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# ON PROBLEMS OF DYNAMIC OPTIMAL NODAL CONTROL FOR GAS NETWORKS

MARTIN GUGAT AND JAN SOKOLOWSKI

**ABSTRACT.** We consider a dynamic optimal control problem for gas pipeline systems. The flow is governed by a quasilinear hyperbolic model. Since in the operation of the gas networks regular solutions without shocks are desirable, we impose appropriate state and control constraint in order to guarantee that a classical solution is generated. Due to a  $W^{2,\infty}$ -regularization term in the objective function, we can show the existence of an optimal control. Moreover, we give conditions that guarantee that the control becomes constant at the end of the control time interval if the weight of the regularization term is sufficiently large.

## 1. INTRODUCTION

In this paper we consider dynamic optimal control problems for gas pipeline systems. The dynamic control for gas pipeline systems has already been studied in a number of papers, see [13] and the references therein. In [13], the linear heat equation has been used as a model for the system dynamics. In this paper, we consider a more precise quasilinear hyperbolic model. We consider regular solutions without shocks that are desirable in the operation of the system. We consider controls that generate classical solutions of the partial differential equation. In order to make sure that the solutions remain regular, we impose an upper bound for the absolute value of the time-derivative of the solution as a pointwise state constraint everywhere. In this way we make sure that the state remains in a neighbourhood of a prescribed stationary reference state. With an additional  $W^{2,\infty}$ -regularization term in the objective function, this allows us to show the existence of a solution of the optimal control problem. Moreover, we show that due to the  $W^{2,\infty}$ -seminorm that appears as a regularization term in the objective function, the control approaches a given reference control.

The gas pipelines form a network, whose structure is described by a graph  $G = (V, E)$ . The flow in the system is governed by the control action of compressors that are located at a finite number of points.

The power that is consumed by the compressors is the objective function that we want to minimize in the dynamic optimal control problem on the

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time interval  $[0, T]$ . Let a finite set  $V_c \subset V$  be given. If we have compressor stations at the nodes  $v \in V_c$  in our network it has the form

$$(1.1) \quad P = \int_0^T \sum_{v \in V_c} A_v q^v(t) \left[ \left( \frac{p_{out,v}(t)}{p_{in,v}(t)} \right)^{R_v} - 1 \right] dt.$$

Here  $A_v > 0$  and  $R_v > 0$  are constants for the compressor at  $v \in V_c$ .

The flow rate through the compressor is given by  $q^v$ . It is well-defined, since at the location  $v \in V_c$  of the compressor, we have the additional coupling condition

$$q_{in,v}(t) = q_{out,v}(t)$$

that models the mass conservation (see [8]). This condition allows us to define

$$q_v(t) = q_{in,v}(t).$$

The notation  $p_{in,v}$  is used for the pressure at the inflow entry of the compressor and  $p_{out,v}$  is used for the pressure at the outflow exit of the compressor.

As a control function for the compressor we will consider  $u^v(t)$  that satisfies

$$u^v(t) + u_{ref}^v = \left( \frac{p_{out,v}(t)}{p_{in,v}(t)} \right)^{R_v}.$$

Here  $u_{ref}^v$  is a given constant reference configuration.

We are interested in controls that generate classical solutions of the system. Therefore, we assume  $u^v \in C^1([0, T])$ . Semi-global classical solutions of quasilinear hyperbolic systems have been studied in [11]. The well-posedness of solutions with  $H^2$ -regularity has been studied recently in the context of stabilization problems see for example [2], [10]. The existence of solutions with  $W^{2,\infty}$ -regularity can be obtained analogously by studying the system in diagonal form such that the evolution of the Riemann invariants along the characteristic curves can be analyzed.

We define a dynamic optimal control problem with a suitable regularization term for the control in the objective function and show that a solution exists. In the regularization term, second order partial derivatives appear. Then we show that if in this regularization also the maximum norm of the controls appears, the optimal controls approach a reference control towards the end of the time interval.

A classical reference about necessary optimality conditions for the control of semilinear hyperbolic boundary value problems is [3]. In [4] an optimal control problem for viscous systems of conservation laws with geometric parameter and application to the shallow-water equations is studied. The evaluation of derivatives of the cost functional for problems of optimal nodal control of networked hyperbolic systems with classical solutions has been studied in [6].

## 2. THE ISOTHERMAL EULER EQUATIONS

The isothermal Euler equations are a well-known model for the flow through gas pipelines, see for example [1]. We use the model for real gas as described in [9]. Let a finite graph  $G = (V, E)$  of a pipeline network be given. Here  $V$  denotes the set of vertices and  $E$  denotes the set of edges. Each edge  $e \in E$  corresponds to an interval  $[0, L^e]$  that represents a pipe of length  $L^e > 0$ . Let  $D^e > 0$  denote the diameter and  $\lambda_{fric}^e > 0$  the friction coefficient in pipe  $e$ . Define  $\theta^e = \frac{\lambda_{fric}^e}{D^e}$ . Let  $\rho^e$  denote the gas density,  $p^e$  the pressure and  $q^e$  the mass flow rate. Since we will linearize the convective part of the equations, we will not specify any pressure law. The isothermal or isotropic Euler equations are hyperbolic provided the pressure  $p = p(\rho)$  is given as a monotone increasing function of the density. We will assume that this monotonicity is strict. Typical examples are isotropic law  $p(\rho) = \rho^\gamma$  with  $\gamma > 1$  and the model of the American Gas Association (AGA)

$$p(\rho) = \frac{R_s T \rho}{1 - R_s T \alpha \rho}$$

where  $T$  is the temperature of the pipe,  $R_s$  is the gas constant and  $\alpha \in (-\frac{1}{4}, 0]$ . Note that for  $\alpha = 0$  the AGA model reduces to the isothermal law  $p(\rho) = RT\rho$ . We study a model that is based upon the isothermal Euler equations

$$(2.1) \quad \begin{cases} \rho_t^e + q_x^e = 0, \\ q_t^e + \left( p^e + \frac{(q^e)^2}{\rho^e} \right)_x = -\frac{1}{2} \theta^e \frac{q^e |q^e|}{\rho^e} \end{cases}$$

that govern the flow through pipe  $e \in E$ .

