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# HOW FAR ARE P-ADIC LIE GROUPS FROM ALGEBRAIC GROUPS?

YVES BENOIST AND JEAN-FRANÇOIS QUINT

ABSTRACT. We show that, in a weakly regular  $p$ -adic Lie group  $G$ , the subgroup  $G_u$  spanned by the one-parameter subgroups of  $G$  admits a Levi decomposition. As a consequence, there exists a regular open subgroup of  $G$  which contains  $G_u$

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*Key words*  $p$ -adic Lie group, algebraic group, unipotent subgroup, Levi decomposition, central extension

## 1. INTRODUCTION

**1.1. Motivations.** When studying the dynamics of the subgroups of a  $p$ -adic Lie group  $G$  on its homogeneous spaces, various assumptions can be made on  $G$ . For instance, one can ask  $G$  to be algebraic as in [10] i.e. to be a subgroup of a linear group defined by polynomial equations.

Another possible assumption on  $G$  is regularity. One asks  $G$  to satisfy properties that are well-known to hold for Zariski connected algebraic  $p$ -adic Lie groups: a uniform bound on the cardinality of the finite subgroups and a characterization of the center as the kernel of the adjoint representation (see Definition 2.5). Ratner's theorems in [13] are written under this regularity assumption.

A more natural and weaker assumption on  $G$  in this context is the weak regularity of  $G$  i.e. the fact that the one-parameter morphisms of  $G$  are uniquely determined by their derivative (see Definition 2.3). Our paper [1] is written under this weak-regularity assumption.

The aim of this text is to clarify the relationships between these three assumptions.

A key ingredient in the proof is Proposition 5.1. It claims

*the finiteness of the center of the universal topological  
central extension of non-discrete simple  $p$ -adic Lie groups.*

This is a fact which is due to Prasad and Raghunathan in [12].

**1.2. Main results.** We will first prove (Proposition 5.5):

*In a weakly regular  $p$ -adic Lie group  $G$ , the subgroup  $G_u$   
spanned by the one-parameter subgroups of  $G$  is closed  
and admits a Levi decomposition,*

i.e.  $G_u$  is a semidirect product  $G_u = S_u \ltimes R_u$  of a group  $S_u$  by a normal subgroup  $R_u$  where  $S_u$  is a finite cover of a finite index subgroup of an algebraic semisimple Lie group and where  $R_u$  is algebraic unipotent.

As a consequence, we will prove (Theorem 5.12):

*In a weakly regular  $p$ -adic Lie group  $G$  there always exists  
an open subgroup  $G_\Omega$  which is regular and contains  $G_u$ .*

This Theorem 5.12 is useful since it extends the level of generality of Ratner's theorems in [13] (see [1, Th 5.15]). More precisely, Ratner's theorems for products of real and  $p$ -adic Lie groups in [13] are proven under the assumption that these  $p$ -adic Lie groups are *regular*. Ratner's theorems can be extended under the weaker assumption that these  $p$ -adic Lie groups are *weakly regular* thanks to our Theorem 5.12.

This Theorem 5.12 has been announced in [1, Prop. 5.8] and has been used in the same paper.

The strategy consists in proving first various statements for weakly regular  $p$ -adic Lie groups which were proven in [13] under the regularity assumption. To clarify the discussion we will reprove also the results from [13] that we need. But we will take for granted classical results on the structure of simple  $p$ -adic algebraic groups that can be found in [3], [6], [7] or [11].

**1.3. Plan.** In the preliminary Chapter 2 we recall a few definitions and examples.

Our main task in Chapters 3 and 4 is to describe, for a weakly regular  $p$ -adic Lie group  $G$ , the subset  $\mathfrak{g}_G \subset \mathfrak{g}$  of derivatives of one-parameter morphisms of  $G$ .

In Chapter 3, the results are mainly due to Ratner. We first study the nilpotent  $p$ -adic Lie subgroups  $N$  of  $G$  spanned by one-parameter subgroups. We will see that they satisfy  $\mathfrak{n} \subset \mathfrak{g}_G$  (Proposition 3.7). Those  $p$ -adic Lie groups  $N$  are called algebraic unipotent. We will simultaneously compare this set  $\mathfrak{g}_G$  with the analogous set  $\mathfrak{g}'_{G'}$  for a quotient group  $G' = G/N$  of  $G$  when  $N$  is a normal algebraic unipotent subgroup (Lemma 3.6). This will allow us to prove that  $G$  contains a largest normal algebraic unipotent subgroup (Proposition 3.8).

In Chapter 4, we will then be able to describe precisely the set  $\mathfrak{g}_G$  using a Levi decomposition of  $\mathfrak{g}$  (Proposition 4.4). A key ingredient is a technic, borrowed from [1], for constructing one-parameter subgroups in a  $p$ -adic Lie group  $G$  (Lemma 4.1).

In the last Chapter 5, we will prove the main results we have just stated: Proposition 5.5 and Theorem 5.12, using Prasad–Raghunathan finiteness theorem in [12] (see Proposition 5.1). We will end this text by an example showing that, when a  $p$ -adic Lie group  $G$  with  $G = G_u$  is not assumed to be weakly regular, it does not always contain a regular open subgroup  $H$  for which  $H = H_u$  (Example 5.14).

## 2. PRELIMINARY RESULTS

We recall here a few definitions and results from [13].

Let  $p$  be a prime number and  $\mathbb{Q}_p$  be the field of  $p$ -adic numbers. When  $G$  is a  $p$ -adic Lie group (see [5]), we will always denote by  $\mathfrak{g}$  the Lie algebra of  $G$ . It is a  $\mathbb{Q}_p$ -vector space. We will denote by  $\text{Ad } \mathfrak{g}$  or  $\text{Ad}$  the adjoint action of  $G$  on  $\mathfrak{g}$  and by  $\text{ad } \mathfrak{g}$  or  $\text{ad}$  the adjoint action of  $\mathfrak{g}$  on  $\mathfrak{g}$ . Any closed subgroup  $H$  of  $G$  is a  $p$ -adic Lie subgroup and its Lie algebra  $\mathfrak{h}$  is a  $\mathbb{Q}_p$ -vector subspace of  $\mathfrak{g}$  (see [13, Prop. 1.5]).

We choose a ultrametric norm  $\|\cdot\|$  on  $\mathfrak{g}$  with values in  $p^{\mathbb{Z}}$ .

### 2.1. One-parameter subgroups.

**Definition 2.1.** *A one-parameter morphism  $\varphi$  of a  $p$ -adic Lie group  $G$  is a continuous morphism  $\varphi : \mathbb{Q}_p \rightarrow G; t \mapsto \varphi(t)$ . A one-parameter subgroup is the image  $\varphi(\mathbb{Q}_p)$  of an injective one-parameter morphism.*

The derivative of a one-parameter morphism is an element  $X$  of  $\mathfrak{g}$  for which  $\text{ad}X$  is nilpotent. This follows from the following Lemma from [13, Corollary 1.2].

**Lemma 2.2.** *Let  $\varphi : \mathbb{Q}_p \rightarrow \text{GL}(d, \mathbb{Q}_p)$  be a one-parameter morphism. Then there exists a nilpotent matrix  $X$  in  $\mathfrak{gl}(d, \mathbb{Q}_p)$  such that  $\varphi(t) = \exp(tX)$  for all  $t$  in  $\mathbb{Q}_p$ .*

Here  $\exp$  is the exponential of matrices :  $\exp(tX) := \sum_{n \geq 0} \frac{t^n X^n}{n!}$

*Proof.* First, we claim that, if  $K$  is a finite extension of  $\mathbb{Q}_p$ , any continuous one-parameter morphism  $\psi : \mathbb{Q}_p \rightarrow K^*$  is constant. Indeed, since the modulus of  $\psi$  is a continuous morphism  $\mathbb{Q}_p$  to a discrete multiplicative subgroup of  $(0, \infty)$ , the kernel of  $|\psi|$  contains  $p^k \mathbb{Z}_p$ , for some integer  $k$ . Since  $\mathbb{Q}_p/p^k \mathbb{Z}_p$  is a torsion group and  $(0, \infty)$  has no torsion,  $\psi$  has constant modulus, that is,  $\psi$  takes values in  $\mathcal{O}^*$ , where  $\mathcal{O}$  is the integer ring of  $K$ . Now, on one hand,  $\mathcal{O}^*$  is a profinite group, that is, it is a compact totally discontinuous group and hence it admits a basis of neighborhoods of the identity which are finite index subgroups (namely, for example, the subgroups  $1 + p^k \mathcal{O}$ ,  $k \geq 0$ ). On the other hand, every closed subgroup of  $\mathbb{Q}_p$  is of the form  $p^k \mathbb{Z}_p$ , for some integer  $k$ , and hence, has infinite index in  $\mathbb{Q}_p$  and therefore any continuous morphism from  $\mathbb{Q}_p$  to a finite group is trivial. Thus,  $\psi$  is constant, which should be proved.

Let now  $\varphi$  be as in the lemma and let  $X \in \mathfrak{gl}(d, \mathbb{Q}_p)$  be the derivative of  $\varphi$ . After having simultaneously reduced the commutative family of matrices  $\varphi(t)_{t \in \mathbb{Q}_p}$ , the joint eigenvalues give continuous morphisms  $\mathbb{Q}_p \rightarrow K^*$  where  $K$  is a finite extension of  $\mathbb{Q}_p$ . By the remark above, these morphisms are constant, that is, there exists  $g$  in  $\text{GL}(d, \mathbb{Q}_p)$ , such that, for any  $t$  in  $\mathbb{Q}_p$ , the matrix  $g\varphi(t)g^{-1}$  is unipotent and upper triangular. We may assume  $g = 1$ . Then  $X$  is nilpotent and upper triangular and it remains to check that the map  $\theta : t \mapsto \varphi(t)\exp(-tX)$  is constant. Since  $\varphi(t)$  commutes with  $X$ , this map  $\theta$  is a one-parameter morphism with zero derivative. Hence, the kernel of  $\psi$  is an open subgroup and the matrices  $\theta(t)$ ,  $t \in \mathbb{Q}_p$ , have finite order. Since they are unipotent, they equal  $e$ , which should be proved.  $\square$

### 2.2. Weakly regular and regular $p$ -adic Lie groups.

**Definition 2.3** (Ratner, [13]). *A  $p$ -adic Lie group  $G$  is said to be weakly regular if any two one-parameter morphisms  $\mathbb{Q}_p \rightarrow G$  with the same derivative are equal.*

Note that any closed subgroup of a weakly regular  $p$ -adic Lie group is also weakly regular.

