsection 2.2, we establish and exploit a relationship between the two parameters ch_Σ and ch_Π , from which we deduce first constant bounds on ch_Π for several graph classes. Finally, in Subsection 2.3, we recall algebraic tools from which improved bounds will be obtained, later in Section 3.

For transparency, let us mention that some of the results from this section, mostly from Subsection 2.1, were already suggested by Seamone in [17], in which the parameter ch_Π is discussed very briefly. To make our contribution clear, we notify properly, through what follows, every remark also mentioned in [17].

2.1 Early remarks on the parameter ch_Π

As remarked in [17], note, given an edge uv of a graph G , that if $\ell(uv) = 0$ by a labelling ℓ of G , then ℓ cannot be p-proper, since this would imply $\pi_\ell(u) = \pi_\ell(v) = 0$. Thus, for any list assignment L of G , a p-proper L -labelling is actually a p-proper L^* -labelling, where L^* is the list assignment of G verifying $L^*(e) = L(e) \setminus \{0\}$ for every edge $e \in E(G)$. Therefore, throughout this work, we consider list assignments not assigning label 0 to the edge lists. To catch this point, we refine the parameter $\text{ch}_\Pi(G)$ of a graph G to the parameter $\text{ch}_\Pi^*(G)$, which is the smallest $k \geq 1$, if any, such that G admits p-proper L -labellings for every k -list assignment L not assigning label 0.

We note that if L is the 1-list assignment of G where $L(e) = \{1\}$ for every edge e , then G admits no p-proper L -labellings, since every such labelling ℓ would verify $\pi_\ell(u) = \pi_\ell(v) = 1$ for every edge $uv \in E(G)$. Thus:

Observation 2.1. *There is no graph G verifying $\text{ch}_\Pi^*(G) = 1$.*

Analogous conclusions can be reached regarding graphs G with $\text{ch}_\Pi^*(G) = 2$. Here, consider the 2-list assignment L of G where $L(e) = \{-1, 1\}$ for every edge e . Then, by an L -labelling ℓ of G , we have $\pi_\ell(v) \in \{-1, 1\}$ for every vertex $v \in V(G)$. This implies that ℓ is p-proper if and only if π_ℓ is a proper 2-vertex-colouring of G . In turn, this yields the following (also mentioned in [17]):

Observation 2.2. *If G is a graph with $\text{ch}_\Pi^*(G) = 2$, then G is bipartite.*

The previous condition is not sufficient, however, as nice connected bipartite graphs G with $\text{ch}_\Pi^*(G) = 2$ must fulfil an additional property.

Proposition 2.3. *Let G be a connected bipartite graph with bipartition (A, B) . If $\text{ch}_{\Pi}^*(G) = 2$, then at least one of $|A|$ and $|B|$ must be even.*

Proof. Assume the claim is wrong, and let G be a connected bipartite graph with $\text{ch}_{\Pi}^*(G) = 2$ in which the two parts A and B are of odd size. Consider L , the 2-list assignment of G where $L(e) = \{-1, 1\}$ for every edge $e \in E(G)$. As mentioned earlier, by every L -labelling ℓ of G , we have $\pi_{\ell}(v) \in \{-1, 1\}$ for every vertex $v \in V(G)$. Thus, because G is connected, for such an ℓ to be p-proper we must have, say, $\pi_{\ell}(a) = -1$ for every $a \in A$ and $\pi_{\ell}(b) = 1$ for every $b \in B$. For the first condition to occur, for every $a \in A$ there must be an odd number of incident edges labelled -1 by ℓ . Since $|A|$ is odd, this means that we must have an odd number of edges of G labelled -1 by ℓ . For the second condition to occur, for every $b \in B$ there must be an even number of incident edges labelled -1 by ℓ . For that, we must have an even number of edges of G labelled -1 by ℓ , which is a contradiction. \square

Thus, connected graphs G with $\text{ch}_{\Pi}^*(G) = 2$ are connected bipartite graphs with at least one part of even cardinality. This condition is necessary but still not sufficient, however, even in simple graph classes such as trees. To see this is true, consider the following easy remarks.

Suppose we have a graph G with a pending path wvu of length 2, where $d(u) = 1$ and $d(v) = 2$, and suppose L is a 2-list assignment to the edges of G . Assume more particularly that $L(wv) = \{1, a\}$ for some $a \neq 1$. Then, note that, in any p-proper L -labelling ℓ of G , we cannot have $\ell(vw) = 1$, as otherwise we would have $\pi_{\ell}(v) = \pi_{\ell}(u)$ whatever $\ell(vu)$ is, a contradiction. In other words, the label of wv by a p-proper L -labelling of G is forced to a .

From this, we can construct arbitrarily many trees T with $\text{ch}_{\Pi}^*(T) = 3$ and any wanted cardinality parity for the parts of its bipartition. As an illustration (which admits obvious generalisations), consider the tree T with vertex set $V(T) = \{v_1, \dots, v_8\}$ and edge set $E(T) = \{v_1v_2, v_2v_5, v_3v_4, v_4v_5, v_5v_6, v_6v_7, v_7v_8\}$, and note that T has no p-proper L -labelling for any list assignment L where $L(v_6v_7) = \{1, a^2\}$ and $L(v_2v_5) = L(v_4v_5) = \{1, a\}$ (for some $a \notin \{1, -1\}$).

2.2 Connections with the sum variant, and first bounds on ch_{Π}

As suggested by Seamone in [17], there is a straight connection between the parameters ch_{Σ} and ch_{Π}^* , which follows from the product rule of logarithms. Despite this fact being easy to visualise, we give a detailed proof to establish the precise relationship between the two.

Theorem 2.4. *If G is a nice graph, then $\text{ch}_{\Pi}^*(G) \leq 2\text{ch}_{\Sigma}(G) - 1$.*

Proof. Assume we have $\text{ch}_{\Sigma}(G) \leq k$ for some nice graph G and $k \geq 2$. We prove that $\text{ch}_{\Pi}^*(G) \leq 2k - 1$. Let L be a $(2k - 1)$ -list assignment to the edges of G , where none of the $L(e)$'s contains label 0. For every $e \in E(G)$, since $|L(e)| = 2k - 1$, there must be $S(e) \subset L(e)$ such that $|S(e)| = k$ and no two elements of $S(e)$ have the same absolute value. We set $X(e) = \{|x| : x \in S(e)\}$ and $L'(e) = \{\log(x) : x \in X(e)\}^1$. Then L' is a k -list assignment of G where each edge e is associated k values that are logarithms of values of $L(e)$ with different absolute values.

Our original assumption $\text{ch}_{\Sigma}(G) \leq k$ implies that G admits an s-proper L' -labelling ℓ' . We now consider an L -labelling ℓ of G obtained as follows. We consider every edge e of G , and we choose, as $\ell(e)$, any label from $L(e)$ that resulted in $L'(e)$ containing $\ell'(e)$. By how L' was obtained, note that, indeed, one such value belongs to $L(e)$. Thus, ℓ is an L -labelling. As a result, for every $v \in V(G)$ with incident edges e_1, \dots, e_d , we get

$$\sigma_{\ell}(v) = \sum_{i=1}^d \ell'(e_i) = \sum_{i=1}^d (\log |\ell(e_i)|) = \log \left(\prod_{i=1}^d |\ell(e_i)| \right) = \log(|\pi_{\ell}(v)|).$$

In particular, ℓ is p-proper since ℓ' is s-proper. \square

¹Throughout this work, any used log function can be in any fixed base.

The connection between ch_Σ and ch_Π^* in Theorem 2.4 implies that, for any constant upper bound on ch_Σ for some graph class, we deduce a constant upper bound on ch_Π^* as well. In the next result, we have listed some constant bounds on ch_Σ from the literature, together with the bounds on ch_Π^* we get as a consequence. It is worth emphasising that we do not claim this list to be exhaustive in any way. Namely, we only list the bounds that seem the most significant to us, and the interested reader has to be aware that more results of the sort below can be established from results mentioned in the references below.

Corollary 2.5. *Let G be a nice connected graph.*

- $\text{ch}_\Sigma(G) \leq \Delta(G) + 1$ (see [7]); thus $\text{ch}_\Pi^*(G) \leq 2\Delta(G) + 1$.
- If G is complete, complete bipartite, or a tree, then $\text{ch}_\Sigma(G) \leq 3$ (see [3]); thus $\text{ch}_\Pi^*(G) \leq 5$.
- If G is 2-degenerate and non-bipartite, then $\text{ch}_\Sigma(G) \leq 3$ (see [20]); thus $\text{ch}_\Pi^*(G) \leq 5$.
- If G is a wheel, then $\text{ch}_\Sigma(G) \leq 3$ (see [16]); thus $\text{ch}_\Pi^*(G) \leq 5$.
- If $\text{mad}(G) \leq \frac{11}{4}$, then $\text{ch}_\Sigma(G) \leq 3$ (see [13]); thus $\text{ch}_\Pi^*(G) \leq 5$.
- If G is outerplanar, then $\text{ch}_\Sigma(G) \leq 4$ (see [16]); thus $\text{ch}_\Pi^*(G) \leq 7$.
- If $\Delta(G) \leq 4$, then $\text{ch}_\Sigma(G) \leq 4$ (see [14]); thus $\text{ch}_\Pi^*(G) \leq 7$.
- If G is 2-connected and chordal, or a line graph, then $\text{ch}_\Sigma(G) \leq 5$ (see [19]); thus $\text{ch}_\Pi^*(G) \leq 9$.
- If G is a planar graph, then $\text{ch}_\Sigma(G) \leq 7$ (see [20]); thus $\text{ch}_\Pi^*(G) \leq 13$.

A consequence of the first item in Corollary 2.5, is that the List 1-2-3 Conjecture itself makes plausible the existence of a general constant upper bound on ch_Π^* . In particular, we currently have no evidence that the following, which would be a legitimate guess, might be false:

List Multiplicative 1-2-3 Conjecture. *If G is a nice graph, then $\text{ch}_\Pi^*(G) \leq 3$.*

Recall that observations raised at the end of Subsection 2.1 establish that this conjecture, if true, would actually be tight.

2.3 Algebraic tools

To improve, in next Section 3, some of the bounds from Corollary 2.5, we will adapt and employ an algebraic approach that was first designed by Bartnicki, Grytczuk and Niwczyk to deal with the List 1-2-3 Conjecture in [3], and which is inspired by polynomial methods developed to deal with list colouring of graphs.

Consider a graph G with edges e_1, \dots, e_m , and a list assignment L to the edges of G . For a vertex u and an edge e of G , we write $e \sim u$ if e is incident to u . Let \vec{G} be any orientation of G . To each edge e_i of G , we associate a variable x_i . Now, we associate to G (through \vec{G}) a polynomial $Q_{\vec{G}}$ with variables x_1, \dots, x_m , being

$$Q_{\vec{G}}(x_1, \dots, x_m) = \prod_{u\vec{v} \in A(\vec{G})} \left(\sum_{e_i \sim u} x_i - \sum_{e_i \sim v} x_i \right).$$

It is easy to see that G has an s -proper L -labelling if and only if there are values $l_1 \in L(e_1), \dots, l_m \in L(e_m)$ such that $Q_{\vec{G}}(l_1, \dots, l_m)$ does not vanish. From this point of view, a powerful tool is the so-called *Combinatorial Nullstellensatz* of Alon [1], which provides sufficient conditions, in terms of the sizes of the lists $L(e_1), \dots, L(e_m)$, for such values l_1, \dots, l_m to be choosable.