## 3. THE NODE CONDITIONS FOR THE NETWORK FLOW

In this section we introduce the coupling conditions that model the flow through the nodes of the network. The node conditions that determine the flow dynamics are given in [1] for the case that all pipes have the same diameter  $D^e$ . Let  $G = (V, E)$  be a finite connected directed graph. At the vertices  $v \in V$ , the flow is governed by the node conditions that require the mass conservation. In order to close the system, additional conditions are necessary.

Let  $E_0(v)$  denote the set of edges in the graph that are incident to  $v \in V$  and  $x^e(v) \in \{0, L^e\}$  denote the end of the interval  $[0, L^e]$  that corresponds to the edge  $e$  that is adjacent to  $v$ . Let  $V_0(e)$  denote the set of nodes adjacent to some edge  $e$ . Define the function

$$(3.1) \quad \mathfrak{s}(v, e) = \begin{cases} -1 & \text{if } x^e(v) = 0 \text{ and } e \in E_0(v), \\ 1 & \text{if } x^e(v) = L^e \text{ and } e \in E_0(v), \\ 0 & \text{if } e \notin E_0(v). \end{cases}$$

We impose the Kirchhoff condition

$$(3.2) \quad \sum_{e \in E_0(v)} \mathfrak{s}(v, e) (D^e)^2 q^e(x^e(v)) = 0$$

that expresses the mass conservation at the nodes.

In order to close the system, additional coupling conditions are needed. A typical choice, leading to well-posed Riemann problems [1] is to require the continuity of the pressure at  $v$ , which means that for all  $e, f \in E_0(v)$  we have

$$(3.3) \quad p(\rho^e(t, x^e(v))) = p(\rho^f(t, x^f(v))).$$

#### 4. DESIRED OUTPUTS

At one of the boundary nodes of the network, say at  $v^* \in V$ , the pressure  $p_{desi}^e$  is prescribed, where  $e$  denotes the edge that is adjacent to  $v^*$ .

At the boundary nodes  $v$  of the network (where  $|E_0(v)| = 1$ ) for  $v \neq v^*$ , for the corresponding edge  $e \in E_0(v)$  the boundary conditions prescribe the desired flow rates  $q_{desi}^e$  that model the amount of gas demanded by the consumers.

For a given stationary compressor action these boundary conditions determine a unique stationary state in the system, see [7].

#### 5. THE NETWORKED SYSTEM

Here we present in a concise form the initial boundary value problem for our networked system for the case of ideal gas where  $p^e = a^2 \rho^e$ . Let  $V_c \subset V$  denote the set of vertices with  $|E_0(v)| = 2$  where the compressors are located. Let  $v^*$  denote the vertex where the pressure is prescribed. Here we present the state equations in terms of physical variables where  $a > 0$  is the sound speed.

The state equation (S) is written below, it includes the initial conditions

$$\begin{cases} q^e(0, x) = q_0^e(x), & x \in (0, L^e), e \in E, \\ \rho^e(0, x) = \rho_0^e(x), & x \in (0, L^e), e \in E, \end{cases}$$

the boundary conditions for  $t \in (0, T)$ ,

$$\begin{cases} q_{out}^e(t, x^e(v)) = q_{desi}^e + u^e(t), & v \in V, e \in E_0(v), \text{ if } |E_0(v)| = 1, v \neq v^*, \\ p_{in}^e(t, x^e(v)) = p_{desi}^e + u^e(t), & v \in V, e \in E_0(v), \text{ if } |E_0(v)| = 1, v = v^*, \end{cases}$$

the Kirchhoff conditions

$$\sum_{e \in E_0(v)} \mathfrak{s}(v, e) (D^e)^2 q^e(t, x^e(v)) = 0, \quad t \in (0, T), \text{ if } |E_0(v)| \geq 2,$$

the continuity conditions

$$p(\rho^e(t, x^e(v))) = p(\rho^f(t, x^f(v))), \quad t \in (0, T), \text{ if } |E_0(v)| \geq 2, e, f \in E_0(v),$$

the compressor conditions for  $t \in (0, T)$ ,  $|E_0(v)| = 2, v \in V_c$ ,

$$\begin{cases} u^v(t) + u_{ref}^v = \left( \frac{p_{out,v}(t)}{p_{in,v}(t)} \right)^{R_v}, |E_0(v)| = 2, v \in V_c, \\ q^e(t, x^e(v)) = q^f(t, x^f(v)), t \in (0, T); e, f \in E_0(v), \end{cases}$$

and the PDEs on  $[0, T] \times [0, L^e]$ ,  $e \in E$ ,

$$\begin{pmatrix} \rho^e \\ q^e \end{pmatrix}_t + \begin{pmatrix} 0 & 1 \\ a^2 - \frac{(q^e)^2}{(\rho^e)^2} & 2 \frac{q^e}{\rho^e} \end{pmatrix}_x \begin{pmatrix} \rho^e \\ q^e \end{pmatrix}_x = \begin{pmatrix} 0 \\ -\frac{1}{2} \theta^e \frac{q^e |q^e|}{\rho^e} \end{pmatrix}.$$

Let a stationary solution  $p_{ref}(x)^e, q_{ref}(x)^e$  ( $e \in E$ ) with the constant controls  $u_{ref}^v$ , ( $v \in V_c$ ) be given. The theory of semi-global solutions (see [11]) asserts that for any given time horizon  $T_0 > 0$  there exists a number  $\varepsilon(T_0) > 0$  such that for all initial states with

$$(5.1) \quad \|q_0^e - q_{ref}\|_{C^1([0, L^e])} \leq \varepsilon(T_0)$$

and

$$(5.2) \quad \|\rho_0^e - \rho_{ref}\|_{C^1([0, L^e])} \leq \varepsilon(T_0)$$

and all controls that satisfy

$$(5.3) \quad \|u^e\|_{C^1([0, T_0])} \leq \varepsilon(T_0)$$

and

$$(5.4) \quad \|u^v\|_{C^1([0, T_0])} \leq \varepsilon(T_0)$$

and are  $C^1$  compatible with the initial state there exists a classical solution of  $(\mathbf{S})$  on  $[0, T_0]$  that satisfies an a priori estimate for the corresponding  $C^1$ -norm. Moreover, the solution is continuous as a function of the controls in  $(C^1([0, T_0])$  and there exists a constant  $C_c(T_0) > 0$  such that if (5.1), (5.2), (5.3) and (5.4) hold for two controls  $u_1$  and  $u_2$ , we have

$$(5.5) \quad \begin{aligned} & \max_{e \in E} \|p(\rho_1^e(t, x)) - p(\rho_2^e(t, x))\|_{C([0, T_0] \times [0, L^e])} \\ & \leq C_c(T_0) \max\left\{ \max_{e \in E} \|u_1^e - u_2^e\|_{C([0, T_0])}, \max_{v \in V_c} \|u_1^v - u_2^v\|_{C([0, T_0])} \right\} \end{aligned}$$

and

$$(5.6) \quad \begin{aligned} & \max_{e \in E} \|q_1^e(t, x) - q_2^e(t, x)\|_{C([0, T_0] \times [0, L^e])} \\ & \leq C_c(T_0) \max\left\{ \max_{e \in E} \|u_1^e - u_2^e\|_{C([0, T_0])}, \max_{v \in V_c} \|u_1^v - u_2^v\|_{C([0, T_0])} \right\} \end{aligned}$$