*Example 2.4* ([13, Cor. 1.3 and Prop. 1.5]). Every closed subgroup of  $\mathrm{GL}(d, \mathbb{Q}_p)$  is weakly regular.

*Proof.* This follows from Lemma 2.2. □

**Definition 2.5** (Ratner, [13]). *A  $p$ -adic Lie group  $G$  is said to be Ad-regular if the kernel of the adjoint map  $\mathrm{Ker}(\mathrm{Ad}_{\mathfrak{g}})$  is equal to the center  $Z(G)$  of  $G$ . It is said to be regular if it is Ad-regular and if the finite subgroups of  $G$  have uniformly bounded cardinality.*

Note that any open subgroup of a regular  $p$ -adic Lie group is also regular.

This definition is motivated by the following example.

*Example 2.6.* a) The finite subgroups of a compact  $p$ -adic Lie group  $K$  have uniformly bounded cardinality.

b) The finite subgroups of a  $p$ -adic linear group have uniformly bounded cardinality.

c) The Zariski connected linear algebraic  $p$ -adic Lie groups  $G$  are regular.

*Proof of Example 2.6.* (see [13])

a) Since  $K$  contains a torsion free open normal subgroup  $\Omega$ , for every finite subgroup  $F$  of  $K$ , one has the bound  $|F| \leq |K/\Omega|$ .

b) We want to bound the cardinality  $|F|$  of a finite subgroup of a group  $G \subset \mathrm{GL}(d, \mathbb{Q}_p)$ . This follows from a) since  $F$  is included in a conjugate of the compact group  $K = \mathrm{GL}(d, \mathbb{Z}_p)$ .

c) It remains to check that  $G$  is Ad-regular. Let  $g$  be an element in the kernel of the adjoint map  $\mathrm{Ad}_{\mathfrak{g}}$ . This means that the centralizer  $Z_g$  of  $g$  in  $G$  is an open subgroup of  $G$ . This group  $Z_g$  is also Zariski closed. Hence it is Zariski open. Since  $G$  is Zariski connected, one deduces  $Z_g = G$  and  $g$  belongs to the center of  $G$ . □

We want to relate the two notions “weakly regular” and “regular”. We first recall the following implication in [13, Cor. 1.3].

**Lemma 2.7.** *Any regular  $p$ -adic Lie group is weakly regular.*

*Proof.* Let  $\varphi_1 : \mathbb{Q}_p \rightarrow G$  and  $\varphi_2 : \mathbb{Q}_p \rightarrow G$  be one-parameter morphisms of  $G$  with the same derivative. We want to prove that  $\varphi_1 = \varphi_2$ .

According to Lemma 2.2, the one-parameter morphisms  $\mathrm{Ad}_{\mathfrak{g}}\varphi_1$  and  $\mathrm{Ad}_{\mathfrak{g}}\varphi_2$  are equal. Since  $G$  is Ad-regular, this implies that, for all  $t$  in  $\mathbb{Q}_p$ , the element  $\theta(t) := \varphi_1(t)^{-1}\varphi_2(t)$  is in the center of  $G$ . Hence  $\theta$  is a one-parameter morphism of the center of  $G$  with zero derivative. Its image is either trivial or an infinite  $p$ -torsion group. This second case is excluded by the uniform bound on the finite subgroups of  $G$ . This proves the equality  $\varphi_1 = \varphi_2$ .  $\square$

The aim of this text is to prove Theorem 5.12 which is a kind of converse to Lemma 2.7.

### 3. ALGEBRAIC UNIPOTENT $p$ -ADIC LIE GROUP

In this chapter, we study the algebraic unipotent subgroups of a weakly regular  $p$ -adic Lie group. The results in this chapter are mainly due to Ratner.

#### 3.1. Definition and closedness.

We first focus on a special class of  $p$ -adic Lie groups.

We say that an element  $g$  of a  $p$ -adic Lie group  $G$  admits a logarithm if one has  $g^{p^n} \xrightarrow[n \rightarrow \infty]{} e$ : indeed, for such a  $g$ , the map  $n \mapsto g^n$  extends as a continuous morphism  $\mathbb{Z}_p \rightarrow G$  and one can define the logarithm  $\log(g) \in \mathfrak{g}$  as being the derivative at 0 of this morphism.

**Definition 3.1.** *A  $p$ -adic Lie group  $N$  is called algebraic unipotent if its Lie algebra is nilpotent, if every element  $g$  in  $N$  admits a logarithm  $\log(g)$ , and if the logarithm map  $\log : N \rightarrow \mathfrak{n}$  is a bijection.*

This implies that every non trivial element  $g$  of  $N$  belongs to a unique one-parameter subgroup of  $N$ . By definition these groups  $N$  are weakly regular. The inverse of the map  $\log$  is denoted by  $\exp$ . These maps  $\exp$  and  $\log$  are  $\mathrm{Aut}(N)$ -equivariant.

The following lemma is in [13, Prop. 2.1].

**Lemma 3.2.** *Let  $N$  be an algebraic unipotent  $p$ -adic Lie group. Then the map*

$$\mathfrak{n} \times \mathfrak{n} \rightarrow \mathfrak{n}; (X, Y) \mapsto \log(\exp X \exp Y)$$

*is polynomial and is given by the Baker-Campbell-Hausdorff formula. In other words, a  $p$ -adic Lie group  $N$  is algebraic unipotent if and only if it is isomorphic to the group of  $\mathbb{Q}_p$ -points of a unipotent algebraic group defined over  $\mathbb{Q}_p$ .*

These groups have been studied in [13, Sect.2] (where they are called *quasiconnected*).

In particular, we have

**Corollary 3.3.** *Let  $N$  be an algebraic unipotent  $p$ -adic Lie group. Then the exponential and logarithm maps of  $N$  are continuous.*

*Proof of Lemma 3.2.* In this proof, we will say that a  $p$ -adic Lie group is *strongly algebraic unipotent* if it is isomorphic to the group of  $\mathbb{Q}_p$ -points of a unipotent algebraic group defined over  $\mathbb{Q}_p$ . Such a group is always algebraic unipotent.

The aim of this proof is to check the converse. Let  $N$  be an algebraic unipotent  $p$ -adic Lie group. We want to prove that  $N$  is strongly algebraic unipotent. Its Lie algebra  $\mathfrak{n}$  contains a flag  $0 \subset \mathfrak{n}_1 \subset \cdots \subset \mathfrak{n}_r = \mathfrak{n}$  of ideals with  $\dim \mathfrak{n}_i = i$ . We will prove, by induction on  $i \geq 1$ , that the set  $N_i := \exp(\mathfrak{n}_i)$  is a closed subgroup of  $N$  which is strongly algebraic unipotent.

By the induction assumption, the set  $N_{i-1}$  is a strongly algebraic unipotent closed subgroup. Since  $\mathfrak{n}_{i-1}$  is an ideal, this subgroup  $N_{i-1}$  is normal. Let  $X_i$  be an element of  $\mathfrak{n}_i \setminus \mathfrak{n}_{i-1}$  and  $N'_i$  the semidirect product  $N'_i := \mathbb{Q}_p \ltimes N_{i-1}$  where the action of  $t \in \mathbb{Q}_p$  by conjugation on  $N_{i-1}$  is given by  $t \exp(X) t^{-1} = \exp(e^{t \text{ad} X_i} X)$ , for all  $X$  in  $\mathfrak{n}_{i-1}$ . By construction this group  $N'_i$  is strongly algebraic unipotent and the map  $\psi : N'_i \rightarrow N; (t, n) \mapsto \exp(tX_i)n$  is a group morphism. Since  $N'_i = \exp(\mathfrak{n}'_i)$ , the set  $N_i = \exp(\mathfrak{n}_i)$  is equal to the image  $\psi(N'_i)$ . Hence, by Lemma 3.4 below, the set  $N_i$  is a closed subgroup which is isomorphic to  $N'_i$  and hence  $N_i$  is strongly algebraic unipotent.  $\square$

The following lemma tells us that an algebraic unipotent Lie subgroup of a  $p$ -adic Lie group is always closed.

**Lemma 3.4.** *Let  $G$  be a totally discontinuous locally compact topological group,  $N$  be an algebraic unipotent  $p$ -adic Lie group and  $\varphi : N \rightarrow G$  be an injective morphism. Then  $\varphi$  is a proper map. In particular,  $\varphi(N)$  is a closed subgroup of  $G$  and  $\varphi$  is an isomorphism of topological groups from  $N$  onto  $\varphi(N)$ .*

*Proof.* If  $\varphi$  was not proper, there would exist a sequence  $Y_n$  in the Lie algebra  $\mathfrak{n}$  of  $N$  such that

$$\lim_{n \rightarrow \infty} \varphi(\exp(Y_n)) = e \quad \text{and} \quad \lim_{n \rightarrow \infty} \|Y_n\| = \infty.$$

We write  $Y_n = p^{-k_n} X_n$  with integers  $k_n$  going to  $\infty$  and  $\|X_n\| = 1$ . Since the group  $G$  admits a basis of compact open subgroups, one also has  $\lim_{n \rightarrow \infty} \varphi(\exp(X_n)) = e$ . Let  $X$  be a cluster point of the sequence  $X_n$ . One has simultaneously,  $\varphi(\exp(X)) = e$  and  $\|X\| = 1$ . This contradicts the injectivity of  $\varphi$ .  $\square$



### 3.2. Lifting one-parameter morphisms.

We now explain how to lift one-parameter morphisms.