Combinatorial Nullstellensatz. *Let \mathbb{F} be an arbitrary field, and let $f = f(x_1, \dots, x_n)$ be a polynomial in $\mathbb{F}[x_1, \dots, x_n]$. Suppose the total degree of f is $\sum_{i=1}^n t_i$, where each t_i is a nonnegative integer, and suppose the coefficient of $\prod_{i=1}^n x_i^{t_i}$ is nonzero. If S_1, \dots, S_n are subsets of \mathbb{F} with $|S_i| > t_i$, then there are $s_1 \in S_1, \dots, s_n \in S_n$ so that $f(s_1, \dots, s_n) \neq 0$.*

Thus, bounds on $\text{ch}_\Sigma(G)$ can be obtained via the Combinatorial Nullstellensatz through studying the monomials in the expansion of $Q_{\vec{G}}$, more precisely monomials with nonzero coefficient, maximum degree, and, preferably, low exponent values. Note that all the monomials of $Q_{\vec{G}}$ share the very convenient property that they are all of maximum degree m , which is one of the prerequisites for the Combinatorial Nullstellensatz to work. The tricky part, actually, is about anticipating the coefficients of the monomials of $Q_{\vec{G}}$ (the nonzero ones, particularly), which are far from being obvious in general. In [3], the authors developed a very nice dedicated approach, which is based on studying the permanent of a particular matrix representing $Q_{\vec{G}}$.

A similar polynomial approach can of course be applied for deducing bounds on $\text{ch}_\Pi(G)$. The main difference is that, this time, we have to consider the products of labels incident to the vertices, instead of their sums. More precisely, the polynomial of interest is here

$$P_{\vec{G}}(x_1, \dots, x_m) = \prod_{\vec{uv} \in A(\vec{G})} \left(\prod_{e_i \sim u} x_i - \prod_{e_i \sim v} x_i \right).$$

Compared to the polynomial $Q_{\vec{G}}$, a big difference is that, in the expansion of $P_{\vec{G}}$, the monomials are likely to have different degrees, which means that the Combinatorial Nullstellensatz might apply to a few of them only. Even worse is that the degree of $P_{\vec{G}}$ is generally bigger than that of $Q_{\vec{G}}$, and, in particular, the exponents of the monomials generally tend to be bigger too. Note indeed that the degree of $Q_{\vec{G}}$ is precisely m , while the degree of $P_{\vec{G}}$ can be as large as $\sum_{uv \in E(G)} \max\{d(u), d(v)\}$ (which can be reached, e.g. when no two adjacent vertices of G have the same degree).

For these reasons, as will be seen in next Section 3, deducing bounds on ch_Π^* via the Combinatorial Nullstellensatz only, seems to be viable in particular contexts only.

3 Improved bounds on ch_Π^* for some graph classes

We here improve some of the bounds on ch_Π^* from Corollary 2.5. We first consider graphs in general, in Subsection 3.1. We then focus, in Subsection 3.2, on particular classes of graphs, including trees, planar graphs with large girth, and subcubic graphs. In the latter subsection, the exhibited improved bounds are optimal, or close to optimal.

3.1 General graphs

The bounds on ch_Π^* we establish in this section are expressed as functions of the maximum degree, our goal being to improve the bound of the first item of Corollary 2.4. We start off by improving that bound slightly for all nice graphs. From the bound we provide, we deduce, towards the List Multiplicative 1-2-3 Conjecture, that the List 1-2-3 Conjecture, if verified, would imply that $\text{ch}_\Pi^*(G) \leq 5$ holds for every nice graph G .

Theorem 3.1. *If G is a nice graph, then $\text{ch}_\Pi^*(G) \leq 2\Delta(G) - 1$.*

Proof. Let us denote by e_1, \dots, e_m the edges of G , and, for every $i \in \{1, \dots, m\}$, let x_i be a variable associated to e_i . Now, let \vec{G} be any orientation of G , and $P_{\vec{G}}$ be the polynomial with variables x_1, \dots, x_m defined as

$$P_{\vec{G}}(x_1, \dots, x_m) = \prod_{\vec{uv} \in A(\vec{G})} \left(\prod_{e_i \sim u} x_i - \prod_{e_i \sim v} x_i \right).$$

As described in Subsection 2.3, if L is a list assignment of G , and $P_{\vec{G}}(l_1, \dots, l_m) \neq 0$ for some $l_1 \in L(e_1), \dots, l_m \in L(e_m)$, then clearly we deduce a p-proper L -labelling of G .

Let $M = cx_1^{t_1} \dots x_m^{t_m}$ be a monomial of maximum degree from the expansion of $P_{\vec{c}}$ with $c \neq 0$. Such an M has to exist, since $\mathbb{R}[x_1, \dots, x_m]$ is an integral domain. We note that for every $i \in \{1, \dots, m\}$, we have $t_i \leq 2\Delta(G) - 1$. This is because variable x_i appears in at most $2\Delta(G) - 1$ factors of $P_{\vec{c}}$: once due to the edge e_i , and at most $2\Delta(G) - 2$ times due to the other edges incident to the two ends of e_i . By earlier arguments, the Combinatorial Nullstellensatz, due to the existence of M , now implies that $\text{ch}_{\Pi}(G) \leq 2\Delta(G)$, thus our conclusion on ch_{Π}^* . \square

The next bound is a significant improvement over Theorem 3.1, in the case of graphs having vertices with convenient neighbourhood properties.

Theorem 3.2. *If G is a nice graph with a vertex u such that $d(u) \geq 2$, $N(u)$ is a stable set, and $\text{ch}_{\Pi}^*(G - u) \leq \Delta(G - u) + 3$, then $\text{ch}_{\Pi}^*(G) \leq \Delta(G - u) + 3$.*

Proof. Let L be a $(\Delta(G - u) + 3)$ -list assignment to the edges of G , and let L' be the restriction of L to the edges of $G' = G - u$. Since $\text{ch}_{\Pi}^*(G') \leq \Delta(G') + 3$, there is a p-proper L' -labelling ℓ' of G' . Our aim is to extend ℓ' to a p-proper L -labelling ℓ of G , by considering the edges uv_1, \dots, uv_d ($d \geq 2$) incident to u , and, for each one uv_i of them, assigning it a label from $L(uv_i)$ so that, eventually, no conflict arises. For every $i \in \{1, \dots, d\}$, we denote by z_i the current product of v_i (i.e., by ℓ').

Because $N(v)$ is a stable set, note that assigning a label to any uv_i completely determines the product of v_i , in the sense that all edges incident to v_i get labelled. Thus, when labelling uv_i , we must ensure that v_i does not get in conflict with its neighbours different from u . Since $|N(v_i) \setminus \{u\}| \leq \Delta(G - u)$, and because $|L(uv_i)| = \Delta(G - u) + 3$, there are at least three distinct values in $L(uv_i)$ that can be assigned to uv_i without causing any conflict between v_i and its neighbours different from u . For every $i \in \{1, \dots, d\}$, we denote by S_i this subset of ‘‘safe’’ values of $L(uv_i)$ for uv_i . Because $|S_i| \geq 3$, there are in S_i at least two values a_i, b_i such that $|a_i| \neq |b_i|$.

We will be done if we can find an assignment of a_i 's and b_i 's to the uv_i 's for which u gets in conflict with none of the v_i 's. Such an assignment is actually guaranteed to exist by the Combinatorial Nullstellensatz. Indeed, for every $i \in \{1, \dots, d\}$, let x_i be a variable associated to the edge uv_i . Consider the polynomial

$$P(x_1, \dots, x_d) = \prod_{i=1}^d \left(\prod_{j=1}^d x_j - x_i z_i \right).$$

For every $i \in \{1, \dots, d\}$, set $y_i = \log(|x_i|)$. Note that considering P is similar to considering

$$P'(y_1, \dots, y_d) = \prod_{i=1}^d \left(\sum_{\substack{j=1 \\ j \neq i}}^d y_j - \log(z_i) \right),$$

which, because the z_i 's are constants, in the current context is similar to studying

$$P''(y_1, \dots, y_d) = \prod_{i=1}^d \sum_{\substack{j=1 \\ j \neq i}}^d y_j.$$

We remark that the monomial $y_1 \dots y_d$ in the expansion of P'' has strictly positive coefficient. Thus, by the Combinatorial Nullstellensatz, we can assign values to the y_i 's so that P'' (and thus P') does not vanish, assuming we have at least two possible values to choose from for each of them. From this, we get that we can assign values to the x_i 's so that P does not vanish as long as we have at least two possible values with distinct absolute values to choose from, for each of them. Particularly, this means that we can assign a label from $\{a_i, b_i\}$ to every uv_i , in such a way that a p-proper L -labelling of G results. \square

Together with checking a few base cases (which is done through results in next Subsection 3.2), Theorem 3.2 implies the following:

Corollary 3.3. *If G is a nice triangle-free graph, then $\text{ch}_{\Pi}^*(G) \leq \Delta(G) + 3$.*

3.2 Particular classes of graphs

Paths and cycles

Note that Theorem 3.1 implies that the List Multiplicative 1-2-3 Conjecture holds for nice graphs G with $\Delta(G) \leq 2$, i.e., paths and cycles. In such simple cases, this can actually be refined to a tightest result. In the sequel, for an $n \geq 2$ (or an $n \geq 3$ in the case of a cycle), we denote by P_n and C_n the path² and cycle, respectively, of length n . Note that, by a labelling of a path or cycle, adjacent vertices being distinguished by their products of incident labels are actually also distinguished by their sets of incident labels, and *vice versa*. Consequently, the next result also follows from results from [9]. Still, we present a full proof for the sake of self-containedness.

Theorem 3.4. *For an $n \geq 2$, we have:*

- $\text{ch}_{\Pi}^*(P_n) = 2$ if n is even or $n = 3$;
- $\text{ch}_{\Pi}^*(P_n) = 3$ otherwise.

For an $n \geq 3$, we have:

- $\text{ch}_{\Pi}^*(C_n) = 2$ if $n \equiv 0 \pmod{4}$;
- $\text{ch}_{\Pi}^*(C_n) = 3$ otherwise.

Proof. We deal with cycles first. Let us denote by e_0, \dots, e_{n-1} the successive edges of C_n , and by v_0, \dots, v_{n-1} its successive vertices, where $e_i = v_i v_{i+1}$ for every $i \in \{0, \dots, n-1\}$ (where, here and further, operations over the indexes are understood modulo n). For any two adjacent vertices v_i and v_{i+1} , note that, in order to get $\pi_{\ell}(v_i) \neq \pi_{\ell}(v_{i+1})$ by a labelling ℓ of C_n , we must have $\ell(v_{i-1}v_i) \neq \ell(v_{i+1}v_{i+2})$. Thus, for ℓ to be p-proper, any two edges of C_n at distance 2 apart must be assigned different labels.

Now consider G , the graph constructed from C_n by adding one vertex v_{e_i} in G for every edge e_i of C_n , and adding an edge $v_{e_i}v_{e_j}$ between any two vertices v_{e_i}, v_{e_j} of G if e_i and e_j are at distance exactly 2 in C_n . By a remark above, we have $\text{ch}_{\Pi}^*(C_n) = \text{ch}(G)$ (where $\text{ch}(G)$ refers to the usual choice number of G). Note that G is an odd-length cycle when n is odd, an union of two odd-length cycles when $n \equiv 2 \pmod{4}$, and an union of two even-length cycles when $n \equiv 0 \pmod{4}$. Since even-length cycles have choice number 2 and odd-length cycles have choice number 3 (see e.g. [8]), the result follows.