**Remark 5.1.** Note that if  $T_0 > 0$  is chosen sufficiently small,  $\varepsilon(T_0)$  can be quite large.

## 6. STATE AND CONTROL CONSTRAINTS

In the operation of gas networks, the pressure should remain between the given bounds

$$p_{\min} < p_{\max}.$$

In our optimal control problem this will be taken into account in the cost functional by a penalty term in the form

$$\eta_p \sum_{e \in E} \|(p_{\min} - p^e(t, x))_+ \|_{C([0, T] \times [0, L^e])} + \|(p^e(t, x) - p_{\max})_+ \|_{C([0, T] \times [0, L^e])}$$

that penalizes a violation of the pressure bounds. Here  $\eta_p > 0$  is a penalty parameter and we use the notation  $(r)_+ = \max\{r, 0\}$  for a real number  $r$ .

The control action in the compressor is also bounded. With given minimum compressor ratio  $\varepsilon_v^{\min}$  and maximum compressor ratio  $\varepsilon_v^{\max}$  that satisfy the inequalities

$$1 \leq \varepsilon_v^{\min} \leq \varepsilon_v^{\max}$$

we have the control constraint constraints

$$(6.1) \quad \varepsilon_v^{\min} \leq u^v(t) + u_{ref} \leq \varepsilon_v^{\max}$$

that gives bounds on the compression ratio. In practice we admit that  $\varepsilon_v^{\min} > 1$ . In this case, we assume that the compressor is switched on. If the compressor is switched off, we have  $u^v(t) = 1$ . Then in the state equations, such a compressor is invisible.

Choose a time  $T_0 \in (0, T)$ . The theory of semi-global solutions implies that we can choose  $\varepsilon = \varepsilon(T_0) > 0$  sufficiently small such that controls  $u$  that satisfy (6.2) below generate a classical solution of the system equation on  $[0, T_0]$ .

To be precise, in order to make sure that a regular solution exists, we prescribe the control constraint

$$(6.2) \quad \max_{t \in [0, T]} \{\psi(t), \max_{v \in V_c} \{|u^v(t)|, |\partial_t u^v(t)|\}\} \leq \varepsilon$$

where

$$(6.3) \quad \psi(t) := \max_{(v, e) \in V \times E: |E_0(v)|=1; e \in E_0(v)} \{|u^e(t)|, |\partial_t u^e(t)|\},$$

and assume that we have

$$(6.4) \quad \max_{e \in E} \max_{t \in [0, T-T_0]} \max_{x \in [0, L^e]} \{\phi_1^e(t, x), \phi_2^e(t, x) | \partial_t p^e(t, x)|, |\partial_t q^e(t, x)|\} \leq \varepsilon,$$

where

$$(6.5) \quad \phi_1^e(t, x) := |p^e(t, x) - p_{ref}^e(x)|$$

and

$$(6.6) \quad \phi_2^e(t, x) := |q^e(t, x) - q_{ref}^e(x)|.$$

Here  $q^{ref}$  and  $p^{ref}$  denote a stationary reference state that is generated by the controls  $u_{ref}$  as a classical steady state of the pde that is time-independent and compatible with the boundary conditions  $u^e = 0$  and the node conditions. We assume that the steady compressor control  $u_{ref}$  satisfies (6.1) for  $u^v = 0$ .

The state constraint (6.4) allows us to make the time horizon  $T$  arbitrarily large without losing the regularity of the solutions. This can be seen as follows: Due to (5.1) and (5.2) and (6.2) with  $\varepsilon = \varepsilon(T_0)$ , the theory of semi-global solutions implies that there exists a classical solution on the time interval  $[0, T_0]$ . At the time  $T_0$ , due to (6.2) and (6.4) with  $\varepsilon = \varepsilon(T_0)$ , we can apply the same result again and obtain a classical solution on  $[T_0, 2T_0]$ .

By proceeding iteratively in this way, we obtain a solution on the whole time interval  $[0, T]$ . Note that for this procedure, we do not need (6.4) on the time interval  $[T - T_0, T]$ .

Since for hyperbolic problems without the constraints (6.2) and (6.4), singularities can evolve in a finite time, they are mandatory for our analysis.

## 7. THE OPTIMAL CONTROL PROBLEM

Let a Banach space  $X(T) \subset (C^1([0, T]))^{|V_c|+N_B}$  (where  $N_B$  denotes the number of boundary nodes) be given. As examples, think of  $X(T) = (H^2(0, T))^{|V_c|+N_B}$  or  $X(T) = (W^{3,1}(0, T))^{|V_c|+N_B}$ . Let  $\|\cdot\|_{sem, X(T)}$  denote a seminorm in  $X(T)$ .

We assume that for all  $e \in E$  the initial state  $(q_0^e, \rho_0^e) \in C^1([0, L^e])$  of the system is given and satisfies (5.1) and (5.2). First, we define the set  $U(T)$  of feasible controls. The set  $U(T)$  contains the control functions

$$u(t) \in X(T)$$

such that the control constraints (6.1) and (6.2), and for the corresponding system state generated by **(S)** the state constraints (6.4) with  $\varepsilon = \varepsilon(T_0)$  are satisfied. The optimal control problem is to find a control function  $u \in U(T)$  such that the objective function

$$(7.1) \quad J(u) = \int_0^T \sum_{v \in V_c} A_v q^v(t) [u^v(t) + u_{ref}^v - 1] dt \\ + \eta_p \sum_{e \in E} \|(p_{\min} - p^e(t, x))_+\|_{C([0, T] \times [0, L^e])} + \|(p^e(t, x) - p_{\max})_+\|_{C([0, T] \times [0, L^e])} \\ + \gamma \|u\|_{sem, X(T)}$$

is minimized. Here  $\eta_p > 0$  and  $\gamma > 0$  are penalty parameters. For this dynamic optimal control problem we use the notation  $\mathbf{P}_{\mathbf{dyn}}(T)$ .

