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# The complexity of the Unit Stop Number Problem and its implications to other related problems

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## Abstract

The Stop Number Problem arises in the management of a dial-a-ride system served by a fleet of autonomous electric vehicles. In such a system, clients request for a ride from an origin station to a destination station, and a fleet of capacitated vehicles must satisfy all requests. The goal is to minimize the number of pick-up/drop-off operations. In this paper we focus on a special case of this problem that was recently conjectured to be NP-Hard. In this regard, we show how such special case relates to other problems known in the literature in order to derive some polynomial-time solvable variants. Moreover, we provide a positive answer to the conjecture by showing that the problem is NP-Hard for any fixed capacity greater than or equal to 2, even for the case where the graph of requests is restricted to the class of planar bipartite graphs. Our proof of NP-Hardness also improves the complexity results known in the literature for the related problems identified.

*Keywords:* Combinatorial optimization, Computational complexity, Dial-a-Ride, Stop Number Problem, Traffic Grooming

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## 1. Introduction

In modern societies, mobility plays a central role in economic and social activities such as commuting, manufacturing, distributing goods, or supplying energy (see Rodrigue et al. (2016)). Mobility is supported and driven by transport systems allowing interactions among individuals and/or institutions which are continuously evolving. Indeed, tightening CO<sub>2</sub> regulations together with an increasing traffic load in urban areas and a consistent growth in the usage of web services are inducing new technology-driven trends in transportation, giving rise to new transport systems such as ride-sharing (e.g., Agatz et al. (2012)), car-sharing (e.g., Boyacı et al. (2015); Shaheen & Cohen (2012)) and dial-a-ride (e.g., Cordeau & Laporte (2007); Cordeau (2006); Cordeau & Laporte (2003); Parragh et al. (2008); Ho et al. (2018)) systems. Moreover, the role of transport systems in the development of the well-being and life satisfaction of a community is highlighted in Delbosc (2012).

The VIPAFLEET project arises from this background and attempts to contribute to sustainable intelligent mobility through the development of models and algorithms for managing fleets of specific autonomous vehicles named VIPA, a French acronym for Autonomous Individual Passenger Vehicle. VIPA is an electrical vehicle designed to operate in fully autonomous manner (i.e., without any driver assistance) notably in closed and semi-closed sites like industrial and commercial areas, medical complexes and campuses.

VIPA shuttles may perform in different modes of operation. In this paper we focus our efforts on the so-called 'tramway mode' (see Bsaybes et al. (2019) for a study on other modes). In this mode of operation, a circuit with predefined stations is fixed and customers use their smartphones or a 'call terminal' to request for a ride from an origin station to some destination station of their choice. For its part, the fleet of identical capacitated vehicles travels around the circuit (always in the same direction) and stops at a station upon request.

It is worth noting that a VIPA shuttle can transport more than one

passenger at the same time. In its latest version, each vehicle is designed to have a capacity of up to 15 passengers. This leads to a Dial-a-Ride Problem (DARP) on which a fleet of capacitated vehicles is responsible for transporting clients or goods that must be moved from certain pickup locations to other delivery locations on a given network.

Due to infrastructure restrictions, stations usually do not belong to the circuit but are attached to it (see Figure 1). This particularity produces a significant impact on the fleet management of such a system. Indeed, in order to respond to a client demand, the vehicle must slow down and make a deviation from its original course. Such deviations increases the travel times of on-board customers. If the deviations lengths are supposed to be approximately the same, then improving the quality of service fairly corresponds to minimizing the total number of stops performed by the fleet of vehicles. In Pimenta et al. (2017), it is also pointed out that minimizing the total number of stops is a good way of improving the system’s reliability by ensuring a steady flow of the vehicles.

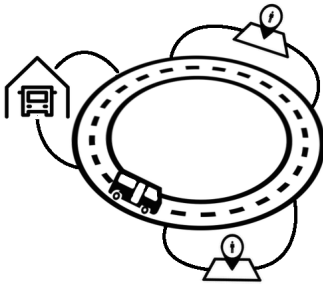


Figure 1: Circuit Scheme

The Stop Number Problem (SNP) consists of assigning each client demand to a vehicle such that no vehicle gets overloaded, and the total number of vehicles’ stops is minimized. For this, one may use as many vehicles as desired. Notice that, in the search for a better solution, a vehicle is allowed to make several tours before serving a demand. Tours performed before serving a demand are called waiting tours. In order to ensure the quality of service and deal with customers time windows, the maximum number

of waiting tours is bounded by a given parameter  $H \geq 0 \in \mathbb{N}$ . Moreover, once a customer is picked up, it cannot stay on the vehicle for a full tour, that is, once loaded it has to be unloaded as soon as the vehicle reaches its destination. Finally, a customer demand may request for more than one seat on a single vehicle. In this sense, a demand is specified by an origin station, a destination station and a load that stands for the number of seats requested.