**Lemma 3.5.** *Let  $G$  be a  $p$ -adic Lie group,  $N \subset G$  a normal algebraic unipotent closed subgroup,  $G' := G/N$ , and  $\pi : G \rightarrow G'$  the projection. Then, for any one-parameter morphism  $\varphi'$  of  $G'$ , there exists a one-parameter morphism  $\varphi$  of  $G$  which lifts  $\varphi'$ , i.e. such that  $\varphi' = \pi \circ \varphi$ .*

*If  $\varphi'$  has zero derivative, one can choose  $\varphi$  to have zero derivative.*

*Proof.* According to Lemma 3.4 the image  $\varphi'(\mathbb{Q}_p)$  is a closed subgroup of  $G'$ . Hence we can assume that  $\varphi'(\mathbb{Q}_p) = G'$ .

**First Case :**  $N$  is central in  $G$ . For  $k \geq 1$ , we introduce the subgroup  $Q'_k$  of  $G'$  spanned by the element  $g'_k := \varphi'(p^{-k})$ . Since  $Q'_k$  is cyclic and  $N$  is central, the group  $Q_k := \pi^{-1}(Q'_k)$  is abelian. Since the increasing union of these groups  $Q_k$  is dense in  $G$ , the group  $G$  is also abelian. Since  $N$  is infinitely  $p$ -divisible, one can construct, by induction on  $k \geq 0$ , a sequence  $(g_k)_{k \geq 0}$  in  $G$  such that

$$\pi(g_k) = g'_k \text{ and } p g_{k+1} = g_k .$$

We claim that  $g_0^{p^k} \xrightarrow[k \rightarrow \infty]{} e$ . Indeed, since  $\pi(g_0)^{p^k} \xrightarrow[k \rightarrow \infty]{} e$  and since every element  $h$  of a  $p$ -adic Lie group that is close enough to the identity element satisfies  $h^{p^k} \xrightarrow[k \rightarrow \infty]{} e$ , one can find  $\ell \geq 0$  and  $n$  in  $N$  such that  $(g_0^{p^\ell} n^{-1})^{p^k} \xrightarrow[k \rightarrow \infty]{} e$ . As  $N$  is algebraic unipotent, we have  $n^{p^k} \xrightarrow[k \rightarrow \infty]{} e$ , and the claim follows, since  $N$  is central.

Now, the formulae  $\varphi(p^{-k}) = g_k$ , for all  $k \geq 0$ , define a unique one-parameter morphism  $\varphi$  of  $G$  which lifts  $\varphi'$ .

Note that when  $\varphi'$  has zero derivative, one can assume, after a reparametrization of  $\varphi'$ , that  $\varphi'(\mathbb{Z}_p) = 0$  and choose the sequence  $g_k$  so that  $g_0 = e$ . Then the morphism  $\varphi$  has also zero derivative.

**General Case :** The composition of  $\varphi'$  with the action by conjugation on the abelianized group  $N/[N, N] \simeq \mathbb{Q}_p^d$  is a one-parameter morphism  $\psi : \mathbb{Q}_p \rightarrow \mathrm{GL}(d, \mathbb{Q}_p)$ . According to Lemma 2.2, there exists a nilpotent matrix  $X$  such that  $\psi(t) = \exp(tX)$  for all  $t \in \mathbb{Q}_p$ . The image of this matrix  $X$  corresponds to an algebraic unipotent subgroup  $N_1$  with  $[N, N] \subset N_1 \subsetneq N$  which is normal in  $G$  and such that  $N/N_1$  is a central subgroup of the group  $G/N_1$ . According to the first case, the morphism  $\varphi'$  can be lifted as a morphism  $\varphi'_1$  of  $G/N_1$ . By an induction argument on the dimension of  $N$ , this morphism  $\varphi'_1$  can be lifted as a morphism of  $G$ .  $\square$

Let  $G$  be a  $p$ -adic Lie group. We recall the notation

$$(3.1) \quad \mathfrak{g}_G := \{X \in \mathfrak{g} \text{ derivative of a one-parameter morphism of } G\}.$$

Note that this set  $\mathfrak{g}_G$  is invariant under the adjoint action of  $G$ .

The following lemma tells us various stability properties by extension when the normal subgroup is algebraic unipotent.

**Lemma 3.6.** *Let  $G$  be a  $p$ -adic Lie group,  $N$  a normal algebraic unipotent subgroup of  $G$ , and  $G' := G/N$ .*

a) *One has the equivalence*

$$G \text{ is algebraic unipotent} \iff G' \text{ is algebraic unipotent.}$$

b) *Let  $X$  in  $\mathfrak{g}$  and  $X'$  its image in  $\mathfrak{g}' = \mathfrak{g}/\mathfrak{n}$ . One has the equivalence*

$$X \in \mathfrak{g}_G \iff X' \in \mathfrak{g}'_{G'}.$$

c) *One has the equivalence*

$$G \text{ is weakly regular} \iff G' \text{ is weakly regular.}$$

Later on in Corollary 5.7 we will be able to improve this Lemma.

*Proof.* We denote by  $\pi : G \rightarrow G/N$  the natural projection.

a) The implication  $\Rightarrow$  is well-known. Conversely, we assume that  $N$  and  $G/N$  are algebraic unipotent and we want to prove that  $G$  is algebraic unipotent. Arguing by induction on  $\dim G/N$ , we can assume  $\dim G/N = 1$ , i.e. that there exists an isomorphism  $\varphi' : \mathbb{Q}_p \rightarrow G/N$ . According to Lemma 3.5, one can find a one-parameter morphism  $\varphi$  of  $G$  that lifts  $\varphi'$ . By Lemma 3.4, the image  $Q := \varphi(\mathbb{Q}_p)$  is closed and  $G$  is the semidirect product  $G = Q \ltimes N$ . By Lemma 2.2, the one-parameter morphism  $t \mapsto \text{Ad}_{\mathfrak{n}}\varphi(t)$  is unipotent, and hence the group  $G$  is algebraic unipotent.

b) The implication  $\Rightarrow$  is easy. Conversely, we assume that  $X'$  is the derivative of a one-parameter morphism  $\varphi'$  of  $G'$ . When  $X' = 0$ , the element  $X$  belongs to  $\mathfrak{n}$  and, since  $N$  is algebraic unipotent,  $X$  is the derivative of a one-parameter morphism  $\varphi$  of  $N$ . We assume now that  $X' \neq 0$  so that the group  $Q' := \varphi'(\mathbb{Q}_p)$  is algebraic unipotent and isomorphic to  $\mathbb{Q}_p$ . According to point a), the group  $H := \pi^{-1}(Q')$  is algebraic unipotent. Since the element  $X$  belongs to the Lie algebra  $\mathfrak{h}$  of  $H$ , it is the derivative of a one-parameter morphism  $\varphi'$  of  $H$ .

c)  $\Rightarrow$  We assume that  $G$  is weakly regular. Let  $\varphi'_1$  and  $\varphi'_2$  be one-parameter morphisms of  $G'$  with the same derivative  $X' \in \mathfrak{g}'$ . We want to prove that  $\varphi'_1 = \varphi'_2$ . If this derivative  $X'$  is zero, by Lemma 3.5, we can lift both  $\varphi'_1$  and  $\varphi'_2$  as one-parameter morphisms of  $G$  with zero derivative. Since  $G$  is weakly regular, both  $\varphi_1$  and  $\varphi_2$  are trivial and  $\varphi'_1 = \varphi'_2$ . We assume now that the derivative  $X'$  is non-zero. As above, for  $i = 1, 2$ , the groups  $Q'_i := \varphi'_i(\mathbb{Q}_p)$  and  $H_i := \pi^{-1}(Q'_i)$  are algebraic unipotent. Since  $G$  is weakly regular, and its algebraic

unipotent subgroups  $H_1$  and  $H_2$  have the same Lie algebra, one gets successively  $H_1 = H_2$ ,  $Q'_1 = Q'_2$ , and  $\varphi'_1 = \varphi'_2$ .

$\Leftarrow$  We assume that  $G/N$  is weakly regular. Let  $\varphi_1$  and  $\varphi_2$  be one-parameter morphisms of  $G$  with the same derivative  $X \in \mathfrak{g}$ . We want to prove that  $\varphi_1 = \varphi_2$ . Since  $G/N$  is weakly regular the one-parameter morphisms  $\pi \circ \varphi_1$  and  $\pi \circ \varphi_2$  are equal and their image  $Q'$  is a unipotent algebraic subgroup of  $G'$ . According to point *a*), the group  $H := \pi^{-1}(Q')$  is algebraic unipotent. Since  $\varphi_1$  and  $\varphi_2$  take their values in  $H$ , one has  $\varphi_1 = \varphi_2$ .  $\square$

### 3.3. Unipotent subgroups tangent to a nilpotent Lie algebra.

Proposition 3.7 below describes the nilpotent Lie subgroups of a weakly regular  $p$ -adic Lie group  $G$  which are spanned by one-parameter morphisms.

The following proposition is due to Ratner in [13, Thm. 2.1].

**Proposition 3.7.** *Let  $G$  be a weakly regular  $p$ -adic Lie group and  $\mathfrak{n} \subset \mathfrak{g}$  be a nilpotent Lie subalgebra. Then the set  $\mathfrak{n}_G := \mathfrak{n} \cap \mathfrak{g}_G$  is an ideal of  $\mathfrak{n}$  and there exists an algebraic unipotent subgroup  $N_G$  of  $G$  with Lie algebra  $\mathfrak{n}_G$ .*

This group  $N_G$  is unique. It is a closed subgroup of  $G$ . By construction, it is the largest algebraic unipotent subgroup whose Lie algebra is included in  $\mathfrak{n}$ .

*Proof.* We argue by induction on  $\dim \mathfrak{n}$ . We can assume  $\mathfrak{n}_G \neq 0$ .