The goal of the control problem is to have a control with minimal control cost such that a regular state without shocks or other singularities is generated by the state equation. For the operation of gas networks it is important to remain within the scenario of classical solutions in order to avoid damages in the system caused by shocks.



## 8. EXISTENCE OF A SOLUTION

In this section we discuss the existence of a solution for  $\mathbf{P}_{\text{dyn}}(T)$  with  $\varepsilon = \varepsilon(T_0)$  in (6.2) and (6.4) for a given  $T_0 > 0$ .

We assume that the feasible set  $U(T)$  is *nonempty*. We make the following assumption on the Banach space  $X(T)$ :

**Assumption 8.1.** Every sequence of controls  $(u_n)_n$  in  $U(T)$  where

$$\sup_n \|u_n\|_{\text{sem}, X(T)} < \infty$$

contains a subsequence that converges strongly in  $(C^1([0, T]))^{|V_c|+N_B}$ .

Assumption 8.1 holds if  $\|u_n\|_{\text{sem}, X(T)}$  is chosen as the norm in  $X(T) = (H^2(0, T))^{|V_c|+N_B}$  or  $X(T) = (W^{3,1}(0, T))^{|V_c|+N_B}$  (see [14]).

Another possibility where only the second order derivatives occur, is  $X(T) = (W^{2,\infty}(0, T))^{|V_c|+N_B}$ .

Note that due to the control constraint (6.2) in the definition of  $U(T)$ , it suffices to take a seminorm  $\|u_n\|_{\text{sem}, X(T)}$  that contains the norms of the second derivatives. As an example, let us consider the latter space. We introduce a special seminorm which depends on a parameter  $n \in \{1, 2, 3, \dots\}$ . It is defined as

$$(8.1) \quad \|u\|_n = \sum_{j=1}^n \left[ \sum_{v \in V_c} \|(u_{\text{opt}}^v)''\|_{L^\infty(\frac{j-1}{n}T, \frac{j}{n}T)} + \|(u_{\text{opt}}^v)\|_{L^\infty(\frac{j-1}{n}T, \frac{j}{n}T)} \right] \\ + \sum_{j=1}^n \left[ \sum_{(v,e) \in V \times E: |E_0(v)|=1} \|(u^e)''\|_{L^\infty(\frac{j-1}{n}T, \frac{j}{n}T)} + \|(u^e)\|_{L^\infty(\frac{j-1}{n}T, \frac{j}{n}T)} \right].$$

We show that under Assumption 8.1, adding the  $X(T)$ -regularization term to the objective function allows us to prove the existence of a solution of the dynamic optimal control problem  $\mathbf{P}_{\text{dyn}}(T)$ .

**Theorem 8.2.** *Assume that the set  $U(T)$  of admissible controls is non-empty. Let Assumption 8.1 hold. Let  $T > T_0 > 0$  be given.*

*Then a solution of the dynamic optimal control problem  $\mathbf{P}_{\text{dyn}}(T)$  with  $\varepsilon = \varepsilon(T_0)$  in (6.2) and (6.4) does exist.*

**Proof:** We consider a minimizing sequence of feasible controls for  $\mathbf{P}_{\text{dyn}}(T)$ . Due to the regularization term, this sequence in  $U(T)$  is bounded with respect to the seminorm  $\|\cdot\|_{\text{sem}, X(T)}$ , hence Assumption 8.1 implies that it contains a subsequence that converges strongly in  $(C^1([0, T]))^{|V_c|+N_B}$  to a limit point  $u^*$ . Due to the theory of semi-global solutions, the strong convergence implies that also the corresponding subsequence of generated states given by the classical solutions of **(S)** converges strongly to the solution that is generated by the limit point  $u^*$ . Moreover, this also implies that  $u^* \in U(T)$  and that the corresponding subsequence of values of the objective function is also convergent.

**Remark 8.3.** Note that in this way we can prove the existence for optimal controls for general graphs. In particular, we obtain this results for graphs with an arbitrarily large number of cycles. Note that results about the exact controllability of networked quasilinear systems are only available for tree-shaped graphs, see for example [5], [11].

The proof of the existence of an optimal control also works if in the regularization term we replace  $\|\cdot\|_{sem, X(T)}$  by  $\|\cdot\|_{sem, X(T)}^2$ , which is natural for  $X(T) = (H^2(0, T))^{|V_c|+N_B}$ .

## 9. THE OPTIMAL CONTROLS APPROACH THE REFERENCE CONTROL

Again we consider the dynamic optimal control problem  $\mathbf{P}_{\text{dyn}}(T)$  with  $\varepsilon = \varepsilon(T_0)$  for a given  $T_0 > 0$ . In this section we present a sufficient condition that implies that for  $T > T_0$  in a subinterval of the form  $[T - \delta_1, T]$  (that is close to the end of the control time) the optimal control is zero if the penalty parameter is sufficiently large. This is related to the turnpike phenomenon that has been studied for example in [15], [16], [12], Turnpike results state that asymptotically when the time horizon gets larger, in the interior of the time interval the optimal control approaches a static optimal control. For our optimal control problem  $\mathbf{P}_{\text{dyn}}(T)$  we give a sufficient conditions that implies that at the end of the control time interval  $[0, T]$ , the optimal controls become static. If this condition is violated, the optimal control has to be decreasing in a certain sense.