When the load of each client demand is unitary and there are no waiting tours, that is, each client may only request for a single seat and the fleet must respond to all requests within a single tour (i.e.,  $H = 0$ ), the problem is called Unit Stop Number Problem (U-SNP).

In practice, the system should be capable of reacting dynamically to the on-going flow of demands through online algorithms. However, in order to better evaluate such reactive procedures, the static case (offline) should be understood and mastered. In return, a good understanding of the properties and difficulties of the static case is essential to the development of better suited online algorithms. A reasonable choice is thus to deeply investigate where resides the complexity of SNP on the static case.

In this study, the combinatorial core of SNP is explored by looking into its constrained unitary version (i.e., U-SNP), which was conjectured to be NP-Hard by Pimenta et al. (2017). The paper is organized as follows. In Section 2, we formally define the U-SNP and provide a literature review of the works related with its computational complexity. Next, in Section 3 we discuss some particular cases where U-SNP can be solved in polynomial time and provide a study on the properties their optimal solutions hold. Such study is then used to derive a proof of NP-Hardness for the U-SNP that is capable of yielding stronger complexity results for other related problems. Finally in Section 5 we draw some conclusions and propose some further research directions.

## 2. Problem Definition and literature review

### 2.1. The Unit Stop Number Problem

For U-SNP imposes that only vehicles in their first tour can pick up customers, we represent the circuit as an ordered line from the depot to the last destination station and the customers' requests as intervals on this line. Notice that this line may correspond to more than a tour of the circuit (i.e., some stations are duplicated) if for some request the depot appears in between its origin and destination stations. So let  $V = \{1, \dots, n\}$  be the set of all the stations but the depot as they appear on the ordered line. Let  $E$  be the set of  $m$  unit-load requests where each request  $e \in E$  is specified by a pick-up station  $o_e \in V$  and a drop-off station  $d_e \in E$  where  $1 \leq o_e < d_e \leq n$ , that is  $e = (o_e, d_e)$ . Finally, let  $K = \{1, \dots, p\}$  denote the set of available vehicles, all having the same capacity  $C \in \mathbb{Z}_+$ . An U-SNP instance is hence represented by a 4-tuple  $(V, E, C, p)$ . Without loss of generality, we suppose that every station in  $V$  is a pick-up or drop-off station of at least one request in  $E$ .

With any U-SNP instance  $\mathcal{I} = (V, E, C, p)$ , an undirected graph  $G_{\mathcal{I}} = (V, E)$  is associated and henceforth, stations and requests may be referred to as nodes and edges, respectively. We use  $G$  whenever the instance  $\mathcal{I}$  can be deduced from the context. Notice that although  $G$  has a natural orientation as  $o_e < d_e$  for any  $e \in E$ , we keep it undirected for the sake of simplicity. Figure 2 depicts the construction of the associated graph  $G$  from an instance composed of five requests represented as intervals over four stations. Several requests may share the same pick-up and drop-off stations (e.g.,  $e_1$  and  $e_2$  in Figure 2) and correspond to parallel edges in  $G$ . They are thus referred to as *parallel* requests. For any two distinct sets  $S_1$  and  $S_2$  of  $V$ , we say that  $S_1$  *precedes*  $S_2$ , and write  $S_1 \prec S_2$ , if any station in  $S_1$  appears before any station in  $S_2$  on the ordered line, that is,  $v_1 < v_2$  for all  $v_1 \in S_1$  and  $v_2 \in S_2$ . Similarly,  $S_1$  *succeeds*  $S_2$ , written  $S_1 \succ S_2$ , if  $S_2$  precedes  $S_1$ .

For any subset  $F \subseteq E$  and any station  $v \in V$ , let

$$\Delta_F(v) = \{e \in F : o_e \leq v \leq d_e - 1\} \quad (1)$$

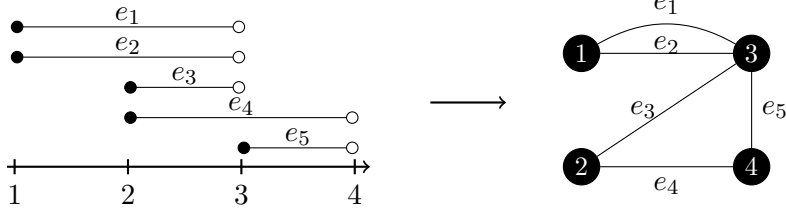


Figure 2: Stations and requests of an instance  $\mathcal{I}$  and the associated graph  $G_{\mathcal{I}}$

denote the set of requests in  $F$  that *cross* or *start* at station  $v$ . Notice that requests *ending* at station  $v$  (i.e., having  $v$  as their drop-off station) do not belong to  $\Delta_F(v)$ . In graph  $G$ ,  $\Delta_E(v)$  corresponds to the cut induced by the node set  $\{1, \dots, v\}$ . Figure 3 illustrates  $\Delta_E(v)$ , for each station  $v \in V$ , for the instance described in Figure 2.

