

L^p -projections on subspaces and quotients of Banach spaces

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Abstract. The aim of this paper is to study L^p -projections, a notion introduced by Cunningham in 1953, on subspaces and quotients of complex Banach spaces. An L^p -projection on a Banach space X , for $1 \leq p \leq +\infty$, is an idempotent operator P satisfying $\|f\|_X = \|(\|P(f)\|_X, \|(I - P)(f)\|_X)\|_{\ell_p}$ for all $f \in X$. This is an L^p version of the equality $\|f\|^2 = \|Q(f)\|^2 + \|(I - Q)(f)\|^2$, valid for orthogonal projections on Hilbert spaces. We study the relationships between L^p -projections on a Banach space X and those on a subspace F , as well as relationships between L^p -projections on X and those on the quotient space X/F . All the results in this paper are true for $1 < p < +\infty$, $p \neq 2$. The cases $p = 1, 2$ or $+\infty$ can exhibit different behaviour. In this regard, we give a complete description of L^∞ -projections on spaces $L^\infty(\Omega)$. For this, we introduce a notion of p -orthogonality for two elements x, y by requiring that $\text{Span}(x, y)$ admits an L^p -projection separating x and y . We also introduce the notion of maximal L^p -projections for X , that is L^p -projections defined on a subspace G of X that cannot be extended to L^p -projections on larger subspaces. We prove results concerning L^p -projections and p -orthogonality of general Banach spaces or on Banach spaces with additional properties. Generalizations of some results to spaces $L^p(\Omega, X)$ as well as some results about L^q -projections on subspaces of $L^p(\Omega)$ are also discussed.

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1. Introduction and Preliminaries

Introduction

Several Banach space generalizations of Hilbert space orthogonal projections are known: contractive and bi-contractive projections, Hermitian projections, circular and bi-circular projections, etc. (see for instance [1, 9, 10, 17, 20] and

the references therein). The class of projections that we study in this paper is that of L^p -projections on subspaces and quotients of complex Banach spaces. Note that an L^p -projection is Hermitian. The notion of L^p -projection has been introduced by Cunningham in 1953 [11] and studied, mainly in the cases $p = 1$ and $p = +\infty$, in the papers [11–14]. The general case $1 < p < +\infty$ has been studied by Alfsen-Effros [2], Sullivan [23] and Fakhoury [15]. The main characterization results, which were obtained in 1973–1976 by Alfsen-Effros, Behrends, Fakhoury, Sullivan and others, were compiled in the book [6, Ch.1,2,6].

In this paper, a *projection* (or an *idempotent*) is a bounded linear operator P on a Banach space X that satisfies $P^2 = P$. We recall that an L^p -projection is an idempotent satisfying the additional condition of the following definition.

1.1. Definition (L^p -projection). Let X be a Banach space, and let $1 \leq p \leq +\infty$. Let P be a bounded linear operator acting on X which is an idempotent ($P^2 = P$). The operator P is said to be an L^p -projection if it satisfies the condition

$$\|f\|_X = \|(\|P(f)\|, \|(I - P)(f)\|)\|_{\ell_p}, \text{ for all } f \in X. \quad (1.1)$$

The condition (1.1) means that

$$\begin{cases} \|f\|_X^p = \|P(f)\|_X^p + \|(I - P)(f)\|_X^p, \forall f \in X & \text{when } 1 \leq p < +\infty \\ \|f\|_X = \max(\|P(f)\|_X, \|(I - P)(f)\|_X), \forall f \in X & \text{when } p = +\infty. \end{cases}$$

We denote by $\mathcal{P}_p(X)$ the set of L^p -projections on X .

The aim of this paper is to study L^p -projections on subspaces and quotients of a complex Banach space. We also introduce a notion of p -orthogonality between two elements x, y by asking that $\text{Span}(x, y)$ admits an L^p -projection separating x and y . This notion of p -orthogonality for vectors implies a corresponding notion of the p -orthogonal of a set. We also introduce and study a notion of *maximal L^p -projection* on a Banach space X . This is an L^p -projection defined on a subspace of X that cannot be extended to L^p -projections on larger subspaces.

The properties of L^p -projections that will be studied here are valid either in general Banach spaces, or in Banach spaces with several additional properties. All these additional properties hold true for the spaces $L^p(\Omega)$. We also stress that the results in this paper are all true for $1 < p < +\infty$, $p \neq 2$ and some of them are even true when $p = 1, 2$ or $+\infty$. However, the Hilbert case ($p = 2$) and the non-reflexive cases ($p = 1, +\infty$) can exhibit different behaviour, or do not work well in some contexts. In this regard, Section 2.3 below deals only with the case $p = +\infty$.

Organization.

In the rest of this introductory section we introduce some of the basic results about L^p -projections. For a complex Banach space X and subspaces F and G , the following sections will focus on relationships between the L^p -projections on X and those on F , the quotient X/F or G/F . In Section 2 we present

relationships and characterizations of L^p -projections on a subspace F . In Section 3 we look at relationships and characterizations of L^p -projections on a quotient X/F . In Section 4 we give generalizations of previous results to some spaces $L^p(\Omega, X)$. In an Appendix (Section 5) we give a proof of Proposition 3.12.

Preliminaries

We start with the following remark.

1.2. *Remark.* We can first see that 0 and I are always L^p -projections, so $\mathcal{P}_p(X)$ is never empty. The class $\mathcal{P}_p(X)$ of L^p -projections is stronger than the usual class of Hermitian projections. We recall (see for instance [8]) that a projection Q is *Hermitian* if $\|e^{i\alpha Q}\| = 1$ for each $\alpha \in \mathbb{R}$ and that a projection Q is Hermitian if and only if $Q + \lambda(I - Q)$ is an isometry for every $\lambda \in \partial\mathbb{D}$ or, equivalently, if $\lambda Q + \gamma(I - Q)$ is an isometry, for any $\lambda, \gamma \in \partial\mathbb{D}$. To see that an L^p -projection P is Hermitian, we note that the L^p -projection condition is also equivalent to

$$\|f + g\|_X = \|(\|f\|, \|g\|)\|_{\ell_p}, \text{ for all } f \in \text{Ran}(P), g \in \text{Ker}(P),$$

where *Ran* and *Ker* denote the range and respectively the kernel. Hence, for any $\lambda, \gamma \in \partial\mathbb{D}$, we have

$$\|\lambda f + \gamma g\|_X = \|(\|f\|, \|g\|)\|_{\ell_p} = \|f + g\|_X, \text{ for all } f \in \text{Ran}(P), g \in \text{Ker}(P).$$

Therefore, $\lambda P + \gamma(I - P)$ is an isometry on X , and the L^p -projection P is a Hermitian projection. However, the main characterization results on Hermitian projections mainly concern L^p and H^p spaces (see [7, 8, 21, 24]) and thus they are of little help in our context.

For the rest of this paper, we will only consider complex Banach spaces. Note that L^p -projections do not behave differently between real and complex cases except when $p = 1$ or $+\infty$ (see Theorem 1.9), so most of the results can be easily generalized to the real case when $1 < p < +\infty, p \neq 2$.

For a set E, A a subset of E, F a vector space and a map $f : E \rightarrow F$, we define $M_{\chi_A}(f)$ as the multiplication by the characteristic function of A :

$$M_{\chi_A}(f)(x) = f\chi_A(x) = \begin{cases} f(x) & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}.$$

The following basic facts are recorded here without proofs.

- 1.3. Proposition.** (i) For $(\Omega, \mathcal{F}, \mu)$ a measure space, $1 \leq p \leq +\infty$, and $A \subset \Omega$ such that $A \cap B \in \mathcal{F}$ for every $B \in \mathcal{F}$ with $\mu(B) < +\infty$, the operator $P = M_{\chi_A}$ is an L^p -projection on $L^p(\Omega, \mathcal{F}, \mu)$.
- (ii) For X, Y Banach spaces, $1 \leq p \leq +\infty$, and $T : X \rightarrow Y$ an isometric isomorphism, we have $\mathcal{P}_p(Y) = T \circ \mathcal{P}_p(X) \circ T^{-1}$.
- (iii) The set of L^2 -projections on a Hilbert space H is the set of orthogonal projections.
- (iv) For K a compact topological space, the L^∞ -projections on $C^0(K)$ have the form $P = M_{\chi_A}$ with A a clopen subset of K .

1.4. Proposition. *Let X be a Banach space. Let p be a real number with $1 \leq p \leq +\infty$. If $P, Q \in \mathcal{P}_p(X)$ are such that $\text{Ran}(P) = \text{Ran}(Q)$, then $P = Q$.*

1.5. Remark. The set of all L^p -projections P on X is in bijective correspondence to decompositions $X = X_1 \oplus_{\ell_p} X_2$ of X . A ℓ_p -direct sum decomposition like this is called a p -summand of X . Hence, studying the p -summands of a Banach space X amounts to studying the L^p -projections on X . Proposition 1.4 indicates that for a p -summand $X = X_1 \oplus_p X_2$, X_2 is the only closed subspace of X that is in ℓ_p -direct sum with X_1 .

We present now several results characterizing L^p -projections on a general Banach space X , mainly when $p \neq 2$. Proofs of these results can be found in [6].

1.6. Lemma. *Let X be a Banach space, and consider two real conjugate numbers p and p' , with $\frac{1}{p} + \frac{1}{p'} = 1$. Let $P \in \mathcal{L}(X)$ be a projection on X . Then P is an L^p -projection on X if and only if P' is an $L^{p'}$ -projection on X' .*

1.7. Proposition. *Let X be a Banach space, and $1 \leq p, p' \leq +\infty$, with $\frac{1}{p} + \frac{1}{p'} = 1$ and $p' \neq 1$.*

Then, every $Q \in \mathcal{P}_{p'}(X')$ is continuous for the weak- $$ topology $\sigma(X', X)$. Hence, there is $P \in \mathcal{P}_p(X)$ such that $P' = Q$, so $(\mathcal{P}_p(X))' = \mathcal{P}_{p'}(X')$.*

The only pathologic case regarding duality for L^p -projections is $p' = 1$, $p = +\infty$. For example $X = C^0([0, 1])$ has trivial L^∞ -projections, whereas X' is an L^1 space that possesses many L^1 -projections.

1.8. Theorem. *Let X be a Banach space. Suppose $1 \leq p \leq +\infty$, $p \neq 2$. The following assertions are true.*

- (i) *All elements of $\mathcal{P}_p(X)$ commute together.*
- (ii) *The set $\mathcal{P}_p(X)$ is a commutative Boolean algebra for the operations $(P, Q) \mapsto PQ$, $(P, Q) \mapsto P + Q - PQ$ and $P \mapsto (I - P)$.*
- (iii) *The relationship $P \leq Q \Leftrightarrow PQ = P$ is an order relationship on $\mathcal{P}_p(X)$.*
- (iv) *When $p \neq +\infty$, every decreasing filtrating net $(P_i)_{i \in I}$ in $\mathcal{P}_p(X)$ is pointwise convergent to an L^p -projection P , with $P = \inf_{i \in I} (P_i)$.*
- (v) *When $p \neq +\infty$, the Boolean algebra $\mathcal{P}_p(X)$ is complete: Every subset $\{P_i, i \in I\}$ admits an infimum $\inf_{i \in I} (P_i)$ in $\mathcal{P}_p(X)$. Furthermore, $\text{Ran}(\inf_{i \in I} (P_i)) = \bigcap_{i \in I} \text{Ran}(P_i)$.*

1.9. Theorem. *Let X be a Banach space. Suppose $1 \leq p, q \leq +\infty$, with $p \neq q$. Then, either $\mathcal{P}_p(X)$ or $\mathcal{P}_q(X)$ is reduced to $\{0, I\}$. The result stays true for a real Banach space Y , unless Y is isometrically isomorphic to $l^1(\mathbb{R}^2) \simeq l^\infty(\mathbb{R}^2)$.*

1.10. Remark. Theorems 1.8 and 1.9 show, for $p \neq 2$, the similarity of the set $\mathcal{P}_p(X)$ with the set $\{M_{\mathcal{X}_A} : A \in \mathcal{F}\}$, for $(\Omega, \mathcal{F}, \mu)$ a measure space with μ σ -finite. Theorem 1.9 also shows that we do not need to study L^q -projections on a Banach space X whenever there are non-trivial L^p -projections on X ,

for $1 \leq p, q \leq +\infty$ with $p \neq q$. However, in certain cases, there may exist L^q -projections on subspaces, quotients, or subspaces of quotients of X , like for $X = L^q(\Omega) \oplus_p L^r(\Omega')$ for example. This question will be discussed later on (see Lemma 4.3).

1.11. Corollary. *Let X be a Banach space. Suppose $1 \leq p < +\infty$, $p \neq 2$. Let E be a subset of X . Then, the set of L^p -projections P on X such that $P(E) = E$ admits an unique minimum with respect to the order relation from Theorem 1.8. Furthermore, for any $Q \in \mathcal{P}_p(X)$ such that $Q(E) = E$, we have $P \leq Q$.*

This minimum in Corollary 1.11 is called the *minimal L^p -projection* for E .

We end this section with some results about L^p -projections on L^p spaces. We start with Clarkson's inequalities for L^p -spaces; a proof of these inequalities can be found in [22, Ch.15-7, Lemma 22, p.416].

1.12. Lemma (Clarkson inequalities). *Let $1 \leq p < +\infty$ and let $(\Omega, \mathcal{F}, \mu)$ be a measure space. For every f, g in $L^p(\Omega, \mathcal{F}, \mu)$, we have*

- (i) $\|f + g\|_p^p + \|f - g\|_p^p \leq 2(\|f\|_p^p + \|g\|_p^p)$ if $1 \leq p \leq 2$
- (ii) $\|f + g\|_p^p + \|f - g\|_p^p \geq 2(\|f\|_p^p + \|g\|_p^p)$ if $2 \leq p$

Moreover, if $p \neq 2$ there is equality in the above inequalities if and only if $\mu(\text{supp}(f) \cap \text{supp}(g)) = 0$, that is if f and g have disjoint supports.

The equality case in Clarkson inequalities is the main ingredient in the proof of the two following characterizations of L^p -projections on L^p -spaces. This equality case has no equivalent for $p = +\infty$, as one can see in Proposition 2.24.

1.13. Theorem. *Suppose $1 \leq p < +\infty$, $p \neq 2$ and let $(\Omega, \mathcal{F}, \mu)$ be a measure space, with μ being σ -finite. Let $P \in \mathcal{P}_p(L^p(\Omega))$. Then there exists $A \in \mathcal{F}$ such that $P = M_{\chi_A}$.*

1.14. Theorem. *Let p be a real number with $1 \leq p < +\infty$, $p \neq 2$ and let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let $P \in \mathcal{P}_p(L^p(\Omega))$. Then we have $P = M_{\chi_A}$, with $A \subset \Omega$ such that $A \cap B \in \mathcal{F}$ for every $B \in \mathcal{F}$ with $\mu(B) < +\infty$.*

1.15. Theorem. *Suppose $1 \leq p < +\infty$, $p \neq 2$ and let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let Y be a finite dimensional Banach space. Let $P \in \mathcal{P}_p(L^p(\Omega, Y))$. Denote $\mathcal{P}_p(Y) = \{T_i, i \in I\}$. Then we have $P = \sum_{i \in I} P_i \otimes T_i$, where P_i are L^p -projections on $L^p(\Omega)$ such that $\sum_{i \in I} P_i = I$, that is the ranges $\text{Ran}(P_i)$ are in direct sum in $L^p(\Omega)$.*

1.16. Remark. We refer to Daniel Li's thesis [19] or to [6] for a proof of Theorems 1.13, 1.14 and 1.15. The proof of Theorem 1.15 uses results about the Stonean space associated to $\mathcal{P}_p(X)$ in order to make an L^p -projection P on $L^p(\Omega, Y)$ corresponds to an L^p -projection P' on a space of continuous functions. The form of the latter L^p -projection is described by noticing that P' composed with any evaluation operator is again an L^p -projection. Most

of the tools used in these proofs behave differently in the case $p = +\infty$, even though the underlying idea is similar (see Section 2.3 below). We prove in this paper that $L^\infty(\Omega)$ cannot admit other L^∞ -projections than multiplication operators (Theorem 2.25).

2. p -orthogonality and maximal L^p -projections

For a Banach space X and F a closed subspace of X , we will focus in this section on relationships between L^p -projections on F and those on X , either for a general complex Banach space X or for a space X satisfying additional properties. Section 2.1 introduces and studies the notions of p -orthogonality for vectors and of p -orthogonal of a set. Two properties for Banach spaces regarding p -orthogonality for vectors are also defined and studied. All the results in this section are valid for $1 \leq p < +\infty$, $p \neq 2$, while some of them are true for $p = +\infty$. Section 2.2 gives some examples and Section 2.3 focuses on the case $p = +\infty$, one of its main results being the description of $\mathcal{P}_\infty(L^\infty(\Omega))$. Lastly, Section 2.4 introduces and studies the notion of maximal L^p -projections: L^p -projections on a subspace of X that cannot be extended to L^p -projections on a larger subspace, for $1 \leq p \leq +\infty$, $p \neq 2$. This section also focuses on the number of maximal L^p -projections for a subspace $F \subset X$, especially in the finite dimensional case.

2.1. p -orthogonality, L^p -projections on subspaces, maximal L^p -projections

We introduce the following definition.

2.1. Definition (p -orthogonality). Let X be a Banach space, and $1 \leq p \leq +\infty$. Let $f, g \in X$. The elements f and g are said to be p -orthogonal, denoted by $f \perp_p g$, if

$$\begin{cases} \|f + zg\|^p = \|f\|^p + |z|^p \|g\|^p, \forall z \in \mathbb{C}, \text{ when } p < +\infty; \\ \|f + zg\| = \max(\|f\|, |z| \|g\|), \forall z \in \mathbb{C}, \text{ when } p = +\infty. \end{cases}$$

This condition is equivalent to the fact that $\text{Span}(f, g)$ has dimension 2 and that the projection on $\text{Span}(f)$ parallel to $\text{Span}(g)$ is an L^p -projection on $\text{Span}(f, g)$.

2.2. *Remark.* This p -orthogonality relationship is symmetric. We also have

$$f \perp_p f \Leftrightarrow f = 0 \text{ and } f \perp_p g \Leftrightarrow f \perp_p wg, \text{ for any } w \in \mathbb{C}^*.$$

However, the two conditions $f \perp_p g$ and $f \perp_p h$ do not imply $f \perp_p (g + h)$, in general (see item 2.20 for a counter-example).

A similar notion of orthogonality can be found in a paper of Berkson [7, Section 3], although it concerns Boolean algebras of Hermitian operators. We also recall the *Birkhoff-James orthogonality*, defined as

$$x \perp_{BJ} y \Leftrightarrow \|x + ty\| \geq \|x\| \text{ for every } t \in \mathbb{R}.$$

If $x \perp_p y$, then one can see that $x \perp_{BJ} y$ and $y \perp_{BJ} x$. Hence the p -orthogonality, defined via L^p -projections, is stronger than the Birkhoff-James orthogonality (the latter is defined using norm one projections). We refer for instance to

[5, 5.f] and [4], and warn the reader that there are many other notions of orthogonality in the literature.

2.3. Corollary. *Let p be a real number with $1 \leq p < +\infty$, $p \neq 2$, and let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Take $X = L^p(\Omega, \mathcal{F}, \mu)$ and let $f, g \in X$. We then have the equivalences*

- (i) $f \perp_p g$;
- (ii) $\|f + e^{it}g\|^p = \|f\|^p + \|g\|^p, \forall t \in \mathbb{R}$;
- (iii) $\|f \pm g\|^p = \|f\|^p + \|g\|^p$;
- (iv) $\|f + g\|^p + \|f - g\|^p = 2(\|f\|^p + \|g\|^p)$;
- (v) f and g have disjoint supports (up to a set of measure 0).

Proof. The implications (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) are immediate.

- (v) \Rightarrow (i) If $\mu(\text{supp}(f) \cap \text{supp}(g)) = 0$, then for any $z \in \mathbb{C}$ we have

$$\|f + zg\|^p = \int_{\Omega} |(f + zg)(x)|^p d\mu(x) = \|f\|^p + \|zg\|^p,$$

so $f \perp_p g$.