We use the following notation: For  $j \in \{1, 2, 3, \dots, n-1\}$  define the interval

$$I^j = \left( \frac{j-1}{n} T, \frac{j}{n} T \right).$$

Before we state the theorem, we define the following property of an optimal control  $u_{\text{opt}}$ :

**Property 9.1.** We say that the optimal control  $u_{\text{opt}} \neq 0$  satisfies Property 9.1 if there exist an integer  $j_s \in \{1, 2, 3, \dots, n-1\}$  such that  $\frac{j_s-1}{n} T \in [T - T_0, T)$  and a point  $t_0$  in  $I^* = I^{j_s}$  such that the following conditions hold with the notation

$$t_s = \frac{j_s}{n} T$$

and  $I_0 = (t_0, t_s) \subset [T_0, T)$ :

We have  $u_{\text{opt}}|_{I_0} = 0$

or (if  $\|u_{\text{opt}}\|_{L^\infty(I_0)} > 0$ ) we have the inequalities

$$(9.1) \quad \|u_{\text{opt}}''\|_{L^\infty(I_0)} < \|u_{\text{opt}}''\|_{L^\infty(I^*)},$$

$$(9.2) \quad \|u_{\text{opt}}'\|_{L^\infty(I_0)} < \varepsilon(T_0),$$

and ((9.3) or (9.4)), with

$$(9.3) \quad \|u_{\text{opt}}\|_{L^\infty(I_0)} \leq \|u_{\text{opt}}\|_{L^\infty(I^{j_s+1})},$$

$$(9.4) \quad \|u_{\text{opt}}\|_{L^\infty((\frac{j_s-1}{n} T, \frac{t_0+t_s}{2}))} < \|u_{\text{opt}}\|_{L^\infty(I^*)}.$$

**Remark 9.2.** Condition (9.1) means that the curvature does not attain the essential supremum at the end of  $I^*$ ,

Condition (9.2) means that the inequality constraint for the control derivative is inactive on a part of  $I^*$ .

Condition (9.3) holds for example if all the components of  $u_{opt}$  are increasing on  $I_0$ . Condition (9.4) holds for example if all the components of  $u_{opt}$  are strictly increasing on  $I^*$ .

For a function  $f \in C^2([0, 1])$  with non-constant curvature there exist points  $t \in (0, 1)$  where we have

$$|f''(t)| < \|f\|_{L^\infty([0,1])}.$$

For twice continuously differentiable controls, (9.1) can be satisfied if for no component the maximal curvature is attained at the right-hand side boundary point of  $I^*$ .

Now we state our result about the structure of the optimal controls.

**Theorem 9.3.** *If  $T > T_0$  be given. Let  $n \in \{1, 2, 3, \dots\}$  be given such that*

$$n > \frac{T}{T_0}.$$

*Let  $X(T) = (W^{2,\infty}(0, T))^{|\mathcal{V}_c|+N_B}$  with the seminorm  $\|\cdot\|_{sem,X(T)} = \|\cdot\|_n$  as defined in (8.1).*

*If the penalty parameter  $\gamma > 0$  is sufficiently large, for any optimal control  $u_{opt}$  that solves the dynamic optimal control problem  $\mathbf{P}_{dyn}(T)$  and satisfies Property 9.1 there exists a number  $t_* \in (0, T)$  such that for all  $t \in [t_*, T]$  we have*

$$u_{opt}(t) = 0.$$

For the proof we need the following Lemma:

**Lemma 9.4.** *Let  $t_0 < t_s$  and  $\lambda \in [0, 1]$  be given. There exists a polynom  $H$  such that*

$$(9.5) \quad H(t_0) = 0, \quad \frac{d}{dt}H(t_0) = 0, \quad \frac{d^2}{dt^2}H(t_0) = 0,$$

$$(9.6) \quad H(t_s) = \lambda, \quad \frac{d}{dt}H(t_s) = 0, \quad \frac{d^2}{dt^2}H(t_s) = 0.$$

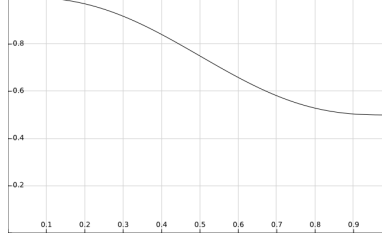
*The polynom  $H$  can be chosen such that for all  $t \in [t_0, t_s]$  we have  $H(t) \geq 0$ ,*

$$(9.7) \quad 0 \leq H'(t) \leq \frac{15}{8} \frac{\lambda}{t_s - t_0}$$

*and*

$$(9.8) \quad |H''(t)| \leq \frac{10}{\sqrt{3}} \frac{\lambda}{(t_s - t_0)^2}.$$

*For all  $t \in [\frac{t_0+t_s}{2}, t_s]$  we have  $H(t) \geq \frac{1}{2} \lambda$ .*

FIGURE 1. The graph of  $1 - H(t)$  on  $[0, 1]$ .

**Proof.** We define

$$H(t) = \lambda \left( \frac{t - t_0}{t_s - t_0} \right)^3 \left[ 1 - 3 \frac{t - t_s}{t_s - t_0} + 6 \left( \frac{t - t_s}{t_s - t_0} \right)^2 \right].$$

Then we have

$$\begin{aligned} H'(t) &= 3\lambda \frac{(t - t_0)^2}{(t_s - t_0)^3} \left[ 1 - 3 \frac{t - t_s}{t_s - t_0} + 6 \left( \frac{t - t_s}{t_s - t_0} \right)^2 \right] \\ &\quad + \lambda \left( \frac{t - t_0}{t_s - t_0} \right)^3 \left[ -3 \frac{1}{t_s - t_0} + 12 \frac{t - t_s}{(t_s - t_0)^2} \right]. \end{aligned}$$

The maximum of  $H'$  on  $[t_0, t_s]$  is attained at the point  $\frac{t_0 + t_s}{2}$ . The maximum of  $H''$  on  $[t_0, t_s]$  is attained at the point  $t_0 + \left( \frac{1}{2} - \frac{1}{2\sqrt{3}} \right) (t_s - t_0)$ .