**First case :**  $\mathfrak{n}$  is abelian. Let  $X_1, \dots, X_r$  be a maximal family of linearly independent elements of  $\mathfrak{n}_G$  and, for  $i \leq r$ , let  $\varphi_i$  be the one-parameter morphism with derivative  $X_i$ . Since  $G$  is weakly regular, the group spanned by the images  $\varphi_i(\mathbb{Q}_p)$  is commutative and the map

$$\psi : \mathbb{Q}_p^d \rightarrow G; (t_1, \dots, t_r) \mapsto \varphi_1(t_1) \dots \varphi_r(t_r)$$

is an injective morphism. Its image is a unipotent algebraic subgroup  $N_G$  of  $G$  whose Lie algebra is  $\mathfrak{n}_G$ .

**Second case :**  $\mathfrak{n}$  is not abelian. Let  $\mathfrak{z}$  be the center of  $\mathfrak{n}$  and  $\mathfrak{z}_2$  the ideal of  $\mathfrak{n}$  such that  $\mathfrak{z}_2/\mathfrak{z}$  is the center of  $\mathfrak{n}/\mathfrak{z}$ . If  $\mathfrak{n}_G$  is included in the centralizer  $\mathfrak{n}'$  of  $\mathfrak{z}_2$ , we can apply the induction hypothesis to  $\mathfrak{n}'$ . We assume now that  $\mathfrak{n}_G$  is not included in  $\mathfrak{n}'$ , i.e. there exists

$$X \in \mathfrak{n}_G \text{ and } Y \in \mathfrak{z}_2 \text{ such that } [X, Y] \neq 0.$$

This element  $Z := [X, Y]$  belongs to the center  $\mathfrak{z}$ .

We first check that  $Z$  belongs also to  $\mathfrak{g}_G$ . Indeed, let  $\mathfrak{m}$  be the 2-dimensional Lie subalgebra of  $\mathfrak{n}$  with basis  $X, Z$ . This Lie algebra is normalized by  $Y$ . For  $\varepsilon \in \mathbb{Q}_p$  small enough, there exists a group morphism  $\psi : \varepsilon\mathbb{Z}_p \rightarrow G$  whose derivative at 0 is  $Y$ , and one has  $\text{Ad}(\psi(\varepsilon))Y = e^{\varepsilon \text{ad}Y}X = X - \varepsilon Z$ . Since  $X$  belongs to  $\mathfrak{g}_G$ , the element  $X - \varepsilon Z$  also belongs to  $\mathfrak{g}_G$ . By the first case applied to the abelian Lie subalgebra  $\mathfrak{m}$ , the element  $Z$  belongs to  $\mathfrak{g}_G$ .

This means that there exists a one-parameter subgroup  $U$  of  $G$  whose Lie algebra is  $\mathfrak{u} = \mathbb{Q}_p Z$ . Let  $C$  be the centralizer of  $U$  in  $G$ . According to Lemma 3.6, the quotient group  $C/U$  is also weakly regular. We apply our induction hypothesis to this group  $C' := C/U$  and the nilpotent Lie algebra  $\mathfrak{n}/\mathfrak{u}$ . There exists a largest algebraic unipotent subgroup  $N'_{C'}$  in  $C'$  whose Lie algebra is included in  $\mathfrak{n}'$ . Hence, using again Lemma 3.6, there exists a largest algebraic unipotent subgroup  $N_C$  of  $C$  whose Lie algebra is included in  $\mathfrak{n}$ . Since  $G$  is weakly regular, any one-parameter subgroup of  $G$  tangent to  $\mathfrak{n}$  is included in  $C$  and  $N_C$  is also the largest algebraic unipotent subgroup of  $G$  whose Lie algebra is included in  $\mathfrak{n}$ .  $\square$

### 3.4. Largest normal algebraic unipotent subgroup.

We prove in this section that a weakly regular  $p$ -adic Lie group contains a largest normal algebraic unipotent subgroup.

Let  $G$  be a  $p$ -adic Lie group. We denote by  $\overline{G}_u$  the closure of the subgroup  $G_u$  of  $G$  generated by all the one-parameter subgroups of  $G$ . This group  $\overline{G}_u$  is normal in  $G$ . We denote by  $\mathfrak{g}_u$  the Lie algebra of  $\overline{G}_u$ . It is an ideal of  $\mathfrak{g}$ .

We recall that the radical  $\mathfrak{r}$  of  $\mathfrak{g}$  is the largest solvable ideal of  $\mathfrak{g}$  and that the nilradical  $\mathfrak{n}$  of  $\mathfrak{g}$  is the largest nilpotent ideal of  $\mathfrak{g}$ . The nilradical is the set of  $X$  in  $\mathfrak{r}$  such that  $\text{ad}X$  is nilpotent and one has  $[\mathfrak{g}, \mathfrak{r}] \subset \mathfrak{n}$  (see [5]).

When  $G$  is weakly regular, we denote by  $R_u$  the largest algebraic unipotent subgroup of  $G$  whose Lie algebra is included in  $\mathfrak{n}$ . It exists by Proposition 3.7.

The following proposition is mainly in [13, Lem. 2.2].

**Proposition 3.8.** *Let  $G$  be a weakly regular  $p$ -adic Lie group.*

- a) *The group  $R_u$  is the largest normal algebraic unipotent subgroup of  $G$ .*
- b) *Its Lie algebra  $\mathfrak{r}_u$  is equal to  $\mathfrak{r}_u = \mathfrak{n} \cap \mathfrak{g}_G = \mathfrak{r} \cap \mathfrak{g}_G$ .*
- c) *One has the inclusion  $[\mathfrak{g}_u, \mathfrak{r}] \subset \mathfrak{r}_u$ .*
- d) *Let  $G' = G/R_u$ . Let  $X$  be in  $\mathfrak{g}$  and  $X'$  be its image in  $\mathfrak{g}' = \mathfrak{g}/\mathfrak{r}_u$ .*

One has the equivalence

$$X \in \mathfrak{g}_G \iff X' \in \mathfrak{g}'_{G'}.$$

*Proof.* a) We have to prove that any normal algebraic unipotent subgroup  $U$  of  $G$  is included in  $R_u$ . Indeed the Lie algebra  $\mathfrak{u}$  of  $U$  is a nilpotent ideal of  $\mathfrak{g}$ , hence it is included in  $\mathfrak{n}$  and  $U$  is included in  $R_u$ .

b) We already know the equality  $\mathfrak{r}_u = \mathfrak{n} \cap \mathfrak{g}_G$  from Proposition 3.7. It remains to check the inclusion  $\mathfrak{r} \cap \mathfrak{g}_G \subset \mathfrak{n}$ . Indeed, let  $X$  be an element in  $\mathfrak{r} \cap \mathfrak{g}_G$ . Since  $X$  is the derivative of a one-parameter morphism, by Lemma 2.2, the endomorphism  $\text{ad}X$  is nilpotent. Since  $X$  is also in the radical  $\mathfrak{r}$ ,  $X$  has to be in the nilradical  $\mathfrak{n}$ .

c) We want to prove that the adjoint action of  $G_u$  on the quotient Lie algebra  $\mathfrak{r}/\mathfrak{r}_u$  is trivial. That is, we want to prove that, for all

$$X \in \mathfrak{g}_G \text{ and } Y \in \mathfrak{r}, \text{ one has } [X, Y] \in \mathfrak{r}_u.$$

By Lemma 2.2, the endomorphism  $\text{ad}X$  is nilpotent. Since  $Y$  is in  $\mathfrak{r}$ , the bracket  $[X, Y]$  belongs to  $\mathfrak{n}$  and the vector space  $\mathfrak{m} := \mathbb{Q}_p X \oplus \mathfrak{n}$  is a nilpotent Lie algebra normalized by  $Y$ . Hence, by Proposition 3.7, the set  $\mathfrak{m} \cap \mathfrak{g}_G$  is a nilpotent Lie algebra. This set is normalized by  $Y$  since it is invariant by  $e^{\varepsilon \text{ad}Y}$  for  $\varepsilon \in \mathbb{Q}_p$  small enough. In particular the element  $[X, Y]$  belongs to  $\mathfrak{g}_G$  and hence to  $\mathfrak{r}_u$ .

d) This is a special case of Proposition 3.7. □

#### 4. DERIVATIVES OF ONE-PARAMETER MORPHISMS

In this chapter, we describe the set  $\mathfrak{g}_G$  of derivatives of one-parameter morphisms of a weakly regular  $p$ -adic Lie group  $G$ .

##### 4.1. Construction of one-parameter subgroups.

We explain first a construction of one-parameter morphisms of  $G$  borrowed from [1] and [2].

**Lemma 4.1.** *Let  $G$  be a  $p$ -adic Lie group and  $g \in G$ . Then the vector space  $\mathfrak{g}_g^+ := \{v \in \mathfrak{g} \mid \lim_{n \rightarrow \infty} \text{Ad}g^{-n}v = 0\}$  is included in  $\mathfrak{g}_G$ .*

Note that  $\mathfrak{g}_g^+$  is a nilpotent Lie subalgebra of  $\mathfrak{g}$ .

The proof relies on the existence of compact open subgroups of  $G$  for which the exponential map satisfies a nice equivariant property. We need some classical definition (see [8]). A  $p$ -adic Lie group  $\Omega$  is said to be a *standard* group if there exists a  $\mathbb{Q}_p$ -Lie algebra  $\mathfrak{l}$  and a compact open sub- $\mathbb{Z}_p$ -algebra  $O$  of  $\mathfrak{l}$  such that the Baker-Campbell-Hausdorff series converges on  $O$  and  $\Omega$  is isomorphic to the  $p$ -adic Lie group  $O$  equipped with the group law defined by this formula.

In this case,  $\mathfrak{l}$  identifies canonically with the Lie algebra of  $\Omega$ , every element of  $\Omega$  admits a logarithm and the logarithm map induces an isomorphism  $\Omega \rightarrow O$ . If  $G$  is any  $p$ -adic Lie group, it admits a standard open subgroup (see [8, Theorem 8.29]). If  $\Omega$  is such a subgroup and if  $O$  is the associated compact open sub- $\mathbb{Z}_p$ -algebra of  $\mathfrak{g}$ , we denote by  $\exp_\Omega : O \rightarrow \Omega$  the inverse diffeomorphism of the logarithm map  $\Omega \rightarrow O$ .