Regarding paths, remark first that if $n \equiv 1 \pmod{4}$, then P_n is a bipartite graph in which the two parts of the bipartition have odd cardinality. As described at the end of Subsection 2.1, we must have $\text{ch}_{\Pi}^*(P_n) > 2$ in such a situation, and we actually have $\text{ch}_{\Pi}^*(P_n) = 3$ by Theorem 3.1.

Let us now consider the remaining values of n . For a given $n \geq 2$, similarly as in the case of cycles, let us denote by e_1, \dots, e_n the successive edges of P_n , and by v_1, \dots, v_{n+1} its vertices, where $e_i = v_i v_{i+1}$ for every $i \in \{1, \dots, n\}$. Note that, contrarily to the case of cycles, labelling P_n is not similar to colouring G , the constraint graph of the edges at distance 2 in P_n , because, when labelling P_n , we must also guarantee that e_2 and e_{n-1} are not assigned label 1, so that v_1 and v_2 and not in conflict, and similarly for v_n and v_{n+1} . Note that G is here the union of two (possibly

²Let us emphasise that in our notation P_n , the subscript refers to the size of the path (i.e., its number of edges). While it is generally more common to have this subscript referring to the order of the path (i.e., its number of vertices), our way makes a bit more sense in the current context, since we are interested in labelling edges.

empty) paths; the new labelling constraint is similar to forbidding one colour for each of v_{e_2} and $v_{e_{n-1}}$. A problem is when these two vertices are the ends of a same path of G .

Indeed, from this remark, we note that in the cases where $n \equiv 3 \pmod{4}$ as well, there are 2-list assignments L of P_n such that P_n has no p-proper L -labelling. Indeed, note that the set of edges $\{e_2, e_4, \dots, e_{n-1}\}$ has odd cardinality due to the value of n . Recall that every two edges at distance 2 in P_n in this set must receive distinct labels by a p-proper labelling. Then note that if $L(e_2) = \{1, a\}$ for some $a \notin \{-1, 1\}$ and $L(e_{2k}) = \{a, b\}$ for some $b \notin \{1, a\}$ for every other edge $e_{2k} \notin \{e_2, e_{n-1}\}$, then, depending on the value of n , for either $L(e_{n-1}) = \{1, a\}$ or $L(e_{n-1}) = \{1, b\}$ label 1 must be assigned to one of e_2 and e_{n-1} by a p-proper L -labelling, which creates a conflict, a contradiction. Then $\text{ch}_{\Pi}^*(P_n) > 2$ for such a value of n , and we have $\text{ch}_{\Pi}^*(P_n) = 3$ by Theorem 3.1.

Let us now consider the remaining cases, i.e., those where $n = 3$ or n is even. Let L be a 2-list assignment of P_n . We deduce a p-proper L -labelling ℓ of P_n is the following way:

- If $n = 3$, then first assign to e_2 a label from $L(e_2)$ different from 1, before assigning distinct labels from $L(e_1)$ and $L(e_3)$ to e_1 and e_3 , respectively. Clearly, ℓ is p-proper.
- Assume n is even. If $n = 2$, then, clearly, we are done when assigning labels from $L(e_1)$ and $L(e_2)$ different from 1 to e_1 and e_2 , respectively. So assume $n \geq 4$. We first label the edges e_1, e_3, \dots, e_{n-1} with odd index with labels from their respective lists, in such a way that 1) $\ell(e_{n-1}) \neq 1$, and that 2) no two of these edges at distance 2 are assigned the same label. These conditions can clearly be achieved by labelling these edges one by one following the ordering $e_{n-1}, e_{n-3}, \dots, e_1$. We then achieve the same thing for the edges e_2, e_4, \dots, e_n with even index, so that 1) $\ell(e_2) \neq 1$, and that 2) no two of these edges at distance 2 are assigned a same label. Again, this can be easily achieved, e.g. by labelling these edges following the ordering e_2, e_4, \dots, e_n . By arguments above, ℓ is eventually p-proper. \square

Trees

We now prove an upper bound on ch_{Π}^* in the case of trees. The exhibited bound is optimal in general, due to some of the remarks at the end of Subsection 2.1. Even some paths attain the upper bound, recall Theorem 3.4.

Theorem 3.5. *If T is a nice tree, then $\text{ch}_{\Pi}^*(T) \leq 3$.*

Proof. The proof is by induction on the number of vertices and edges of T . The base case is when T is a path of length 2, in which situation the claim holds by Theorem 3.4. Thus, we can focus on proving the general case. Let L be a 3-list assignment to the edges of T .

We can assume that T has no pending path of length at least 3, i.e., a path uvw such that $d(u) = 1$, $d(v) = d(w) = 2$, and $d(x) \geq 2$. Indeed, assume T has such a path. Let $T' = T - \{u, v\}$. Clearly T' is nice (as otherwise T would be a path, a case for which Theorem 3.4 yields the desired conclusion), and thus T' admits a p-proper L' -labelling ℓ' , where L' denotes the restriction of L to the edges of T' . To extend ℓ' to a p-proper L -labelling of T , we have to assign to uv and vw labels from their lists, so that no conflict arises. To that aim, we first assign to vw a label different from 1 and from $\frac{\pi_{\ell'}(x)}{\ell'(xw)}$ so that w does not get in conflict with x . Note that this is possible since $|L(vw)| = 3$. Note that, now, because $\ell'(vw) \neq 1$, whatever label we assign to uv , we cannot get a conflict between u and v . Thus, when labelling uv , we just need to make sure that v does not get in conflict with w , which can easily be ensured since $|L(uv)| = 3$.

We may also assume that T has branching vertices, i.e., vertices with degree at least 3. Indeed, if T has no branching vertex, then T is a path, $\Delta(T) = 2$, and the claim follows from Theorem 3.4. So assume that T has branching vertices. Root T at any branching vertex r . This defines the usual root-to-leaf orientation, through which every non-root vertex has a unique *parent*, i.e., a neighbour that is closer to r , and every non-leaf vertex v has *sons*, i.e., neighbours that are farther from r , and, more generally, *descendants*, i.e., vertices for which the unique path to r goes through v .

Let u be a branching vertex of T that is at farthest distance from r . Note that we have $u = r$ if r is the unique branching vertex of T . By this choice, u has at least two descendants, all of which have degree at most 2. In other words, the descendants of u form $k \geq 2$ disjoint pending paths, none of which has length more than 2, as mentioned earlier. There are then $k = p + q \geq 2$ pending paths attached at u formed by its descendants, where $p \geq 0$ of these paths have length 2, while $q \geq 0$ of them have length 1. We denote by $v_1, \dots, v_p, w_1, \dots, w_q$ the sons of u , where v_1, \dots, v_p belong to pending paths of length 2, while w_1, \dots, w_q are leaves. We also denote by v'_1, \dots, v'_p the neighbour of v_1, \dots, v_p , respectively, different from u . Thus, the v_i 's have degree 2, while the v'_i 's and the w_i 's have degree 1. Lastly, we denote by t the parent of u , if it exists (recall that we have $u = r$ when T has only one branching vertex, in which case u has no parent).

Let $T' = T - \{v_1, \dots, v_p, v'_1, \dots, v'_p, w_1, \dots, w_q\}$. The tree T' is nice, because either r is a branching vertex (case where $u \neq r$) or T' consists in only one vertex (case where $u = r$), and thus T' admits a p -proper L' -labelling ℓ' , where L' denotes the restriction of L to the edges of T' . To extend ℓ' to a p -proper L -labelling of T , we just have to assign labels from their lists to the edges incident to the descendants of u , so that no conflict arises.

We distinguish several cases, based mainly on the value of q .

- Suppose that $q = 0$. Label every edge uv_i with $i \in \{1, \dots, p-1\}$ with an arbitrary label from $L(uv_i)$ different from 1. Now, label uv_p with a label from $L(uv_p)$ different from 1 so that u does not get in conflict with t , if it exists (in case it does not, just assign any label different from 1 to uv_p). Note that this is possible since $|L(uv_p)| = 3$. Lastly, consider every edge $v_i v'_i$. Since $\ell(uv_i) \neq 1$, note that v_i and v'_i cannot get in conflict, whatever label from $L(v_i v'_i)$ is assigned to $v_i v'_i$. Thus, when labelling $v_i v'_i$, we just need to ensure that v_i and u do not get in conflict, which can be avoided since $|L(v_i v'_i)| = 3$.
- Suppose now that $q = 1$. Recall that $p \geq 1$ since $k = p + q \geq 2$. We start by labelling, for every $i \in \{1, \dots, p-1\}$, the edge uv_i with any label different from 1, chosen from $L(uv_i)$. We then consider uv_p , and assign to this edge a label from $L(uv_p)$ different from 1 so that the resulting partial product of u is different from 1. Note that this is possible since $|L(uv_p)| = 3$. Now, note that, by this choice of $\ell(uv_p)$, no matter what $\ell(uw_1)$ is, we cannot get a conflict between u and w_1 . We then assign as $\ell(uw_1)$ a label from $L(uw_1)$ so that u does not get in conflict with t (if it exists). Lastly, we consider every $i \in \{1, \dots, p\}$, and, to every edge $v_i v'_i$, we assign a value from $L(v_i v'_i)$ so that v_i and u do not get in conflict. This results in ℓ being p -proper. Recall, in particular, that any two v_i, v'_i cannot be in conflict since $\ell(uv_i) \neq 1$.

Suppose now that $q \geq 2$. We start by stating the following general claim:

Claim 3.6. *Let S be a star with center u and $q + 1 \geq 3$ leaves t, w_1, \dots, w_q . Assume we have a partial labelling ℓ' of S where ut is the only edge being assigned a label, a , and that t has (virtual) product $\pi_{\ell'}(t) = A$. If L is a 3-list assignment to the uw_i 's, then, for every $i \in \{1, \dots, q\}$, we can assign a label from $L(uw_i)$ to uw_i , so that ℓ' is extended to a labelling ℓ of S verifying $\pi_{\ell}(u) \notin \{A, \pi_{\ell}(w_1), \dots, \pi_{\ell}(w_q)\}$.*

Proof of the claim. Suppose first that $q = 2$. We first assign to uw_1 a label from $L(uw_1)$ different from $1/a$. This way, no matter what label is assigned to uw_2 , note that u and w_2 cannot get in conflict. We now assign a label from $L(uw_2)$ to uw_2 so that the resulting product of u is different from A and the product of w_1 . This is possible since $|L(uw_2)| = 3$.