**Remark 9.5.** Figure 1 shows the graph of  $1 - H(t)$  for  $t_0 = 0$ ,  $t_s = 1$  and  $\lambda = \frac{1}{2}$ . In this case we have

$$H(t) = \lambda [t^3 (1 - 3(t - 1) + 6(t - 1)^2)].$$

**Proof of Theorem 9.3.** Let  $u_{opt}$  denote a solution of  $\mathbf{P}_{\text{dyn}}(T)$ . Assume that Property 9.1 holds. Suppose that for all  $\tilde{T} \in (0, T)$  we have

$$\sum_{v \in V_c} \|u_{opt}^v\|_{W^{2,\infty}(\tilde{T}, T)} + \sum_{(v,e) \in V \times E: |E_0(v)|=1} \|u_{opt}^e\|_{W^{2,\infty}(\tilde{T}, T)}^2 > 0.$$

Choose a point  $t_s = \frac{j_s}{n} T \in [T - T_0, T)$  ( $j_s \in \{1, 2, 3, \dots\}$ ) and the point  $t_0$  as in the definition of Property 9.1. For  $\lambda \in (0, 1)$  and the polynomial  $H$  as defined in Lemma 9.4, we define the function  $H_w \in C^2([0, T])$  as

$$(9.9) \quad H_w(t) = \begin{cases} 0 & \text{for } t < t_0, \\ H(t) & \text{for } t_0 \leq t \leq t_s, \\ \lambda & \text{for } t > t_s. \end{cases}$$

For  $t \in [0, T]$ , we define the control

$$(9.10) \quad \tilde{u}(t) = [1 - H_w(t)] u_{opt}(t).$$

Since

$$1 - H_w(t) \in [0, 1].$$

for all  $t \in [0, T]$  we have

$$(9.11) \quad |\tilde{u}(t)| \leq |u_{opt}(t)|.$$

Moreover,  $\tilde{u}$  satisfies the control constraint (6.1).

For a vector  $(z_1, z_2, \dots, z_{dim})$ , define the maximum norm

$$\|z\|_\infty = \max\{|z_1|, |z_2|, \dots, |z_{dim}|\}.$$

For any  $t \in [0, T]$  such that  $\|u_{opt}(t)\|_\infty = 0$  or  $H'_w(t) = 0$  we have

$$\begin{aligned} \|\tilde{u}'(t)\|_\infty &\leq |1 - H_w(t)| \|u'_{opt}(t)\|_\infty + |H'_w(t)| \|u_{opt}(t)\|_\infty \\ &\leq \|u'_{opt}(t)\|_\infty \leq \varepsilon(T_0). \end{aligned}$$

For all  $t \in [0, T]$  where  $\|u_{opt}(t)\|_\infty > 0$  and  $H'_w(t) > 0$  we have

$$(9.12) \quad \|\tilde{u}'(t)\|_\infty \leq |1 - H_w(t)| \|u'_{opt}(t)\|_\infty + |H'_w(t)| \|u_{opt}(t)\|_\infty.$$

Due to (9.2), we can assume that  $\lambda > 0$  is chosen sufficiently small such that for all  $t \in [0, T]$  we have

$$(9.13) \quad 0 \leq H'_w(t) \leq \frac{\varepsilon(T_0) - \|u'_{opt}\|_{L^\infty(I_0)}}{\|u_{opt}\|_{L^\infty(I^*)}}.$$

With (9.12) this implies

$$(9.14) \quad \|\tilde{u}'\|_{L^\infty(I_0)} \leq \varepsilon(T_0).$$

Hence  $\tilde{u}$  satisfies (6.2). Since on the time interval  $[0, T - T_0]$  the states generated by the controls  $\tilde{u}$  and  $u_{opt} \in U(T)$  are identical, the state constraint (6.4) is satisfied for  $\tilde{u}$ . Therefore,  $\tilde{u}$  is a feasible control for  $\mathbf{P}_{dyn}(T)$ .

Now we consider the second order derivative. For  $t \in [0, T]$  almost everywhere we have

$$(9.15) \quad \|\tilde{u}''(t)\|_\infty \leq C_2(t) |1 - H_w(t)| + 2C_1(t) |H'_w(t)| + C_0(t) |H''_w(t)|,$$

where the norms of vector function  $u_{opt}(t)$  and of its derivatives  $u'_{opt}(t), u''_{opt}(t)$  are denoted by

$$(9.16) \quad C_i(t) := \|u_{opt}^{(i)}(t)\|_\infty.$$

If  $u_{opt}|_{I_0} = 0$ , this implies

$$(9.17) \quad \|\tilde{u}''\|_{L^\infty(I_0)} \leq \|u''_{opt}\|_{L^\infty(I^*)}.$$

Otherwise, due to (9.1), we can assume that  $\lambda > 0$  is chosen sufficiently small such that for all  $t \in [0, T]$  we have

$$(9.18) \quad 0 \leq H''_w(t) \leq \frac{\|u''_{opt}\|_{L^\infty(I^*)} - \|u''_{opt}\|_{L^\infty(I_0)}}{3\|u_{opt}\|_{L^\infty(I^*)}}.$$

and

$$(9.19) \quad 0 \leq H'_w(t) \leq \frac{\|u''_{opt}\|_{L^\infty(I^*)} - \|u''_{opt}\|_{L^\infty(I_0)}}{3\|u'_{opt}\|_{L^\infty(I^*)}}.$$

With (9.15) this implies (9.17). Define

$$(9.20) \quad \|u\|_{j_s} = \sum_{j=j_s+1}^n \left[ \sum_{v \in V_c} \|(u^v)''\|_{L^\infty(I^j)} + \|u^v\|_{L^\infty(I^j)} \right. \\ \left. + \sum_{(v,e) \in V \times E: |E_0(v)|=1} \|(u^e)''\|_{L^\infty(I^j)} + \|u^e\|_{L^\infty(I^j)} \right]$$

and

$$(9.21) \quad \|u\|_{<j_s} = \sum_{j=1}^{j_s} \left[ \sum_{v \in V_c} \|(u^v)''\|_{L^\infty(I^j)} + \|u^v\|_{L^\infty(I^j)} \right. \\ \left. + \sum_{(v,e) \in V \times E: |E_0(v)|=1} \|(u^e)''\|_{L^\infty(I^j)} + \|u^e\|_{L^\infty(I^j)} \right].$$

Let  $A_{opt} = \|u_{opt}\|_{<j_s}$  and  $B_{opt} = \|u_{opt}\|_{j_s}$ .