Note that if  $\Omega$  and  $\Omega'$  are standard open subgroups of  $G$ , the maps  $\exp_\Omega$  and  $\exp_{\Omega'}$  coincide in some neighborhood of 0 in  $\mathfrak{g}$ .

**Lemma 4.2.** *Let  $G$  be a  $p$ -adic Lie group,  $\Omega \subset G$  a standard open subgroup and  $\exp_\Omega : O \rightarrow \Omega$  the corresponding exponential map. For every compact subset  $K \subset G$ , there exists an open subset  $O_K \subset \mathfrak{g}$  which is contained in  $O$  and in all the translates  $\text{Ad}g^{-1}(O)$ ,  $g \in K$ , and such that one has the equivariance property*

$$\exp_\Omega(\text{Ad}g(v)) = g \exp_\Omega(v) g^{-1} \quad \text{for any } v \in O_K, g \in K.$$

*Proof.* We may assume that  $K$  contains  $e$ . The intersection  $\Omega_K := \bigcap_{g \in K} g^{-1}\Omega g$  is an open neighborhood of  $e$  in  $G$ . We just choose  $O_K$  to be the open set  $O_K := \log(\Omega_K)$ .  $\square$

*Proof of Lemma 4.1.* This is [1, Lem. 5.4]. For the sake of completeness, we recall the proof. Fix  $g \in G$ . Let  $\Omega$  be a standard open subgroup of  $G$  with exponential map  $\exp_\Omega : O \rightarrow \Omega$ . By Lemma 4.2, there exists an open additive subgroup  $U \subset O \cap \mathfrak{g}_g^+$  such that  $\text{Ad}g^{-1}U \subset O$  and that

$$\exp_\Omega(u) = g \exp_\Omega(\text{Ad}g^{-1}u)g^{-1} \quad \text{for any } u \text{ in } U.$$

After eventually replacing  $U$  by  $\bigcap_{k \geq 0} \text{Ad}g^k U$ , we can assume  $\text{Ad}g^{-1}U \subset U$ . Now, for  $k \geq 0$ , let  $U_k := \text{Ad}g^k U$  and define a continuous map  $\psi_k : U_k \rightarrow G$  by setting

$$\psi_k(u) = g^k \exp_\Omega(\text{Ad}g^{-k}u)g^{-k} \quad \text{for any } u \text{ in } U_k.$$

We claim that, for any  $k$ , one has  $\psi_k = \psi_{k-1}$  on  $U_{k-1} = \text{Ad}g^{-1}U_k$ . Indeed, let  $u$  be in  $U_k$ . As  $u_k := \text{Ad}g^{-k}u$  belongs to  $U$ , we have

$$\psi_{k+1}(u) = g^k (g \exp_\Omega(\text{Ad}g^{-1}u_k)g^{-1})g^{-k} = g^k \exp_\Omega(u_k)g^{-k} = \psi_k(u).$$

Therefore, as  $\mathfrak{g}_g^+ = \bigcup_{k \geq 0} U_k$ , one gets a map  $\psi : \mathfrak{g}_g^+ \rightarrow G'$  whose restriction to any  $U_k$ ,  $k \geq 0$ , is  $\psi_k$ . For every  $v$  in  $\mathfrak{g}_g^+$ , the map  $t \mapsto \psi(tv)$  is a one-parameter morphism of  $G$  whose derivative is equal to  $v$ .  $\square$

#### 4.2. The group $G_{nc}$ and its Lie algebra $\mathfrak{g}_{nc}$ .

We introduce in this section a normal subgroup  $G_{nc}$  of  $G$  which contains  $G_u$ .

Let  $G$  be a  $p$ -adic Lie group and  $\mathfrak{g}_{nc}$  be the smallest ideal of the Lie algebra  $\mathfrak{g}$  such that the Lie algebra  $\mathfrak{s}_c := \mathfrak{g}/\mathfrak{g}_{nc}$  is semisimple and such that the adjoint group  $\mathrm{Ad}_{\mathfrak{s}_c}(G)$  is bounded in the group  $\mathrm{Aut}(\mathfrak{s}_c)$  of automorphisms of  $\mathfrak{s}_c$ . Let  $G_{nc}$  be the kernel of the adjoint action in  $\mathfrak{s}_c$ , i.e.

$$G_{nc} := \{g \in G \mid \text{for all } X \text{ in } \mathfrak{g}, \mathrm{Ad}g(X) - X \in \mathfrak{g}_{nc}\}$$

By construction  $G_{nc}$  is a closed normal subgroup of  $G$  with Lie algebra  $\mathfrak{g}_{nc}$ .

**Lemma 4.3.** *Let  $G$  be a  $p$ -adic Lie group. Any one-parameter morphism of  $G$  takes its values in  $G_{nc}$*

In other words, the group  $G_u$  is included in  $G_{nc}$ .

*Proof.* Let  $\varphi$  be a one-parameter morphism of  $G$ . Then  $\mathrm{Ad}_{\mathfrak{s}_c} \circ \varphi$  is a one-parameter morphism of  $\mathrm{Aut}(\mathfrak{s}_c)$  whose image is relatively compact. By Lemma 2.2, this one-parameter morphism is trivial and  $\varphi$  takes its values in  $G_{nc}$ .  $\square$

#### 4.3. Derivatives and Levi subalgebras.

We can now describe precisely which elements of  $\mathfrak{g}$  are tangent to one-parameter subgroups of  $G$ .

We recall that an element  $X$  of a semisimple Lie algebra  $\mathfrak{s}$  is said to be nilpotent if the endomorphism  $\mathrm{ad}_{\mathfrak{s}}X$  is nilpotent. In this case, for any finite dimensional representation  $\rho$  of  $\mathfrak{s}$ , the endomorphism  $\rho(X)$  is also nilpotent.

We recall that a Levi subalgebra  $\mathfrak{s}$  of a Lie algebra  $\mathfrak{g}$  is a maximal semisimple Lie subalgebra, and that one has the Levi decomposition  $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{r}$ .

The following proposition is proven in [13, Th. 2.2] under the additional assumption that  $G$  is Ad-regular.

**Proposition 4.4.** *Let  $G$  be a weakly regular  $p$ -adic Lie group,  $\mathfrak{r}$  be the radical of  $\mathfrak{g}$ ,  $\mathfrak{s}$  a Levi subalgebra of  $\mathfrak{g}$  and  $\mathfrak{s}_u := \mathfrak{s} \cap \mathfrak{g}_{nc}$ . One has the equality :  $\mathfrak{g}_G = \{X \in \mathfrak{s}_u \oplus \mathfrak{r}_u \mid \mathrm{ad}X \text{ is nilpotent}\}$ .*

It will follow from Lemma 4.7 that the Lie algebra  $\mathfrak{s}_u \oplus \mathfrak{r}_u$  does not depend on the choice of  $\mathfrak{s}$ .

The key ingredient in the proof of Proposition 4.4 will be Lemma 4.1. We will begin by three preliminary lemmas. The first two lemmas are classical.

**Lemma 4.5.** *Let  $V = \mathbb{Q}_p^d$  and  $G$  be a subgroup of  $\mathrm{GL}(V)$  such that  $V$  is an unbounded and irreducible representation of  $G$ . Then  $G$  contains an element  $g$  with at least one eigenvalue of modulus not one.*

*Proof.* Let  $A$  be the associative subalgebra of  $\mathrm{End}(V)$  spanned by  $G$ . Since  $V$  is irreducible, the associative algebra  $A$  is semisimple and the bilinear form  $(a, b) \mapsto \mathrm{tr}(ab)$  is non-degenerate on  $A$  (see [9, Ch. 17]). If all the eigenvalues of all the elements of  $G$  have modulus 1, this bilinear form is bounded on  $G \times G$ . Since  $A$  admits a basis included in  $G$ , for any  $a$  in  $A$ , the linear forms  $b \mapsto \mathrm{tr}(ab)$  on  $A$  are bounded on the subset  $G$ . Hence  $G$  is a bounded subset of  $A$ .  $\square$

**Lemma 4.6.** *Let  $\mathfrak{s}_0$  be a simple Lie algebra over  $\mathbb{Q}_p$  and  $S_0 := \mathrm{Aut}(\mathfrak{s}_0)$ . All open unbounded subgroups  $J$  of  $S_0$  have finite index in  $S_0$ .*

*Proof.* Since  $J$  is unbounded, by Lemma 4.5, it contains an element  $g_0$  with at least one eigenvalue of modulus not one. Since  $J$  is open, the unipotent Lie subgroups

$$U^+ = \{g \in S_0 \mid \lim_{n \rightarrow \infty} g_0^{-n} g g_0^n = e\} \text{ and } U^- = \{g \in S_0 \mid \lim_{n \rightarrow \infty} g_0^n g g_0^{-n} = e\}$$

are included in  $J$ . By [4, 6.2.v and 6.13],  $J$  has finite index in  $S_0$ .  $\square$

The third lemma contains the key ingredient.

**Lemma 4.7.** *Let  $G$  be a weakly regular  $p$ -adic Lie group,  $\mathfrak{r}$  the radical of  $\mathfrak{g}$ ,  $\mathfrak{s}$  a Levi subalgebra of  $\mathfrak{g}$  and  $\mathfrak{s}_u := \mathfrak{s} \cap \mathfrak{g}_{nc}$ .*

- a) *One has the inclusion  $[\mathfrak{s}_u, \mathfrak{r}] \subset \mathfrak{r}_u$ .*
- b) *Every nilpotent element  $X$  in  $\mathfrak{s}_u$  belongs to  $\mathfrak{g}_G$ .*

Note that Lemma 4.7.a does not follow from Proposition 3.8.c, since with the definitions of  $\mathfrak{g}_u$  and  $\mathfrak{s}_u$  that we have given, we do not know yet that  $\mathfrak{s}_u = \mathfrak{s} \cap \mathfrak{g}_u$ .