Assume now that $q \geq 3$. We distinguish the following cases:

- Assume, w.l.o.g., that the three values in $L(uw_1)$ have pairwise distinct absolute values. To each edge uw_i , we associate a variable x_i , and we consider the polynomial

$$P(x_1, \dots, x_q) = \left(a \prod_{i=1}^q x_i - A \right) \cdot \prod_{i=1}^q \left(a \prod_{j=1}^q x_j - x_i \right).$$

For every $i \in \{1, \dots, q\}$, we set $y_i = \log x_i$. Now the polynomial P becomes equivalent to

$$P'(y_1, \dots, y_q) = \left(\log(a) + \sum_{i=1}^q y_i - \log(A) \right) \cdot \prod_{i=1}^q \left(\log(a) + \sum_{j=1}^q y_j - y_i \right).$$

Note that, in the expansion of P' , the monomial $y_1^2 y_2 \dots y_q$ has strictly positive coefficient. Thus, by the Combinatorial Nullstellensatz, we can assign values to the y_i 's so that P' does not vanish, as long as we are given a set of at least three possible distinct values as y_1 , and a set of at least two possible distinct values as each of y_2, \dots, y_q . In turn, this means we can assign values to the x_i 's so that P does not vanish, as long as we have a set of at least three possible values with pairwise distinct absolute values as x_1 , and a set of at least two possible values with distinct absolute values as each of x_2, \dots, x_q . Recall that we made the assumption that the three values in $L(uw_1)$ have pairwise distinct absolute values, while, for every $i \in \{2, \dots, q\}$, there must be at least two values in $L(uw_i)$ with distinct absolute values, since $|L(uw_i)| = 3$. Thus, ℓ' can correctly be extended to ℓ , in the desired way.

- Now assume that every $L(uw_i)$ is of the form $\{\alpha_i, \beta_i, -\beta_i\}$, where α_i and β_i are distinct values with the same sign. Let us start from the labelling ψ of S obtained from ℓ' after setting $\ell(uw_i) = \alpha_i$ for every $i \in \{1, \dots, q\}$. We denote by $s \in \{-, +\}$ the sign of $\pi_\psi(u)$, while, for every sign $\epsilon \in \{-, +\}$, we denote by W^ϵ the set of vertices w_i for which the sign of $\pi_\psi(w_i)$ (thus, of α_i and β_i) is ϵ . Note that W^- and W^+ partition the w_i 's.

To conclude the proof, we consider two last main cases.

- Suppose that $s = +$ and $W^- = \emptyset$. We start by assigning label $-\beta_1$ from $L(uw_1)$ to uw_1 . Note that, as long as each uw_i with $i \in \{2, \dots, q\}$ is assigned a label from $\{\alpha_i, \beta_i\}$, we cannot get a conflict between u and w_i due to their products having different signs. Thus, under that convention, the only conflicts we must pay attention to, are along the edges uw_1 and, possibly, ut (in case A is negative).

We here assign a variable x_i to each edge uw_i with $i \in \{2, \dots, q\}$, and consider

$$P(x_2, \dots, x_q) = \left(-\beta_1 a \prod_{i=2}^q x_i - A \right) \cdot \left(-\beta_1 a \prod_{i=2}^q x_i - \beta_1 \right).$$

For every $i \in \{1, \dots, q\}$, we again set $y_i = \log x_i$. Then P is equivalent to

$$P'(y_2, \dots, y_q) = \left(\log(-\beta_1 a) + \sum_{i=2}^q y_i - \log(A) \right) \cdot \left(\log(-\beta_1 a) + \sum_{i=2}^q y_i - \log(\beta_1) \right).$$

Recall that $q \geq 3$. Then, whatever q is, in the expansion of P' the monomial $y_2 y_3$ has strictly positive coefficient. The Combinatorial Nullstellensatz then implies that we can assign values to y_2, \dots, y_q so that P' does not vanish, assuming we have at least two values to choose from for each of y_2 and y_3 , and at least one value to choose from for each of y_4, \dots, y_q . From this, we deduce that we can assign values to vw_2, \dots, vw_q from $\{\alpha_2, \beta_2\}, \{\alpha_3, \beta_3\}, \{\alpha_4\}, \{\alpha_5\}, \dots, \{\alpha_q\}$, respectively, so that u is in conflict with none of w_1 and t . Recall that the resulting sign of $\pi_\ell(u)$ is negative, while the sign of all vertices w_i with $i \in \{2, \dots, q\}$ is positive. Thus, these vertices also cannot be in conflict.

- Suppose that $s = +$ and $W^- \neq \emptyset$. Assume w.l.o.g. that $w_1 \in W^-$. Recall that, as long as u and w_1 get products with different signs by a labelling, they cannot be in conflict. Thus, we here get our conclusion through the Combinatorial Nullstellensatz, by not modelling the possible conflict between u and w_1 . The precise details are as follows. For every

$i \in \{1, \dots, q\}$, let x_i be a variable associated to uw_i . We consider the polynomial

$$P(x_1, \dots, x_q) = \left(a \prod_{i=1}^q x_i - A \right) \cdot \prod_{i=2}^q \left(a \prod_{j=1}^q x_j - x_i \right).$$

For every $i \in \{1, \dots, q\}$, we set $y_i = \log x_i$. Then P is equivalent to

$$P'(y_1, \dots, y_q) = \left(\log(a) + \sum_{i=1}^q y_i - \log(A) \right) \cdot \prod_{i=2}^q \left(\log(a) + \sum_{j=1}^q y_j - y_i \right).$$

In the expansion of P' , the monomial $y_1 \dots y_q$ has strictly positive coefficient, and, thus, by the Combinatorial Nullstellensatz, we can assign labels from $\{\alpha_1, \beta_1\}, \dots, \{\alpha_q, \beta_q\}$ to uw_1, \dots, uw_q , respectively, resulting in a labelling ℓ of S where u gets in conflict with none of w_2, \dots, w_q, t . Proceeding that way, recall that the sign of $\pi_\ell(u)$ is positive, while that of $\pi_\ell(w_1)$ is negative. Then, also u and w_1 cannot be in conflict, and ℓ is p-proper.

To conclude the proof, let us point out that the cases where $s = -$ can be treated in a symmetric way, by considering whether W^+ is empty or not. \diamond

We are now ready to conclude the proof of Theorem 3.5. Recall that we have obtained a labelling ℓ' of $T' = T - \{v_1, \dots, v_p, v'_1, \dots, v'_p, w_1, \dots, w_q\}$ by induction, and that we are in the case where u is adjacent to $q \geq 2$ leaves (and, possibly, p v_i 's and one parent t). We start extending ℓ' to T by considering every edge uv_i (if such edges exist) and assigning to it a label from $L(uv_i)$ different from 1. This is clearly possible, since $|L(uv_i)| = 3$. We now apply Claim 3.6 to the uw_i 's to get all edges incident to u labelled, in such a way that u is not in conflict with any of t (if it exists; if it does not, then note that the claim applies in a very close way) and the w_i 's. The main difference here, is that, though we do not have to care about possible conflict between u and the v_i 's for now, the claim must be employed with taking into consideration the contribution of the uv_i 's to the product of u . Lastly, it remains to label every $v_i v'_i$ with a label from $L(v_i v'_i)$ so that v_i and u do not get into conflict, which is possible since we have three possible labels. Recall in particular that v_i and v'_i cannot be in conflict since $\ell(uv_i) \neq 1$. Eventually, ℓ is p-proper, as desired. \square

Planar graphs with large girth

Recall that a *planar graph* is a graph that can be embedded in the plane so that no two edges cross, and that, for any graph G , the *girth* $g(G)$ of G refers to the length of its shortest cycles. In case G has no cycle, we set $g(G) = \infty$.

Planar graphs with large enough girth are known to be 2-degenerate and to have low maximum average degree. Thus, the third and fifth items of Corollary 2.5 establish 5 as a constant upper bound on $\text{ch}_{\Pi}^*(G)$ when G is indeed a nice planar graph with large girth. In what follows, we improve this upper bound down to 4 when $g(G) \geq 16$, getting closer to the List Multiplicative 1-2-3 Conjecture for this class of graphs. Our proof involves arguments that are reminiscent to those used to prove Theorem 3.5, combined together with the following structural result:

Theorem 3.7 (e.g. Nešetřil, Raspaud, Sopena [15]). *If G is a planar graph with girth $g(G) \geq 5\ell + 1$ for some $\ell \geq 1$, then either:*

- $\delta(G) = 1$, or
- G contains an ℓ -thread, i.e., a path $uv_1 \dots v_\ell w$ where $d(u), d(w) \geq 2$, and $d(v_i) = 2$ for every $i \in \{1, \dots, \ell\}$.

We are now ready to prove our result.

Theorem 3.8. *If G is a nice planar graph with girth $g(G) \geq 16$, then $\text{ch}_\Pi^*(G) \leq 4$.*

Proof. Assume the claim is wrong, and let G be a minimal counterexample to the claim. We may assume that G is connected, and, due to Theorems 3.4 and 3.5, that $\Delta(G) \geq 3$ and that G is not a tree. Let L be a 4-list assignment to the edges of G . We prove the result by contradicting the existence of G , i.e., by showing that G admits p-proper L -labellings, whatever L is.

If $\delta(G) \geq 2$, then, by Theorem 3.7, we can find a 3-thread $uv_1v_2v_3w$ in G . In that case, we consider $G' = G - v_2$. Note that G' may consist in up to two connected components, each of which has at least two edges (since $d(u), d(w) \geq 2$, by the assumption on $\delta(G)$) and girth at least 16 (in case there is only one connected component, G' might be a tree; in that case, $g(G') = \infty$, and the girth condition remains true). So G' is nice and planar, and, by minimality of G , there is a p-proper L' -labelling ℓ' of G' , where L' denotes the restriction of L to the edges of G' . To obtain a contradiction, it now suffices to extend ℓ' to a p-proper L -labelling of G , and, for this, we just have to assign labels from $L(v_1v_2)$ and $L(v_2v_3)$ to v_1v_2 and v_2v_3 , respectively, so that no conflict arises. This can clearly be done since $|L(v_1v_2)| = |L(v_2v_3)| = 4$, by first assigning to v_1v_2 a label different from $\ell'(v_3w)$ for which v_1 and u get different partial products, and then assigning to v_2v_3 a label so that v_1 and v_2 are not in conflict, and similarly for v_3 and w .

We may thus assume that $\delta(G) = 1$. Since G is not a tree, this means that, by repeatedly removing vertices of degree 1 while there are some, we end up with a planar connected graph G^- such that $\delta(G^-) \geq 2$ and $g(G^-) \geq 16$. More precisely, for every $v \in V(G) \cap V(G^-)$, we can denote by T_v the pending tree rooted at v in G , which, if $d_G(v) = d_{G^-}(v)$, is reduced to the single vertex v . Then G^- is obtained from G by contracting every T_v to v . For every $v \in V(G) \cap V(G^-)$, we deal, in G , with T_v through the terminology introduced in the proof of Theorem 3.5 (in particular, the notions of parent, son, descendant and branching vertex have the exact same meaning).

Because $g(G^-) \geq 16$, then, by Theorem 3.7, we deduce that G^- has a 3-thread $P = uv_1v_2v_3w$. Note that P also exists back in G , the difference being that v_1, v_2, v_3 might each be the root of a pending tree (denoted $T_{v_1}, T_{v_2}, T_{v_3}$, respectively, following our terminology) that might have edges. In case we have $V(T_{v_i}) = \{v_i\}$ for every $i \in \{1, 2, 3\}$, then note that P is actually a 3-thread in G , in which case a contradiction can be obtained in the similar way as in the previous case $\delta(G) \geq 2$. Thus, in what follows, we assume that some of $T_{v_1}, T_{v_2}, T_{v_3}$ are not reduced to a single vertex.

By arguments similar to some used in the proof of Theorem 3.5, we may assume that none of $T_{v_1}, T_{v_2}, T_{v_3}$ has 1) a non-root branching vertex, or 2) a pending path of length at least 3 (remind, in particular, that in the current context there is even more room for labelling extensions, due to L being a 4-list assignment). This means that each T_{v_i} is a subdivided star with center v_i , where the pending paths attached to v_i (if any) have length 1 or 2.