Let  $\tilde{A} = \|\tilde{u}\|_{<j_s}$ . On account of (9.17) and (9.11) we have

$$\tilde{A} \leq A_{opt}.$$

Let  $\tilde{B} = \|\tilde{u}\|_{j_s}$ . The definition of  $\tilde{u}$  implies that

$$\tilde{B} = (1 - \lambda) B_{opt}.$$

Hence we have

$$(\tilde{A} + \tilde{B}) - (A_{opt} + B_{opt}) \leq -\lambda B_{opt}.$$

In the regularization term in the objective function  $\tilde{J} = J(\tilde{u})$  we have the term

$$\gamma \|\tilde{u}\|_n = \gamma (\tilde{A} + \tilde{B}).$$

Now we consider the term in the objective function that models the control cost for the compressors. For  $u \in U(T)$ , define

$$(9.22) \quad P(u) = \int_0^T \sum_{v \in V_c} A_v q^v(t) [u^v(t) + u_{ref}^v - 1] dt.$$

We have

$$P(\tilde{u}) - P(u_{opt}) \leq \int_0^T \sum_{v \in V_c} A_v |\tilde{q}^v(t) - q_{opt}^v(t)| [\tilde{u}^v(t) + u_{ref}^v - 1] + A_v |q_{opt}^v(t)| |\tilde{u}^v(t) - u_{opt}^v(t)| dt$$

Due to (5.6) and the state constraint (6.4), this yields

$$P(\tilde{u}) - P(u_{opt}) \leq \sum_{v \in V_c} T A_v C_c(T_0) \|\tilde{u}^v - u_{opt}^v\|_{C([t_0, T])} [|u_{ref}^v| + \varepsilon(T_0)] \\ + T A_v [|q_{ref}| + C_{priori}(T_0)] \|\tilde{u}^v - u_{opt}^v\|_{C([t_0, T])}$$

where  $C_{priori}(T_0)$  is a constant from an priori bound on  $[0, T_0]$  for the maximum norm. This yields the inequality

$$(9.23) \quad \begin{aligned} & P(\tilde{u}) - P(u_{opt}) \\ & \leq TA_v [C_c(T_0) [|u_{ref}^v| + \varepsilon(T_0)] + [|q_{ref}| + C_{priori}(T_0)]] \|\lambda u_{opt}^v\|_{C([t_0, T])}. \end{aligned}$$

**Case 1:** If (9.3) holds we obtain

$$(9.24) \quad \begin{aligned} & P(\tilde{u}) - P(u_{opt}) \\ & \leq TA_v [C_c(T_0) [|u_{ref}^v| + \varepsilon(T_0)] + [|q_{ref}| + C_{priori}(T_0)]] \lambda \|u_{opt}\|_{j_s}. \end{aligned}$$

Inequality (5.5) implies that for the penalty term for the pressure we have the upper bound

$$\begin{aligned} \eta_p & \left| \sum_{e \in E} \|(p_{\min} - p_{opt}^e(t, x))_+\|_{C([0, T] \times [0, L^e])} + \|(p_{opt}^e(t, x) - p_{\max})_+\|_{C([0, T] \times [0, L^e])} \right. \\ & \quad \left. - \sum_{e \in E} \|(p_{\min} - \tilde{p}^e(t, x))_+\|_{C([0, T] \times [0, L^e])} + \|(\tilde{p}^e(t, x) - p_{\max})_+\|_{C([0, T] \times [0, L^e])} \right| \\ & \leq \eta_p C_c(T_0) \max\{\max_{e \in E} \|\tilde{u}^e - u_{opt}^e\|_{C([t_0, T])}, \max_{v \in V_c} \|\tilde{u}^v - u_{opt}^v\|_{C([t_0, T])}\} \\ & \leq \eta_p C_c(T_0) \lambda \|u_{opt}\|_{j_s}. \end{aligned}$$

With the notation  $J_{opt} = J(u_{opt})$  and  $\tilde{J} = J(\tilde{u})$  for the values of the objective function due to (9.24) this yields

$$\begin{aligned} \tilde{J} & \leq J_{opt} + [\eta_p C_c(T_0) - \gamma] \lambda B_{opt} \\ & \quad + TA_v [C_c(T_0) [|u_{ref}^v| + \varepsilon(T_0)] + [|q_{ref}| + C_{priori}(T_0)]] \lambda B_{opt}. \end{aligned}$$

If  $\gamma$  is sufficiently large we have

$$(9.25) \quad \eta_p C_c(T_0) + TA_v [C_c(T_0) [|u_{ref}^v| + \varepsilon(T_0)] + [|q_{ref}| + C_{priori}(T_0)]] - \gamma < 0.$$

This implies

$$\tilde{J} < J_{opt}$$

which is a contradiction to the optimality of  $u_{opt}$ . Hence for Case 1, the assertion follows.

Now we consider the second case.

**Case 2:** If (9.4) holds instead of (9.24) we obtain

$$(9.26) \quad \begin{aligned} & P(\tilde{u}) - P(u_{opt}) \\ & \leq TA_v [C_c(T_0) [|u_{ref}^v| + \varepsilon(T_0)] + [|q_{ref}| + C_{priori}(T_0)]] \lambda \|u_{opt}\|_{C([t_0, T])}. \end{aligned}$$

For the penalty term for the pressure we have the upper bound

$$\begin{aligned} \eta_p & \left| \sum_{e \in E} \|(p_{\min} - p_{opt}^e(t, x))_+\|_{C([0, T] \times [0, L^e])} + \|(p_{opt}^e(t, x) - p_{\max})_+\|_{C([0, T] \times [0, L^e])} \right. \\ & \quad \left. - \sum_{e \in E} \|(p_{\min} - \tilde{p}^e(t, x))_+\|_{C([0, T] \times [0, L^e])} + \|(\tilde{p}^e(t, x) - p_{\max})_+\|_{C([0, T] \times [0, L^e])} \right| \end{aligned}$$

$$\leq \eta_p C_c(T_0) \max\{\max_{e \in E} \|\tilde{u}^e - u_{opt}^e\|_{C([t_0, T])}, \max_{v \in V_c} \|\tilde{u}^v - u_{opt}^v\|_{C([t_0, T])}\}.$$

Due to (9.4), we can define  $\delta > 0$  as

$$\delta = \frac{\|u_{opt}\|_{L^\infty(I^*)} - \|u_{opt}\|_{L^\infty([\frac{j_s-1}{n}T, \frac{t_0+t_s}{2}] )}}{\|u_{opt}\|_{L^\infty(I^*)}}$$

Then for all  $t \in [\frac{j_s-1}{n}T, \frac{t_0+t_s}{2}]$  we have

$$(9.27) \quad |\tilde{u}(t)| \leq |u_{opt}(t)| \leq (1 - \delta) \|u_{opt}\|_{L^\infty(I^*)}.$$