*Proof.* a) By Proposition 3.8, we can assume  $\mathfrak{r}_u = 0$ . We want to prove that  $[\mathfrak{s}_u, \mathfrak{r}] = 0$ . Let  $\mathfrak{s}_i$ ,  $i = 1, \dots, \ell$  be the simple ideals of  $\mathfrak{s}_u$ . Replacing  $G$  by a finite index subgroup, we can also assume that the ideals  $\mathfrak{s}_i \oplus \mathfrak{r}$  are  $G$ -invariant. Similarly, let  $\mathfrak{r}_j$ ,  $j = 1, \dots, m$  be the simple subquotients of a Jordan-Hölder sequence of the  $G$ -module  $\mathfrak{r}$ .

On the one hand, by assumption, for all  $i \leq \ell$ , the group  $\mathrm{Ad}_{\mathfrak{s}_i \oplus \mathfrak{r}/\mathfrak{r}}(G)$  is unbounded. Hence, by Lemma 4.5, there exists an element  $g_i$  in  $G$  and  $X_i$  in  $\mathfrak{g}_{g_i}^+ \cap (\mathfrak{s}_i \oplus \mathfrak{r})$  whose image in  $\mathfrak{g}/\mathfrak{r}$  is non zero. By Lemma 4.1, there exists a one-parameter morphism  $\varphi_i$  of  $G$  whose derivative is  $X_i$ .

On the other hand, since  $\mathfrak{r}_u = 0$ , by the same Lemma 4.1, for every  $g$  in  $G$ , all the eigenvalues of  $\mathrm{Ad}_{\mathfrak{r}}(g)$  have modulus 1. By Lemma 4.5, for all  $j \leq m$ , the image  $\mathrm{Ad}_{\mathfrak{r}_j}(G)$  of  $G$  in any simple subquotient



$\mathfrak{r}_j$  is bounded. In particular, the one-parameter morphisms  $\text{Ad}_{\mathfrak{r}_j} \circ \varphi_i$  are bounded. Hence, by Lemma 2.2, one has  $\text{ad}_{\mathfrak{r}_j}(X_i) = 0$ . Since  $\mathfrak{r}_j$  is a simple  $\mathfrak{g}$ -module, the Lie algebra  $\text{ad}_{\mathfrak{r}_j}(\mathfrak{g})$  is reductive and contains  $\text{ad}_{\mathfrak{r}_j}(\mathfrak{s}_i)$  as an ideal. Since  $\mathfrak{s}_i$  is a simple Lie algebra, this implies  $\text{ad}_{\mathfrak{r}_j}(\mathfrak{s}_i) = 0$ . Since the action of  $\mathfrak{s}_u$  on  $\mathfrak{r}$  is semisimple, this implies the equality  $[\mathfrak{s}_u, \mathfrak{r}] = 0$ .

b) As in a), we can assume that  $G$  preserves the ideals  $\mathfrak{s}_i \oplus \mathfrak{r}$  and that  $\mathfrak{r}_u = 0$ . According to this point a), the Lie algebras  $\mathfrak{s}_u$  and  $\mathfrak{r}$  commute and hence  $\mathfrak{s}_u$  is the unique Levi subalgebra of  $\mathfrak{g}_{nc}$  (see [5, §6]). In particular,

$$(4.1) \quad \text{for all } g \text{ in } G, \text{ one has } \text{Ad}g(\mathfrak{s}_u) = \mathfrak{s}_u.$$

Let  $X$  be a nilpotent element of  $\mathfrak{s}_u$ . We want to prove that  $X$  is the derivative of a one parameter morphism of  $G$ . By Jacobson-Morozov theorem, there exists an automorphism  $\psi$  of  $\mathfrak{s}_u$  such that  $\psi(X) = p^{-1}X$ . Since, for every simple ideal  $\mathfrak{s}_i$  of  $\mathfrak{s}$ , the subgroup  $\text{Ad}_{\mathfrak{s}_i \oplus \mathfrak{r}/\mathfrak{r}}(G_{nc}) \subset \text{Aut}(\mathfrak{s}_i)$  is unbounded and open, this subgroup has finite index. Hence, remembering also (4.1), there exists  $k \geq 1$  and  $g$  in  $G_{nc}$  such that  $\text{Ad}g(X) = p^{-k}X$ . Then, by Lemma 4.1,  $X$  is the derivative of a one-parameter morphism of  $G$ .  $\square$

*Proof of Proposition 4.4.* We just have to gather what we have proved so far. By Proposition 3.8, we can assume  $\mathfrak{r}_u = 0$ . Let  $X \in \mathfrak{g}$ . We write  $X = X_{\mathfrak{s}} + X_{\mathfrak{r}}$  with  $X_{\mathfrak{s}} \in \mathfrak{s}$  and  $X_{\mathfrak{r}} \in \mathfrak{r}$ .

Proof of the inclusion  $\subset$ . Assume that  $X$  is the derivative of a one-parameter morphism  $\varphi$  of  $G$ . By Lemma 4.3,  $X$  belongs to  $\mathfrak{g}_{nc}$  and hence  $X_{\mathfrak{s}}$  belongs to  $\mathfrak{s}_u$ . By Lemma 2.2, the endomorphism  $\text{ad}_{\mathfrak{g}}X$  is nilpotent, and hence  $X_{\mathfrak{s}}$  is a nilpotent element of the semisimple Lie algebra  $\mathfrak{s}$ . According to Lemma 4.7,  $X_{\mathfrak{s}}$  and  $X_{\mathfrak{r}}$  commute and  $X_{\mathfrak{s}}$  is the derivative of a one-parameter morphism  $\varphi_{\mathfrak{s}}$  of  $G$ . Then  $X_{\mathfrak{r}}$  is also the derivative of a one-parameter morphism  $\varphi_{\mathfrak{r}}$  of  $G$ , the one given by  $t \mapsto \varphi_{\mathfrak{s}}(t)^{-1}\varphi(t)$ . Hence  $X_{\mathfrak{r}}$  belongs to  $\mathfrak{r}_u$ .

Proof of the inclusion  $\supset$ . Assume that  $X_{\mathfrak{s}}$  belongs to  $\mathfrak{s}_u$ ,  $X_{\mathfrak{r}}$  belongs to  $\mathfrak{r}_u$  and  $\text{ad}X$  is nilpotent. By Lemma 4.7,  $X_{\mathfrak{s}}$  and  $X_{\mathfrak{r}}$  commute and  $X_{\mathfrak{s}}$  is the derivative of a one-parameter morphism  $\varphi_{\mathfrak{s}}$  of  $G$ . By assumption  $X_{\mathfrak{r}}$  is the derivative of a one-parameter morphism  $\varphi_{\mathfrak{r}}$  of  $G$ . Hence  $X$  is also the derivative of a one-parameter morphism  $\varphi$  of  $G$ , the one given by  $t \mapsto \varphi_{\mathfrak{s}}(t)\varphi_{\mathfrak{r}}(t)$ . Hence  $X$  belongs to  $\mathfrak{g}_G$ .  $\square$

5. GROUPS SPANNED BY UNIPOTENT SUBGROUPS

In this chapter, we prove the two main results Proposition 5.5 and Theorem 5.12 that we announced in the introduction.

5.1. Semisimple regular  $p$ -adic Lie groups.

We recall first a nice result due to Prasad-Raghunathan which is an output from the theory of congruence subgroups

Let  $\mathfrak{s}$  be a semisimple Lie algebra over  $\mathbb{Q}_p$ ,  $\text{Aut}(\mathfrak{s})$  the group of automorphisms of  $\mathfrak{s}$  and  $S_+ := \text{Aut}(\mathfrak{s})_u \subset \text{Aut}(\mathfrak{s})$  the subgroup spanned by the one-parameter subgroups of  $\text{Aut}(\mathfrak{s})$ . We will say that  $\mathfrak{s}$  is *totally isotropic* if  $\mathfrak{s}$  is spanned by nilpotent elements. In this case  $S_+$  is an open finite index subgroup of  $\text{Aut}(\mathfrak{s})$ , see [4, 6.14]. Since  $\mathfrak{s} = [\mathfrak{s}, \mathfrak{s}]$ , the group  $S_+$  is perfect i.e.  $S_+ = [S_+, S_+]$ . In particular this group admits a universal central topological extension  $\tilde{S}_+$ , i.e. a group which is universal among the central topological extension of  $S_+$ . In [12] Prasad-Raghunathan were able to describe this group  $\tilde{S}_+$  (special cases were obtained before by Moore, Matsumoto and Deodhar). We will only need here the fact that this group is a finite extension of  $S_+$ .

**Proposition 5.1. (Prasad–Raghunathan)** *Let  $\mathfrak{s}$  be a totally isotropic semisimple  $p$ -adic Lie algebra. Then the group  $S_+$  admits a universal topological central extension*

$$1 \longrightarrow Z_0 \longrightarrow \tilde{S}_+ \xrightarrow{\pi_0} S_+ \longrightarrow 1$$

and its center  $Z_0$  is a finite group.

The word universal means that for all topological central extension

$$1 \longrightarrow Z_E \longrightarrow E \xrightarrow{\pi} S_+ \longrightarrow 1$$

where  $E$  is a locally compact group and  $Z_E$  is a closed central subgroup, there exists a unique continuous morphism  $\psi : \tilde{S}_+ \rightarrow E$  such that  $\pi_0 = \pi \circ \psi$ .

*Proof.* See [12, Theorem 10.4]. □

*Remark 5.2.* This result does not hold for real Lie groups: indeed, the center  $Z_0$  of the universal cover of  $\text{SL}(2, \mathbb{R})$  is isomorphic to  $\mathbb{Z}$ .

**Corollary 5.3.** *Let  $\mathfrak{s}$  be a totally isotropic semisimple  $p$ -adic Lie algebra. For every topological central extension*

$$1 \longrightarrow Z_E \longrightarrow E \xrightarrow{\pi} S_+ \longrightarrow 1$$

with  $E = \overline{[E, E]}$ , the group  $Z_E$  is finite.