- (iv) \Rightarrow (v) If $\|f + g\|^p + \|f - g\|^p = 2(\|f\|^p + \|g\|^p)$, then f and g satisfy the equality case in Clarkson inequalities 1.12, which implies that $\mu(\text{supp}(f) \cap \text{supp}(g)) = 0$. \square

2.4. Definition (p -orthogonal). Let X be a Banach space, and $1 \leq p \leq +\infty$. Let E be a subset of X . We define E^{\perp_p} the p -orthogonal of E as the set

$$E^{\perp_p} := \{f \in X: f \perp_p g, \forall g \in E\}.$$

2.5. Remark. Since the map $f \mapsto \|f + zg\| - (\|f\|, |z|\|g\|)_p$ is continuous for every $z \in \mathbb{C}$, the set E^{\perp_p} is closed. We also have $E \subset (E^{\perp_p})^{\perp_p}$, and $E \subset F \Rightarrow F^{\perp_p} \subset E^{\perp_p}$, similarly to orthogonal sets in Hilbert spaces or in dual spaces. Hence, for P an L^p -projection leaving E invariant, we have $\text{Ker}(P) \subset E^{\perp_p}$. However, E^{\perp_p} is not a linear subspace and is not always equal to $\text{Ker}(P)$, as we will see in Counter-examples 2.20 and 2.21.

2.6. Lemma ([6]). *Let X be a Banach space. Suppose that $1 \leq p \leq +\infty$. Let $Q \in \mathcal{P}_p(X)$ be an L^p -projection. We have $\text{Ran}(Q)^{\perp_p} = \text{Ker}(Q)$ and $\text{Ker}(Q)^{\perp_p} = \text{Ran}(Q)$.*

Proof. We can first notice that for any $h_1 \in \text{Ran}(Q)$, $h_2 \in \text{Ker}(Q)$, we have $h_1 \perp_p h_2$. This implies that $\text{Ker}(Q) \subset \text{Ran}(Q)^{\perp_p}$. Let $f \in \text{Ran}(Q)^{\perp_p}$.

Suppose first that $p < +\infty$. We have $f = Q(f) + (I - Q)(f)$, so

$$\|f\|^p = \|Q(f)\|^p + \|(I - Q)(f)\|^p.$$

As $(I - Q)(f) = f - Q(f)$, we get

$$\|(I - Q)(f)\|^p = \|f\|^p + \|Q(f)\|^p.$$

Hence, $Q(f) = 0$, so $f \in \text{Ker}(Q)$ and $\text{Ran}(Q)^{\perp_p} = \text{Ker}(Q)$.

Suppose now that $p = +\infty$. Let $z \in \mathbb{C}$. We have

$$f + zQ(f) = (1 + z)Q(f) + (I - Q)(f).$$

Therefore

$$\max(\|f\|, \|zQ(f)\|) = \|f + zQ(f)\| = \max(\|(1+z)Q(f)\|, \|(I-Q)(f)\|).$$

As this must be true for every $z \in \mathbb{C}$, we have $Q(f) = 0$. Thus $f \in \text{Ker}(Q)$ and $\text{Ran}(Q)^{\perp p} = \text{Ker}(Q)$.

The other equality is obtained by taking $Q' = (I - Q)$. \square

We introduce now some properties that vectors of, or L^p -projections on, a Banach space X can have, for $1 \leq p \leq +\infty$. These properties hold true in the case of L^p -spaces, as we will see in Proposition 2.12 below.

2.7. Property (Extension of p -orthogonality to X). For any $f, g \in X$ such that $f \perp_p g$, there exists $P \in \mathcal{P}_p(X)$ such that $P(f) = f$ and $P(g) = 0$.

2.8. Property (Linearity of p -orthogonality on X). For any $f, g, h \in X$ such that $f \perp_p g$ and $f \perp_p h$, we have $f \perp_p (g + h)$.

2.9. Proposition. *Let X be a Banach space, and $1 \leq p \leq +\infty$, $p \neq 2$. Then, Property 2.7 implies Property 2.8.*

Proof. Let $f, g, h \in X$ be such that $f \perp_p g$ and $f \perp_p h$. Then there exists P_1 and P_2 two L^p -projections such that $P_1(f) = f$, $P_1(g) = 0$, $P_2(f) = f$, $P_2(h) = 0$. Let $Q = P_1 P_2$. Since $p \neq 2$, P_1 and P_2 commute, so we have $Q(f) = P_1 P_2(f) = f$, $Q(g) = P_1 P_2(g) = P_2 P_1(g) = 0$, and $Q(h) = P_1 P_2(h) = 0$. Hence, $g, h \in \text{Ker}(Q)$, so $g + h \in \text{Ker}(Q)$. Since $f \in \text{Ran}(Q)$, this implies that $f \perp_p g + h$. \square

2.10. Proposition. *Let X be a Banach space, and suppose that $1 \leq p < +\infty$, $p \neq 2$. The following are equivalent:*

- (i) X satisfies Property 2.7 for p ;
- (ii) For any subsets E_1, E_2 of X , such that $f \perp_p g$ for every $f \in E_1, g \in E_2$, there exists $P \in \mathcal{P}_p(X)$ such that $P(E_1) = E_1$ and $P(E_2) = \{0\}$.

Furthermore, if one of them is true, then for any subspace F of X and for any $P \in \mathcal{P}_p(F)$, there exists $Q \in \mathcal{P}_p(X)$ such that $P = Q|_F$.

Proof. By definition, item (ii) implies Property 2.7 for p . For the converse implication, let E_1, E_2 be subsets of X that are p -orthogonal. Recall that we assume $p < +\infty$. Let $f \in E_1$ and denote by P_f the minimal L^p -projection for f . For any $g \in E_2$, we have an L^p -projection Q such that $Q(f) = f$ and $Q(g) = 0$. Hence, $P_f \leq Q$ by Corollary 1.11, so $P_f Q = P_f$. This implies that $P_f(g) = P_f Q(g) = 0$, so $g \in \text{Ker}(P_f)$. Therefore, $E_2 \in \text{Ker}(P_f)$.

Let us now denote by P the minimal L^p -projection for E_2 . By minimality of P , we have $P \leq (I - P_f)$ that is $P(I - P_f) = P$. This implies that $P(f) = P(I - P_f)(f) = 0$. Therefore, we have $E_1 \in \text{Ker}(P)$, so $E_1 \in \text{Ran}(I - P)$ and $E_2 \in \text{Ker}(I - P)$. Thus the conditions of item (ii) are satisfied.

If we now take a subspace F of X and an L^p -projection R on F , we can apply the condition of item (ii) to $E_1 = \text{Ran}(R)$ and $E_2 = \text{Ker}(R)$ to get $S \in \mathcal{P}_p(X)$ such that $S(E_1) = E_1$ and $S(E_2) = \{0\}$. As $F = \text{Ran}(R) \oplus \text{Ker}(R)$ we have $S(F) \subset F$, and S coincides with R on the subspace F , so $S|_F = R$. \square

2.11. *Remark.* Proposition 2.10 turns out to be also true for $p = +\infty$, but its proof requires additional information for this case; see Proposition 2.29. If X satisfies Property 2.8, then so does every subspace F of X . This is however not true in general for Property 2.7 (see the example 2.21).

2.12. Proposition. *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and suppose that $1 \leq p < +\infty$, $p \neq 2$. Then $L^p(\Omega, \mathcal{F}, \mu)$ satisfies Property 2.7 for p .*

Proof. Let $f, g \in L^p(\Omega)$ be such that $f \perp_p g$. According to Corollary 2.3, f and g have disjoint supports. Hence, if we take $A = \text{supp}(f)$ and $P = M_{\chi_A}$, then P is an L^p -projection on $L^p(\Omega)$ such that $P(f) = f$ and $P(g) = 0$. \square

Proposition 2.12 is not true in general for L^∞ -spaces (see Corollary 2.27). The result in the case $p = 1$ can also be found in the book of Harmand, Werner and Werner [16, Prop 1.21]. It allows us to fully describe L^p -projections on subspaces of L^p -spaces using $\mathcal{P}_p(L^p(\Omega))$.

2.13. Corollary (L^p -projections on subspaces of $L^p(\Omega)$). *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Suppose that $1 \leq p < +\infty$, $p \neq 2$. Let F be a subspace of $L^p(\Omega, \mathcal{F}, \mu)$ and let $P \in \mathcal{P}_p(F)$. Then there exists $A \in \Omega$ satisfying $A \cap B \in \mathcal{F}$ for every $B \in \mathcal{F}$ with $\mu(B) < +\infty$ such that $P = M_{\chi_A}$. Moreover, we have $F = M_{\chi_A}(F) \oplus_p M_{\chi_{A^c}}(F)$. Furthermore, F admits non-trivial L^p -projections if and only if it admits a non-trivial decomposition of this form.*

Proof. Proposition 2.12 tells us that $L^p(\Omega)$ satisfies Property 2.7 for p . We can then apply Proposition 2.10 and Theorem 1.14 in order to obtain that $P = Q|_F$ with $Q = M_{\chi_A}$. Thus we have $P = M_{\chi_A}$, which proves the Corollary. \square

For a general Banach space X we have no notion of support for an element x , unlike L^p -spaces. However, when X satisfies Property 2.7, the minimal L^p -projection for x plays a similar role regarding p -orthogonality. The following Lemma proves an equivalence similar with (iv) \Leftrightarrow (v) from Corollary 2.3.

2.14. Lemma. *Let X be a Banach space. Suppose $1 \leq p < +\infty$, $p \neq 2$, and that X satisfies Property 2.7 for p . Let $x, y \in X$ and let P, Q be the minimal L^p -projections for x, y respectively. Then, the following are equivalent:*

- (i) $x \perp_p y$;
- (ii) $PQ = 0$;
- (iii) $\text{Ran}(P) \cap \text{Ran}(Q) = \{0\}$.

Proof. (i) \Rightarrow (ii) Since X satisfies Property 2.7 for p and $x \perp_p y$, we can use Proposition 2.10 to obtain $R \in \mathcal{P}_p(X)$ such that $R(x) = x$ and $R(y) = 0$. Since P and Q are minimal, Corollary 1.11 gives $PR = P$ and $Q(I - R) = Q$. Thus

$$PQ = PRQ(I - R) = PR(I - R)Q = 0.$$

- (ii) \Rightarrow (i) We have $P(x) = x$ and $Q(y) = y$. Therefore $P(y) = P(Q(y)) = 0$, and thus $x \perp_p y$.

- (ii) \Leftrightarrow (iii) This equivalence comes from the fact that P and Q are commuting projections. \square

2.15. Proposition (L^p -projections on specific subspaces and quotients of X , [19]). *Let X be a Banach space and suppose $1 \leq p \leq +\infty$, $p \neq 2$. Let $P \in \mathcal{P}_p(X)$. Then, we have*

- (i) *The Boolean algebra $\mathcal{P}_p(\text{Ran}(P))$ is isomorphic to $\mathcal{P}_p(X) \circ P$;*
- (ii) *The canonical projection $\pi : X \rightarrow X/\text{Ran}(P)$ induces an isometric isomorphism between $\text{Ker}(P)$ and $X/\text{Ran}(P)$. Hence, the Boolean algebra $\mathcal{P}_p(X/\text{Ran}(P))$ is isomorphic to $\mathcal{P}_p(X) \circ (I - P)$.*

Proof. (i) Let $T \in \mathcal{P}_p(X)$. Since $p \neq 2$, the projection T commutes with P . Thus $T(\text{Ran}(P)) \subset \text{Ran}(P)$. Hence, the map $Q_T : y \in \text{Ran}(P) \mapsto T(y)$ is well-defined and is an L^p -projection on $\text{Ran}(P)$. Let $Q \in \mathcal{P}_p(\text{Ran}(P))$. We then have

$$\begin{aligned} X &= \text{Ker}(P) \oplus_p \text{Ran}(P) = \text{Ker}(P) \oplus_p (\text{Ker}(Q) \oplus_p \text{Ran}(Q)) \\ &= (\text{Ker}(P) \oplus_p \text{Ker}(Q)) \oplus_p \text{Ran}(Q). \end{aligned}$$

Thus, the projection T on $\text{Ran}(Q)$ parallel to $(\text{Ker}(P) \oplus \text{Ker}(Q))$ is an L^p -projection on X . Since $p \neq 2$, T commutes with P and $TP = T$. By construction we get $Q_T = Q$. Therefore, there is a bijection between $\mathcal{P}_p(\text{Ran}(P))$ and $\{T \circ P, T \in \mathcal{P}_p(X)\}$, and the commutativity of these sets makes this bijection an isomorphism of commutative Boolean algebras.

(ii) As $X = \text{Ker}(P) \oplus_p \text{Ran}(P)$, the quotient map $\pi : X \rightarrow X/\text{Ran}(P)$ induces a linear bijection T between $\text{Ker}(P)$ and $X/\text{Ran}(P)$. For any $x \in \text{Ker}(P)$, we have

$$\begin{aligned} \|T(x)\| &= \|\pi(x)\| = \inf_{h \in \text{Ran}(P)} \{\|x - h\|\} \\ &= \inf_{h \in \text{Ran}(P)} \{\|((I - P)(x), \| - h\|)\|_p\} = \|x\|. \end{aligned}$$

Therefore, the linear bijection T is isometric. Hence, using item (ii) of Proposition 1.3, we have that $\mathcal{P}_p(X/\text{Ran}(P)) = T\mathcal{P}_p(\text{Ker}(P))T^{-1}$. Thus there is an isomorphism of Boolean algebras between $\mathcal{P}_p(\text{Ker}(P))$ and $\mathcal{P}_p(X/\text{Ran}(P))$. Note now that $\mathcal{P}_p(\text{Ker}(P))$ is in turn isomorphic to $\mathcal{P}_p(X) \circ (I - P)$. \square

2.16. Remark. When $p = 2$, the same arguments show that $\mathcal{P}_2(\text{Ran}(P))$ is in bijective correspondence with the set $\{TP, T \in \mathcal{P}_2(X) : TP = PT\}$, which is, in general, neither commutative nor a Boolean algebra.

2.17. Proposition. *Let $(X_i)_{i \in I}$ be a family Banach spaces. Suppose $1 \leq p \leq +\infty$, $p \neq 2$ and let $Y = \prod_{i \in I}^{\ell_p} X_i$. Let $P_i \in \mathcal{L}(Y)$ be the projection on X_i parallel to the product of X_j , $j \neq i$. Then the map $\phi : P \in \mathcal{P}_p(Y) \mapsto (PP_i|_{X_i})_{i \in I} \in \prod_{i \in I} \mathcal{P}_p(X_i)$ is well defined and is an isomorphism of Boolean algebras.*

Proof. Each operator P_i is an L^p -projection on Y . Let $P \in \mathcal{P}_p(Y)$. As $p \neq 2$, P commutes with each P_i , so $P(X_i) \subset X_i$, and $PP_i|_{X_i}$ defines an L^p -projection on X_i . The commutativity also implies that for $P' \in \mathcal{P}_p(Y)$, we have $(PP')P_i|_{X_i} = (P'P_i|_{X_i})(PP_i|_{X_i})$, so $\phi(PP') = \phi(P)\phi(P')$. Thus ϕ turns to be a morphism of Boolean algebras. Now, let $Q = (Q_i)_{i \in I} \in \prod_{i \in I} \mathcal{P}_p(X_i)$.

The map $P_Q : (x_i)_i \in Y \mapsto (Q_i(x_i)) \in Y$ is well defined and is a norm one projection on Y . We also have

$$\begin{aligned} \|(x_i)_i\|_Y &= \|(\|x_i\|_{X_i})_i\|_{\ell_p} = \|(\|Q_i(x_i)\|, \|(I_{X_i} - Q_i)(x_i)\|)\|_{\ell_p}\|_i\|_{\ell_p} \\ &= \|(\|Q_i(x_i)\|)_i\|_{\ell_p}, \|(\|(I_{X_i} - Q_i)(x_i)\|)_i\|_{\ell_p}\|_{\ell_p} \\ &= \|(\|Q_i(x_i)\|)_i\|_Y, \|(\|(I_{X_i} - Q_i)(x_i)\|)_i\|_Y\|_{\ell_p}. \end{aligned}$$

Hence Q is an L^p -projection on Y . By construction of ϕ and Q we have $\phi(Q) = (Q_i)_i$, so ϕ is bijective. This concludes the proof. \square

2.18. Proposition (Finite dimensional p -orthogonal decomposition). *Let X be a finite dimensional normed space, with $X \neq 0$ and $1 \leq p \leq +\infty$, $p \neq 2$. Then, we have*

- (i) $Card(\mathcal{P}_p(X)) = 2^m$, with $1 \leq m \leq dim(X)$, where $Card(E)$ is the cardinality of E ;
- (ii) There exist subspaces X_1, \dots, X_m of X such that

$$X = X_1 \oplus_p \dots \oplus_p X_m \text{ with } X_i \neq \{0\} \text{ and } \mathcal{P}_p(X_i) = \{0, I\};$$

- (iii) Denote P_i the projection on X_i parallel to $\oplus_{j \neq i} X_j$. Then $\mathcal{P}_p(X)$ is generated as a Boolean algebra by the family $\{P_1, \dots, P_m\}$;
- (iv) If X satisfies Property 2.7, then every X_i satisfies

$$x \perp_p y, x, y \in X_i \Rightarrow x = 0 \text{ or } y = 0.$$

Proof. (i) Recall first that each finite dimensional normed space is complete. Let $n = dim(X)$. Operators acting on X can then be identified to $n \times n$ matrices with complex entries (for a choice of a basis for X). Thus, $\mathcal{P}_p(X)$ identifies to a set of commutative and diagonalizable matrices with eigenvalues in $\{0, 1\}$. These matrices are then jointly diagonalizable: there exists a change of basis of X that turns all these matrices into diagonal ones. This means that there can be at most $Card(\{0, 1\})^n = 2^n$ elements in this set. As the cardinal of a finite commutative Boolean algebra is 2^m for some $m \geq 0$, we obtain the conclusion of item (i).

- (ii) We will prove the result by induction over $m \geq 1$. If $m = 1$, then $\mathcal{P}_p(X) = \{0, I\}$ and the result is true. Suppose that the result is true for any Banach space Y with $Card(\mathcal{P}_p(Y)) \leq 2^m$, and let X be a Banach space with $Card(\mathcal{P}_p(X)) = 2^{m+1}$. As $m + 1 > 1$, X possesses non-trivial L^p -projections. Let $P \in \mathcal{P}_p(X)$ and suppose that P is non-trivial. We then have $X = Ran(P) \oplus_p Ker(P)$, and Proposition 2.17 gives

$$2^{m+1} = Card(\mathcal{P}_p(X)) = Card(\mathcal{P}_p(Ran(P)))Card(\mathcal{P}_p(Ker(P))).$$

Since both $Ran(P)$ and $Ker(P)$ are different from $\{0\}$, these subspaces possess at least two L^p -projections. The previous equation then implies that they cannot possess more than 2^m L^p -projections. As

$$Card(\mathcal{P}_p(Ran(P))) = 2^k \text{ and } Card(\mathcal{P}_p(Ker(P))) = 2^{m+1-k},$$

we can apply the induction hypothesis to $Ran(P)$ and $Ker(P)$ to get subspaces F_1, \dots, F_k and G_{k+1}, \dots, G_{m+1} such that $F_i, G_j \neq \{0\}$, $\mathcal{P}_p(F_i) = \{0, I\}$,

$\mathcal{P}_p(G_j) = \{0, I\}$, and

$$X = \text{Ran}(P) \oplus_p \text{Ker}(P) = (F_1 \oplus_p \dots \oplus_p F_k) \oplus_p (G_{k+1} \oplus_p \dots \oplus_p G_{m+1}).$$

This completes the proof by induction.

- (iii) Since $X = X_1 \oplus_p \dots \oplus_p X_m$, every P_i is an L^p -projection on X . Since every X_i is not reduced to $\{0\}$, every P_i is non-zero. For any set $E \subset \{1, \dots, m\}$, we can see that the projection on $\bigoplus_{i \in E} X_i$ parallel to $\bigoplus_{j \notin E} X_j$ is well-defined and lies in the Boolean sub-algebra generated by the family $\{P_1, \dots, P_m\}$. Since all these projections are distinct, the Boolean sub-algebra generated by the family $\{P_1, \dots, P_m\}$ has at least $2^m = \text{Card}(\mathcal{P}_p(X))$ elements, so it is equal to $\mathcal{P}_p(X)$.