We start by handling a very particular case, which is when every T_{v_i} has only one edge $v_iv'_i$, i.e., is a star with a single edge $v_iv'_i$. In this case, we consider $G' = G - v_2$. A p-proper L' -labelling of G' (where, again, L' denotes the restriction of L to G'), which exists by minimality of G , can then be extended to a p-proper L -labelling of G , a contradiction, by first labelling v_1v_2 with a label from $L(v_1v_2)$ so that no conflict between v_1 and its two neighbours different from v_2 arises, then labelling v_2v_3 with a label from $L(v_2v_3)$ so that 1) no conflict between v_3 and its two neighbours different from v_2 arises, and 2) v_2 gets partial product different from 1; and lastly labelling the edge $v_2v'_2$ of T_{v_2} with a label from $L(v_2v'_2)$ so that no conflict between v_2 and its two neighbours different from v'_2 arises. Recall, in particular, that v_2 and v'_2 cannot be in conflict due to how v_2v_3 was labelled. Note also that lists of four labels are indeed sufficient to achieve this whole process.

In the more general case, let us consider the graph $G' = G - (V(T_{v_1}) \setminus \{v_1\}) - V(T_{v_2}) - (V(T_{v_3}) \setminus \{v_3\})$ (obtained by removing the non-root vertices of T_{v_1} and T_{v_3} , and the whole of T_{v_2}). By arguments used earlier in the case where $\delta(G) \geq 2$, there is a p-proper L' -labelling ℓ' of G' , where L' denotes the restriction of L to the edges of G' . Our goal, to get a final contradiction, is

to extend ℓ' in a p-proper way to the edges v_1v_2 , v_2v_3 and those in $T_{v_1}, T_{v_2}, T_{v_3}$, assigning labels from their respective lists, so that a p-proper L -labelling of G results.

We start by assigning labels from $L(v_1v_2)$ and $L(v_2v_3)$ to v_1v_2 and v_2v_3 , respectively, in such a way that, for the resulting partial products of v_1, v_2, v_3 , 1) v_2 is in conflict with none of v_1 and v_3 , 2) v_1 is not in conflict with u , 3) v_3 is not in conflict with w , and 4) none v_i of v_1, v_2, v_3 for which T_{v_i} contains only one edge, gets product 1 as a result. This is possible to achieve since $|L(v_1v_2)| = |L(v_2v_3)| = 4$. More precisely, this can be achieved by labelling v_1v_2 first and v_2v_3 second if T_{v_1} has only one edge, or by labelling v_2v_3 first and v_1v_2 second otherwise. Recall, in particular, that we have treated separately the case where all of $T_{v_1}, T_{v_2}, T_{v_3}$ have only one edge, so we are not in that case; the fourth condition must thus be fulfilled for at most two of the v_i 's.

It now remains to label the edges from the T_{v_i} 's. We achieve this by considering T_{v_1}, T_{v_2} and T_{v_3} in turn, so that, once every T_{v_i} has been treated, no vertex in $V(T_{v_1}) \cup \dots \cup V(T_{v_i})$ is involved in conflicts, and none of the vertices in $V(T_{v_{i+1}}) \cup \dots \cup V(T_{v_3})$ had its product altered. This way, the desired p-proper L -labelling of G will result once T_{v_3} has been treated. In what follows, we focus on T_{v_1} , but the arguments apply similarly for T_{v_2} and T_{v_3} .

Recall that T_{v_1} consists of some (possibly none) pending paths of length 1 or 2 attached to v_1 . Let us assume that $p \geq 0$ of these paths have length 2, while $q \geq 0$ of them have length 1. We denote by b_1, \dots, b_p the sons of v_1 that belong to the pending paths of length 2, while we denote by c_1, \dots, c_q those from the pending paths of length 1. Finally, for every $i \in \{1, \dots, p\}$, we denote by b'_i the son of b_i in T_{v_1} . By how v_1v_2 was labelled earlier, note that we already have the desired conclusion around v_1 if $p = q = 0$. We thus focus on the cases where $p + q > 0$.

- The cases where $q \in \{0, 1\}$ can be treated quite similarly as the cases $q = 0$ and $q = 1$ in the proof of Theorem 3.5. Namely, we first label the edges $v_1b_1, \dots, v_1b_{p-1}$ (if such exist) with labels different from 1 from their respective lists. If $q = 0$, then we label v_1b_p with a label different from 1 from its list, with making sure that the resulting product of v_1 is different from that of u and v_2 . Otherwise, if $q = 1$, then we label v_1b_p with a label different from 1 from its list, with making sure that the resulting partial product of v_1 does not get equal to 1 (if $p = 0$, then recall that this property is already verified at v_1 , due to how v_1v_2 and v_2v_3 have been labelled). Still in the case where $q = 1$, this guarantees that v_1 and c_1 cannot get in conflict no matter how v_1c_1 is labelled; thus, we can label v_1c_1 with a label from its list so that v_1 does not in conflict with u and v_2 . Note that lists of size 4 are sufficient to achieve these conditions in all cases. We lastly label every edge $b_ib'_i$ (if any) with a label from its list, with making sure that b_i does not get in conflict with v_1 . Because v_1b_i was assigned a label different from 1, recall that b_i and b'_i cannot be in conflict.
- The cases where $q = 2$ can be treated quite similarly. Start by labelling every edge v_1b_i (if there are some) with a label different from 1 from its list. Then, label v_1c_1 with a label from its list, so that the resulting partial product of v_1 does not get equal to 1. Last, label v_1c_2 with a label from its list, so that v_1 gets in conflict with none of u , v_2 and c_1 . Note that this is possible, since we do not have to care about a possible conflict between v_1 and c_2 , and $|L(v_1c_2)| = 4$. To conclude, we can eventually label the $b_ib'_i$'s just as in the previous case.

The general case is when $q \geq 3$. We need a generalisation of Claim 3.6 to the current context.

Claim 3.9. *Let S be a star with center u and $q + 2 \geq 5$ leaves t, t', w_1, \dots, w_q . Assume we have a partial labelling ℓ' of S where ut and ut' are the only edges being assigned a label, a and a' , respectively, and that t and t' have (virtual) product $\pi_{\ell'}(t) = A$ and $\pi_{\ell'}(t') = A'$. If L is a 4-list assignment to the uw_i 's, then, for every $i \in \{1, \dots, q\}$, we can assign a label from $L(uw_i)$ to uw_i , so that ℓ' is extended to a labelling ℓ of S verifying $\pi_{\ell}(u) \notin \{A, A', \pi_{\ell}(w_1), \dots, \pi_{\ell}(w_q)\}$.*

Proof of the claim. Note that each $L(uw_i)$ contains two, three or four values with pairwise distinct absolute values. We consider several cases based on that fact.

- Assume, w.l.o.g., that the four values in $L(uw_1)$ have pairwise distinct absolute values. To each edge uw_i , we associate a variable x_i , and we consider the polynomial

$$P(x_1, \dots, x_q) = \left(aa' \prod_{i=1}^q x_i - A \right) \cdot \left(aa' \prod_{i=1}^q x_i - A' \right) \cdot \prod_{i=1}^q \left(aa' \prod_{j=1}^q x_j - x_i \right).$$

For every $i \in \{1, \dots, q\}$, we set $y_i = \log x_i$. Then P gets equivalent to

$$P'(y_1, \dots, y_q) = \left(\log(aa') + \sum_{i=1}^q y_i - \log(A) \right) \cdot \left(\log(aa') + \sum_{i=1}^q y_i - \log(A') \right) \\ \cdot \prod_{i=1}^q \left(\log(aa') + \sum_{j=1}^q y_j - y_i \right).$$

In the expansion of P' , the monomial $y_1^3 y_2 \dots y_q$ has strictly positive coefficient. Thus, by the Combinatorial Nullstellensatz, we can assign values to the y_i 's so that P' does not vanish, as long as we are given a set of at least four possible distinct values as y_1 , and a set of at least two possible distinct values as each of y_2, \dots, y_q . Regarding P , this implies we can assign values to the x_i 's so that P does not vanish, assuming we have a set of at least four possible values with pairwise distinct absolute values as x_1 , and a set of at least two possible values with distinct absolute values as each of x_2, \dots, x_q . This is met in the current case, since $L(uw_1)$ is assumed to have four values with pairwise distinct absolute values, and $|L(uw_i)| = 4$ for every $i \in \{2, \dots, q\}$. Thus, ℓ' can be extended to ℓ as desired.

- Assume now that, w.l.o.g., both $L(uw_1)$ and $L(uw_2)$ include three values with pairwise distinct absolute values. Then the same conclusion as in the previous case can be reached from considering the monomial $y_1^2 y_2^2 y_3 \dots y_q$ in the expansion of P' .
- We can thus assume that none of the two previous cases applies, i.e., that, w.l.o.g., $L(uw_1)$ includes two or three values with pairwise distinct absolute values, while $L(uw_2), \dots, L(uw_q)$ include each exactly two values with pairwise distinct absolute values. In other words, we have $L(uw_i) = \{\alpha_i, -\alpha_i, \beta_i, -\beta_i\}$ for every $i \in \{2, \dots, q\}$, for some distinct α_i, β_i , while $L(uw_1) = \{\alpha_1, -\alpha_1, \beta_1, -\beta_1\}$ or $L(uw_1) = \{\alpha_1, -\alpha_1, \beta_1, \gamma_1\}$, for some distinct $\alpha_1, \beta_1, \gamma_1$. To conclude the proof, we consider a few more cases:

- Assume first that A and A' have the same sign $s \in \{-, +\}$. For every $i \in \{1, \dots, q-2\}$, let us assign to uw_i a label with sign s from its list. Then:
 - * If s and the sign of the partial product of u are the same, then we assign to uw_{q-1} a label with sign s from its list, chosen so that the partial product of u gets different from 1. Note that this is possible, since $L(uw_{q-1})$ contains two values with sign s . This guarantees that u and w_q cannot be in conflict, whatever the label of uw_q is. We then assign to uw_q a label with sign $-s$ from its list, so that all edges are labelled and no conflict remains. In particular, u gets product with sign $-s$, while only w_q has this property.
 - * Otherwise, i.e., if s and the sign of the partial product of u are different, then we assign to uw_{q-1} and uw_q a label with sign s from their lists. As a result, no conflict remains, since u is the only vertex with product being of sign $-s$.
- Now assume that A and A' have different signs, say A is positive while A' is negative. We here start by assigning, for every $i \in \{1, \dots, q-2\}$, a positive label to uw_i from its list $L(uw_i)$. Now:

- * If currently u has negative product, then we assign to uw_{q-1} and uw_q a positive label from their respective lists, with making sure that the product of u gets different from A' . This is possible since $L(uw_{q-1})$ and $L(uw_q)$ have two positive values each. Since only u and t' have negative product, no conflict remains.
- * Otherwise, i.e., u currently has positive product, then we first assign a positive label to uw_{q-1} from its list, chosen so that the current product of u does not get equal to 1. This is possible, since $L(uw_{q-1})$ contains two positive values. This guarantees that u and w_q cannot get in conflict. We then assign to uw_q a negative label from $L(uw_q)$, chosen so that u gets product different from A' . This is possible since $L(uw_q)$ contains two negative values. Since only A' and the products of u and w_q are negative, no conflict remains.