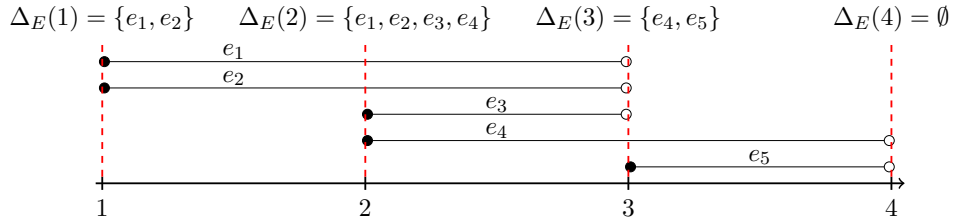


Figure 3: Illustration of  $\Delta_E(v)$ .

We can now define a feasible solution to U-SNP as a partition of  $E$  into  $p$  subsets  $E_1, \dots, E_p$  that satisfy  $|\Delta_{E_i}(v)| \leq C$  for each  $i = 1, \dots, p$  and each  $v \in V$ . The cost of a feasible solution is expressed as the total number of stations the  $p$  vehicles visit, that is, as

$$\sum_{i=1}^p |V[E_i]|, \quad (2)$$

where  $V[E_i]$  represents the set of stations vehicle  $i \in \{1, \dots, p\}$  stops at. U-SNP thus consists of finding a feasible partition  $\{E_1, \dots, E_p\}$  that minimizes (2).

To guarantee the existence of feasible solutions, we hereafter assume

that the number  $p$  of available vehicles is at least the minimum number  $p_{\min}$  of vehicles necessary to serve the  $m$  requests. Indeed for the requests correspond to half-closed intervals on an ordered line,  $p_{\min}$  clearly is bounded from below by

$$\max_{v \in V} \left\{ \left\lceil \frac{|\Delta_E(v)|}{C} \right\rceil \right\}. \quad (3)$$

Moreover any feasible solution output by a first-fit type algorithm where the requests are served according to a nondecreasing order of their pick-up stations uses a number of vehicles equal to (3). Consequently,  $p_{\min}$  can be computed in polynomial time and from now on we assume  $p \geq p_{\min} = \max_{v \in V} \left\{ \left\lceil \frac{|\Delta_E(v)|}{C} \right\rceil \right\}$ .

## 2.2. Literature review

SNP was introduced in Pimenta et al. (2017) and proved to be weakly NP-Hard. They showed that the classic Partition Problem (see, e.g., Garey & Johnson (2002)) indeed corresponds to SNP where all the requests share the same pick-up and drop-off stations and the vehicles' capacity  $C$  equals

$$\frac{\sum_{e \in E} l_e}{2},$$

where  $l_e$  denotes the load of request  $e \in E$ . Notice however that this proof is fully based on different request loads and hence cannot be directly applied to determine the computational complexity of U-SNP. After having proved that U-SNP is solvable in polynomial time through dynamic programming providing the number of available vehicles and their capacity are considered as fixed parameters, Pimenta et al. (2017) also conjectured that U-SNP is otherwise NP-Hard.

Pimenta et al. (2017) also introduced an integer linear-programming formulation for the SNP with request-vehicle assignment variables and vehicle-station stop variables. For this formulation provides a weak linear relaxation, a set-partitioning reformulation yielding better bounds but containing an exponential number of variables was also proposed in Pimenta et al. (2017).



The associated pricing problem was shown to be NP-Hard and a GRASP-type procedure was proposed to heuristically solve it.