- (iv) Let $1 \leq i \leq m$. Let $x, y \in X_i$ be such that $x \perp_p y$, and let $Q \in \mathcal{P}_p(X)$ be such that $Q(x) = x$ and $Q(y) = 0$. For every $1 \leq j \leq m$, we have $P_j(X_i) = X_i$ or $\{0\}$. Item (iii) then implies that for any $P \in \mathcal{P}_p(X)$, we have $P(X_i) = X_i$ or $\{0\}$. If we have $Q(X_i) = X_i$, then $0 = Q(y) = y$. If we have $Q(X_i) \neq X_i$, then $Q(X_i) = \{0\}$, so $x = Q(x) = 0$. Therefore, we have $x = 0$ or $y = 0$. \square

2.19. *Remark.* Concerning item (iv) of Proposition 2.18, we have not been able to answer the following question. For X a Banach space of the form $X = X_1 \oplus_p \dots \oplus_p X_m$ satisfying

$$x \perp_p y, x, y \in X_i \Rightarrow x = 0 \text{ or } y = 0,$$

does X satisfy Property 2.7 for p ?

2.2. Counter-examples for p -orthogonality

We collect in this section several examples.

2.20. Counter-Example (A Banach space not satisfying Property 2.8). Let $X = \mathbb{C}^3$, and let $\{e_1, e_2, e_3\}$ be its canonical basis. Let $1 \leq p \leq +\infty$. Take

$$\begin{aligned} E &= \{e^{ia}te_1 + e^{ib}(1-t^p)^{\frac{1}{p}}e_2; e^{ia}te_3 + e^{ib}(1-t^p)^{\frac{1}{p}}e_2; \\ &\quad e^{ic}(e_1 + e_2 + e_3), a, b, c \in \mathbb{R}\} \\ K &= \text{Conv}(E). \end{aligned}$$

Then, K is a compact convex set that contains 0 in its interior and that is invariant under multiplication by λ , for any $\lambda \in \partial\mathbb{D}$. Hence there exists a norm N on X whose closed unit ball is K . By construction, $\text{Span}(e_1, e_2)$ and $\text{Span}(e_2, e_3)$ are isometric to $\ell_p(\mathbb{C}^2)$, therefore we have $e_2 \perp_p e_1$ and $e_2 \perp_p e_3$. Since $e_1 + e_2 + e_3$, e_2 and $\frac{e_1 + e_3}{2}$ are in the boundary of K , we have

$$N(e_1 + e_2 + e_3) = N(e_2) = N\left(\frac{e_1 + e_3}{2}\right) = 1.$$

Thus, we have

$$N(e_2 + (e_1 + e_3)) = 1 < 2 = N(e_1 + e_3) \leq \|(N(e_2), N(e_1 + e_3))\|_p.$$

Hence, e_2 is not p -orthogonal to $e_1 + e_3$, so X does not satisfy Property 2.7.

2.21. Counter-Example (A Subspace not satisfying Property 2.7, nor Proposition 2.10). Let $1 \leq p < +\infty$ and $X = \ell_p(\mathbb{C}^4)$. Take $f = (1, -1, 0, 0)$, $g = (0, 0, 1, -1)$, $h = (1, 1, 1, 1)$, and $F = \text{Span}(f, g, h)$. As X is an L^p -space, it satisfies Property 2.7. However, its closed subspace F does not. Indeed, on F we have $f \perp_p h$, but since $h^{\perp_p} = \text{Span}(f)$ and $f^{\perp_p} = \text{Span}(h)$, F cannot possess any L^p -projection P such that $P(f) = f$ and $P(g) = 0$ as we would have

$$\dim(\text{Ran}(P)) + \dim(\text{Ker}(P)) \leq 1 + 1 = 2 < 3 = \dim(F).$$

Furthermore, every element x in F that is not in $\text{Span}(f)$ nor $\text{Span}(g)$ satisfies $x^{\perp_p} = \{0\}$. Indeed, there is no element in F outside 0 that has a support disjoint with $\text{supp}(x)$. Hence, $\mathcal{P}_p(F) = \{0, I\}$, whereas the subspace $\text{Span}(f, g)$ clearly possesses non-trivial L^p -projections. Thus, the second part of Proposition 2.10 is also false for F and F does not satisfy Property 2.7.

2.22. Counter-Example (A Banach Space satisfying Property 2.7 but not Clarkson's equality case). Suppose $1 \leq p < +\infty$. Let $X = \mathbb{C}^2$ and let $\{e_1, e_2\}$ be its canonical basis. Take

$$K = \text{Conv}(\{e^{ia}e_1; e^{ia}e_2; \frac{e^{ia}e_1 + e^{ib}e_2}{2^{1/p}}, a, b \in \mathbb{R}\}).$$

Then, K is a compact convex set that contains 0 in its interior and that is invariant under multiplication by λ , for any $\lambda \in \partial\mathbb{D}$. Hence there exists a norm N on X whose closed unit ball is K . By construction, e_1, e_2 and $\frac{1}{2^{1/p}}(e_1 + e^{ib}e_2)$ are in the boundary of K , so these elements are of norm one. Thus, we have

$$N(e_1 + e^{it}e_2)^p = 2 = N(e_1)^p + N(e_2)^p, \forall t \in \mathbb{R}.$$

Since the unit ball for N is not uniformly convex, it cannot be equal to a ℓ_p unit ball for some $1 < p < +\infty$. Hence, $\text{Span}(e_1, e_2)$ cannot be isometrically isomorphic to $\ell_p(\mathbb{C}^2)$, so e_1 and e_2 are not p -orthogonal. Thus, $\mathcal{P}_p(X)$ is trivial, as well as $\mathcal{P}_p(F)$ for every subspace F of X . Therefore, X satisfies Property 2.7 but it does not possess anything similar to the equality case in Clarkson inequalities as the implication (iii) \Rightarrow (iv) of Corollary 2.3 is false for X .

2.23. Counter-Example (A Banach space not satisfying Property 2.8, of finite dimension, but with $\alpha(X)$ not finite). Let $X = \mathbb{C}^3$, and let $\{e_1, e_2, e_3\}$ be its canonical basis. Suppose $1 \leq p \leq +\infty$. We refer to Definition 2.30 for the definition of the cardinality $\alpha(X)$. Take

$$\begin{aligned} E = & \{e^{ia}t.e_2 + e^{ib}(1 - t^p)^{\frac{1}{p}} [\cos(s)e_1 + \sin(s)e_3]; \\ & e^{ia}(\frac{1}{2}e_2 + (1 - \frac{1}{2^p})^{\frac{1}{p}}(e_1 - e_3)); e^{ia}te_1 + e^{ib}\sqrt{1 - t^2}e_3, \\ & a, b \in \mathbb{R}, 0 \leq t \leq 1, 0 \leq s \leq \frac{\pi}{2}\} \\ K = & \text{Conv}(E). \end{aligned}$$

Then, K is a compact convex set that contains 0 in its interior and that is invariant under multiplication by λ , for any $\lambda \in \partial\mathbb{D}$. Hence there exists a norm N on X whose closed unit ball is K . By construction, $\text{Span}(e_1, e_3)$ is isometric to $\ell_2(\mathbb{C}^2)$ and $\text{Span}(e_2, \cos(s)e_1 + \sin(s)e_3)$ is isometric to $\ell_p(\mathbb{C}^2)$ for every $0 \leq s \leq \frac{\pi}{2}$. Since $\frac{\sqrt{2}}{2}(e_1 - e_3)$, e_2 and $\frac{1}{2}e_2 + (1 - \frac{1}{2^p})^{\frac{1}{p}}(e_1 - e_3)$ are in the boundary of K , we have

$$N(e_1 - e_3) = \sqrt{2}, N(e_2) = N\left(\frac{1}{2}e_2 + \left(1 - \frac{1}{2^p}\right)^{\frac{1}{p}}(e_1 - e_3)\right) = 1.$$

This gives

$$N\left(\frac{1}{2}e_2\right)^p + N\left(\left(1 - \frac{1}{2^p}\right)^{\frac{1}{p}}(e_1 - e_3)\right)^p = \frac{1}{2^p} + \left(1 - \frac{1}{2^p}\right)\sqrt{2^p} > 1 = 1^p,$$

so the vectors e_2 and $e_1 - e_3$ are not p -orthogonal. Since e_2 is p -orthogonal to e_1 and e_3 , we obtain that X does not satisfy Property 2.8 for p . Since e_2 is not p -orthogonal to $\text{Span}(e_1, e_3)$, then for almost every s with $0 \leq s \leq \frac{\pi}{2}$, any non-trivial L^p -projection on $\text{Span}(e_2, \cos(s)e_1 + \sin(s)e_3)$ does not extend to $\text{Span}(e_2, e_1, e_3) = X$. Therefore, according to Definition 2.30, almost every non-trivial L^p -projection on these spaces is maximal, so $\alpha(X)$ is not finite.

2.3. The case of L^∞ -projections

This section focuses on results regarding L^∞ -projections. We describe ∞ -orthogonality in $L^\infty(\Omega)$ and we use it to determine $\mathcal{P}_\infty(L^\infty(\Omega))$, along with an equivalent form of Proposition 2.10 for $p = +\infty$.

2.24. Proposition. *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and let $f \in L^\infty(\Omega)$, $f \neq 0$. Then,*

- (i) $f^\perp = \{g : |g(x)| \leq \frac{\|f\| - |f(x)|}{\|f\|} \|g\|, \forall_{a.e.} x \in \Omega\} = \{g \neq 0 : \frac{|g|}{\|g\|} + \frac{|f|}{\|f\|} \leq_{a.e.} 1\} \cup \{0\}$;
- (ii) For $g \in f^\perp$ and $B_n \in \mathcal{F}$ with $\mu(B_n) > 0$ and $\|(\|f\| - |f|)\chi_{B_n}\| \rightarrow_n 0$, we have $\|g\chi_{B_n}\| \rightarrow_n 0$;
- (iii) We have $f^\perp = \{0\}$ if 1 and only if $\| \|f\| - |f| \| < \|f\|$;
- (iv) f^\perp is a non-zero subspace if and only if $|f| =_{a.e.} \|f\| \chi_A$ for $A \in \Omega$ with $\mu(A^C) > 0$.

Proof. Every element of $L^\infty(\Omega)$ that we will consider here will be associated to a representative that takes finite values everywhere.

- (i) Let $g \in f^\perp$. There is nothing to prove when $g = 0$. Suppose that $g \neq 0$. Since the ∞ -orthogonality is homogeneous, we can divide f and g by their respective norms and suppose that $\|f\| = \|g\| = 1$. As \mathbb{C} is separable, there is $A \in \mathcal{F}$ with $\mu(A^C) = 0$ such that for every $x \in A$ and for every $z \in \mathbb{C}$, we have $|f(x) + zg(x)| \leq \|f + zg\|$. We also recall that for $x \in A$ and for any

¹We mention the following examples in order to better understand this condition. When $\Omega = \{1, \dots, n\}$, (iii) is equivalent to $|f(i)| \neq 0$ for every $i \in \Omega$. When $\Omega = \mathbb{N}$, (iii) is equivalent to 0 not being in the closure of $\{f(n), n \geq 0\}$.

$r \geq 0$, we have $z \in \mathbb{C}$ with $|z| = r$ such that $|f(x) + zg(x)| = |f(x)| + r|g(x)|$. Therefore, for $r = 1$, there exists $z \in \mathbb{C}$ such that

$$|f(x) + zg(x)| = |f(x)| + |g(x)| \leq \|f + zg\| = \max(\|f\|, \|zg\|) = \max(1, 1) = 1.$$

Hence, we obtain

$$|g(x)| \leq 1 - |f(x)|, \forall x \in A,$$

which is the condition in the statement when $\|f\| = \|g\| = 1$. We now need to prove that for every g non-zero that satisfies this condition, g lies in f^\perp . By homogeneity of both sets, we can suppose that $\|g\| = 1$. Hence, we need to show that

$$\|f + zg\| = \max(\|f\|, \|zg\|) = \max(1, |z|), \forall z \in \mathbb{C}.$$

Let $z \in \mathbb{C}$. For almost every $x \in \Omega$, we have

$$\begin{aligned} |f(x) + zg(x)| &\leq |f(x)| + |z||g(x)| = (|f(x)| + |g(x)|) + (|z| - 1)|g(x)| \\ &\leq 1 + (|z| - 1)|g(x)|. \end{aligned}$$

This is no greater than $1 + (|z| - 1) = |z|$ if $|z| \geq 1$ and no greater than 1 if $|z| \leq 1$. Therefore we have $\|f + zg\| \leq_{a.e.} \max(1, |z|)$.

Suppose that $|z| \geq 1$. The properties of $\|g\|_\infty$ imply the existence of sets $A_n \in \mathcal{F}$ such that $\mu(A_n) > 0$ and $\alpha(n) := \|(\|g\| - |g|)\chi_{A_n}\| \rightarrow_n 0$. Therefore, for almost every $x \in A_n$, the condition on g gives us

$$\begin{aligned} |f(x) + zg(x)| &\geq |z||g(x)| - |f(x)| = |z|\|g\| - |z|(\|g\| - |g(x)|) - |f(x)| \\ &\geq |z| - |z|\alpha(n) - (1 - |g(x)|) \\ &\geq |z| - |z|\alpha(n) - \alpha(n). \end{aligned}$$

Hence we obtain

$$|z| \geq \|f + zg\| \geq \|(f + zg)\chi_{A_n}\| \geq |z| - |z|\alpha(n) - \alpha(n) \rightarrow_n |z|,$$

so $\|f + zg\| = |z|$.

Suppose now that $|z| \leq 1$. The properties of $\|f\|_\infty$ imply the existence of sets $B_n \in \mathcal{F}$ such that $\mu(B_n) > 0$ and $\beta(n) := \|(\|f\| - |f|)\chi_{B_n}\| \rightarrow_n 0$. Therefore, for almost every $x \in B_n$, the condition on g gives us

$$\begin{aligned} |f(x) + zg(x)| &\geq |f(x)| - |z||g(x)| = \|f\| - (\|f\| - |f(x)|) - |z||g(x)| \\ &\geq 1 - \beta(n) - |z|(1 - |f(x)|) \\ &\geq 1 - \beta(n) - |z|\beta(n). \end{aligned}$$

Hence we obtain

$$1 \geq \|f + zg\| \geq \|(f + zg)\chi_{B_n}\| \geq 1 - \beta(n) - |z|\beta(n) \rightarrow_n 1,$$

so $\|f + zg\| = 1$. This proves that $f \perp_\infty g$ and concludes the proof of item (i).
- (ii) Let $B_n \in \mathcal{F}$ be such that $\mu(B_n) > 0$ and $\|(\|f\| - |f|)\chi_{B_n}\| \rightarrow_n 0$ and $g \in f^\perp$. Then, for almost every $x \in B_n$ we have

$$|g(x)| \leq \frac{\|g\|}{\|f\|} (\|f\| - |f(x)|) \leq \frac{\|g\|}{\|f\|} \|(\|f\| - |f|)\chi_{B_n}\|,$$

so $\|g\chi_{B_n}\| \rightarrow_n 0$.

- (iii) Suppose that we have a vector g in f^\perp that is non-zero. Then we have $f \in g^\perp$. We also have $A_n \in \mathcal{F}$ such that $\mu(A_n) > 0$ and $\|(\|g\| - |g|)\chi_{A_n}\| \rightarrow_n 0$. Since g is non-zero, item (ii) implies that $\|f\chi_{A_n}\| \rightarrow_n 0$. Therefore, we have

$$\|f\| \geq \| \|f\| - |f| \| \geq \|(\|f\| - |f|)\chi_{A_n}\| = \|f\| - \|f\chi_{A_n}\| \rightarrow_n \|f\|,$$

so we obtain $\| \|f\| - |f| \| = \|f\|$.

Conversely, suppose now that $\| \|f\| - |f| \| = \|f\|$. Take $g = \frac{\|f\| - |f|}{\|f\|}$. Then $g \in L^\infty(\Omega)$ and $\|g\| = \frac{\|f\|}{\|f\|} = 1$. Thus,

$$|g| =_{a.e} \frac{\|f\| - |f|}{\|f\|} = \frac{\|f\| - \| \|f\| - |f| \|}{\|f\|} \|g\|,$$

so $g \in f^\perp$ according to item (i), and the set f^\perp is not reduced to 0.

- (iv) When $|f| =_{a.e} \|f\|$, item (i) tells us that $f^\perp = M_{\chi_{A^C}}(L^\infty)$, which is a subspace. Such a subspace is non-zero if and only if $\mu(A^C) > 0$. We now take f such that $|f|$ does not have the form $\|f\|$. If $\| \|f\| - |f| \| < \|f\|$, then item (iii) tells us that $f^\perp = \{0\}$. Thus we may assume that $\| \|f\| - |f| \| = \|f\|$. Then, as seen previously, the function $g = \frac{\|f\| - |f|}{\|f\|}$ has norm 1 and lies in f^\perp . Since $|f|$ is not equal to any $\|f\|$, we can find $\epsilon > 0$ and $B \in \mathcal{F}$ such that $\mu(B) > 0$ and $\epsilon \leq |f| \leq \|f\| - \epsilon$ almost everywhere on B . We now take $h = \frac{\|f\| - |f|}{\|f\|} - \frac{\epsilon}{\|f\|} \chi_B$. As we have $|f| + \epsilon \leq \|f\|$ a.e. on B , we can see that $h \geq 0$ a.e. on Ω . As h coincides with g a.e. on B^C , the properties of g and B also imply that $\|h\| = \|g\| = 1$. Hence, for almost every x we have

$$|h(x)| = h(x) \leq \frac{\|f\| - |f(x)|}{\|f\|} = \frac{\|f\| - |f|}{\|f\|} \|h\|,$$

so h is also in f^\perp . However, $g - h$ does not lie in f^\perp . Indeed, if $f \perp_\infty g - h$, then f would belong to $(g - h)^\perp = L^\infty(B^C)$, which is not true since $f > 0$ a.e. on B . Therefore f^\perp is not a subspace in this case. \square

L^∞ -projections have been completely characterized on certain subspaces of L^∞ such as $C^0(\Omega)$ (continuous maps on a locally compact and Hausdorff set, [16, Ex.1.4(a)]); $L^\infty(\Omega)$ when Ω is σ -finite ([14] and [16, Thm.1.9]) ; $Lip([0, 1])$, $AC([0, 1])$ or $C^1([0, 1])$ (Lipschitz, absolutely continuous, or C^1 maps over $[0, 1]$, [8]); $l^\infty(\mathbb{N})$, $c_0(\mathbb{N})$, or $c(\mathbb{N})$ (see [24]). These results either come from a characterization of Hermitian operators on the said spaces, from duality, or from a characterization of surjective isometries. Recall that $P \in \mathcal{L}(X)$ is Hermitian if $\|e^{i\alpha P}\| = 1$ for each $\alpha \in \mathbb{R}$ and that a projection P is Hermitian if and only if $P + \lambda(I - P)$ is an isometry for every $\lambda \in \partial\mathbb{D}$.

The following theorem characterizes L^∞ -projections on every L^∞ space. The proof uses the ∞ -orthogonality relationship.

2.25. Theorem (L^∞ -projections on $L^\infty(\Omega)$). *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Then*

$$\mathcal{P}_\infty(L^\infty(\Omega)) = \{M_{\chi_A}, A \in \mathcal{F}\}.$$

Proof. Notice first that for every $A \in \mathcal{F}$ the projection M_{χ_A} is an L^∞ -projection. Let $P \in \mathcal{P}_\infty(L^\infty(\Omega))$ that is non-trivial. Let $f \in \text{Ker}(P)$ and $g \in \text{Ran}(P)$ that are non-zero. We will start by showing that f and g have disjoint supports (up to a set of measure zero). Take $A = \text{supp}(f) \cap \text{supp}(g)$ and suppose that $\mu(A) > 0$. Denote $f_1 = \chi_A f$ and $g_1 = \chi_A g$. The L^∞ -projection $Q = PM_{\chi_A}$ gives

$$Q(f_1) = PM_{\chi_A}M_{\chi_A}(f) = PM_{\chi_A}(f) = M_{\chi_A}P(f) = M_{\chi_A}(f) = f_1.$$

Similarly we obtain $Q(g_1) = M_{\chi_A}P(g) = 0$, so f_1 and g_1 are ∞ -orthogonal and have as support A (up to a set of measure zero). As the ∞ -orthogonality is linear and as f_1, g_1 are non-zero, we can divide them by their respective norm to suppose that $\|f_1\| = \|g_1\| = 1$. Hence, there exists $\epsilon > 0$ small enough such that the set

$$B = \{x \in A: \epsilon \leq |f_1(x)|\}$$

has a strictly positive measure.