In all cases, we end up with the desired labelling ℓ , which concludes the proof. \diamond

We can now conclude the case $q \geq 3$ of the proof of Theorem 3.8, thus proving the whole statement. We start by labelling every edge v_1b_i (if any) with any label different from 1 from its list $L(v_1b_i)$. We now apply Claim 3.9 to get all v_1c_i 's labelled with labels from their lists, so that v_1 is not in conflict with any of u , v_2 and the c_i 's. This can be done by applying Claim 3.9 with v_1 , u and v_2 playing the role of u , t and t' , respectively, $\pi_{\ell'}(u)$ and $\pi_{\ell'}(v_2)$ playing the role of A and A' , respectively, $\ell'(uv_1) \prod_{i=1}^p \ell(ub_i)$ and $\ell'(v_1v_2)$ playing the role of a and a' , respectively, and the c_i 's playing the role of the w_i 's. It remains to label the $b_ib'_i$'s (if any), and, for each such edge $b_ib'_i$, it suffices to assign a label from its list so that b_i and v_1 do not get in conflict. Recall that we do not have to mind about a possible conflict between b_i and b'_i , since $\ell(v_1b_i) \neq 1$. \square

3.2.1 Subcubic graphs

We now consider subcubic graphs, i.e., graphs with maximum degree 3. Note that, at this point, the best upper bound we have on ch_{Π}^* for these graphs is 5, obtained from Theorem 3.1. We get one step closer to the List Multiplicative 1-2-3 Conjecture for this class of graphs, by lowering the upper bound down to 4 in the next result.

Theorem 3.10. *If G is a nice subcubic graph, then $\text{ch}_{\Pi}^*(G) \leq 4$.*

Proof. Assume the claim is wrong, and consider G a minimal counterexample to the claim. Clearly, G is connected. Let L be a 4-list assignment to the edges of G . We prove below that G admits a p-proper L -labelling whatever L is, a contradiction. To that aim, we first show that G is cubic:

- Assume first that $\delta(G) = 1$, and consider u a degree-1 vertex of G with unique neighbour v .
 - Assume first that $d(v) = 2$, and let w denote the second neighbour of v . Set $G' = G - \{u, v\}$. We can assume that G' is nice, as otherwise G would be the path of length 3, in which case even $\text{ch}_{\Pi}^*(G) \leq 3$ holds by Theorem 3.4, a contradiction. Then, by minimality of G , there is a p-proper L' -labelling ℓ' of G' , where L' denotes the restriction of L to the edges of G' . We extend ℓ' to a p-proper L -labelling of G , getting a contradiction, by correctly assigning labels to uv and vw from their respective lists. We first label vw , by assigning a label from $L(vw)$ that is different from 1, and so that w does not get in conflict with any of its at most two other neighbours different from v . Note that this is possible since $|L(vw)| = 4$. We can now extend the labelling to uv by assigning a label from $L(uv)$ so that v does not get in conflict with w . Note that by how vw was labelled, u and v cannot get in conflict.
 - Assume now that $d(v) = 3$, and let w_1, w_2 denote the two neighbours of v different from u . Set $G' = G - \{u, v\}$. We can assume that G' is nice, as otherwise either 1) one of the w_i 's is a degree-2 vertex adjacent to a 1-vertex, or 2) w_1w_2 exists and both w_1 and

w_2 have degree 2. In the former case, we fall into the previous case (where $d(v) = 2$) we have handled. In the latter case, G has only four edges and the claim can be checked by hand. So G' is nice, and, by minimality of G , there is a p-proper L' -labelling ℓ' of G' , where L' denotes the restriction of L to the edges of G' . To extend it to one of G , thus getting a contradiction, we proceed as follows. For every $i \in \{1, 2\}$, note that there are at least two values $a_i, b_i \in L(uw_i)$ that can be assigned to vw_i without causing any conflict between w_i and its at most two neighbours different from v . We assign labels to vw_1 and vw_2 from $\{a_1, b_1\}$ and $\{a_2, b_2\}$, respectively, so that the product of these two labels is different from 1. It then suffices to assign to uv a label from $L(uv)$ so that v gets in conflict with none of w_1 and w_2 , which is possible since $|L(uv)| = 4$. Again, u and v cannot be in conflict due to how vw_1 and vw_2 have been labelled.

- Assume now that $\delta(G) = 2$, and consider u a degree-2 vertex of G with neighbours v_1, v_2 . By the minimum degree assumption, each of v_1 and v_2 has one or two neighbours different from u . We here consider $G' = G - u$. We can assume that G' is nice, as, because $\delta(G) = 2$, otherwise it would mean that v_1v_2 is the only other edge, thus that G is C_3 , the cycle of length 3, in which case $\text{ch}_{\Pi}^*(G) \leq 3$ holds by Theorem 3.4, a contradiction. So G' admits a p-proper L' -labelling ℓ' , where L' is the restriction of L to the edges of G' . We show that this p-proper labelling can be extended to uv_1 and uv_2 by assigning labels from their lists, thereby getting a contradiction.

Let x_1, x_2 be variables associated to uv_1 and uv_2 , respectively. Let us denote by y_1, y_2 the values $\pi_{\ell'}(v_1), \pi_{\ell'}(v_2)$, respectively. Let us now consider the polynomial

$$P(x_1, x_2) = (x_1x_2 - x_1y_1) \cdot (x_1x_2 - x_2y_2) \cdot \prod_{w \in N(v_1) \setminus \{u\}} (x_1y_1 - \pi_{\ell'}(w)) \cdot \prod_{w \in N(v_2) \setminus \{u\}} (x_2y_2 - \pi_{\ell'}(w)).$$

If x_1 and x_2 can be assigned values in $L(uv_1)$ and $L(uv_2)$, respectively, so that P does not vanish, then we get a p-proper L -labelling of G . Since x_1 and x_2 are the only variables of P , it is easy to see that, in the expansion of P , the monomial M with largest degree is either $x_1^4x_2^4$ (when $d(v_1) = d(v_2) = 3$), $x_1^3x_2^4$ (when $d(v_1) = 2$ and $d(v_2) = 3$), $x_1^4x_2^3$ (when $d(v_1) = 3$ and $d(v_2) = 2$) or $x_1^3x_2^3$ (when $d(v_1) = d(v_2) = 2$). In all case, since M has nonzero coefficient, then, by the Combinatorial Nullstellensatz, desired values for x_1 and x_2 can be chosen from lists of size at least 5, thus from lists of size at least 4 if we are guaranteed that they do not include 0 (due to the first two factors of P). From this, we deduce that a p-proper L -labelling of G can be obtained from ℓ' , a contradiction.

Thus, from now on, G can be assumed to be cubic. Let $C = u_1 \dots u_p u_1$ be a smallest induced cycle of G . For every $i \in \{1, \dots, p\}$, we denote by u'_i the neighbour of u_i which does not belong to C . Let $G' = G - E(C)$. Note that G' is nice, since the u_i 's have degree 1 and are not adjacent in G' , while all other vertices have degree 3. Thus, by minimality of G , there is a p-proper L' -labelling ℓ' of G' , where L' denotes the restriction of L to the edges of G' . Our goal is to extend it to the edges of C in a p-proper way to an L -labelling of G , thereby getting a final contradiction.

To ease the exposition of the upcoming arguments, let us introduce some notation. For every $i \in \{1, \dots, p\}$, we set $L_i = L(u_i u_{i+1})$, $a'_i = \ell'(u_i u'_i)$ and $A'_i = \frac{\pi_{\ell'}(u'_i)}{a'_i}$ (where, here and further, we set $u_{p+1} = u_1$ and $u_0 = u_p$). For some set X of values and $\lambda \in \mathbb{R}^*$, we define $\lambda X = \{\lambda x : x \in X\}$ and $\frac{\lambda}{X} = \{\frac{\lambda}{x} : x \in X\}$. For two sets X and Y , we define $XY = \{xy : x \in X, y \in Y\}$.

The proof goes by distinguishing several cases depending on some lists by L and on the structure of G . In each considered case, it is implicitly assumed that none of the previous cases applies.

1. *There are $i_0 \in \{1, \dots, p\}$ and $\alpha \in L_{i_0-1}$ such that, for all $\alpha' \in L_{i_0}$, we have $\alpha\alpha' \neq A'_{i_0}$.*

W.l.o.g., assume that $i_0 = 1$. The assumption implies that u_1 and u'_1 can never be in conflict in an extension of ℓ' assigning label α to $u_p u_1$. Let us thus start by assigning label α to $u_p u_1$.

We then consider the other edges $u_{p-1}u_p, u_{p-2}u_{p-1}, \dots, u_1u_2$ of C one by one, following this exact ordering. For every edge u_iu_{i+1} considered that way, we assign a label from $L(u_iu_{i+1})$ chosen in the following manner:

- If $i \in \{3, \dots, p-1\}$, then we assign to u_iu_{i+1} a label so that u_{i+1} is in conflict with neither u_{i+2} nor u'_{i+1} . Note that this is possible since $|L(u_iu_{i+1})| = 4$. In the case where $i = p-1$, we note that $u_{i+2} = u_1$ is a vertex whose product is not fully determined yet; the conflict between u_p and u_1 will actually be taken care of in a later stage of the extension process.
- If $i = 2$, then we assign to u_2u_3 a label so that u_3 is in conflict with neither u_4 nor u'_3 , and the resulting partial product of u_2 gets different from the partial product of u_1 . This is possible, since $|L(u_2u_3)| = 4$. In case $p = 3$ and, thus, $u_4 = u_1$, the possible conflict between u_3 and u_1 will be handled during the next step of the process.
- If $i = 1$, then we assign to u_1u_2 a label so that u_2 gets in conflict with neither u_3 nor u'_2 , and u_1 and u_p are not in conflict. Again, this is possible because $|L(u_1u_2)| = 4$. Recall further that u_1 and u_2 cannot be in conflict due to the choice of the label assigned to u_2u_3 . Also, u_1 and u'_1 cannot be in conflict by the initial assumption on α .

Thus, once the whole process has been carried out, we get an L -labelling of G which is p -proper, a contradiction.

Since Case 1 does not apply, then, throughout what follows, for every $i \in \{1, \dots, p\}$, we have

$$L_{i-1} = \frac{A'_i}{L_i} \text{ and } L_i = \frac{A'_i}{L_{i-1}}. \quad (1)$$

2. *There are $i_0 \in \{1, \dots, p\}$ and $\alpha \in L_{i_0}$ such that, for all $\alpha' \in L_{i_0+2}$, we have $\alpha\alpha'_{i_0+1} \neq \alpha'a'_{i_0+2}$.* W.l.o.g., assume that $i_0 = 1$. The assumption implies that u_2 and u_3 can never be in conflict in an extension of ℓ' assigning label α to u_1u_2 . Let us thus assign label α to u_1u_2 . We then consider the other edges of C , and label them with labels from their respective lists so that no conflict arises. We consider a special value of p , before considering the general case.