For all  $t \in [\frac{t_0+t_s}{2}, t_s]$  we have  $H_w(t) \geq \frac{1}{2} \lambda$  and hence

$$|\tilde{u}(t)| \leq (1 - \frac{1}{2} \lambda) \|u_{opt}\|_{L^\infty(I^*)}.$$

Thus we have

$$\|\tilde{u}\|_{L^\infty(I^*)} \leq \max\left\{(1 - \frac{1}{2} \lambda), (1 - \delta)\right\} \|u_{opt}\|_{L^\infty(I^*)}.$$

We assume that  $\lambda \leq 2\delta$ . Then we have

$$\|\tilde{u}\|_{L^\infty(I^*)} - \|u_{opt}\|_{L^\infty(I^*)} \leq -\frac{1}{2} \lambda \|u_{opt}\|_{L^\infty(I^*)}.$$

Thus for the regularization terms in the objective function in Case 2 we have

$$(\tilde{A} + \tilde{B}) - (A_{opt} + B_{opt}) \leq -\lambda B_{opt} - \frac{1}{2} \lambda \|u_{opt}\|_{L^\infty(I^*)}.$$

This yields

$$\begin{aligned} \tilde{J} &\leq J_{opt} \\ &+ [\eta_p C_c(T_0) + T A_v [C_c(T_0) [|u_{ref}^v| + \varepsilon(T_0)] + [|q_{ref}| + C_{priori}(T_0)]]] \\ &\quad (\lambda \|u_{opt}\|_{L^\infty(I_0)} + \lambda B_{opt}) \\ &\quad - \gamma \frac{1}{2} \lambda \|u_{opt}\|_{L^\infty(I^*)} - \gamma \lambda B_{opt}. \end{aligned}$$

If  $\gamma$  is sufficiently large we have

$$\eta_p C_c(T_0) + T A_v [C_c(T_0) [|u_{ref}^v| + \varepsilon(T_0)] + [|q_{ref}| + C_{priori}(T_0)]] - \frac{1}{2} \gamma < 0.$$

This implies

$$\tilde{J} < J_{opt}$$

which is a contradiction to the optimality of  $u_{opt}$ . Hence also for Case 2, the assertion follows.

Thus we have shown that if Property 9.1 holds, the optimal controls for  $\mathbf{P}_{\text{dyn}}(T)$  are equal to zero in the last part of the time interval  $[0, T]$ . This finishes the proof of Theorem 9.3.



**Remark 9.6.** A natural choice for  $u_{ref}$  is the solution of the static problem  $\mathbf{P}_{stat}$  that corresponds to our dynamic optimal control problem  $\mathbf{P}_{dyn}(T)$ . In  $\mathbf{P}_{stat}$  the control values at each control location are constant so the controls are given by

$$(9.28) \quad u \in \mathbb{R}^{M+N_B}.$$

The static system equations ( $\mathbf{S}_{stat}$ ) include the boundary conditions

$$\begin{cases} q_{out}^e(x^e(v)) = q_{desi}^e, & v \in V, e \in E_0(v), \text{ if } |E_0(v)| = 1, v \neq v^* \\ p_{in}^e(x^e(v)) = p_{desi}^e, & v \in V, e \in E_0(v), \text{ if } |E_0(v)| = 1, v = v^* \end{cases}$$

the Kirchhoff conditions

$$\sum_{e \in E_0(v)} \mathfrak{s}(v, e) (D^e)^2 q^e(x^e(v)) = 0, \quad t \in (0, T), \text{ if } |E_0(v)| \geq 2,$$

the continuity conditions

$$p(\rho^e(x^e(v))) = p(\rho^f(x^f(v))), \quad \text{if } |E_0(v)| \geq 2, e, f \in E_0(v),$$

the compressor conditions

$$\begin{cases} u^v = \left( \frac{p_{out,v}}{p_{in,v}} \right)^{R_v}, & \text{if } |E_0(v)| = 2, v \in V_c \\ q^e(x^e(v)) = q^f(x^f(v)), & \text{if } |E_0(v)| = 2, v \in V_c; e, f \in E_0(v) \end{cases}$$

and the PDEs

$$\begin{pmatrix} 0 & 1 \\ a^2 - \frac{(q^e)^2}{(\rho^e)^2} & 2 \frac{q^e}{\rho^e} \end{pmatrix}_x \begin{pmatrix} \rho^e \\ q^e \end{pmatrix}_x = \begin{pmatrix} 0 \\ -\frac{1}{2} \theta^e q^e \frac{|q^e|}{\rho^e} \end{pmatrix} \quad \text{on } [0, L^e], e \in E.$$

The feasible set  $U_{stat}$  for the static optimal control problem  $\mathbf{P}_{stat}$  contains the static controls  $u$  as in (9.28), such that the control constraints

$$(9.29) \quad \varepsilon_v^{\min} \leq u^v \leq \varepsilon_v^{\max}$$

and for the corresponding state generated with the system equations ( $\mathbf{S}_{stat}$ ), the state constraints

$$(9.30) \quad p^e(x) \in [p_{\min}, p_{\max}].$$

are satisfied. We are looking for a control in  $U_{stat}$  such that the objective function

$$(9.31) \quad J_{stat}(u) = \sum_{v \in V_c} A_v q^v [u^v - 1]$$

is minimized.

Since the controls are in a finite dimensional space, the existence of a solution of the static optimal control problem ( $\mathbf{S}_{stat}$ ) is standard.

Note that the structure of ( $\mathbf{S}_{stat}$ ) is much simpler than the structure of the dynamic optimal control problem. The optimal control and the corresponding stationary states give a reference state for which the desired flow rates are achieved with minimal compressor cost. So a natural choice for  $u_{ref}$  is a static optimal control.

## 10. CONCLUSION

We have presented a dynamic optimal control problem that models the optimal operation of gas pipeline networks with a quasilinear system and allows to obtain controls that generate regular solutions of the system with minimal cost.

We have shown that for sufficiently large penalty parameters  $\gamma$  for the  $W^{2,\infty}$  regularization term, under generic assumptions we obtain optimal controls that approach the reference control towards the end of the time interval. In particular, Theorem 9.3 implies that if the optimal controls vanish on a subinterval of  $[T_0, T]$  that is longer than  $\frac{1}{n}T$ , then they must also vanish at the end of the time interval.

We expect that if the system is controlled by such constant controls, it will converge to the corresponding stationary state. However, a proof of this statement is out of the scope of the present paper and a subject of future work.

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