Let us now consider $f_2 = f_1\chi_B$ and $g_2 = g_1\chi_B$. Since A is a support for f_1 and g_1 and since $B \subset A$, f_2 and g_2 are non-zero and the set B is a support of these maps. With the L^∞ -projection $R = QM_{\chi_B}$, a small computation gives $R(f_2) = f_2$ and $R(g_2) = 0$, so f_2 and g_2 are ∞ -orthogonal. However, since $|f_2| \geq \epsilon$ almost everywhere on B , we obtain that

$$\| \|f_2\| - |f_2| \|_{L^\infty(B)} \leq \|f_2\|_{L^\infty(B)} - \epsilon < \|f_2\|_{L^\infty(B)}.$$

Considering now $L^\infty(B)$, we can apply item (iii) of Proposition 2.24 to obtain $f_2^{\perp\infty} = \{0\}$ in $L^\infty(B)$. This contradicts the fact that in $L^\infty(B)$ we have $g_2 \neq 0$ and $f_2 \perp_\infty g_2$. Thus, we have $\mu(A) = 0$, so the initial functions f and g have disjoint supports.

Let us now take $f = P(\chi_\Omega)$, and $g = (I - P)(\chi_\Omega)$. We now know that the supports of f and g are disjoint. Since $f + g = \chi_\Omega$ and $\text{supp}(\chi_\Omega) = \Omega$, the set $\text{supp}(f)^C$ is a support for g (up to a set of measure zero). For every $g' \in \text{Ker}(P)$, the previous argument tells us that $\text{supp}(g')$ is contained in $\text{supp}(f)^C$. For every $f' \in \text{Ran}(P)$, $\text{supp}(f')$ is then contained in

$$\text{supp}(g)^C = (\text{supp}(f)^C)^C = \text{supp}(f).$$

Thus, the L^∞ -projection $M_{\chi_{\text{supp}(f)}}$ coincides with P on $\text{Ran}(P)$ and on $\text{Ker}(P)$, hence $P = M_{\chi_{\text{supp}(f)}}$. This concludes the proof. \square

2.26. Remark. The ideas from the proof of this theorem plus some extra computations can be used to show that for $X = c_0(\mathbb{N}), c(\mathbb{N})$, or $c_{00}(\mathbb{N})$, we have $\mathcal{P}_\infty(X) = \{P \in \mathcal{P}_{+\infty}(l^\infty(\mathbb{N})) : P(X) \subset X\}$.

2.27. Corollary. *Let X be a Banach space such that $\text{Card}(\mathcal{P}_\infty(X)) > 4$. Then X does not satisfy Property 2.8 for $p = +\infty$. This is in particular true for $X = \ell^\infty(\{0, \dots, n-1\})$, with $n \geq 3$.*

Proof. We will show that X possesses a subspace isometrically isomorphic to $\ell^\infty(\{0, 1, 2\})$. As the cardinal of $\mathcal{P}_\infty(X)$ is greater than 4, we can find L^∞ -projections P, Q on X such that $P \notin \{0, I\}$ and $Q \notin \{0, I, P, (I - P)\}$. Since P

and Q commute, we have $Q(\text{Ran}(P)) \subset \text{Ran}(P)$ and $Q(\text{Ker}(P)) \subset \text{Ker}(P)$, so $Q|_{\text{Ker}(P)}$ and $Q|_{\text{Ran}(P)}$ are also L^∞ -projections, with at least one of them non-trivial. Hence, we have

$$\begin{aligned} X &= \text{Ran}(P) \overset{\infty}{\oplus} \text{Ker}(P) \\ &= (Q(\text{Ran}(P)) \overset{\infty}{\oplus} (I - Q)(\text{Ran}(P))) \overset{\infty}{\oplus} (Q(\text{Ker}(P)) \overset{\infty}{\oplus} (I - Q)(\text{Ker}(P))) \\ &= E_1 \overset{\infty}{\oplus} E_2 \overset{\infty}{\oplus} E_3 \overset{\infty}{\oplus} E_4. \end{aligned}$$

The choice of P and Q implies that at most one E_i is reduced to $\{0\}$. Up to reordering, let us suppose that the subspaces E_1, E_2, E_3 are not reduced to $\{0\}$, and take $f_i \in E_i$ with $\|f_i\| = 1$. Thus, we have

$$\begin{aligned} \|af_1 + bf_2 + cf_3\| &= \max(\|af_1 + bf_2\|, \|cf_3\|) = \max(\max(\|af_1\|, \|bf_2\|), |c|) \\ &= \max(|a|, |b|, |c|) = \|(a, b, c)\|_\infty. \end{aligned}$$

Hence, $\text{Span}(f_1, f_2, f_3)$ is isometrically isomorphic to $\ell^\infty(\{0, 1, 2\})$. By using Proposition 2.24 for $f = (2, 1, 0)$ on $\ell^\infty(\{0, 1, 2\})$ we can see that $f^{\perp+\infty}$ is not a subspace. Hence $\text{Span}(f_1, f_2, f_3)$ does not satisfy Property 2.8 for $p = +\infty$ and Remark 2.11 implies that X does not satisfy Property 2.8 too. \square

2.28. *Remark.* Using Corollary 2.27, we can see that for $p = +\infty$, the question in Remark 2.19 turns out to be false in general. Indeed, we can see that the ∞ -orthogonality is trivial on \mathbb{C} but $\ell^\infty(\{0, 1, 2\}) = \mathbb{C} \overset{\infty}{\oplus} \mathbb{C} \overset{\infty}{\oplus} \mathbb{C}$ does not satisfy Property 2.8, so it does not satisfy Property 2.7.

Corollary 2.27 also allows us to extend the results of Proposition 2.10 to the case $p = +\infty$.

2.29. Proposition. *Let X be a Banach space, and $p = +\infty$. The following items are equivalent*

- (i) X satisfies Property 2.7 for p ;
- (ii) For any subsets E_1, E_2 of X , such that $f \perp_p g$ for every $f \in E_1, g \in E_2$, there exists $P \in \mathcal{P}_p(X)$ such that $P(E_1) = E_1$ and $P(E_2) = \{0\}$.

Furthermore, if one item is true, then there exist closed subspaces X_1, X_2 satisfying $f \perp_\infty g \Rightarrow f = 0$ or $g = 0$, such that $X = X_1 \overset{\infty}{\oplus} X_2$.

Proof. We can see that item (ii) implies Property 2.7. Hence, suppose that Property 2.7 is true for X and $+\infty$. Thus, Property 2.8 is true for X and $+\infty$, according to Proposition 2.9. Therefore, $\text{Card}(\mathcal{P}_\infty(X)) \leq 4$, according to Corollary 2.27. We then have $\mathcal{P}_\infty(X) = \{0, I\}$ or $\{0, I, P, (I - P)\}$. If $\mathcal{P}_\infty(X) = \{0, I\}$, then $f \perp_\infty g$ implies that $f = 0$ or $g = 0$, which means that for any set E we have $E^{\perp+\infty} = \{0\}$. Hence X satisfies item (ii) as every ∞ -orthogonality between sets is trivial, and for $X_1 = X, X_2 = \{0\}$ we also have the desired decomposition.

Suppose now that $\mathcal{P}_\infty(X) = \{0, I, P, (I - P)\}$. Denote $X_1 = \text{Ker}(P), X_2 = \text{Ran}(P)$. Let $E_1, E_2 \subset X$ be such that $E_1 \perp_\infty E_2$. If $E_1 = \{0\}$, then E_1 and E_2 are separated by $Q = 0$. If $E_2 = \{0\}$, then E_1 and E_2 are separated by $Q = I$. Suppose now that $E_1 \neq \{0\}$ and $E_2 \neq \{0\}$. Let $f \in E_1$ and $g \in E_2$

that are non-zero. We have $f \perp_{\infty} g$. Hence, there exists $Q \in \mathcal{P}_{\infty}(X)$ such that $Q(f) = f$ and $Q(g) = 0$. Since f and g are non-zero, we must have $Q = P$ or $Q = (I - P)$, that is $f \in X_1, g \in X_2$ or $f \in X_2, g \in X_1$. Up to reordering, suppose that $f \in X_1$ and $g \in X_2$. Thus, for any $f' \in E_1$ we have $f' \perp_{\infty} g$, so the L^{∞} -projection Q such that $Q(f') = f'$ and $Q(g) = 0$ is either 0 or P . Similarly, for any $g' \in E_2$ we have $f \perp_{\infty} g'$. Thus, the L^{∞} -projection Q such that $Q(f) = f$ and $Q(g') = 0$ is either 0 or P . This implies that $E_1 \subset X_1$ and $E_2 \subset X_2$ so $P(E_1) = E_1$ and $P(E_2) = \{0\}$. Therefore, X satisfies item (ii).

We can also notice that the subspaces $X_1 = Ker(P)$ and $X_2 = Ran(P)$ are non-trivial and that they have no L^{∞} -projection other than 0 and I , as it is stated by item (i) of Proposition 2.15. Thus, similarly to the beginning of the proof, for $f, g \in X_1$ (resp. X_2) that are ∞ -orthogonal, we have $f = 0$ or $g = 0$. Then $X = X_1 \oplus_{\infty} X_2$ gives the desired decomposition. \square

2.4. Maximal L^p -projections for a Banach space

In this section we introduce the notion of maximal L^p -projections for a Banach space X . These are L^p -projections on a subspace of X that cannot be extended to L^p -projections on a larger subspace. This section will study these projections and focus on upper bounds for their number in certain situations, especially in the finite dimensional case. Most of the results are stated for $1 \leq p \leq +\infty, p \neq 2$. The results presented at the beginning of the section (up to Lemma 2.37) are also true for $p = 2$, but not the later ones since they rely on properties of $\mathcal{P}_p(X)$. As the previous section highlighted differences in the case $p = +\infty$, (see Proposition 2.24 and Proposition 2.29), some of these later results are not stated for $p = +\infty$. However, some information about the case $p = +\infty$ is given in item (ii) of Proposition 2.50 below.

2.30. Definition (Maximal L^p -projections). Let X be a Banach space. Suppose that $1 \leq p \leq +\infty$. Let F be a closed subspace of X , and let $P \in \mathcal{P}_p(F)$. The L^p -projection P is said to be *maximal for X* if there exist no subspace G and no L^p -projection Q on G such that $F \subsetneq G$ and $Q|_F = P$.

We denote

$$\alpha(F) := Card(\{P : P \text{ is a maximal } L^p\text{-projection for } F\}),$$

where $Card()$ refers to the cardinality.

Let us consider P a maximal L^p -projection on X , defined on a subspace G , that is non-trivial. Then for every x in $Ker(P)$ (resp. $Ran(P)$) we have $Ran(P) \subset x^{\perp p}$ (resp. $Ker(P) \subset x^{\perp p}$). As G is spanned by $Ran(P)$ and $Ker(P)$, we obtain that G is spanned by vectors x of X whose p -orthogonal is not reduced to $\{0\}$. This fact led us to the following definition.

2.31. Definition. Let X be a Banach space, and let p be such that $1 \leq p \leq +\infty$. We define the subspace

$$X_{(p)} := Span(\{x \in X : x^{\perp p} \neq \{0\}\}).$$

Thus $X_{(p)}$ is the subspace of X spanned by all the vectors whose p -orthogonal is not reduced to $\{0\}$.

2.32. *Remark.* When F satisfies Property 2.7 for p , every L^p -projection that is maximal for F is an element of $\mathcal{P}_p(F)$ according to Proposition 2.10. Thus $\alpha(F) = \text{Card}(\mathcal{P}_p(F))$. In this case, we also have $F_{(p)} = F$ if $\mathcal{P}_p(F) \neq \{0, I\}$, or $F_{(p)} = \{0\}$ if not. As we saw in Counter-Example 2.21, we can construct a subspace G of F possessing L^p -projections and find some elements $(f_i)_{i \in I}$ that do not "behave" well regarding the L^p -projections of G . Using these elements, we can construct a subspace H of F that contains G but that only possesses trivial L^p -projections. Hence, we focus our study on the L^p -projections that may exist on subspaces of H . We give below some general results concerning L^p -projections that are maximal for H , and give an upper bound of $\alpha(H)$ when $\dim(H)$ is finite.

The next lemmas and corollaries show how the subspace $X_{(p)}$ appears in the study of maximal L^p -projections on X .

2.33. Lemma. *Let X be a Banach space. Suppose $1 \leq p \leq +\infty$.*

- (i) *Let F be a closed subspace of X and let $P \in \mathcal{P}_p(F)$. Then, there exists a subspace G and $Q \in \mathcal{P}_p(G)$ such that $F \subset G$, Q is a maximal L^p -projection for X , and $Q|_F = P$;*
- (ii) *Let $x \in X$ be such that $x^{\perp p} \neq \{0\}$. Then, there exists Q a maximal L^p -projection for X such that $Q(x) = x$;*
- (iii) *Let $H \subset F$ be closed subspaces of X and let $P \in \mathcal{P}_p(H)$ be a maximal L^p -projection for F . Then, for any G and $Q \in \mathcal{P}_p(G)$ such that $Q|_H = P$, we have*

$$\text{Ran}(P) = (F \cap Q(F)) \text{ and } \text{Ker}(P) = (F \cap (I - Q)(F));$$

- (iv) *Let F be a closed subspace of X . Then $\alpha(F) \leq \alpha(X)$.*

Proof. (i) Let G_1 be a vector subspace included in $\text{Ran}(P)^{\perp p}$, that contains $\text{Ker}(P)$, and that is maximal for the inclusion of sets. As $\text{Ran}(P)^{\perp p}$ is closed, it contains $\overline{G_1}$, so $\overline{G_1} = G_1$ by maximality and G_1 is closed. Similarly, let G_2 be a vector subspace included in $G_1^{\perp p}$, that contains $\text{Ran}(P)$, and that is maximal for the inclusion of sets. As $G_1^{\perp p}$ is closed, it contains $\overline{G_2}$, so $\overline{G_2} = G_2$ by maximality and G_2 is closed. Hence, the subspace $G = G_1 \oplus G_2$ admits an L^p -projection Q such that $Q(G_1) = \{0\}$ and $Q(G_2) = G_2$. We also have $F \subset G$ and $Q|_F = P$. The maximality of G_1 and G_2 implies that Q cannot be extended on a larger subspace, so it is a maximal L^p -projection for X .

- (ii) If $x = 0$, then choose $Q = 0$. If not, since $x^{\perp p} \neq \{0\}$, there is $y \in X$ that is non-zero such that $x \perp_p y$, and the subspace $\text{Span}(x, y)$ admits a L^p -projection P such that $P(x) = x$ and $P(y) = 0$. We then apply item (i) to P in order to get the desired result.

- (iii) As $F \cap \text{Ran}(Q)$ is a closed subspace of F , containing $\text{Ran}(P)$, and contained in $\text{Ker}(P)^{\perp p} \cap F$, we have $F \cap \text{Ran}(Q) = \text{Ran}(P)$ by maximality of

P . A similar reasoning for $Ker(Q)$ and $Ker(P)$ gives $F \cap Ker(Q) = Ker(P)$. Since Q is a projection, we have $F \cap Ran(Q) = F \cap Q(F)$ and $F \cap Ker(Q) = F \cap (I - Q)(F)$.

- (iv) Let P be a maximal L^p -projection for F , defined on the subspace H of F . Using item (i), we get that Q is a maximal L^p -projection for X such that $Q|_H = P$. Item (iii) gives us $Ran(P) = (F \cap Q(F))$ and $Ker(P) = (F \cap (I - Q)(F))$. Therefore, for P_1, P_2 two different maximal L^p -projections for F , their extensions Q_1, Q_2 as maximal L^p -projections for X are different. Hence, $\alpha(F) \leq \alpha(X)$. \square

2.34. *Remark.* The construction in the proof of Lemma 2.33 does not give every maximal L^p -projection on a Banach space X . For example, let $(\Omega, \mathcal{F}, \mu)$ a measure space that is not σ -finite such that $\Omega = \Omega_1 \sqcup \Omega_2$, with $\Omega_i \in \mathcal{F}$ that are not σ -finite. Then, for any $f \in L^p(\Omega)$, the L^p -projection P that is built as in the proof will be equal to $M_{\chi_{supp(f)}}$. Such an L^p -projection cannot be equal to the L^p -projection $M_{\chi_{\Omega_1}}$ as $supp(f)$ is σ -finite whereas Ω_1 is not. The projection $I - P$ is also not equal to $M_{\chi_{\Omega_1}}$, as $\Omega_2 = \Omega_1^C$ is not σ -finite.

2.35. Corollary. *Let X be a Banach space. Suppose $1 \leq p \leq +\infty$. The following are equivalent:*

- (i) *The p -orthogonality on X is trivial, that is $x \perp_p y \Rightarrow x = 0$ or $y = 0$;*
- (ii) $X_{(p)} = \{0\}$;
- (iii) $\alpha(X) = 2$ if $X \neq \{0\}$ or $\alpha(X) = 1$ if $X = \{0\}$.

In such a case, X satisfies Property 2.7 for p .

Proof. The equivalence (i) \Leftrightarrow (ii) follows immediately from the definition of the subspace $X_{(p)}$. We can also see that $\alpha(X) = 1$ can only happen when $X = \{0\}$, in which case the equivalences are immediate. Therefore we suppose that $X \neq \{0\}$ for the rest of the proof.

- (i) \Rightarrow (iii) Let P be a maximal L^p -projection for X . Suppose that P is non-trivial. Thus $Ran(P)$ and $Ker(P)$ are not reduced to $\{0\}$ and we can find x, y that are non-zero such that $P(x) = x$ and $P(y) = 0$. This implies that $x \perp_p y$, a contradiction. Therefore the maximal L^p -projections for X are only the trivial ones, and $\alpha(X) = 2$.

- (iii) \Rightarrow (i) Let $x, y \in X$ be such that $x \perp_p y$. We apply item (ii) of Lemma 2.33 to obtain an L^p -projection Q that is maximal for X and such that $Q(x) = x, Q(y) = 0$. Since $\alpha(X) = 2$, we either have $Q = 0$ or $Q = I$, which is equivalent to $x = 0$ or $y = 0$. In such a case, every p -orthogonality relationship has the form $0 \perp_p y$ and $x \perp_p 0$, so it is extended by either $P = 0$ or $P = I$. Thus X satisfies Property 2.7. Lastly, the subspace $Y = \{0\}$ possesses 0 as its single L^p -projection, so $\alpha(Y) = 1$, which concludes the proof. \square

2.36. Lemma. *Let X be a Banach space and let F be a subspace of X . Suppose $1 \leq p \leq +\infty$.*

- (i) *Let P be a non-trivial L^p -projection that is maximal for F . Then P is maximal for $F_{(p)}$;*

- (ii) Let Q be a non-trivial L^p -projection that is maximal for $F_{(p)}$. Then Q is maximal for F ;
- (iii) If $F_{(p)} \neq \{0\}$ or $F = F_{(p)}$, then $\alpha(F) = \alpha(F_{(p)})$.

Proof. (i) Since P is non-trivial, for any element x in $\text{Ran}(P)$ or $\text{Ker}(P)$ we can see that $x^{\perp p} \cap F \neq \{0\}$. Hence, such an element x lies in $F_{(p)}$. Since $F_{(p)}$ is a subspace, $\text{Ran}(P) \oplus_p \text{Ker}(P)$ is included in $F_{(p)}$. As we have $F_{(p)} \subset F$, the maximality of P for F implies that P is maximal for $F_{(p)}$.

- (ii) By applying item (i) of Lemma 2.33 to Q and F , we obtain a subspace H and $R \in \mathcal{P}_p(H)$ such that $\text{Ran}(Q) \oplus_p \text{Ker}(Q) \subset H \subset F$, $R|_{\text{Ran}(Q) \oplus_p \text{Ker}(Q)} = Q$, and R is maximal for F . Item (i) of this Lemma implies that $H \subset F_{(p)}$. Since Q is maximal for $F_{(p)}$, we obtain $H = \text{Ran}(Q) \oplus_p \text{Ker}(Q)$ and $R = H$. Thus, Q is maximal for F .