- Assume first that $p = 3$, i.e., C is a triangle. We start by assigning a label from $L(u_2u_3)$ to u_2u_3 so that u_2 does not get in conflict with u'_2 , and the partial product of u_3 gets different from the partial product of u_1 . Note that this is possible since $|L(u_2u_3)| = 4$. We then assign a label from $L(u_1u_3)$ to u_1u_3 so that u_1 gets in conflict with neither u'_1 nor u_2 , and u_3 does not get in conflict with u'_3 . Again, such a label exists since $|L(u_1u_3)| = 4$. Recall that u_1 and u_3 cannot be in conflict due to how u_2u_3 was labelled. Also, u_2 and u_3 cannot be in conflict by the assumption on α .
- Otherwise, i.e., $p \geq 4$, we start by assigning a label from $L(u_2u_3)$ to u_2u_3 so that u_2 and u'_2 do not get in conflict. We then consider the remaining edges $u_pu_1, u_{p-1}u_p, \dots, u_3u_4$ of C one by one, following this exact ordering. For every edge u_iu_{i+1} considered that way, we assign a label from $L(u_iu_{i+1})$ chosen in the following way:
 - If $i \in \{5, \dots, p\}$, then we assign to u_iu_{i+1} a label chosen so that u_{i+1} gets in conflict with neither u_{i+2} nor u'_{i+1} . This is possible since $|L(u_iu_{i+1})| = 4$.
 - If $i = 4$, then we assign to u_4u_5 a label chosen so that u_5 gets in conflict with neither u_6 nor u'_5 , and the partial product of u_4 does not get equal to the partial product of u_3 . This is possible since $|L(u_4u_5)| = 4$.
 - If $i = 3$, then we assign to u_3u_4 a label so that u_4 gets in conflict with neither u_5 nor u'_4 , and u_3 does not get in conflict with u'_3 . Again, this is possible since $|L(u_3u_4)| = 4$. Recall that u_4 and u_3 cannot be in conflict due to how u_4u_5 has been labelled. Also, u_2 and u_3 cannot be in conflict by the assumption on α .

Thus, in all cases, we get a p -proper L -labelling of G , a contradiction.

Since Case 2 does not apply in what follows, then, for every $i \in \{1, \dots, p\}$, we have

$$L_i = \frac{a'_{i+2}}{a'_{i+1}} L_{i+2}. \quad (2)$$

3. G is K_4 , the complete graph on four vertices.

Here, C is a cycle $u_1 u_2 u_3 u_1$ of length 3, and we have $u' = u'_1 = u'_2 = u'_3$. Also, ℓ' assigns labels to the three edges incident to u' , since G' is a star. Note that, as long as we label the edges of C last and handle all conflicts at that point, then, prior to labelling C , we might actually change the labels assigned to $u_1 u'$, $u_2 u'$, $u_3 u'$ by ℓ' for other labels from their respective lists. Note now that, for any choice of label a'_3 from $L(u_3 u')$ assigned to $u_3 u'$, Identity (2) must apply, i.e., we must have $L_1 = \frac{a'_3}{a'_2} L_3$, as otherwise previous Case 2 would apply the very same way. This implies that $|L_3| \geq 5$, a contradiction, by the following arguments. Since $|L(u_3 u')| = 4$, there are at least two values $x, y \in L(u_3 u')$ with distinct absolute values, say $|x| < |y|$. Start by assigning label x to $L(u_3 u')$; because Identity (2) applies, we deduce that for every $\alpha \in L_1$ we have $x\alpha \in L_3$. The other way around, we have $L_3 = L'_3 = \{x\alpha : \alpha \in L_1\}$ and $|L'_3| = |L_3| = 4$. Now change the label of $u_3 u'$ to y . Because $|x| < |y|$, we deduce that, for an $\alpha \in L_1$ with largest absolute value, $y\alpha \notin L'_3$. This implies that L_3 must contain a fifth value not in L'_3 for Identity (2) to apply with y .

4. $p = 3$ and C shares an edge with another triangle.

Assume $u_1 u_2$ belongs to a triangle $u' u_1 u_2 u'$ different from C , where $u' = u'_1 = u'_2$ is the common neighbour of u_1 and u_2 different from u_3 . Because we are not in Case 3, we have $u'_3 \neq u'$, and u' has a neighbour $w \notin V(C)$. Note that, by ℓ' , there are actually three possible values in $L(u'_2 u')$ that can be assigned to $u'_2 u'$ without causing u' to be in conflict with w , thus two such values x, y , with, say, $|x| < |y|$. Start by setting $a'_2 = y$. By an application of Identity (2) (which applies as otherwise Case 2 would), we deduce that $L_1 = \frac{a'_2}{a'_2} L_3$, which reveals the exact four values in L_3 . Now, just as in previous Case 3, we note that by changing the value of a'_2 to x and applying Identity (2) again, we deduce that L_3 must contain a fifth value not among the previous four revealed ones. This is a contradiction.

At this point, note that if we modify the label a'_i assigned to any edge $u_i u'_i$ by ℓ' , then this has no impact on the value A'_{i+1} (and, symmetrically, on A'_{i-1}). Indeed, if modifying a'_i also modified A'_{i+1} , then this would imply that $u_i u'_i$ is incident to u'_{i+1} , thus that $u'_i = u'_{i+1}$. But, in this case, we would deduce that $u_i u_{i+1} u'_i u_i$ is a triangle sharing an edge with C , thereby getting a contradiction to the fact that none of Cases 3 and 4 applies.

By manipulating Identities (1) and (2), note that we can establish the relationship

$$L_i = \frac{a'_{i+2} A'_{i+2}}{a'_{i+1} A'_{i+1}} L_i = \frac{a'_{i+1} A'_{i+1}}{a'_{i+2} A'_{i+2}} L_i \quad (3)$$

between any list L_i and some of the a'_i 's and A'_i 's. For every $i \in \{1, \dots, p\}$, we define $\lambda_i = \frac{A'_{i+1}}{a'_{i+2} A'_{i+2}}$; then, $L_i = a'_{i+1} \lambda_i L_i$ by the above.

5. There are $i \in \{1, \dots, p\}$ and a p -proper L -labelling ℓ of G' matching ℓ' on all edges but possibly $u_{i+1} u'_{i+1}$, and such that $|\ell(u_{i+1} u'_{i+1}) \lambda_i| \neq 1$.

The definition of ℓ and the fact previous Cases 3 and 4 do not apply, imply that A'_{i+1} , A'_{i+2} and a'_{i+2} are the same by both ℓ' and ℓ . From Identity 3, we deduce that $L_i = \ell(u_{i+1} u'_{i+1}) \lambda_i L_i$, where λ_i is the same by both ℓ' and ℓ . Now consider $x_0 \in L_i$; from what we have just deduced, we now get that

$$\{(\ell(u_{i+1} u'_{i+1}) \lambda_i)^j x_0\}_{j \in \mathbb{N}} \subseteq L_i.$$

Because $|\ell(u_{i+1}u'_{i+1})\lambda_i| \neq 1$, we then deduce that the set $\{(\ell(u_{i+1}u'_{i+1})\lambda_i)^j x_0\}_{j \in \mathbb{N}}$ has infinite cardinality and is included in L_i , which has size 4; a contradiction.

Note that, by ℓ' , there are actually at least two values in $L(u_i u'_i)$ that could be assigned to $u_i u'_i$ without breaking p-properness. This is because $|L(u_i u'_i)| = 4$, and, when labelling $u_i u'_i$, we only have to make sure that u'_i gets product different from that of its at most two neighbours different from u_i in G' (in particular, note that we must have $A'_i \neq 1$ by ℓ' so that $\pi_{\ell'}(u_i) \neq \pi_{\ell'}(u'_i)$, and thus we do not have to care about u_i and u'_i getting in conflict when relabelling $u_i u'_i$). Because Case 5 does not apply, this actually implies that there are exactly two such values from every $L(u_i u'_i)$, and that these two values are precisely a_i and $-a_i$.

6. *There exists $i \in \{1, \dots, p\}$ such that $L_i \neq \{\alpha, -\alpha, \beta, -\beta\}$ for some distinct $\alpha, \beta \in \mathbb{R}^*$.*

Let us consider the identity $L_i = a'_{i+1} \lambda_i L_i$ again. Since Case 5 does not apply, we have $|\ell'(u_{i+1}u'_{i+1})\lambda_i| = 1$ for any possible value as $\ell'(u_{i+1}u'_{i+1})$ from $L(u_{i+1}u'_{i+1})$. Since u'_{i+1} has, in G' , two neighbours different from u_{i+1} , there are, in $L(u_{i+1}u'_{i+1})$, two possible values for $u_{i+1}u'_{i+1}$ that make u'_{i+1} being not in conflict with these two neighbours, and these at least two possibilities must include a'_{i+1} and $-a'_{i+1}$. Now, by considering the p-proper L' -labelling of G' obtained from ℓ' by changing the label of $u_{i+1}u'_{i+1}$ to $-a_{i+1}$, the same reasoning process leads us to deduce that $L_i = -a'_{i+1} \lambda_i L_i$. This implies that $L_i = -L_i$, a contradiction.

We are now ready to conclude the proof, by considering a few cases on the length of C . The crucial points to keep in mind from now on, are that L verifies, for every $i \in \{1, \dots, p\}$, that 1) $a'_i, -a'_i \in L(u_i u'_i)$ and, in ℓ' , changing the label of $u_i u'_i$ from a'_i to $-a'_i$ cannot raise a conflict in G' , and that 2) there are nonzero real numbers α_i, β_i such that $L_i = \{\alpha_i, -\alpha_i, \beta_i, -\beta_i\}$.

7. *p is even.*

For every $i \in \{1, \dots, p\}$, we associate a variable x_i to the edge $u_i u_{i+1}$. We consider the polynomial

$$P(x_1, \dots, x_p) = \prod_{i=1}^p (x_{i-1} x_i - A'_i),$$

which is equivalent to considering

$$P'(y_1, \dots, y_p) = \prod_{i=1}^p (y_{i-1} + y_i - \log(A'_i))$$

where $y_i = \log x_i$ for every $i \in \{1, \dots, p\}$. Note that the monomial $y_1 \dots y_p$ has maximum degree and nonzero coefficient in the expansion of P' . Thus, by the Combinatorial Nullstellensatz, we can assign values to the y_i 's so that P' does not vanish, assuming we have at least two possible values to choose from for each of the y_i 's. This implies that we can assign values to the x_i 's so that P does not vanish, assuming we have at least two possible values with distinct absolute values to choose from, for each of the x_i 's. Particularly, since $|L(u_i u_{i+1})| = 4$ for every edge $u_i u_{i+1}$, this implies that ℓ' can be extended to the edges of C , resulting in an L -labelling ℓ of G where $\pi_\ell(u_i)$ and $\pi_\ell(u'_i)$ have distinct absolute values for every $i \in \{1, \dots, p\}$. Now, the only possible remaining conflicts are between the u_i 's. Due to all the assumptions made this far, recall, for every $i \in \{1, \dots, p\}$, that ℓ assigns label a'_i to every edge $u_i u'_i$, that $-a'_i \in L(u_i u'_i)$, and that switching $\ell(u_i u'_i)$ from a'_i to $-a'_i$ cannot raise a conflict between u'_i and its neighbours. Thus, to get a p-proper L -labelling of G , we can just consider each of the $u_i u'_i$'s in turn, and for each $u_i u'_i$ of them, switch, if necessary, its label to $-a'_i$ so that u_i gets positive product if i is, say, even, or negative product otherwise.