It is worth mentioning that the so-called traffic grooming problem (see Dutta et al. (2000)) that arises in Wavelength-Division-Multiplexing (WDM) optical networks is closely related to U-SNP. To the best of our knowledge, no connections between that problem and U-SNP have ever been made before. An instance  $(H, R, g)$  of the traffic grooming problem is composed of a simple connected graph  $H$  describing the WDM optical network, a simple graph  $R = (V(H), E)$  where the edges of  $E$  represent the traffic requests, and a positive integer  $g$  called the grooming factor. The traffic grooming problem consists of assigning a lightpath (i.e., a path in  $H$ ) and a wavelength from a set of  $\Lambda$  available wavelengths to each traffic request in  $R$  such that (i) a wavelength gets at most  $g$  lightpaths that share an edge and (ii) the total number of Add-Drop Multiplexers (ADMs) is minimized where each lightpath requires an ADM at each endnode and two lightpaths having a common endnode and being assigned to the same wavelength can share an ADM. The reader is referred to Dutta & Rouskas (2002) for a survey on this problem.

The traffic grooming problem has also been considered when the optical network is restricted to some specific topologies such as star, path, and ring networks (see, e.g., Huang et al. (2006); Amini et al. (2009); Bermond et al. (2007)). In the so-called *Path Traffic Grooming Problem* (PTGP), graph  $H$  is assumed to be a path and the only decisions to make are the wavelength assignments for there exists a unique lightpath for each request in  $R$ . PTGP thus consists of finding a partition  $\{R_1, \dots, R_\Lambda\}$  of  $R$  such that (i) on no edges in  $H$  the number of traffic requests in  $R_i$ ,  $i \in \{1, \dots, \Lambda\}$ , traversing that edge does exceed  $g$  and (ii) the total number of ADMs  $\sum_{i=1}^{\Lambda} |V(R_i)|$  is minimized. It is straightforward to see that there exist one-to-one correspondences between the U-SNP set of stations and the PTGP set of nodes and between the U-SNP and PTGP sets of requests as long as there are no parallel requests in U-SNP. The grooming factor is equivalent to the vehicle capacity. U-SNP therefore is a generalization of PTGP.

PTGP was proved to be NP-Hard if the grooming factor  $g$  is at least 2 by

Huang et al. (2006). Moreover Bermond et al. (2007) presented polynomial-time algorithms to solve PTGP when  $g = 1$  or  $g = 2$  and  $R$  is a complete graph. Some of the computational-complexity results that were obtained for PTGP can thus be extended to U-SNP as stated in the next section.

### 3. Polynomial Cases and Optimal Solution Properties

In this section, we first study some properties optimal solutions to U-SNP satisfy and then present some polynomial-time solvable variants of U-SNP mainly based on the relations between U-SNP and PTGP.

U-SNP predominantly consists of minimizing the total number of stops the  $p$  vehicles have to make to serve all the requests in  $E$ . To lay emphasis on this objective, the number of available vehicles usually is assumed to not correspond to a critical resource, that is, the decision maker can use as many vehicles as needed. Notice however that the number  $m$  of requests is a trivial upper bound on the number of needed vehicles and we thus always consider  $p = m$  from now on.

Pimenta et al. (2017) pointed out that all the optimal solutions to U-SNP they empirically obtained use  $p_{min}$  vehicles. The next result states that it might not be necessarily the case.

**Proposition 1.** *Some instances of U-SNP require strictly more than  $p_{min}$  nonempty vehicles in any of their optimal solutions.*

*Proof.* Consider the instance of U-SNP depicted in Figure 4 with  $C = 2$ .

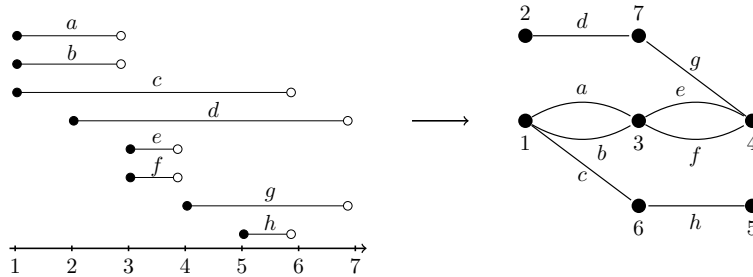


Figure 4: U-SNP Instance needing more than  $p_{min} + 1$  vehicles in optimal solutions

For  $\max_{v \in V} \Delta_E(v) = 4$ , we have  $p_{min} = 2$ . It is easy to see that the partition  $\{\{a, b, e, f\}, \{c, h\}, \{d, g\}\}$  of  $E$  corresponds to a feasible solution using 3 vehicles for a total of 9 stops. We now show that no feasible solutions whose total number of stops is less than 10 only use 2 vehicles.

Suppose first that requests  $c$  and  $d$  are assigned to the same vehicle, say vehicle 1, and only two vehicles are used. All the other requests must be assigned to the second vehicles for otherwise vehicle 1 would be overloaded. The partition  $\{\{c, d\}, \{a, b, e, f, g, h\}\}$  corresponds to a feasible solution using 2 vehicles for a total of 10 stops.