- (iii) The previous items show that the set of L^p -projections that are non-trivial and maximal for F is equal to the set of L^p -projections that are non-trivial and maximal for $F_{(p)}$. If $F_{(p)} \neq \{0\}$ or $F = F_{(p)}$, then F and $F_{(p)}$ have the same amount of trivial L^p -projections (either 2 or 1). Thus their sets of maximal L^p -projections are in bijective correspondence, so $\alpha(F) = \alpha(F_{(p)})$. \square

2.37. Lemma (Maximal L^p -projections on a subspace). *Let X be a Banach space and let F and G be closed subspaces of X . Suppose $1 \leq p \leq +\infty$. Let $R \in \mathcal{P}_p(G)$. The following are equivalent:*

- (i) R defines a non-trivial L^p -projection on a subspace H of F ;
- (ii) $(F \cap R(F)) \neq \{0\}$ and $(F \cap (I - R)(F)) \neq \{0\}$.

When X satisfies Property 2.7 for p , it suffices to look at L^p -projections on $G = X$.

Proof. (i) \Rightarrow (ii) Since $R|_H$ is a projection, we have $R(H) \subset H$. As $R(H)$ and $(I - R)(H)$ are included in H and H is included in F , we have $R(H) \subset (R(F) \cap F)$ and $(I - R)(H) \subset ((I - R)(F) \cap F)$. Since $R|_H$ is non-trivial, both of these subspaces are not reduced to $\{0\}$.

- (ii) \Rightarrow (i) Conversely, take $H = (F \cap R(F)) \oplus_p (F \cap (I - R)(F))$. We have $H \subset F$ and $R(H) \subset H$, so $R|_H$ is an L^p -projection. It is also non-trivial since $R(H) \neq \{0\}$ and $(I - R)(H) \neq \{0\}$.

When X satisfies Property 2.7 for p , every L^p -projection on a subspace G extends to an L^p -projection on X , so looking at every $R \in \mathcal{P}_p(X)$ is enough. \square

2.38. Remark. Every maximal L^p -projection for F can be obtained with this construction. However, even if $(F \cap R(F)) \neq \{0\}$, $(F \cap (I - R)(F)) \neq \{0\}$ and $H = (F \cap R(F)) \oplus_p (F \cap (I - R)(F))$, the projection $P|_H$ may not be a maximal L^p -projection for F . For example, take $X = \ell_p(\mathbb{C}^4)$, $f_1 = (0, 0, 0, 1)$, $f_2 = (0, -1, 1, 0)$, $f_3 = (1, 1, 1, 0)$ and $F = \text{Span}(f_1, f_2, f_3)$. For $P = M_{\chi_{\{2,3\}}}$ we can see that $H = \text{Span}(f_2) \oplus_p \text{Span}(f_1)$, but $F = \text{Span}(f_2, f_3) \oplus_p \text{Span}(f_1)$ so $P|_H$ is not a maximal L^p -projection for F .

2.39. Lemma. *Suppose $1 \leq p \leq +\infty$, $p \neq 2$. Let X be a Banach space satisfying Property 2.7 for p and let F be a closed subspace of X . Let $x, y \in F$ with $x \perp_p y$. Let $P \in \mathcal{P}_p(F)$. Then, we have $P(x) \perp_p P(y)$.*

Proof. Since X satisfies Property 2.7 for p , there exist $Q, R \in \mathcal{P}_p(X)$ such that $Q|_F = P$ and $R(x) = x$, $R(y) = 0$. As $p \neq 2$, the projections Q and R commute. Hence, we have

$$P(x) = Q(x) = Q(R(x)) = R(Q(x)) = R(P(x)).$$

Thus, $P(x) \in \text{Ran}(R)$, so $P(x) \perp_p y$. Since Property 2.7 implies the linearity of the p -orthogonality on X according to Proposition 2.9 and since $P(x) \perp_p (I - P)(y)$, we obtain $P(x) \perp_p y - (I - P)(y) = P(y)$. \square

2.40. *Remark.* Is Lemma 2.39 always true if we only suppose that F satisfies Property 2.8 for p , or for any F ? This question is analogous to the one in Remark 2.19, albeit in a more general context.

2.41. Corollary. *Let X be a Banach space, and let p be areal number with $1 \leq p \leq +\infty$, $p \neq 2$. Suppose that X satisfies Property 2.7 for p . Let F be a closed subspace of X . Suppose that $F = F_1 \oplus_p \dots \oplus_p F_k$ and let $x = x_1 + \dots + x_k \in F$. Then we have*

$$x^{\perp p} = \{y_1 + \dots + y_k : y_i \in x_i^{\perp p} \cap F_i\} = \sum_{j=1}^k x_j^{\perp p} \cap F_j.$$

Proof. Let $1 \leq i \leq k$, and $y_i \in x_i^{\perp p} \cap F_i$. Then, for any $1 \leq j \leq k$, we have $y_i \perp_p x_j$. Hence $y_i \perp_p (x_1 + \dots + x_k)$, i.e. $y_i \perp_p x$. Thus, we have $x \perp_p (y_1 + \dots + y_k)$.

Let us look now at the converse assertion. If we take $y \in F$ such that $y \perp_p x$, then, by denoting P_i the projection on F_i parallel to $\bigoplus_{j \neq i} F_j$, P_i is an L^p -projection on F . Lemma 2.39 tells us that $y_i = P_i(y) \perp_p P_i(x) = x_i$. Hence, y_i lies in $x_i^{\perp p} \cap F_i$. \square

2.42. Corollary (Maximal L^p -projections for p -orthogonal sums). *Suppose $1 \leq p \leq +\infty$, $p \neq 2$. Let X be a Banach space satisfying Property 2.7 for p . Let F be a closed subspace of X with a decomposition $F = F_1 \oplus_p \dots \oplus_p F_k$. Denote P_i the L^p -projection on F_i parallel to $\bigoplus_{j \neq i} F_j$. Let $G \subset F$ and $Q \in \mathcal{P}_p(G)$. Denote $G_i = P_i(G)$ and $Q_i = P_i Q : G \rightarrow G_i$. Then:*

- (i) *If Q is a maximal L^p -projection for F , then $G = G_1 \oplus_p \dots \oplus_p G_k$, $Q = Q_1 + \dots + Q_k$, and all Q_i are maximal L^p -projections for F_i ;*
- (ii) *If $G = G_1 \oplus_p \dots \oplus_p G_k$ and all $Q_i|_{G_i}$ are maximal L^p -projections for F_i , then Q is a maximal L^p -projection for F ;*
- (iii) $\alpha(F) = \prod_{i=1}^k \alpha(F_i)$.

Proof. (i) By construction we have $G_i \subset F_i$ and $I_F = P_1 + \dots + P_k$. Suppose first that Q is maximal for F . Let i such that $1 \leq i \leq k$. Since X satisfies Property 2.7, there are $\tilde{Q}, \tilde{P}_i \in \mathcal{P}_p(X)$ such that $\tilde{Q}|_G = Q$ and $\tilde{P}_i|_F = P_i$. Let $y \in P_i(G)$. There is $z \in G$ such that $y = P_i(z)$. We then have

$$\tilde{Q}(y) = \tilde{Q}(P_i(z)) = \tilde{Q}(\tilde{P}_i(z)) = \tilde{P}_i(\tilde{Q}(z)) = P_i(Q(z)) \in P_i(G).$$

Denote $G' = P_1(G) \oplus_p \dots \oplus_p P_k(G)$. Thus $G \subset G' \subset F$ and the projection \tilde{Q} leaves G' invariant, so $\tilde{Q}|_{G'} \in \mathcal{P}_p(G')$. As we also have $\tilde{Q}|_G = Q$, the maximality of Q for F implies that $G = G'$. Hence, $G = G_1 \oplus_p \dots \oplus_p G_k$ and $Q = Q_1 + \dots + Q_k$.

Suppose now that there is a i such that Q_i is not maximal for F_i . Up to reordering, we can choose $i = 1$. Using item (i) of Lemma 2.33, we get a subspace H_1 such that $G_1 \subsetneq H_1 \subset F_1$, and $R_1 \in \mathcal{P}_p(H_1)$ such that $R_1|_{G_1} = Q_1$. Thus, if we now consider $H = H_1 \oplus_p G_2 \oplus_p \dots \oplus_p G_k$ and define $R \in \mathcal{L}(H)$ with $R(x_1 + \dots + x_k) = R_1(x_1) + \sum_{j=2}^k Q_j(x_j)$, we can see that H strictly contains G , that R is an L^p -projection on H , and that $R|_G = Q$, which contradicts the maximality of Q for F .

- (ii) Suppose that Q is not maximal for F . Then, according to item (i) of Lemma 2.33, there exists a subset H such that $G \subsetneq H \subset F$, and $R \in \mathcal{P}_p(H)$ such that $R|_G = Q$ and that R is maximal for F . Item (i) of this Corollary implies that $H = P_1(H) \oplus_p \dots \oplus_p P_k(H)$. As H strictly contains G , this decomposition implies that there must be at least one i such that $P_i(H)$ strictly contains $P_i(G)$. We obtain that $P_i(G) \subsetneq P_i(H) \subset F_i$, that $R|_{P_i(H)}$ is an L^p -projection on $P_i(H)$ and that $(R|_{P_i(H)})|_{P_i(G)} = Q|_{P_i(G)} = Q_i|_{G_i}$. This contradicts the maximality of Q_i , so Q is maximal for F .

- (iii) Items (i) and (ii) give a bijection between L^p -projections Q that are maximal for F and k -tuples (Q_1, \dots, Q_k) of L^p -projections such that Q_i is maximal for F_i . Therefore, we get $\alpha(F) = \prod_{i=1}^k \alpha(F_i)$. \square

2.43. *Remark.* Producing examples using subspaces of an L^p -space as above means that the statements of Lemma 2.39 and Corollary 2.41 will always be valid. Hence, these subspaces seem to only be usable as good models for the behavior of L^p -projections on subspaces F for a space X satisfying Property 2.7, whereas L^p -projections on subspaces F for a general Banach space X (like ones that would only satisfy Property 2.8) could exhibit different behaviors.

For the following propositions, we recall that for a subspace F , $F_{(p)}$ denotes the subspace of F spanned by every vector x whose p -orthogonal $x^{\perp p}$ is not reduced to $\{0\}$.

2.44. Proposition. *Let n and p be such that $1 \leq p < +\infty$, $p \neq 2$ and $n \geq 4$. Let $X = \ell_p(\mathbb{C}^n)$. Denote by $(e_i)_i$ the canonical basis of X . For $1 \leq i < n$, denote $f_i = e_i + e_{i+1}$. Take $F = \text{Span}(f_i, 1 \leq i < n)$. Then, we have $F = F_{(p)}$, $\dim(F) = n - 1$, and*

$$\alpha(F) = \text{Card}(\{P: P \text{ is a maximal } L^p\text{-projection for } F\}) = 2^n - 2n.$$

Thus, for $n \geq 5$, there are (strictly) more than $2^{\dim(F)}$ L^p -projections that are maximal for F .

Proof. By construction, we have $\dim(F) \leq n - 1$. We can see that $F + \mathbb{C}e_1$ contains every e_i , so $F + \mathbb{C}e_1 = X$. Hence $\dim(F) = n - 1$ and $e_1 \notin F$. Similarly, we can see that $e_i \notin F$ for every $1 \leq i \leq n$. For every $1 \leq i < j \leq n$, denote $g_{i,j} = e_i + (-1)^{j-i}e_j$. A computation gives $g_{i,j} = \sum_{k=0}^{j-i-1} (-1)^k f_{i+k}$,

thus $g_{i,j} \in F$.

As $n \geq 4$, for any $1 \leq i < n$ there exist two indices k and l such that $k < l$ and $k, l, i, i + 1$ are all distinct. Thus, $f_i \perp_p g_{k,l}$. Since both elements are non-zero and are in F , we deduce that $f_i \in F_{(p)}$ by definition of $F_{(p)}$. As $F_{(p)}$ is a subspace of F , we get $F_{(p)} = F$.

Let $P \in \mathcal{P}_p(X)$, with $P \neq 0, I$. Then there is $A \subset \{1, \dots, n\}$ such that $P = M_{\chi_A}$. Write A as $A = \{m_1, \dots, m_r\}$ with $1 \leq r \leq n - 1$ and $m_1 < m_2 < \dots < m_r$. We will prove that $\dim(F \cap P(F)) = r - 1$ and

$$F \cap P(F) = F \cap \text{Ran}(P) = \{0\} \text{ if } r = 1,$$

$$F \cap P(F) = F \cap \text{Ran}(P) = \text{Span}(g_{m_1, m_2}, \dots, g_{m_{r-1}, m_r}) \text{ if } 2 \leq r \leq n.$$

We have $F \cap P(F) = F \cap \text{Ran}(P)$ (since P is a projection). If $r = 1$ we have $\text{Ran}(P) = \text{Span}(e_{m_1})$, so $F \cap \text{Ran}(P) = \{0\}$ as $e_{m_1} \notin F$.

Suppose now that $r \geq 2$. We have $\text{Ran}(P) = \text{Span}(e_{m_1}, \dots, e_{m_r})$, so we obtain

$$\text{Span}(g_{m_1, m_2}, \dots, g_{m_{r-1}, m_r}) \subset F \cap \text{Ran}(P).$$

We also see that $F \neq \text{Ran}(P)$, as every e_{m_i} does not belong to F . Thus

$$\dim(F \cap \text{Ran}(P)) \leq \dim(\text{Ran}(P)) - 1 = r - 1.$$

We have that $\text{Span}(g_{m_1, m_2}, \dots, g_{m_{r-1}, m_r}) + \mathbb{C}e_{m_1}$ contains every e_{m_i} , so this subspace is equal to $\text{Ran}(P)$ and $\dim(\text{Span}(g_{m_1, m_2}, \dots, g_{m_{r-1}, m_r})) \geq r - 1$. Therefore, we have

$$F \cap \text{Ran}(P) = \text{Span}(g_{m_1, m_2}, \dots, g_{m_{r-1}, m_r}) \text{ and } \dim(F \cap \text{Ran}(P)) = r - 1.$$

Using this result, we obtain

$$\dim((F \cap P(F)) \oplus (F \cap (I - P)(F))) = (r - 1) + ((n - r) - 1) = n - 2.$$

Since X satisfies Property 2.7 for p , for any non-trivial L^p -projection Q on F there exists a non-trivial projection $R \in \mathcal{P}_p(X)$ that extends Q . This implies that $F = (F \cap P(F)) \oplus (F \cap (I - P)(F))$, with $\dim(F) = n - 1$, which contradicts the previous result. Therefore, we have $\mathcal{P}_p(F) = \{0, I\}$.

We now consider $P \in \mathcal{P}_p(X)$ that is non-trivial, with $P = M_{\chi_A}$, $A = \{m_1, \dots, m_r\}$. If $r = 1$ or $r = n - 1$, then we either have $P \cap P(F) = \{0\}$, or $P \cap (I - P)(F) = \{0\}$. If $2 \leq r \leq n - 2$, we have $P \cap P(F) \neq \{0\}$ and $P \cap (I - P)(F) \neq \{0\}$, so P defines a non-trivial L^p -projection on the subspace

$$F_{(p)} := (F \cap P(F)) \oplus_p (F \cap (I - P)(F)).$$

As $\dim(F_{(p)}) = n - 2 = \dim(F) - 1$, and as F does not possess a non-trivial L^p -projection, we obtain that $P|_{F_{(p)}}$ is a maximal L^p -projection for F . Furthermore, all these L^p -projections $P|_{F_{(p)}}$ are different. Since $g_{i,j} \in \text{Ran}(M_{\chi_B})$ if and only if $i, j \in B$, we get that the only L^p -projection R on X such that $F \cap R(F) = \text{Span}(g_{m_1, m_2}, \dots, g_{m_{r-1}, m_r})$ is $R = M_{\chi_B} = P$ with $B = \{m_1, \dots, m_r\}$. By using Lemma 2.37 and Remark 2.38 we can conclude that there are as many maximal L^p -projections (that are non-trivial for F)

as there are subsets A of $\{1, \dots, n\}$ with $A = \{m_1, \dots, m_r\}$ and $2 \leq r \leq n - 2$. This quantity is equal to

$$2^n - \binom{0}{n} - \binom{n}{n} - \binom{1}{n} - \binom{n-1}{n} = 2^n - 2 - 2n.$$

If we add 0 and I , we conclude that there are $2^n - 2n$ maximal L^p -projections for F . When $n = 4$ we have $2^n - 2n = 8 = 2^{n-1}$ and when $n \geq 5$ we have $2n < 2^{n-1}$ so $2^n - 2n > 2^{n-1}$. \square

2.45. Remark. We can generalize the result of Proposition 2.44 by replacing $\ell_p(\mathbb{C}^n) = \text{Span}(e_1, \dots, e_n)$ with $\text{Span}(h_1, \dots, h_n)$, with h_1, \dots, h_n vectors of a Banach space Y , such that $h_i \perp_p h_j$ for every $i \neq j$. Indeed, we can build an isometric isomorphism between this space and $\ell_p(\mathbb{C}^n)$ (so the space satisfies Property 2.7 for p) and the proof of Proposition 2.44 only used the p -orthogonality of the family $\{e_1, \dots, e_n\}$.

2.46. Proposition. *Suppose $1 \leq p < +\infty$, $p \neq 2$. Let X be a Banach space satisfying Property 2.7 and let F be a subspace of X with $\dim(F_{(p)}) \leq 3$. Then we have $\alpha(F) \leq 2^{\max(\dim(F_{(p)}), 1)}$.*

Proof. Item (iii) of Lemma 2.36 gives $\alpha(F) = \alpha(F_{(p)})$ whenever $F_{(p)} \neq \{0\}$. Corollary 2.35 gives $\alpha(F) = 2$ or 1 when $F_{(p)} = \{0\}$. We will now consider all possible cases depending on $\dim(F_{(p)})$.

If $F_{(p)} = \{0\}$, then we have $\alpha(F) \leq 2 = 2^{\max(0, 1)}$.

If $F_{(p)} \neq \{0\}$, then let $x \in F$ be a vector which is non-zero and such that $x^{\perp_p} \cap F \neq \{0\}$. Let y be an element in F that is non-zero and such that $y \perp_p x$. Then, $y^{\perp_p} \cap F \neq \{0\}$. Therefore both x and y lie in $F_{(p)}$. As y cannot be collinear to x , we cannot have $\dim(F_{(p)}) = 1$.

If $\dim(F_{(p)}) = 2$, the vectors x and y chosen previously span the subspace $F_{(p)}$. Hence $F_{(p)} = \text{Span}(x, y) = \text{Span}(x) \oplus_p \text{Span}(y)$ and $F_{(p)}$ is isometrically isomorphic to $\ell_p(\mathbb{C}^2)$. Item (iii) of Corollary 2.42 then implies

$$\alpha(F_{(p)}) = \alpha(\text{Span}(x))\alpha(\text{Span}(y)) = 2 \cdot 2 = 2^{\max(\dim(F_{(p)}), 1)}.$$

If $\dim(F_{(p)}) = 3$, the vectors x and y chosen previously only form a linearly independent family in $F_{(p)}$. By definition of $F_{(p)}$, there exists a vector $z \in F$ such that $z^{\perp_p} \cap F \neq \{0\}$ and such that $z \notin \text{Span}(x, y)$. Thus, $\{x, y, z\}$ is a basis for $F_{(p)}$. We will discuss cases depending on the vectors in $z^{\perp_p} \cap F$. For this, take $P \in \mathcal{P}_p(X)$ an L^p -projection that extends the p -orthogonality between x and y , with $P(x) = x$ and $P(y) = 0$. If $z \perp_p a x$, $a \neq 0$, we then have $F_{(p)} = \text{Span}(x) \oplus_p \text{Span}(y, z)$. Item (iii) of Corollary 2.42 then implies

$$\alpha(F_{(p)}) = \alpha(\text{Span}(x))\alpha(\text{Span}(y, z)) \leq 2 \cdot 2^{\max(2, 1)} \leq 2^3 = 2^{\max(\dim(F_{(p)}), 1)}.$$

If $z \perp_p b y$, $b \neq 0$, we get the same upper bound by symmetry between x and y . If $z \perp_p a x + b y$, $a, b \neq 0$, then Lemma 2.39 implies that $P(z) \perp_p a P(x)$. Since $a \neq 0$ and since the p -orthogonality is linear on X , we obtain

$$x = P(x) \perp_p P(z) + (I - P)(z) = z,$$

which brings us to the case $z \perp_p x$ and gives us the desired result.