8. *$p = 3$.*

Because Cases 3 and 4 do not apply, recall that u'_1, u'_2, u'_3 are pairwise different. We extend ℓ' as follows. We start by assigning any label from $L(u_1u_2)$ to u_1u_2 . Next, we assign to u_3u_1 a label from $L(u_3u_1)$ so that no conflict between u_1 and u'_1 arises, and the resulting partial products of u_2 and u_3 have different absolute values. Note that this is possible, since L_3 is of the form $\{\alpha, -\alpha, \beta, -\beta\}$. We finally assign to u_2u_3 a label from $L(u_2u_3)$ so that there is no conflict between u_2 and u'_2 , u_3 and u'_3 , and u_1 and u_3 . Recall that u_2 and u_3 cannot be in conflict due to how u_3u_1 was labelled. Thus, the only potential conflict that can remain is between u_2 and u_1 , and, if it occurs, then we can get rid of it by simply changing the label of u_2u_2 from a'_2 to $-a'_2$. Recall that this cannot make u'_2 get in conflict with its neighbours different from u_2 , and that u_2 and u'_2 also cannot get in conflict unless they already were before switching the label of $u_2u'_2$.

9. p is odd at least 5.

We first use the Combinatorial Nullstellensatz similarly as in Case 7, to label the edges of C in such a way that, for certain pairs of vertices, the resulting products have distinct absolute values. More precisely, we want to achieve this for the pairs $\{u_1, u'_1\}$, $\{u_1, u_2\}$, $\{u_2, u_3\}$, $\{u_3, u'_3\}$, $\{u_4, u'_4\}$, $\{u_5, u'_5\}$, \dots , $\{u_{p-2}, u'_{p-2}\}$ and $\{u_p, u'_p\}$. We denote by \mathcal{S} the set of those pairs. In order to show that such an extension exists, for every $i \in \{1, \dots, p\}$ we associate a variable x_i to the edge u_iu_{i+1} , and consider the polynomial

$$P(x_1, \dots, x_p) = (x_px_1 - A'_1) \cdot (x_pa'_1 - x_2a'_2) \cdot (x_1a'_2 - x_3a'_3) \\ \cdot \left(\prod_{i=3}^{p-2} (x_{i-1}x_i - A'_i) \right) \cdot (x_{p-1}x_p - A'_p),$$

which, if $y_i = \log |x_i|$ for every $i \in \{1, \dots, p\}$, is the same as considering

$$P'(y_1, \dots, y_p) = (y_p + y_1 - \log(A'_1)) \cdot (y_p + \log(a'_1) - y_2 - \log(a'_2)) \cdot (y_1 + \log(a'_2) - y_3 - \log(a'_3)) \\ \cdot \left(\prod_{i=3}^{p-2} (y_{i-1} + y_i - \log(A'_i)) \right) \cdot (y_{p-1} + y_p - \log(A'_p)).$$

It can be checked that, in the expansion of P' , the monomial $y_1 \dots y_p$ has maximum degree and nonzero coefficient -2 . Thus, by the Combinatorial Nullstellensatz we deduce that there is a way to label the edges of C with labels from their respective lists, so that the desired conflicts (between the adjacent vertices in the pairs of \mathcal{S}) are avoided. In particular, this is possible because all these lists are of the form $\{\alpha, -\alpha, \beta, -\beta\}$, and, in particular, contain two values with distinct absolute values.

The resulting labelling might be not p -proper, and, to turn it into a p -proper one, we will *switch* some edges incident to the vertices in C , and, by that, we mean changing the current label l of an edge to $-l$. More particularly, we will switch edges of the form u_iu_{i+1} and $u_iu'_i$; due to some of the assumptions made this far, recall that for every such edge e with current label l , we do have $-l \in L(e)$.

We start by switching, if necessary, $u_2u'_2$ and $u_{p-1}u'_{p-1}$ so that the products of u'_2 and u'_{p-1} get positive and negative, respectively. Next, we switch u_1u_2 , if necessary, so that the product of u_2 gets negative. Now, we consider the edges $u_3u_4, u_4u_5, \dots, u_pu_1$ one by one following this ordering, and, for every such considered edge u_iu_{i+1} , we switch it, if necessary, so that the product of u_i gets negative if i is odd, and positive otherwise. Lastly, we switch $u_1u'_1$, if necessary, so that the product of u_1 gets negative.

We claim that the eventual labelling of G is p -proper, our final contradiction. First recall, as mentioned earlier, that the switching operation guarantees that the resulting labelling is an L -labelling. Its p -properness follows from the following arguments. First, for all the pairs of

adjacent vertices in \mathcal{S} , the products are different due to distinct absolute values (preserved under the switching operation). Regarding the two adjacent vertices in the pair $\{u_{p-1}, u'_{p-1}\}$, the products have different signs and are thus different. Now, for every two adjacent vertices in the pairs $\{u_3, u_4\}, \{u_4, u_5\}, \dots, \{u_p, u_1\}$, the products are different due to their signs being different. \square

4 Conclusion

In this work, we have considered a problem being a combination of the Multiplicative 1-2-3 Conjecture and of the List 1-2-3 Conjecture, standing as a List Multiplicative 1-2-3 Conjecture. In particular, we have exhibited a few bounds on the parameter ch_{Π}^* , both for graphs in general and for more specific classes of graphs. While some of these bounds are tight, some others remain a bit distant from what we believe should be optimal.

An interesting point stemming from our proofs, is the methods we have used to establish our bounds. In the context of the List 1-2-3 Conjecture, the algebraic approach, through, in particular, the polynomial method and tools such as the Combinatorial Nullstellensatz, is definitely the best approach we know of at the moment to establish bounds on ch_{Σ} . As described notably in Subsection 2.3, and seen throughout this work, the potential of this method is a bit less obvious for exhibiting bounds on ch_{Π}^* . Recall that, in the current work, we have mainly exploited the connection between ch_{Σ} and ch_{Π}^* established in Theorem 2.4. It might be, however, that there are dedicated ways to better exploit the algebraic approach, and get better bounds on ch_{Π}^* .

As a main perspective for further work on the topic, it would be nice to obtain a constant upper bound on ch_{Π}^* for graphs in general. Recall that, due to Theorem 2.4, this could be obtained through establishing a constant upper bound on ch_{Σ} . This apart, it would be interesting to verify the List Multiplicative 1-2-3 Conjecture for more classes of graphs. For instance, it would be interesting to improve any of the upper bounds in Corollary 2.5, some of which we have already improved in Subsection 3.2. Notably, it is worth mentioning that the arguments used to prove Theorems 3.8 and 3.10 are tight, and, as a result, it seems that our proofs would be hard to improve to lower the bound of 4. From this, we would be interested in having a proof of the List Multiplicative 1-2-3 Conjecture for planar graphs with girth at least 16 or for subcubic graphs.

Addendum: We would like to report that a number of new significant results appeared in the literature after the first version of the current paper was submitted for publication. While the main results in the current paper (Theorems 3.5, 3.8, and 3.10) are not impacted by these new results, some of these actually improve some others of our results. Let us start by mentioning that we provided a proof of the Multiplicative 1-2-3 Conjecture in [5], which brings more support to our List Multiplicative 1-2-3 Conjecture. Next, the first constant upper bound, 17, on the parameter ch_{Σ} was provided by Cao in [6], which, in conjunction with Theorem 2.4, implies that we have $\text{ch}_{\Pi}^*(G) \leq 33$ for every nice graph G . An improvement down to 5 on the bound of Cao was recently claimed by Zhu [21], which would imply that we even have $\text{ch}_{\Pi}^*(G) \leq 9$ for every nice graph G . Note that, due to such bounds, some of our results, such as Theorem 3.1 and Corollary 3.3, become mostly interesting for graphs with low maximum degree.

References

- [1] N. Alon. Combinatorial Nullstellensatz. *Combinatorics, Probability and Computing*, 8:7-29, 1999.
- [2] M. Anholcer. Product irregularity strength of graphs. *Discrete Mathematics*, 309(22):6434-6439, 2009.

- [3] T. Bartnicki, J. Grytczuk, S. Niwczyk. Weight choosability of graphs. *Journal of Graph Theory*, 60:242–256, 2009.
- [4] J. Bensmail, H. Hocquard, D. Lajou, É. Sopena. Further Evidence Towards the Multiplicative 1-2-3 Conjecture. *Discrete Applied Mathematics*, 307:135-144, 2022.
- [5] J. Bensmail, H. Hocquard, D. Lajou, É. Sopena. A Proof of the Multiplicative 1-2-3 Conjecture. In *Balachandran N., Inkulu R. (eds) Algorithms and Discrete Applied Mathematics. CALDAM 2022. Lecture Notes in Computer Science*, vol 13179. Springer, Cham. 2022.
- [6] L. Cao. Total weight choosability of graphs: Towards the 1-2-3-conjecture. *Journal of Combinatorial Theory, Series B*, 149:109-146, 2021.
- [7] L. Ding, G.-H. Duh, G. Wang, T.-L. Wong, J. Wu, X. Yu, X. Zhu. Graphs are $(1, \Delta + 1)$ -choosable. *Discrete Mathematics*, 342:279-284, 2019.
- [8] P. Erdős, A.L. Rubin, H. Taylor. Choosability in graphs. Proceeding of the West Coast Conference on Combinatorics, Graph Theory and Computing, Arcata. *Congressus Numerantium*, 26:125–157, 1979.
- [9] M. Horňák, M. Woźniak. On neighbour-distinguishing colourings from lists, *Discrete Mathematics & Theoretical Computer Science*, 14(2):21-28, 2012.
- [10] M. Kalkowski, M. Karoński, F. Pfender. Vertex-coloring edge-weightings: towards the 1-2-3 Conjecture. *Journal of Combinatorial Theory, Series B*, 100:347-349, 2010.
- [11] M. Karoński, T. Łuczak, A. Thomason. Edge weights and vertex colours. *Journal of Combinatorial Theory, Series B*, 91:151–157, 2004.
- [12] T. Li, C. Qu, G. Wang, X. Yu. Neighbor product distinguishing total colorings. *Journal of Combinatorial Optimization*, 33:237–253, 2017.
- [13] Y.-C. Liang, T.-L. Wong, X. Zhu. Graphs with maximum average degree less than $\frac{11}{4}$ are $(1, 3)$ -choosable. *Discrete Mathematics*, 341(10):2661-2671, 2018.
- [14] Y. Lu, C. Li, Z.K. Miao. Weight Choosability of Graphs with Maximum Degree 4. *Acta Mathematica Sinica, English Series*, 36(6):723-732, 2020.
- [15] J. Nešetřil, A. Raspaud, É. Sopena. Colorings and girth of oriented planar graphs. *Discrete Mathematics*, 165-166:519-530, 1997.
- [16] H. Pan, D. Yang. On total weight choosability of graphs. *Journal of Combinatorial Optimization*, 25(4):766–783, 2013.
- [17] B. Seamone. The 1-2-3 Conjecture and related problems: a survey. Preprint, 2012. Available online at <http://arxiv.org/abs/1211.5122>.
- [18] J. Skowronek-Kaziów. Multiplicative vertex-colouring weightings of graphs. *Information Processing Letters*, 112(5):191-194, 2012.
- [19] T.-L. Wong. 2-connected chordal graphs and line graphs are $(1, 5)$ -choosable. *European Journal of Combinatorics*, 91:103227, 2021.
- [20] T.-L. Wong, X. Zhu. Total weight choosability of d -degenerate graphs. Manuscript, 2013.
- [21] X. Zhu. Every nice graph is $(1, 5)$ -choosable. Preprint, 2021. Available online at <http://arxiv.org/abs/2104.05410>.

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