*Proof of Corollary 5.3.* Let  $\psi : \tilde{S}_+ \rightarrow E$  be the morphism given by the universal property. Since, by Proposition 5.1, the group  $Z_0$  is finite, the projection  $\pi_0 = \pi \circ \psi$  is a proper map, hence  $\psi$  is also a proper map and the image  $\psi(\tilde{S}_+)$  is a closed subgroup of  $E$ . Since  $E = \psi(\tilde{S}_+)Z_E$ , one has the inclusion  $[E, E] \subset \psi(\tilde{S}_+)$ , and the assumption  $E = \overline{[E, E]}$  implies that the morphism  $\psi$  is onto. Hence the group  $Z_E = \psi(Z_0)$  is finite.  $\square$

*Remark 5.4.* For real Lie groups, the center  $Z_E$  might even be non discrete. Such an example is given by the quotient  $E$  of the product  $\mathbb{R} \times \widetilde{\mathrm{SL}(2, \mathbb{R})}$  by a discrete subgroup of  $\mathbb{R} \times Z_0$  whose projection on  $\mathbb{R}$  is dense.

## 5.2. The Levi decomposition of $G_u$ .

We prove in this section that in a weakly regular  $p$ -adic Lie group  $G$ , the subgroup  $G_u$  is closed and admits a Levi decomposition.

Let  $G$  be a weakly regular  $p$ -adic Lie group and  $\mathfrak{r}$  be the solvable radical of  $\mathfrak{g}$ . We recall that  $\mathfrak{g}_{nc}$  is the smallest ideal of  $\mathfrak{g}$  containing  $\mathfrak{r}$  such that the group  $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{g}_{nc}}(G)$  is bounded. Let  $\mathfrak{s}$  be a Levi subalgebra of  $\mathfrak{g}$  and  $\mathfrak{s}_u := \mathfrak{s} \cap \mathfrak{g}_{nc}$ .

We recall that  $G_u$  is the subgroup of  $G$  spanned by all the one-parameter subgroups of  $G$ , that  $R_u$  is the subgroup of  $G$  spanned by all the one-parameter subgroups of  $G$  tangent to  $\mathfrak{r}$ , and we define  $S_u$  as the subgroup of  $G$  spanned by all the one-parameter subgroups of  $G$  tangent to  $\mathfrak{s}$ . Note that we don't know yet, but we will see it in the next proposition, that  $S_u$  is indeed a closed subgroup with Lie algebra equal to  $\mathfrak{s}_u$ .

**Proposition 5.5.** *Let  $G$  be a weakly regular  $p$ -adic Lie group.*

- a) *The group  $R_u$  is closed. It is the largest normal algebraic unipotent subgroup of  $G$ .*
- b) *The group  $S_u$  is closed. Its Lie algebra is  $\mathfrak{s}_u$ , and the morphism  $\mathrm{Ad}_{\mathfrak{s}_u} : S_u \rightarrow (\mathrm{Aut} \mathfrak{s}_u)_u$  is onto and has finite kernel.*
- c) *The group  $G_u$  is closed. One has  $G_u = S_u R_u$  and  $S_u \cap R_u = \{e\}$ .*

*Remark 5.6.* In a real Lie group, the group tangent to a Levi subalgebra is not necessary closed, as for example, if  $G = (S \times \mathbb{T})/Z$  where  $S$  is the universal cover of  $\mathrm{SL}(2, \mathbb{R})$ ,  $\mathbb{T} = \mathbb{R}/\mathbb{Z}$  and  $Z$  is the cyclic subgroup spanned by  $(z_0, \alpha_0)$  with  $z_0$  a generator of the center of  $S$  and  $\alpha_0$  an irrational element of  $\mathbb{T}$ .

*Proof of Proposition 5.5.* a) This follows from Proposition 3.8.

b) Let  $E := \overline{S_u}$  be the closure of  $S_u$  and  $S_+ := (\text{Aut } \mathfrak{s}_u)_u$ . Note that this group  $E$  normalizes  $\mathfrak{s}_u$ . We want to apply Corollary 5.3 to the morphism

$$E \xrightarrow{\pi} S_+$$

where  $\pi$  is the adjoint action  $\pi := \text{Ad}_{\mathfrak{s}_u}$ .

We first check that the assumptions of Corollary 5.3 are satisfied. Since  $E$  is weakly regular the kernel  $Z_E$  of this morphism  $\pi$  commutes with all the one-parameter subgroups tangent to  $\mathfrak{s}_u$ . Hence  $Z_E$  is equal to the center of  $E$ . Since  $S_+$  is spanned by one-parameter subgroups, and since by Proposition 4.4 any nilpotent element  $X$  of  $\mathfrak{s}_u$  is tangent to a one parameter subgroup  $\varphi$  of  $E$ , this morphism  $\pi$  is surjective. Now, by Jacobson Morozov Theorem, for any nilpotent element  $X$  of  $\mathfrak{s}_u$ , there exists an element  $H$  in  $\mathfrak{s}_u$  such that  $[H, X] = X$ . Let  $\varphi : \mathbb{Q}_p \rightarrow G$  be the one-parameter subgroup tangent to  $X$ , which exists by Proposition 4.4. Since  $G$  is weakly regular, one has, for  $t$  in  $\mathbb{Q}_p$  and  $g_\varepsilon := \exp(\varepsilon H)$  with  $\varepsilon$  small,

$$g_\varepsilon \varphi(t) g_\varepsilon^{-1} \varphi(t)^{-1} = \varphi(e^\varepsilon t) \varphi(-t) = \varphi((e^\varepsilon - 1)t).$$

This proves that  $\varphi(\mathbb{Q}_p)$  is included in the derived subgroup  $[E, E]$ . In particular one has  $E = \overline{[E, E]}$ .

According to Corollary 5.3, the kernel  $Z_E$  is finite. In particular, one has  $\dim E = \dim \mathfrak{s}_u$ . Since  $\mathfrak{s}_u$  is totally isotropic, one can find a basis of  $\mathfrak{s}_u$  all of whose elements are nilpotent. By Lemma 4.7, all these elements are in  $\mathfrak{g}_G$ . Hence, by the implicit function theorem, the group  $S_u$  is open in  $E$ . Therefore,  $S_u$  is also closed and  $S_u = E$ .

c) Since the adjoint action of  $R_u$  on  $\mathfrak{g}_{nc}/\mathfrak{r}$  is trivial, the intersection  $S_u \cap R_u$  is included in the kernel  $Z_E$  of the adjoint map  $\text{Ad}_{\mathfrak{s}_u}$ . Since, by b), this kernel is finite, and since the algebraic unipotent group  $R_u$  does not contain finite subgroups, one gets  $S_u \cap R_u = \{e\}$ .

It remains to check that  $S_u R_u$  is closed and that  $G_u = S_u R_u$ . Thanks to Propositions 3.8 and 4.4, we can assume that  $R_u = \{e\}$ . In this case, we know from point b) that  $S_u$  is closed and from Proposition 4.4 that  $G_u = S_u$ .  $\square$

Here are a few corollaries. The first corollary is an improvement of Lemma 3.6.

**Corollary 5.7.** *Let  $G$  be a  $p$ -adic Lie group,  $H$  a normal weakly regular closed subgroup of  $G$  such that  $H = H_u$ , and  $G' := G/H$ .*

a) *One has the equality  $G'_u = G_u/H$ .*

b) *Let  $X$  in  $\mathfrak{g}$  and  $X'$  its image in  $\mathfrak{g}' = \mathfrak{g}/\mathfrak{h}$ . One has the equivalence*

$$X \in \mathfrak{g}_G \iff X' \in \mathfrak{g}'_{G'}.$$

c) *One has the equivalence*

$$G \text{ is weakly regular} \iff G' \text{ is weakly regular.}$$

*Remark 5.8.* The assumption that  $H = H_u$  is important. For instance the group  $G = \mathbb{Q}_p$  and its normal subgroup  $H = \mathbb{Z}_p$  are weakly regular while the quotient  $G/H$  is not weakly regular.

*Proof.* We prove these three statements simultaneously. Since  $H = H_u$ , according to Proposition 5.5, the group  $H$  admits a Levi decomposition  $H = SR$  where  $R$  is a normal algebraic unipotent Lie subgroup and where  $S$  is a Lie subgroup with finite center  $Z$  whose Lie algebra is semisimple, totally isotropic, and such that the adjoint map  $\text{Ad}_{\mathfrak{s}} : S \rightarrow \text{Aut}(\mathfrak{s})_u$  is surjective. Note that  $R$  is also a normal subgroup of  $G$ . Using Lemma 3.6, we can assume that  $R = \{e\}$ .

Let  $C$  be the centralizer of  $H = S$  in  $G$ . Since  $H$  is normal in  $G$ , since  $H = H_u$  and  $H$  is weakly regular,  $C$  is also the kernel of the adjoint action of  $G$  on  $\mathfrak{h} = \mathfrak{s}_u$ . Therefore, by Proposition 5.5, the image of the group morphism

$$H \times C \rightarrow G; (h, c) \mapsto hc$$

has finite index in  $G$ . Its kernel is isomorphic to  $H \cap C = Z$  and hence is finite. When this morphism  $H \times C \rightarrow G$  is an isomorphism, our three statements are clear. The general case reduces to this one thanks to Lemma 5.9 below.  $\square$

**Lemma 5.9.** *Let  $G$  be a locally compact topological group,  $Z$  be a finite central subgroup of  $G$  and  $\varphi : \mathbb{Q}_p \rightarrow G/Z$  be a continuous morphism. Then  $\varphi$  may be lifted as a continuous morphism  $\tilde{\varphi} : \mathbb{Q}_p \rightarrow G$ .*

*Proof.* Let  $H$  be the inverse image of  $\varphi(\mathbb{Z}_p)$  in  $G$ . Then  $H$  is totally discontinuous. In particular it contains an open compact subgroup  $U$  such that  $U \cap Z = \{e\}$ , so that  $U$  maps injectively in  $G/Z$ . Let  $\ell$  be an integer such that  $\varphi(p^\ell \mathbb{Z}_p) \subset UZ/Z$ . After rescaling, we can assume that  $\ell = 0$ . We let  $g_0$  be the unique element of  $U$  such that  $\varphi(1) = g_0 Z$ . Since  $U$  map injectively in  $G/Z$ , we have  $g_0^{p^k} \xrightarrow[k \rightarrow \infty]{} e$ .