We are then left with the case where $z^{\perp p} \cap \text{Span}(x, y) = \{0\}$. Thus, an element in F that is non-zero and p -orthogonal to z has the form $w = ax + by + cz$, with $c \neq 0$. Up to dividing by c we may assume that $c = 1$. Lemma 2.39 then gives

$$P(z) \perp_p P(w) = ax + P(z) \text{ and } (I - P)(z) \perp_p (I - P)(w) = by + (I - P)(z).$$

If we have $a = 0$ we get $P(z) = 0$, so $z = (I - P)(z) \perp_p x$, which is impossible. Similarly, if $b = 0$ we get $(I - P)(z) = 0$, so $z = P(z) \perp_p y$, a contradiction. As $a, b \neq 0$, up to replacing x by ax and y by by we may assume that $a = b = 1$, so $w = x + y + z$. Take

$$G = \text{Span}(P(z), P(z) + x, (I - P)(z), (I - P)(z) + y).$$

As in the previous use of Lemma 2.39, these vectors form a p -orthogonal family. Denote

$$h_1 = P(z) + x, h_2 = -P(z), h_3 = -(I - P)(z), h_4 = (I - P)(z) + y.$$

We then have $G = \text{Span}(h_1, h_2, h_3, h_4)$ and $x = h_1 + h_2, y = h_4 + h_3$ and $z = -(h_2 + h_3)$, so G contains $F_{(p)} = \text{Span}(x, y, z)$ and $F_{(p)} = \text{Span}(h_1 + h_2, h_3 + h_4, h_2 + h_3)$. By using Remark 2.45, and applying Proposition 2.44 to G and $F_{(p)}$, we get

$$\alpha(F_{(p)}) = 2^4 - 2.4 = 8 = 2^{\max(\dim(F_{(p)}), 1)}.$$

The proof is complete. □

2.47. Proposition. *Suppose $1 \leq p \leq +\infty, p \neq 2$. Let X be a Banach space satisfying Property 2.7 for p and let F be a subspace of X of finite dimension. Then, there exists a subspace H of X such that $F \subset H, \dim(H) \leq 2^{\max(\dim(F)-1, 0)}$, and H satisfies Property 2.7 for p .*

Proof. We will prove the result by induction on $n = \dim(F)$.

If $n = 1$ then for any $x, y \in F$ such that $x \perp_p y$, we must have $x = 0$ or $y = 0$ as x and y are collinear. Thus F satisfies Property 2.7 for p and we can take $H = F$.

Let $n \geq 2$. Suppose that the result is true for any subspace of X of dimension less or equal to $n - 1$. If all the maximal L^p -projections for F are defined on F , then Corollary 2.35 implies that F satisfies Property 2.7 for p and we can take $H = F$. If not, let P be a maximal L^p -projection for F that is not defined on F . Such a projection P is defined on $\text{Ran}(P) \oplus \text{Ker}(P) \subsetneq F$. Since X satisfies Property 2.7, there exists $Q \in \mathcal{P}_p(X)$ that extends P . Denote $G = Q(F) \oplus (I - Q)(F)$. Item (i) of Lemma 2.37 gives us

$$\text{Ker}(P) = F \cap \text{Ker}(Q) \text{ and } \text{Ran}(P) = F \cap \text{Ran}(Q).$$

By using the rank-nullity Theorem for the linear maps $Q : F \rightarrow Q(F)$ and $(I - Q) : F \rightarrow (I - Q)(F)$, we get

$$\begin{aligned} \dim(Q(F)) &= \dim(F) - \dim(F \cap \text{Ker}(Q)) = \dim(F) - \dim(\text{Ker}(P)) \\ \dim((I - Q)(F)) &= \dim(F) - \dim(F \cap \text{Ran}(Q)) = \dim(F) - \dim(\text{Ran}(P)). \end{aligned}$$

As P is non-trivial, we have $\text{Ker}(P), \text{Ran}(P) \neq \{0\}$, so $\dim(Q(F)) \leq n - 1$ and $\dim((I - Q)(F)) \leq n - 1$. We can then apply the induction hypothesis to $Q(F)$ and $(I - Q)(F)$ to obtain subspaces H_1, H_2 such that $Q(F) \subset H_1$, $(I - Q)(F) \subset H_2$ and H_1, H_2 satisfy Property 2.7. Denote $H = H_1 \oplus_p H_2$. We then have $F \subset Q(F) \oplus (I - Q)(F) \subset H$, so we need to show that H satisfies Property 2.7. Let $x, y \in H$ be such that $x \perp_p y$. Lemma 2.39 tells us that

$$Q(x) \perp_p Q(y) \text{ and } (I - Q)(x) \perp_p (I - Q)(y).$$

Since $Q(x), Q(y)$ are in H_1 , there is $P_1 \in \mathcal{P}_p(H_1)$ such that

$$P_1(Q(x)) = Q(x) \text{ and } P_1(Q(y)) = 0.$$

Since $(I - Q)(x), (I - Q)(y)$ are in H_2 , there is $P_2 \in \mathcal{P}_p(H_2)$ such that

$$P_2((I - Q)(x)) = (I - Q)(x) \text{ and } P_2((I - Q)(y)) = 0.$$

Proposition 2.17 gives us $R \in \mathcal{P}_p(H)$ such that $R(h_1 + h_2) = P_1(h_1) + P_2(h_2)$ for any $h_1 \in H_1, h_2 \in H_2$. Therefore, we have

$$\begin{aligned} R(x) &= R(Q(x) + (I - Q)(x)) = Q(x) + (I - Q)(x) = x, \\ R(y) &= R(Q(y) + (I - Q)(y)) = 0 + 0 = 0, \end{aligned}$$

and H satisfies Property 2.7 for p . We also have

$$\begin{aligned} \dim(H) &= \dim(H_1) + \dim(H_2) \\ &\leq 2^{\max(\dim(Q(F))-1, 0)} + 2^{\max(\dim((I-Q)(F))-1, 0)} \\ &\leq 2^{\dim(Q(F))-1} + 2^{\dim((I-Q)(F))-1} \\ &\leq 2^{\dim(F)-1-1} + 2^{\dim(F)-1-1} = 2^{\dim(F)-1}, \end{aligned}$$

which proves the upper bound on $\dim(H)$ and concludes the proof. \square

2.48. Remark. Proposition 2.47 gives us a first upper bound for $\alpha(F)$, that is $\alpha(F) \leq 2^{\max(\dim(F)-1, 0)}$. But its main interest is to allow us to look at F as a subspace of G , with G satisfying Property 2.7 and of finite dimension. Therefore we can apply Proposition 2.18 and Corollary 2.42 to write G as $G = G_1 \oplus_p \dots \oplus_p G_m$, with $\alpha(G_i) = 2$ and $\alpha(G) = \text{Card}(\mathcal{P}_p(G)) = 2^m$. Then, F as a subspace of G is generated by elements of the form $f = f_{i_1} + \dots + f_{i_r}$, with $1 \leq r \leq n$, $1 \leq i_1 < \dots < i_r \leq n$, and $f_{i_k} \in G_{i_k}$. This is similar to Propositions 2.44 and 2.46, where we can count the maximal L^p -projections for F by looking at the behavior of every L^p -projection of G with respect to F .

The previous results and examples motivate the following conjecture.

2.49. Conjecture. Let p be a real number with $1 \leq p < +\infty$, $p \neq 2$. Let X be a Banach space satisfying Property 2.7 and let F be a subspace of X of finite dimension such that $\mathcal{P}_p(F_{(p)}) = \{0, I\}$. Then, we have

$$\alpha(F) = \text{Card}(\{P: P \text{ is a maximal } L^p\text{-projection for } F\}) \leq 2^{\dim(F_{(p)})+1}.$$

If this conjecture is true, we can then obtain a better upper bound for $\alpha(F)$ when F has finite dimension and F is a subspace of a Banach space X that satisfies Property 2.7. We can also treat the case $p = +\infty$, which gives a simpler result.

2.50. Proposition. *Let X be a Banach space. Let F be a subspace of X of finite dimension.*

- (i) *(conditional on Conjecture 2.49) Let $1 \leq p < +\infty$, $p \neq 2$. Suppose that X satisfies Property 2.7 for p . Suppose also that the Conjecture 2.49 is true. Denoting $\text{Card}(\mathcal{P}_p(F_{(p)})) = 2^m$, we then have*

$$\alpha(F) \leq 2^{\min(\frac{5}{4}\dim(F_{(p)}), \dim(F_{(p)})+m)}.$$

- (ii) *Let $p = +\infty$. Suppose that X satisfies Property 2.7 for p . For $X = X_1 \oplus_\infty X_2$, we then have*

$$\begin{aligned} \alpha(F) &= 1 \text{ if } F = \{0\}; \\ \alpha(F) &= 2 \text{ if } F = \{0\} \text{ and } F \cap X_1 = \{0\} \text{ or } F \cap X_2 = \{0\}; \\ \alpha(F) &= 4 \text{ if } F \cap X_1 \neq \{0\} \text{ and } F \cap X_2 \neq \{0\}. \end{aligned}$$

Proof. (i) We have $\alpha(F) = \alpha(F_{(p)})$, so we only need to prove the result when $F = F_{(p)}$. Since F has a finite dimension, $\text{Card}(\mathcal{P}_p(F))$ is finite and is a power of 2 according to Proposition 2.18. Furthermore, for m such that $\text{Card}(\mathcal{P}_p(F_{(p)})) = 2^m$, we then have subspaces F_1, \dots, F_m of F such that

$$F = F_1 \oplus_p \dots \oplus_p F_m, F_i \neq \{0\}, \text{ and } \mathcal{P}_p(F_i) = \{0, I\}.$$

Furthermore, Corollary 2.42 tells us that

$$\alpha(F) = \prod_{i=1}^m \alpha(F_i).$$

Now, if $1 \leq \dim(F_i) \leq 3$, we can apply the result of Proposition 2.46 to get

$$\alpha(F_i) \leq 2^{\max(\dim((F_i)_p), 1)} \leq 2^{\max(\dim(F_i), 1)} = 2^{\dim(F_i)}.$$

If $\dim(F_i) \geq 4$, Conjecture 2.49 can be applied to F_i to obtain

$$\alpha(F_i) \leq 2^{\dim((F_i)_p)+1} \leq 2^{\dim(F_i)+1}.$$

Denote

$$E = \{i \in \{1, \dots, m\} : \dim(F_i) \geq 4\}.$$

Since $\dim(F_{(p)}) = \dim(F) = \sum_{i=1}^m \dim(F_i)$, we have $\text{Card}(E) \leq \frac{\dim(F_{(p)})}{4}$, so

$$\text{Card}(E) \leq \min\left(\frac{\dim(F_{(p)})}{4}, m\right).$$

Combining the previous results we obtain

$$\begin{aligned} \alpha(F) &= (\prod_{i \in E} \alpha(F_i)) (\prod_{j \notin E} \alpha(F_j)) \leq (\prod_{i \in E} 2^{\dim(F_i)+1}) (\prod_{j \notin E} 2^{\dim(F_j)}) \\ &\leq 2^{\text{Card}(E) + \sum_{k=1}^m \dim(F_k)} = 2^{\text{Card}(E) + \dim(F_{(p)})} \\ &\leq 2^{\dim(F_{(p)}) + \min(\frac{\dim(F_{(p)})}{4}, m)} = 2^{\min(\frac{5}{4}\dim(F_{(p)}), \dim(F_{(p)})+m)}, \end{aligned}$$

which concludes the proof.

- (ii) We apply Corollary 2.27 to obtain that $\alpha(X) = \mathcal{P}_p(X) = 1, 2$ or 4 . Since $\alpha(F) \leq \alpha(X) \leq 4$, we have $\alpha(F) = 1, 2$ or 4 . Proposition 2.29 allows us to write X as $X = X_1 \oplus_\infty X_2$, with $\alpha(X_i) = 1$ if $X_i = \{0\}$ or $\alpha(X_i) = 2$. If $F = \{0\}$, then $\alpha(F) = 1$. If $F \cap X_1 \neq \{0\}$ and $F \cap X_2 \neq \{0\}$, then $F \neq \{0\}$ and F has non-zero elements that are ∞ -orthogonal, so F possesses non-trivial maximal L^p -projections. Therefore $\alpha(F) = 4$. If $F \neq \{0\}$ and $F \cap X_1 = \{0\}$ or $F \cap X_2 = \{0\}$, then Proposition 2.29 tells us that the ∞ -orthogonality is trivial on F , thus $\alpha(F) = 2$. \square

3. L^p -projections on quotient spaces and on $L^p(X)$

The aim of this section is to link p -orthogonality and L^p -projections between a Banach space X and quotient spaces X/F . We will introduce a third property for X about p -orthogonality between X/F and X . All the results will be valid for $1 < p < +\infty$, $p \neq 2$. Unlike Section 2, the case $p = 1$ does not behave well with respect to quotients (see Counter-example 3.7). The first results of this section (up to Lemma 3.11) are also true for $p = 2$, but the Hilbertian case is excluded as soon as properties of the Boolean algebra $\mathcal{P}_p(X)$ are required. We quickly drop the case $p = +\infty$ as the properties that we can add to X, F or $P(F)$ require special care (in a similar way to Proposition 2.29 or item (ii) of Proposition 2.50).

3.1. Definition (Representative of minimal norm of a quotient, metric projection on a subspace). Suppose $1 < p < +\infty$, $p \neq 2$. Let F be a closed subspace of X such that every element of X/F admits a unique representative of minimal norm. For $\bar{x} \in X/F$ we define $Rep_F(\bar{x}) \in X$ the representative of \bar{x} of minimal norm. For $x \in X$ we denote $Proj(x, F) \in F$ the metric projection of x onto the closed convex set F .

We then have $Rep_F(\bar{x}) = x - Proj(x, F)$ and

$$\|Rep_F(\bar{x})\| = \|\bar{x}\| = \inf_{a \in F} (\|x - a\|) = \|x - Proj(x, F)\|.$$

Also, $Rep(X/F) = \{x \in X : Proj(x, F) = 0\}$.

If there is no ambiguity regarding the quotient space, the map $Rep_F : X/F \rightarrow X$ will be abbreviated as Rep in the rest of the article.

3.2. Remark. While some results can be applied to subspaces F of a Banach space X with no condition on the quotient X/F , most of them will require the unicity of any representative of minimal norm of $\bar{x} \in X/F$, or equivalently the existence and unicity of a metric projection of any $x \in X$ onto F . Even though L^p -projections give decompositions into p -orthogonal subspaces, it is not true that for any subspace F and any L^p -projection P , the existence and uniqueness of the metric projection on F implies the existence and uniqueness of the metric projection on $P(F)$. It is however true with additional conditions between P and F (see Lemma 3.11). We also remark that in general the set $Rep_F(X/F)$ is not a subspace of X . Further results will show that in certain

conditions the set $\text{Rep}_F(X/F)$ possesses two subsets A, B such that $A + B = \text{Rep}_F(X/F)$ and $B \subset A^{\perp p}$. The p -orthogonality between A and B ensures that every element of $\text{Rep}_F(X/F)$ has a unique decomposition as a sum $A+B$. Hence, such a decomposition will be denoted as $\text{Rep}_F(X/F) = A \oplus_p B$, even though the set A and B are not subspaces, in order to match the p -orthogonal decompositions for subspaces.

3.3. Lemma. *Suppose $1 < p \leq +\infty$. Let X be a Banach space and let F be a closed subspace of X . Let $x \in X$, let G be a subspace of X containing F and x , and let $P \in \mathcal{P}_p(G)$ be such that $P(x) = x$. The following are equivalent:*

- (i) $\inf_{a \in F} \|x - a\| = \|x\|$;
- (ii) $\inf_{a \in F} \|x - P(a)\| = \|x\|$.

If the metric projections on F and $P(F)$ are well-defined, then we also have equivalence between:

- (1) $\text{Proj}(x, F) = 0$;
- (2) $\text{Proj}(x, P(F)) = 0$.

Proof.

- (ii) \Rightarrow (i). Suppose first that $p < +\infty$. For any $a \in F$, we have

$$\|x - a\|^p = \|x - P(a)\|^p + \|(I - P)(a)\|^p \geq \|x\|^p + \|(I - P)(a)\|^p \geq \|x\|^p.$$

Thus, $\|x - 0\|^p = \inf_{a \in F} (\|x - a\|^p)$. When $p = +\infty$, we similarly have

$$\|x - a\| = \max(\|x - P(a)\|, \|(I - P)(a)\|) \geq \|x - P(a)\| \geq \|x\|,$$

so we get the same conclusion.

- (i) \Rightarrow (ii). Suppose first that $p < +\infty$, and that $\inf_{a \in F} \|x - P(a)\| < \|x\|$. Hence, there exists a non-zero $a \in F$ such that

$$\|x - P(a)\|^p < \|x\|^p.$$

We will show by a convexity argument that there exists $0 < \lambda < 1$ such that $\|x - \lambda a\|^p < \|x\|^p$. In order to do this, we define the following maps

$$g(\lambda) := \|x - \lambda P(a)\|^p \text{ and } h(\lambda) := \|x - \lambda a\|^p = g(\lambda) + |\lambda|^p \|(I - P)(a)\|^p.$$

Now, for every $b, c \in X$, the map $r \in \mathbb{R} \mapsto \|b + cr\| \in [0, +\infty[$ is convex. The map $y \in [0, +\infty[\mapsto y^p \in \mathbb{R}$ is convex with a positive derivative on $[0, +\infty[$. Thus, the composed map $r \mapsto \|b + cr\|^p$ is convex on \mathbb{R} , so g and h are convex on \mathbb{R} . It follows that these maps have right derivatives at every point, denoted by g'_r and h'_r . As we have

$$g(1)\|x - P(a)\|^p < \|x\|^p = g(0),$$

we must have $g'_r(0) < 0$. Also, since $p > 1$, the map $r \mapsto |r|^p$ has a derivative of 0 at 0. Thus $h'_r(0)' = g'_r(0)' + 0 < 0$, which means that h is decreasing on a neighborhood of 0. This gives a number λ with $0 < \lambda < 1$ such that

$$\|x - \lambda a\|^p = h(\lambda) < h(0) = \|x\|^p,$$

which proves the equivalence in this case.

When $p = +\infty$, if $\inf_{a \in F} \|x - P(a)\| < \|x\|$, then there exists a non-zero $a \in F$ such that

$$\|x - P(a)\| < \|x\|.$$

Since the map $s : r \in \mathbb{R} \mapsto \|x - rP(a)\|$ is convex with $s(0) > s(1)$, it is decreasing on an interval $[0, t]$, for some $0 < t \leq 1$. Let $r > 0$ be such that $r\|(I - P)(a)\| < \|x\|$ and $r < t$. Then, we have

$$\|x - ra\| = \max(\|x - rP(a)\|, r\|(I - P)(a)\|) = \max(s(r), r\|(I - P)(a)\|) < \|x\|,$$

which proves the equivalence in this case.

- (1) \Leftrightarrow (2) When the metric projection on a subspace G is well-defined, $\text{Proj}(x, G)$ is the unique $g \in G$ such that $\inf_{a \in G} \|x - a\| = \|x - g\|$. Since it is the case for F and $P(F)$, the equivalence (i) \Leftrightarrow (ii) concludes the proof. \square

We can now use Lemma 3.3 to transfer some p -orthogonality properties of X to X/F .

3.4. Corollary. *Suppose $1 < p < +\infty$. Let X be a Banach space and let F be a closed subspace of X . Let $x, y \in X$ be such that $\|\bar{x}\| = \|x\|$, $\|\bar{y}\| = \|y\|$ and $x \perp_p y$. Suppose that there exists $P \in \mathcal{P}_p(X)$ such that $P(x) = x$ and $P(y) = 0$. Then*

- (i) *We have $\bar{x} \perp_p \bar{y}$ and $\|\overline{ax + by}\| = \|ax + by\|$ for all $a, b \in \mathbb{C}$;*
- (ii) *If the metric projection onto F is well-defined, then we also have $\text{Proj}(ax + by, F) = 0$ for all $a, b \in \mathbb{C}$.*

Proof. (i) Up to changing x by ax and y by by , it is enough to show that $\|\bar{x} + \bar{y}\| = \|x + y\|$. Let $u \in F$. By applying Lemma 3.3 we obtain

$$\|x + y - u\|^p = \|x - P(u)\|^p + \|y - (I - P)(u)\|^p \geq \|x\|^p + \|y\|^p = \|x + y\|^p.$$

Thus, $\|x + y - 0\|^p = \inf_{u \in F} (\|x + y - u\|^p)$, so $\|\overline{x + y}\| = \|x + y\|$. Therefore, for any $z \in \mathbb{C}$, we get

$$\|\bar{x} + z\bar{y}\|^p = \|x + zy\|^p = \|x\|^p + \|zy\|^p = \|\bar{x}\|^p + \|z\bar{y}\|^p,$$

so \bar{x} and \bar{y} are p -orthogonal.

- (ii) If the metric projection onto F is well-defined, then item (i) implies that $\text{Proj}(ax + by, F) = 0$. \square

The next corollary generalizes the second equivalence of Lemma 3.3.

3.5. Corollary. *Suppose $1 < p < +\infty$. Let X be a Banach space and let F be a closed subspace of X . Let $x \in X$ and let $P \in \mathcal{P}_p(X)$ be such that $P(x) = x$. Suppose that the metric projections onto F and $P(F)$ are well-defined. If metric projections onto F exist, then the following are equivalent :*

- (i) $\inf_{a \in F} \|x - a\| = \|x - \alpha\|$, for $\alpha \in F \cap P(F)$;
- (ii) $\inf_{a \in F} \|x - a\| = \inf_{b \in P(F)} \|x - b\|$;
- (iii) $\inf_{b \in P(F)} \|x - b\| = \|x - \beta\|$, for $\beta \in F$.

If the metric projections onto F and $P(F)$ are well-defined, then the following are equivalent :

- (1) $\text{Proj}(x, F) \in P(F)$;

- (2) $Proj(x, F) = Proj(x, P(F)) \in (F \cap P(F))$;
- (3) $Proj(x, P(F)) \in F$.

Proof. (i) \Rightarrow (ii) Take $y = x - \alpha$. Since $\alpha \in F \cap P(F)$ we have $P(y) = y$ and $\inf_{a \in F} \|y - a\| = \|y\|$. Hence, Lemma 3.3 gives

$$\inf_{b \in P(F)} \|x - b\| = \inf_{b \in P(F)} \|y - b\| = \|y\| = \inf_{a \in F} \|y - a\| = \inf_{a \in F} \|x - a\|.$$

- (ii) \Rightarrow (iii) Since $\inf_{a \in F} \|x - a\|$ is attained for some $\beta \in F$, we obtain item (iii).

- (iii) \Rightarrow (i) Since for any $a \in F$ we have

$$\|x - a\|^p = \|x - P(a)\|^p + \|(I - P)(a)\|^p \geq \|x - P(a)\|^p,$$

we obtain $\inf_{a \in F} \|x - a\| \geq \inf_{b \in P(F)} \|x - b\|$. Thus, item (iii) implies

$$\inf_{b \in P(F)} \|x - b\| = \|x - \beta\| = \inf_{a \in F} \|x - a\|.$$

Since $\|x - \beta\|^p = \|x - P(\beta)\|^p + \|(I - P)(\beta)\|^p \geq \inf_{b \in P(F)} \|x - b\|^p + 0$, we must have $\|(I - P)(\beta)\|^p = 0$, that is $P(\beta) = \beta$, which implies that $\beta \in F \cap P(F)$ and proves item (i).

We suppose now that the metric projections onto F and $P(F)$ are well-defined. The implications (2) \Rightarrow (1) and (2) \Rightarrow (3) are immediate.

- (1) \Rightarrow (2) As $Proj(x, F) \in F$, we have $Proj(x, F) \in F \cap P(F)$. Thus item (2) comes from the implication (i) \Rightarrow (ii).

- (3) \Rightarrow (2) The implication (iii) \Rightarrow (ii) tells us that $Proj(x, P(F)) = Proj(x, F)$, so $Proj(x, F)$ belongs to $P(F) \cap F$, which concludes the proof. \square

3.6. Remark. The Counter-example 3.7 below shows that Lemma 3.3, which is central in this section, is not true for $p = 1$. We also remark that similarly to Lemma 3.3, Corollaries 3.4 and 3.5 are true for $p = +\infty$. The only difference in the proofs between $1 < p < +\infty$ and $p = +\infty$ is the fact that $\|a + b\|^p = \|a\|^p + \|b\|^p$ must be replaced by $\|a + b\| = \max(\|a\|, \|b\|)$.

3.7. Counter-Example. Lemma 3.3 is not true when $p = 1$. Indeed, if we take $X = \ell_1(\mathbb{C}^3)$, $M \geq 1$, $a = (0, 1, M)$, $x = (1, 1, 0)$, $F = \text{Span}(a)$, and the L^1 -projection $P = M_{\chi_{\{1,2\}}}$, then we have

$$\inf_{v \in F} (\|x - v\|) = 2 = \|x\| \text{ but } \inf_{u \in P(F)} (\|x - u\|) = 1 < \|x\|.$$

This is because the right derivative of $r \mapsto |r|$ at 0 is not zero.

The next proposition shows a converse to Corollary 3.4 when $X = L^p(\Omega)$. This is obtained via a generalization of the equality case in Clarkson's inequalities for quotients of $L^p(\Omega)$.

3.8. Proposition (equality case in Clarkson's inequality for quotients of L^p). Suppose $1 < p < +\infty$, $p \neq 2$. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let F be a closed subspace of $L^p(\Omega)$. Then $L^p(\Omega)$ satisfies Property 3.10 for p and F . Also, for any $x, y \in L^p$ such that $Proj(x, F) = 0$ and $Proj(y, F) = 0$, the following are equivalent:

- (i) $\|\overline{x+y}\|^p + \|\overline{x-y}\|^p = 2(\|\overline{x}\|^p + \|\overline{y}\|^p)$;
- (ii) $\|\overline{x \pm y}\|^p = \|\overline{x}\|^p + \|\overline{y}\|^p$;
- (iii) $\overline{x} \perp_p \overline{y}$;
- (iv) $x \perp_p y$;
- (v) $\text{Proj}(ax + by, F) = 0, \forall a, b \in \mathbb{C}$ and $x \perp_p y$.

Proof. Since $1 < p < +\infty$, every class in $L^p(\Omega)/F$ admits an unique representative of minimal norm and $\text{Proj}(\cdot, F)$ is well defined.

- (iii) \Rightarrow (ii) \Rightarrow (i) is immediate.

- (iv) \Leftrightarrow (v) has already been obtained in Corollary 3.5.

- (v) \Rightarrow (iii) Since $\text{Proj}(ax + by, F) = 0$, we have $\|\overline{ax + by}\| = \|ax + by\|$ for any $a, b \in \mathbb{C}$. As $x \perp_p y$, we obtain

$$\|\overline{x + zy}\|^p = \|x + zy\|^p = \|x\|^p + |z|^p \|y\|^p = \|\overline{x}\|^p + |z|^p \|\overline{y}\|^p.$$

- (i) \Rightarrow (iv) We will discuss separately the cases $1 < p < 2$ and $2 < p$.

Suppose first that $1 < p < 2$. Since $\|\overline{h}\| \leq \|h\|$, Clarkson's inequality gives

$$\|\overline{x+y}\|^p + \|\overline{x-y}\|^p \leq \|x+y\|^p + \|x-y\|^p \leq 2(\|x\|^p + \|y\|^p).$$

Since the leftmost and rightmost terms are equal, we are in the equality case of Clarkson's inequalities. Thus, x and y are p -orthogonal.

Suppose now that $2 < p$. Let h, k be the representatives of $\overline{x+y}, \overline{x-y}$ of minimal norm. Then, $\frac{h+k}{2}, \frac{h-k}{2}$ are representatives of $\overline{x}, \overline{y}$. We will show that they are the ones of minimal norm, i.e. x and y . We have

$$\begin{aligned} 2\left(\inf_{u \in F} \left\| \frac{h+k}{2} - u \right\|^p + \inf_{v \in F} \left\| \frac{h-k}{2} - v \right\|^p\right) &= 2(\|\overline{x}\|^p + \|\overline{y}\|^p) \\ &= \|\overline{x+y}\|^p + \|\overline{x-y}\|^p \\ &= \|h\|^p + \|k\|^p. \end{aligned}$$

and Clarkson's inequality gives

$$\|h\|^p + \|k\|^p \geq 2\left(\left\| \frac{h-k}{2} \right\|^p + \left\| \frac{h+k}{2} \right\|^p\right).$$

Thus,

$$\left\| \frac{h-k}{2} \right\|^p + \left\| \frac{h+k}{2} \right\|^p \leq \inf_{u \in F} \left\| \frac{h+k}{2} - u \right\|^p + \inf_{v \in F} \left\| \frac{h-k}{2} - v \right\|^p.$$

Since we have an equality, this means that $\frac{h+k}{2}, \frac{h-k}{2}$ are the representatives of $\overline{x}, \overline{y}$ of minimal norm, i.e. x and y . Hence, $x = \frac{h+k}{2}$ and $y = \frac{h-k}{2}$, so $x+y = h$ and $x-y = k$. Therefore we obtain

$$\|x+y\|^p + \|x-y\|^p = 2(\|x\|^p + \|y\|^p),$$

and the equality case in the Clarkson inequality implies that $x \perp_p y$. \square

3.9. Remark. The equivalences (i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv) of Proposition 3.8 are also true when $p = 2$. Indeed, this case amounts to orthogonality in a Hilbert space setting, where such equivalences can be easily proven using

orthogonality and inner products. However, as the quotient of a Hilbert space is a Hilbert space, these spaces have no significant interest in this section.

The equivalence (iii) \Leftrightarrow (iv) proves to be very useful in studying the L^p -projections on $L^p(\Omega)/F$. It turns out that we can study $\mathcal{P}_p(X/F)$ for a larger class of Banach spaces X as long as they satisfy a similar property.

Let X be a Banach space, let F be a closed subspace of X , and suppose that $1 \leq p \leq +\infty$. Similarly to Property 2.7, we introduce a property linking the p -orthogonality between the quotient X/F and X .

3.10. Property (Extension of p -orthogonality from a quotient). For any $\bar{x}, \bar{y} \in X/F$ such that $\bar{x} \perp_p \bar{y}$, there exists $x, y \in X$ representatives of \bar{x}, \bar{y} of minimal norm such that $x \perp_p y$.

3.11. Lemma. *Let X be a Banach space and suppose that $1 < p < +\infty$. Let F be a closed subspace such that every element of X/F admits an unique representative of minimal norm. Let $P \in \mathcal{P}_p(X)$ be such that $P(F) \subset F$. Then,*

- (i) *The metric projection on $P(F)$ is well-defined, so the maps $Rep_{P(F)} : X/P(F) \rightarrow X$ and $Proj(\cdot, P(F)) : X \rightarrow P(F)$ are well-defined. The same is true for $(I - P)$;*
- (ii) *For any $x \in X$ we have*

$$Proj(x, P(F)) = Proj(P(x), P(F)) = Proj(P(x), F);$$

- (iii) *For any $x \in X$ we have $Proj(x, F) = Proj(x, P(F)) + Proj(x, (I - P)(F))$.*

Proof. Since $P(F) \subset F$, we have $F = P(F) \oplus_p (I - P)(F)$. We recall that every element of X/F admits unique representatives of minimal norm if and only if for every element x of X , the metric projection of x on F is unique.

Let $x \in X$. We have

$$\begin{aligned} \|P(x) - Proj(P(x), F)\|^p &= \inf_{f \in F} \|P(x) - f\|^p \\ &= \inf_{a+b \in P(F) \oplus_p (I-P)(F)} (\|P(x) - a\|^p + \|b\|^p) \\ &= \inf_{a \in P(F)} \|P(x) - a\|^p. \end{aligned}$$

Since we also have

$$\begin{aligned} \|P(x) - Proj(P(x), F)\|^p &= \|P(x) - P(Proj(P(x), F))\|^p + \\ &\quad \|(I - P)(Proj(P(x), F))\|^p \\ &\geq \inf_{a \in P(F)} \|P(x) - a\|^p + 0, \end{aligned}$$

we then get $(I - P)(Proj(P(x), F)) = 0$, that is $Proj(P(x), F) \in P(F)$. Thus $P(x)$ possesses a metric projection on $P(F)$. Therefore,

$$\begin{aligned} \|x - Proj(P(x), F)\|^p &= \|P(x) - Proj(P(x), F)\|^p + \|(I - P)(x)\|^p \\ &= \inf_{a \in P(F)} \|P(x) - a\|^p + \|(I - P)(x)\|^p \\ &= \inf_{a \in P(F)} \|x - a\|^p, \end{aligned}$$

so x possesses a metric projection on $P(F)$.

Now, let $f_1 \in P(F)$ be such that $\|x - f_1\| = \inf_{a \in P(F)} \|x - a\|$. We can see that

$$\begin{aligned} \|P(x) - f_1\|^p &= \|x - f_1\|^p - \|(I - P)(x)\|^p \\ &= \|x - Proj(P(x), F)\|^p - \|(I - P)(x)\|^p \\ &= \|P(x) - Proj(P(x), F)\|^p, \end{aligned}$$

So $f_1 = Proj(P(x), F)$ by unicity of the metric projection on F . Thus, the metric projection of x on $P(F)$ is unique. As also $Proj(x, P(F)) = Proj(P(x), F) = Proj(P(x), P(F))$, item (ii) is proved. Since we have $(I - P)(F) \subset F$, we can mimic the proof for $(I - P)$ to prove that the metric projection on $(I - P)(F)$ is well-defined, and that $Proj(x, (I - P)(F)) = Proj((I - P)(x), F) = Proj((I - P)(x), (I - P)(F))$. Thus item (i) is now proved.

- (iii) As the metric projections on $P(F)$ and $(I - P)(F)$ are well-defined, we obtain

$$\begin{aligned} \|x - Proj(x, F)\|^p &= \inf_{a \in F} \|x - a\|^p \\ &= \inf_{a \in P(F) \oplus (I - P)(F)} \|(P(x) + (I - P)(x)) - \\ &\quad (P(a) + (I - P)(a))\|^p \\ &= \inf_{c \in (I - P)(F)} \|(I - P)(x) - c\|^p + \inf_{b \in P(F)} \|P(x) - b\|^p \\ &= \|(I - P)(x) - Proj((I - P)(x), (I - P)(F))\|^p + \\ &\quad \|P(x) - Proj(P(x), P(F))\|^p \\ &= \|x - Proj((I - P)(x), (I - P)(F)) - \\ &\quad Proj(P(x), P(F))\|^p. \end{aligned}$$

Hence, by unicity of the metric projection on F , we get

$$\begin{aligned} Proj(x, F) &= Proj(P(x), P(F)) + Proj((I - P)(x), (I - P)(F)) \\ &= Proj(x, P(F)) + Proj(x, (I - P)(F)). \end{aligned}$$

The proof is now complete. \square

The following proposition gives, for a subspace F , the form of some L^p -projections on X/F .

3.12. Proposition. *Let X be a Banach space and let $1 \leq p < +\infty$, $p \neq 2$. Let F be a closed subspace of X , and let $P \in \mathcal{P}_p(X)$ be such that $P(F) \subset F$. Then,*

- (i) $X/F \simeq P(X)/P(F) \oplus_p (I - P)(X)/(I - P)(F)$;
If the metric projection on F is well-defined, then

$$\text{Rep}(X/F) = \text{Rep}(P(X)/P(F)) \oplus_p \text{Rep}((I - P)(X)/(I - P)(F)).$$

- (ii) *There exists an L^p -projection P' on X/F such that $P'(\bar{x}) = \overline{P(x)}$;*
- (iii) *P' is non-trivial if and only if $P(F) \neq P(X)$ and $(I - P)(F) \neq (I - P)(X)$;*
- (iv) *Let $P_F \in \mathcal{P}_p(X)$ be the maximal L^p -projection such that $\text{Ran}(P_F) \subset F$. Then X/F is isometrically isomorphic to $(I - P_F)(X)/(I - P_F)(F)$.*
- (v) *Denote $\phi : P \in \{Q \in \mathcal{P}_p(X) : Q(F) \subset F\} \mapsto P' \in \mathcal{P}_p(X/F)$. Then ϕ is a morphism of commutative Boolean algebras, and $\text{Ker}(\phi) = \mathcal{P}_p(X) \circ P_F$. Hence, $\phi(P_1) = \phi(P_2)$ if and only if $(I - P_F)P_1 = (I - P_F)P_2$, and ϕ is injective if and only if $P_F = 0$.
In general, $\text{Ran}(\phi) = \phi(\{Q \in \mathcal{P}_p(X) : QP_F = 0, P(F) \subset F\})$ and ϕ is injective on this set.*

The proof of this Proposition is postponed to the Appendix (Section 5) in order not to interrupt the flow of the exposition.

3.13. Remark. The maximality condition of P_F implies that $P_{(I - P_F)(F)} = 0$. By considering $(I - P_F)(X)/(I - P_F)(F)$, we can reduce the study to a quotient for which the map ϕ is injective. The set $\{P \in \mathcal{P}_p(X) : PP_F = 0, P(F) \subset F\}$ is the Boolean algebra of L^p -projections that have a "disjoint support" with P_F . Hence, one can wonder if the map ϕ is bijective. The answer is false in general, but it turns out to be true when X satisfies Properties 2.7 and 3.10 and when quotients of X have unique representatives of minimal norm (see Counter-example 3.14 and Proposition 3.15 below).

3.14. Counter-Example (Non-surjectivity of the morphism ϕ). Suppose $1 < p < +\infty$, $p \neq 2$. In $\ell_p(\mathbb{C}^4)$, take $x = (-1, 1, 0, 0)$, $y = (0, 0, -1, 1)$, $f = (1, 1, 1, 1)$. Denote $G = \text{Span}(x, y, f)$ and $F = \text{Span}(f)$ as in Counter-example 2.21. A computation shows that $\text{Proj}(x, F) = 0$ and $\text{Proj}(y, F) = 0$. As we have $x \perp_p y$, we also have $\text{Proj}(ax + by, F) = 0$, according to Corollary 3.4. Hence, G/F is isometrically isomorphic to $\text{Span}(x, y)$, which possesses non-trivial L^p -projections. However, G has trivial L^p -projections as its only elements who have a non-zero p -orthogonal are scalar multiples of x and y . Therefore the morphism ϕ of Boolean algebras of Proposition 3.12 is not surjective in this case, even though Properties 2.8 and 3.10 are satisfied for G, F and p . Furthermore, since f has a maximal support in $\ell_p(\mathbb{C}^4)$, we can use item (i) of Proposition 3.15 to obtain $\mathcal{P}_p(\ell_p(\mathbb{C}^4)/F) = \{0, I\}$. This implies that $\ell_p(\mathbb{C}^4)/F$ does not satisfy Property 2.7 as $\mathcal{P}_p(G/F)$ is non-trivial, even though $\ell_p(\mathbb{C}^4)$ satisfies Property 2.7 and 3.10.

3.15. Proposition. *Suppose $1 < p < +\infty$, $p \neq 2$. Let X be a Banach space satisfying Property 2.7 and let F be a closed subspace of X such that every element of X/F admits a representative of minimal norm. Suppose that Property 3.10 is satisfied for X, F and p . Denote*

$$\phi : P \in \{Q \in \mathcal{P}_p(X) : Q(F) \subset F\} \mapsto \phi(P) \in \mathcal{P}_p(X/F)$$

the morphism of commutative Boolean algebras from Proposition 3.12, with $\phi(P)$ satisfying $\phi(P)(\bar{x}) = \overline{P(x)}$ for every $x \in X$. Then,

- (i) *The morphism ϕ is surjective, every L^p -projection of X/F can be associated to an L^p -projection P on X such that $P(F) \subset F$;*
- (ii) *The Boolean algebra $\mathcal{P}_p(X/F)$ is isomorphic to $\{P \in \mathcal{P}_p(X) : PP_F = 0, P(F) \subset F\}$;*
- (iii) *Denote P_F the maximal L^p -projection of X such that $\text{Ran}(P_F) \subset F$. The space X/F admits non-trivial L^p -projections if and only if there exist L^p -projections P such that $P_F < P < I$ and $P(F) \subset F$.*

Proof. (i) Let $Q \in \mathcal{P}_p(X/F)$. Then $X/F = \text{Ran}(Q) \oplus_p \text{Ker}(Q)$. Denote $\text{Rep} : X/F \rightarrow X$ the map sending \bar{f} to its representative of minimal norm. As Property 3.10 is satisfied for X, F and p , we have $\text{Rep}_F(\text{Ran}(Q)) \perp_p \text{Rep}(\text{Ker}(Q))$ and Corollary 3.4 implies that $\text{Rep}(X/F) = \text{Rep}(\text{Ran}(Q)) \oplus_p \text{Rep}(\text{Ker}(Q))$. Property 2.7 and Proposition 2.10 imply the existence of $P \in \mathcal{P}_p(X)$ such that

$$P(\text{Rep}(\text{Ran}(Q))) = \text{Rep}(\text{Ran}(Q)) \quad \text{and} \quad P(\text{Rep}(\text{Ker}(Q))) = \{0\}.$$

We will show that $P(F) \subset F$ and that $\phi(P) = Q$. Let $x \in P(F)$. We can write $x = g_1 + g_2 + f$, with $g_1 \in \text{Rep}(\text{Ran}(Q))$, $g_2 \in \text{Rep}(\text{Ker}(Q))$, $f \in F$. Thus, $P(x) = x = g_1 + P(f)$, so $g_1 = x - P(f) \in P(F)$. Since $P(g_1) = g_1$ and $\text{Proj}(g_1, F) = 0$, we have $\text{Proj}(g_1, P(F)) = 0$ according to Lemma 3.3. This implies that $g_1 = 0$. Similarly, we have $(I - P)(x) = 0 = g_2 + (I - P)(f)$, so $g_2 = -(I - P)(f) \in (I - P)(F)$. Since $(I - P)(g_2) = g_2$ and $\text{Proj}(g_2, F) = 0$, we have $\text{Proj}(g_2, (I - P)(F)) = 0$. This implies that $g_2 = 0$. Therefore, $x = f \in F$, hence $P(F) \subset F$ and $\phi(P)$ is well defined.