Let  $X$  be the group of elements of  $p$ -torsion in  $\tilde{Z}$  and  $Y$  be the group of elements whose torsion is prime to  $p$ . For any  $k \geq 0$  pick some  $g_k$  in  $G$  such that  $\varphi(p^{\ell-k}) = g_k Z$  and let  $x_k$  and  $y_k$  be the elements of  $X$  and  $Y$  such that  $g_k^{p^k} = g_0 x_k y_k$ . We let  $z_k$  be the unique element of  $Y_k$  such that  $z_k^{p^k} = y_k$ . Replacing  $g_k$  by  $g_k z_k^{-1}$ , we can assume that  $z_k = e$ . Since  $x_k$  only takes finitely many values, we can find a  $x$  in  $X$  and an increasing sequence  $(k_n)$  such that, for any  $n$ ,  $x_{k_n} = x$ . Now, since  $x$  is a central  $p$ -torsion element, one has  $(g_0 x)^{p^k} \xrightarrow[k \rightarrow \infty]{} e$ . Since, for any  $n$ ,

$g_{k_n}^{p^n} = g_0x$ , there exists a unique morphism  $\tilde{\varphi} : \mathbb{Q}_p \rightarrow G$  such that, for any  $n$ ,  $\tilde{\varphi}(p^{-k_n}) = g_{k_n}$  and  $\tilde{\varphi}$  clearly lifts  $\varphi$ .  $\square$

The second corollary is an improvement of Proposition 4.4.

**Corollary 5.10.** *Let  $G$  be a weakly regular  $p$ -adic Lie group. One has the equality :  $\mathfrak{g}_G = \{X \in \mathfrak{g}_u \mid \text{ad}X \text{ is nilpotent}\}$ , where  $\mathfrak{g}_u$  is the Lie algebra of  $G_u$ .*

*Proof.* This follows from Proposition 4.4 since, by Proposition 5.5, one has the equality  $\mathfrak{g}_u = \mathfrak{s}_u \oplus \mathfrak{r}_u$ .  $\square$

The last corollary tells us that a weakly regular  $p$ -adic Lie group  $G$  with  $G = G_u$  is “almost” an algebraic Lie group.

**Corollary 5.11.** *Let  $G$  be a weakly regular  $p$ -adic Lie group such that  $G = G_u$ . Then there exists a Lie group morphism  $\psi : G \rightarrow H$  with finite kernel and cokernel where  $H$  is the group of  $\mathbb{Q}_p$ -points of a linear algebraic group defined over  $\mathbb{Q}_p$ .*

*Proof.* According to Proposition 4.4,  $G = G_u$  is a semidirect product  $G_u = S_u \ltimes R_u$ . We choose  $H$  to be the semi direct product  $H := S' \ltimes R_u$  where  $S'$  is the Zariski closure of the group  $\text{Ad}(S_u)$  in  $\text{Aut}(\mathfrak{g}_u)$ . Note that, since  $G$  is weakly regular, any automorphism of  $\mathfrak{g}_u$  induces an automorphism of  $R_u$ . We define the morphism  $\psi : G_u \rightarrow H$  by  $\psi(g) = (\text{Ad}(s), r)$  for  $g = sr$  with  $s \in S_u, r \in R_u$ . Proposition 4.4 tells us also that this morphism  $\psi$  has finite kernel and cokernel.  $\square$

### 5.3. Regular semiconnected component.

We are now ready to prove the following theorem which was the main motivation of our paper.

**Theorem 5.12.** *Let  $G$  be a weakly regular  $p$ -adic Lie group. Then, there exists an open regular subgroup  $G_\Omega$  of  $G$  which contains all the one-parameter subgroups of  $G$ .*

*Remark 5.13.* Let  $\Omega$  be a *standard* open subgroup of  $G$ . We define the  $\Omega$ -*semiconnected component* of  $G$  as its open subgroup  $G_\Omega := \Omega G_u$  (see [13]). In this language, Theorem 5.12 states that, the  $\Omega$ -semiconnected component of a weakly regular  $p$ -adic Lie group is regular, if the standard subgroup  $\Omega$  is small enough.

*Proof of Theorem 5.12.* We will need some notations. Let  $\mathfrak{s}$  be a Levi subalgebra of  $\mathfrak{g}$ ,  $\mathfrak{s}_u := \mathfrak{s} \cap \mathfrak{g}_{nc}$ ,  $\mathfrak{s}'$  the centralizer of  $\mathfrak{s}_u$  in  $\mathfrak{s}$ ,  $\mathfrak{r}$  the radical of  $\mathfrak{g}$ , and  $\mathfrak{r}'$  the centralizer of  $\mathfrak{s}_u$  in  $\mathfrak{r}$ .

We can choose a standard subgroup  $\Omega'_S$  of  $G$  with Lie algebra  $\mathfrak{s}'$ , and a standard subgroup  $\Omega'_R$  of  $G$  with Lie algebra  $\mathfrak{r}'$  such that  $\Omega'_S$

normalizes  $\Omega'_R$  and the semidirect product  $\Omega' := \Omega'_S \Omega'_R$  is a standard subgroup of  $G$  with Lie algebra  $\mathfrak{s}' \oplus \mathfrak{t}'$ . Since  $G$  is weakly regular, by shrinking  $\Omega'$ , we can assume that it commutes with  $S_u$  and normalizes  $R_u$ .

We claim that, if  $\Omega'$  is small enough, the group

$$G_\Omega := \Omega' G_u$$

is an open regular subgroup of  $G$ .

**First step** : Openness. One has the equalities  $\mathfrak{s} = \mathfrak{s}' \oplus \mathfrak{s}_u$  and, according to Proposition 3.8,  $\mathfrak{t} = \mathfrak{t}' + \mathfrak{t}_u$ . Hence  $G_\Omega$  is open in  $G$ .

**Second step** : Ad-regularity. Let  $J \subset G_\Omega$  be the kernel of  $\text{Ad}_{\mathfrak{g}}$ . We want to prove that  $J$  is the center of  $G_\Omega$ . Since  $J$  acts trivially on the quotient  $\mathfrak{g}/\mathfrak{t} \simeq \mathfrak{s}' \oplus \mathfrak{s}_u$ , by Proposition 5.5, one has the inclusion  $J \subset J' := Z_E \Omega'_R R_u$  where  $Z_E$  is a finite subgroup of  $S_u$ . One has  $J_u = J \cap R_u$  and this group is algebraic unipotent. Hence, by Lemma 3.6, the quotient  $R_u/J_u$  is also algebraic unipotent. By Lemma 3.4, the group  $R_u/J_u$  is closed in the group  $J'/J$ . This means that  $R_u J$  is closed in  $J'$ , hence that  $J/J_u$  is closed in the compact group  $J'/R_u$ . In particular,  $J/J_u$  is compact. Therefore, there exists a compact set  $K \subset J$  such that

$$J = K J_u.$$

We can now prove that if  $\Omega'$  is small enough the group  $J$  commutes with  $G_\Omega$ . This is a consequence of the following three facts.

(i) Since  $J$  is the kernel of  $\text{Ad}_{\mathfrak{g}}$  and  $G$  is weakly regular,  $J$  commutes with  $G_u$ .

(ii) Since  $J_u$  is a subgroup of  $R_u$  whose Lie algebra  $\mathfrak{j}_u$  is included in the center of  $\mathfrak{g}$ , if we choose  $\Omega'$  small enough, one has  $\text{Ad}_{\mathfrak{j}_u}(\Omega') = \{e\}$ , and the group  $J_u$  commutes with  $\Omega'$ .

(iii) Since  $K$  is compact and  $\text{Ad}_{\mathfrak{g}}(K) = \{e\}$ , by Lemma 4.2, if we choose  $\Omega'$  small enough, the group  $K$  commutes with  $\Omega'$ .

**Third step** : Size of finite subgroups. We want a uniform upper bound on the cardinality of the finite subgroups of  $G_\Omega$ . This follows from the inclusions  $R_u \subset G_u \subset G_\Omega$  of normal subgroups and from the following three facts.

(i) Since the group  $R_u$  is algebraic unipotent, it does not contain finite groups.

(ii) Since, by Proposition 5.5, the group  $G_u/R_u$  is a finite extension of a linear group, by Example 2.6.b, its finite subgroups have bounded cardinality.

(iii) Since the group  $G_\Omega/G_u$  is a compact  $p$ -adic Lie group, by Example 2.6.a, its finite subgroups have bounded cardinality.  $\square$

#### 5.4. Non weakly regular $p$ -adic Lie groups.

Not every  $p$ -adic Lie group is weakly regular. Here is a surprising example.

*Example 5.14.* There exists a  $p$ -adic Lie group  $G$  with  $G = G_u$  which does not contain any open weakly regular subgroup  $H$  with  $H = H_u$ .

We will give the construction of such a group  $G$  with Lie algebra  $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{Q}_p)$ , but we will leave the verifications to the reader.

We recall that the group  $G_0 := \mathrm{SL}(2, \mathbb{Q}_p)$  is an amalgamated product  $G_0 = K \star_{I_0} K$  where  $K := \mathrm{SL}(2, \mathbb{Z}_p)$  and  $I_0 := \{k \in K \mid k_{21} \equiv 0 \pmod{p}\}$  is an Iwahori subgroup of  $K$ . We define  $G$  as the amalgamated product  $G = K \star_I K$  where  $I \subset I_0$  is an open subgroup such that  $I \neq I_0$ .

The morphism  $G \rightarrow G_0$  is a non central extension. Using the construction in Lemma 4.1 one can check that  $G$  is spanned by one-parameter subgroups. However, one can check that the universal central extension  $\tilde{G}_0 \rightarrow G_0$  can not be lifted as a morphism  $\tilde{G}_0 \rightarrow G$ .



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