Now, let $x \in X$. Property 3.10 and Corollary 3.4 imply that

$$\begin{aligned} (x - \text{Proj}(x, F)) &= \text{Rep}(\bar{x}) = \text{Rep}(Q(\bar{x}) + (I - Q)(\bar{x})) \\ &= \text{Rep}(Q(\bar{x})) + \text{Rep}((I - Q)(\bar{x})). \end{aligned}$$

Therefore $P(x - \text{Proj}(x, F)) = \text{Rep}(Q(\bar{x}))$. Since we have $P(\text{Proj}(x, F)) \in P(F) \subset F$, we obtain

$$Q(\bar{x}) = \overline{\text{Rep}(Q(\bar{x}))} = \overline{P(x) - P(\text{Proj}(x, F))} = \overline{P(x)}.$$

As this is true for every $x \in X$, we get $\phi(P) = Q$, so the morphism ϕ is surjective.

- (ii) Since ϕ is surjective, we can use item (v) of Proposition 3.12 to see that ϕ is an isomorphism of Boolean algebras between $\{P \in \mathcal{P}_p(X) : PP_F =$

$0, P(F) \subset F\}$ and $\mathcal{P}_p(X/F)$.

- (iii) A non-trivial L^p -projection on X/F identifies with a non-trivial element

$$Q \in \{P \in \mathcal{P}_p(X): PP_F = 0, P(F) \subset F\}.$$

By taking $P = Q + P_F - QP_F = Q + P_F$, we get $P_F < P < I$ and $P(F) \subset Q(F) + P_F(F) \subset F$ since Q is non-trivial and $Q(F) \subset F$. The converse is obtained by noticing that $Q = P(I - P_F)$. \square

3.16. Example. Let p with $1 < p < +\infty, p \neq 2$ and let X be a Banach space. Let F be a closed subspace of X such that the metric projection on F is well-defined. Let $P \in \mathcal{P}_p(X)$ be such that $P(F) \neq P(X)$ and $(I - P)(F) \neq (I - P)(X)$. Take $x \in \text{Ran}(P) \setminus P(F)$. We then have $x - \text{Proj}(x, P(F)) \in \text{Ran}(P)$ and $\text{Proj}(x - \text{Proj}(x, P(F)), P(F)) = 0$, so $\text{Proj}(x - \text{Proj}(x, P(F)), F) = 0$ according to Lemma 3.3. Thus, $\text{Rep}(X/F) \cap \text{Ran}(P)$ is not reduced to $\{0\}$. Similarly, $\text{Rep}(X/F) \cap \text{Ker}(P)$ is also not reduced to $\{0\}$. Since for any $y \in \text{Rep}(X/F) \cap \text{Ran}(P)$ and any $z \in \text{Rep}(X/F) \cap \text{Ker}(P)$ we have $\bar{y} \perp_p \bar{z}$ according to Corollary 3.4, X/F possesses elements that have non-trivial p -orthogonal sets. This also means that certain subspaces of X/F admit non-trivial L^p -projections.

4. Generalizations for $L^p(\Omega, X)$ and L^q -projections in L^p -spaces

4.1. Generalizations for $L^p(\Omega, X)$

In the previous two sections we have obtained several results for some classes of Banach spaces sharing common properties with L^p -spaces, mainly Property 2.7 and Property 3.10. In this section we exhibit conditions on a Banach space X that allow us to generalize some previous results to vector-valued spaces $L^p(\Omega, X)$.

4.1. Proposition. *Suppose $1 \leq p < +\infty, p \neq 2$. Let X be a Banach space that satisfies the conditions*

- (i) $\|f + g\|_p^p + \|f - g\|_p^p \leq 2(\|f\|_p^p + \|g\|_p^p), \forall f, g \in X$ if $1 \leq p < 2$
and there is equality if and only if $f = 0$ or $g = 0$;
- (ii) $\|f + g\|_p^p + \|f - g\|_p^p \geq 2(\|f\|_p^p + \|g\|_p^p), \forall f, g \in X$ if $2 < p$
and there is equality if and only if $f = 0$ or $g = 0$.

Then all the results regarding L^p -projections on subspaces or quotients of $L^p(\Omega)$ are true for $L^p(\Omega, X)$.

Proof. The space X satisfies the same Clarkson inequality and the same equality case as the scalar field \mathbb{C} . We can thus prove a version of Lemma 1.12 for $L^p(X)$ by mimicking on with an analogue proof as the classical one [22, Ch.15-7,p.416]. Then, every proof regarding L^p -projections on subspaces or quotients of $L^p(\Omega)$ in this paper or in [6], [19] can be mimicked for $L^p(\Omega, X)$ as it possesses a similar Banach lattice structure and satisfies the same Clarkson inequalities. \square

4.2. *Remark.* Let p be a real number with $1 \leq p < +\infty$, $p \neq 2$. Let X be a Banach space that is non-zero.

- (i) Recall that X is said to be a *strictly convex* Banach space if it satisfies the property

$$\|x + y\| = \|x\| + \|y\| \Rightarrow x = \alpha y \text{ or } y = \alpha x \text{ for some } \alpha \geq 0.$$

If $p = 1$ and X is strictly convex, then the condition of Proposition 4.1 is satisfied, as a short computation shows.

- (ii) If X is a subspace, quotient, or subspace of quotient of $L^p(\Omega)$, then X satisfies Clarkson inequalities from either Lemma 1.12 or Proposition 3.8. Both lemmas also say that

$$x \perp_p y \Leftrightarrow \|x + y\|^p + \|x - y\|^p = 2(\|x\|^p + \|y\|^p).$$

Thus, X satisfies the equality condition of Proposition 4.1 if and only if the p -orthogonality is trivial on X , which is in turn equivalent to $X = \{0\}$ or $\alpha(X) = 2$, according to Corollary 2.35.

- (iii) It was proved in [18] that a Banach space satisfies Clarkson's inequalities if and only if its "type or cotype constant" is 1. We refer to [18] for more information.

4.2. L^q -projections in L^p -spaces

When a Banach space X possesses non-trivial L^p -projections, it cannot have non-trivial L^q -projections if $q \neq p$. However, when X is a subspace or quotient of an L^p -space that has trivial L^p -projections, it is not known if X can possess non-trivial L^q -projections. The following lemma gives a partial answer to this question.

4.3. Lemma. *Suppose $1 \leq p < +\infty$. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let X be a subspace, quotient, or subspace of a quotient of $L^p(\Omega)$. Suppose that X possesses a non-trivial L^q -projection P , for some $1 \leq q < +\infty$.*

- (i) *If $p = 2$, then $q = 2$;*
- (ii) *If all Hermitian projections on X have the form M_{χ_A} , then $q = p$;*
- (iii) *If $1 \leq p < 2$, then $p \leq q$;*
- (iv) *If $2 < p$, then $1 < q \leq p$.*

Proof.

- (i) Suppose that $p = 2$. Since an L^q -projection is of norm 1, P is a norm 1 projection on a Hilbert space, so P is an orthogonal projection and $q = 2$.

- (ii) Since an L^q -projection is a Hermitian projection, if X is a subspace of $L^p(\Omega)$ where all Hermitian projections have the form M_{χ_A} , then $P = M_{\chi_B}$ for some $B \subset \Omega$ and thus $q = p$.

- (iii) If $p \neq 1$, we cannot have $q = 1$ by the strict convexity of the L^p norm on X . This property implies that an L^1 -projection on X must be trivial.

- (iv) Since P is non-trivial, choose $x \in \text{Ran}(P)$, $y \in \text{Ker}(P)$, with $x, y \neq 0$. We have $\|x \pm y\|^q = \|x\|^q + \|y\|^q$. Denote

$$a = \left(\frac{\|y\|}{\|x\|} \right)^p > 0.$$

Thus,

$$\|x \pm y\|^p = (\|x\|^q + \|y\|^q)^{p/q} = \|x\|^p \left(1 + \left(\frac{\|y\|}{\|x\|}\right)^q\right)^{p/q} = \|x\|^p (1 + a^{q/p})^{p/q},$$

and $\|x\|^p + \|y\|^p = \|x\|^p (1 + a)$. If $1 \leq p < 2$, we will have

$$(1 + a^{q/p})^{p/q} \leq (1 + a) \Rightarrow 1 + a^{q/p} \leq (1 + a)^{q/p}.$$

If $2 < p$, we will have

$$(1 + a^{q/p})^{p/q} \geq (1 + a) \Rightarrow 1 + a^{q/p} \geq (1 + a)^{q/p}.$$

Let us study the continuous map $h : r \in \mathbb{R}_+ \mapsto (1 + a)^r - (1 + a^r)$. We have $h(0) = -1$, $h(1) = 0$, $\lim_{r \rightarrow +\infty} h(r) = +\infty$, and for $r > 0$,

$$h'(r) = \log(1 + a) \cdot (1 + a)^r - \log(a) a^r.$$

Using the elementary inequalities $\log(a) < \log(1 + a)$ and $0 < a^r < (1 + a)^r$, we obtain $h'(r) > 0$ for all $r > 0$. Therefore h is increasing on \mathbb{R}_+ . Thus, $h(r) \leq 0$ on $[0, 1]$ and $h(r) \geq 0$ on $[1, +\infty[$. Hence, if $1 \leq p < 2$, then we must have $h(q/p) \geq 0$, so $p \leq q$. If $2 < p$, then we must have $h(q/p) \leq 0$, so $q \leq p$. \square

We do not have presently a better result about the values of q , but we think that $q = p$ is the only possible choice. The argument in the proof also tells us that for all $x \in \text{Ran}(P)$, $y \in \text{Ker}(P)$ with $x \neq 0$ and $y \neq 0$, we have

$$\|x + y\|^p < \|x\|^p + \|y\|^p \text{ if } 1 \leq p < 2,$$

and that

$$\|x + y\|^p > \|x\|^p + \|y\|^p \text{ if } p > 2.$$

This also means that we are always in the strict case of the Clarkson inequalities for $x \in \text{Ran}(P)$, $y \in \text{Ker}(P)$ unless $x = 0$ or $y = 0$. Maybe this fact could be exploited to bring out an example or to rule out the cases $q \neq p$.

5. Appendix: Proof of Proposition 3.12

Proof. (i) As $P(F) \subset F$, we also have $(I - P)(F) \subset F$. Thus F can be decomposed as

$$F = P(F) \oplus_p (I - P)(F).$$

Since the same is true for X , we have $X = P(X) \oplus_p (I - P)(X)$. As $P(F) \subset P(X)$ and $(I - P)(F) \subset (I - P)(X)$, the p -orthogonal decompositions of X and F imply that

$$\begin{aligned} X/F &= (P(X) \oplus_p (I - P)(X)) / (P(F) \oplus_p (I - P)(F)) \\ &\simeq P(X)/P(F) \oplus_p (I - P)(X)/(I - P)(F). \end{aligned}$$

Suppose now that the metric projection on F is well-defined. Thus, every element of X/F admits a representative of minimal norm. With item (i) of

Lemma 3.11, the same is true for $P(F)$ and $(I - P)(F)$. Let $x \in \text{Rep}(X/F)$. Item (iii) of Lemma 3.11 gives

$$\begin{aligned} 0 &= \text{Proj}(x, F) = \text{Proj}(x, P(F)) + \text{Proj}(x, (I - P)(F)) \\ &= \text{Proj}(P(x), P(F)) + \text{Proj}((I - P)(x), (I - P)(F)). \end{aligned}$$

Hence, $\text{Proj}(P(x), P(F)) = 0 = \text{Proj}((I - P)(x), (I - P)(F))$. Thus $P(x)$ and $(I - P)(x)$ belong to $\text{Rep}(X/F)$ according to Lemma 3.3. This implies that

$$P(\text{Rep}(X/F)) \subset \text{Rep}(X/F) \text{ and } (I - P)(\text{Rep}(X/F)) \subset \text{Rep}(X/F).$$

Therefore, we get $\text{Rep}(X/F) = P(\text{Rep}(X/F)) + (I - P)(\text{Rep}(X/F))$. For $x \in \text{Rep}(X/F)$, using the previous computation we get $\text{Proj}(P(x), P(F)) = 0$. Therefore $P(x) \in \text{Rep}(P(X)/P(F))$, so $P(\text{Rep}(X/F)) \subset \text{Rep}(P(X)/P(F))$. Conversely, let $y \in \text{Rep}(P(X)/P(F))$. We then have

$$P(y) = y \text{ and } \text{Proj}(y, P(F)) = 0,$$

so $\text{Proj}(y, F) = 0$. This gives us $y \in \text{Ran}(P)$ and $y \in \text{Rep}(X/F)$, so y lies in $P(\text{Rep}(X/F))$. The same reasoning can be applied to $I - P$ to obtain $(I - P)(\text{Rep}(X/F)) = \text{Rep}((I - P)(X)/(I - P)(F))$. Putting all together we obtain

$$\begin{aligned} \text{Rep}(X/F) &= P(\text{Rep}(X/F)) \oplus_p (I - P)(\text{Rep}(X/F)) \\ &= \text{Rep}(P(X)/P(F)) \oplus_p \text{Rep}((I - P)(X)/(I - P)(F)). \end{aligned}$$

- (ii) Let us define P' the projection on $P(X)/P(F) \oplus \{0\}$ parallel to $\{0\} \oplus (I - P)(X)/(I - P)(F)$. Then P' is an L^p -projection on X/F . Now, let $x \in X$. As we have $x = P(x) + (I - P)(x)$ and $F = P(F) \oplus (I - P)(F)$, the class $\bar{x} = (x \bmod(F))$ is equal to the class

$$P(x) \bmod(P(F)) + (I - P)(x) \bmod((I - P)(F)).$$

Hence, the class $\overline{P(x)} = (P(x) \bmod(F))$ is equal to $P(x) \bmod(P(F)) + 0 \bmod((I - P)(F))$ which is equal to

$$P'[P(x) \bmod(P(F)) + (I - P)(x) \bmod((I - P)(F))] = P'(\bar{x}),$$

by definition of P' . Therefore, we have $P'(\bar{x}) = \overline{P(x)}$.

- (iii) The L^p -projection P' is non-trivial on X/F if and only if $P(X)/P(F) \oplus \{0\} = \text{Ran}(P') \neq \{0\}$ and $\{0\} \oplus (I - P)(X)/(I - P)(F) = \text{Ker}(P') \neq \{0\}$. This is equivalent to $P(F) \neq P(X)$ and $(I - P)(F) \neq (I - P)(X)$.

- (iv) The set $\mathcal{P}_{p,F}(X) := \{Q \in \mathcal{P}_p(X) : \text{Ran}(Q) \subset F\}$ contains 0 so it is non-empty. Let $(Q_i)_{i \in I}$ be net converging to Q in the strong operator topology. Thus, for any $x \in X$, the net $(Q_i(x))_{i \in I}$ converges to $Q(x)$. Since $Q_i(x)$ lies in F and F is closed, we get $Q(x) \in F$, therefore $\text{Ran}(Q) \subset F$.

As $1 \leq p < +\infty$, $p \neq 2$, the set $\mathcal{P}_{p,F}(X)$ admits a supremum P_F that is unique, according to Theorem 1.8. Since this set is closed for the net convergence with the strong operator topology, we have $P_F \in \mathcal{P}_{p,F}(X)$. Since

$P_F(X) \subset F$, we have $P_F(F) \subset P_F(X) \subset P_F(F)$, so $P_F(F) = P_F(X)$ and $P_F(F) \subset F$. Hence, item (i) gives

$$\begin{aligned} X/F &\simeq P_F(X)/P_F(X) \oplus_p (I - P_F)(X)/(I - P_F)(F) \\ &\simeq (I - P_F)(X)/(I - P_F)(F). \end{aligned}$$

- (v) The set $\{Q \in \mathcal{P}_p(X) : Q(F) \subset F\}$ is a Boolean sub-algebra of $\mathcal{P}_p(X)$. The linearity of the quotient map $x \mapsto \bar{x}$ and the relationship of item (ii) between P and $\phi(P)$ ensure that $\phi(I) = I$, and that ϕ is invariant for the operations $Q \mapsto I - Q$, $(Q, R) \mapsto QR$, $(Q, R) \mapsto Q + R - QR$. Thus, ϕ is a morphism of Boolean algebras.

Let $Q \in \mathcal{P}_p(X)$ be such that $Q(F) \subset F$. From the construction of item (ii), $\phi(Q)$ identifies as the projection on $Q(X)/Q(F) \oplus \{0\}$ parallel to $\{0\} \oplus (I - Q)(X)/(I - Q)(F)$. As Q and P_F commute and leave F invariant, and as $F = P_F(X) \oplus_p (I - P_F)(F)$, we can decompose the spaces $Q(X)$ and $Q(F)$ as follows :

$$\begin{aligned} Q(X) &= Q(P_F(X) \oplus_p (I - P_F)(X)) = QP_F(X) \oplus_p Q(I - P_F)(X), \\ Q(F) &= Q(P_F(X) \oplus_p (I - P_F)(F)) = QP_F(X) \oplus_p Q(I - P_F)(F). \end{aligned}$$

Hence, we get

$$Ran(\phi(Q)) \simeq Q(X)/Q(F) \simeq Q((I - P_F)(X))/Q((I - P_F)(F)).$$

Therefore, $\phi(Q) = 0$ if and only if $Q((I - P_F)(X)) = Q((I - P_F)(F))$. If this condition is true, then the projection $R = Q + P_F - QP_F$ satisfies

$$\begin{aligned} Ran(R) &= P_F(X) + Q(X) = P_F(X) + Q((I - P_F)(X)) \\ &= P_F(F) + Q((I - P_F)(F)) \subset F. \end{aligned}$$

Thus R lies in $\mathcal{P}_{p,F}(X)$. As we also have $P_FR = P_F$, that is $P_F \leq R$, this implies that $R = P_F$ by maximality of P_F in $\mathcal{P}_{p,F}(X)$. The condition $P_F + Q - QP_F = P_F$ is equivalent $Q(I - P_F) = 0$, which is in turn equivalent to $Q = SP_F$ for some $S \in \mathcal{P}_p(X)$. Therefore, $Ker(\phi) = \mathcal{P}_p(X) \circ P_F$. Since ϕ is a morphism of commutative Boolean algebras, we have $\phi(P_1) = \phi(P_2)$ if and only if $(I - P_F)P_1 = (I - P_F)P_2$. For any Q such that $Q(F) \subset F$, take $S = Q(I - P_F)$. Then, $SP_F = 0$, $S(F) \subset F$, and $S(I - P_F) = Q(I - P_F)$, so $\phi(S) = \phi(Q)$. Thus,

$$Ran(\phi) = \phi(\{Q \in \mathcal{P}_p(X) : QP_F = 0, Q(F) \subset F\}).$$

For P_1, P_2 in this set such that $\phi(P_1) = \phi(P_2)$, we have $P_1(I - P_F) = P_2(I - P_F)$. Since $P_1P_F = 0 = P_2P_F$, we get $P_1 = P_1(I - P_F) + P_1P_F = P_2$, so ϕ is injective on this set. \square

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