Suppose now that requests  $c$  and  $d$  are assigned to different vehicles, say  $c$  to the first vehicle and  $d$  to the second, and only two vehicles are used. Requests  $a$  and  $b$  must also be assigned to different vehicles and so must requests  $e$  and  $f$ . Without loss of generality let  $a$  and  $e$  be assigned to the first vehicle and  $b$  and  $f$  to the second. So far the first vehicle already has to stop 4 times while the second vehicle stops 5 times. For none of the two vehicles stops at station 5 and request  $h$  still needs to be assigned to a vehicle, at least 10 stops would be made by the two vehicles.  $\square$

Assigning up to  $C$  parallel requests to a vehicle might seem intuitive when seeking optimal solutions to U-SNP. The next proposition invalidates this intuition.

**Proposition 2.** *Some instances of U-SNP require assigning parallel requests to different vehicles in any of their optimal solutions*

*Proof.* Consider the instance of U-SNP described in Figure 5 with  $C = 2$ .

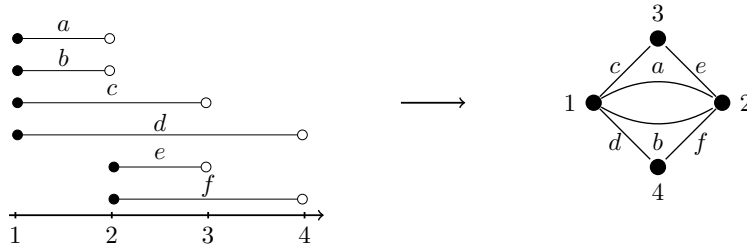


Figure 5: U-SNP instance with parallel requests in different vehicles in optimal solutions

The partition  $\{\{a, c, e\}, \{b, d, f\}\}$  clearly corresponds to a feasible solution where parallel requests  $a$  and  $b$  are assigned to different vehicles and the total number of stops is 6. We now show that if requests  $a$  and  $b$  are assigned to the same vehicle, at least 7 stops then are needed. Suppose that requests  $a$  and  $b$  are assigned to the same vehicle, say vehicle 1. To respect the capacity limit of vehicle 1, requests  $c$  and  $d$  cannot be assigned to vehicle 1.

If they both are assigned to different vehicles, say  $c$  to vehicle 2 and  $d$  to vehicle 3, these three vehicles already stop 6 times. For assigning request  $e$  to any vehicle would increase the number of stops by at least one, no feasible solutions with  $a$  and  $b$  assigned to the same vehicle yield less than 7 stops.

Suppose now that requests  $c$  and  $d$  are assigned to the same vehicle, say vehicle 2. Vehicles 1 and 2 already stop 5 times. Notice that neither request  $e$  nor request  $f$  can be assigned to vehicle 2. If they both are assigned to vehicle 1, the total number of stops would be 7. Any assignment of requests  $e$  and  $f$  to vehicles other than 1 and 2 would increase the number of stops by at least 3, yielding a total number of stops greater than or equal to 8.  $\square$

For a vehicle can serve at most  $C$  requests among those starting or ending at a given station, the total number of stops made by the  $p$  vehicles clearly is bounded from below by

$$\sum_{v \in V} \left\lceil \frac{\max\{|\delta^+(v)|, |\delta^-(v)|\}}{C} \right\rceil, \quad (4)$$

where  $\delta^+(v)$  and  $\delta^-(v)$  denote the set of requests having station  $v$  as their pick-up and drop-off station, respectively. Notice that for the instances depicted in Figures 4 and 5 the optimal values do not match (4). However, this lower bound will be helpful when devising polynomial-time algorithms for U-SNP cases.

We first consider two polynomially-solvable cases of U-SNP that do not depend on any structural properties of graph  $G$ .

**Proposition 3.** *If  $p_{\min} = 1$  or  $C = 1$  then U-SNP is polynomially solvable.*

*Proof.* If  $p_{\min} = 1$ , that is,  $C \geq \max_{v \in V} |\Delta_E(v)|$ , then all the requests fit in a single vehicle. The number of stops this vehicle then makes equals  $n$ . For  $n$  is a trivial lower bound on the total number of stops made by the  $p$  vehicles, the solution that consists of serving all the requests by the same vehicle, and leaving the other vehicles in the depot, is optimal to U-SNP.

If  $C = 1$  then two parallel requests cannot be served by the same vehicle. The greedy algorithm given by Bermond et al. (2007) for solving PTGP with a unitary grooming ratio  $g = 1$  can hence be easily adapted to optimally solve U-SNP with  $C = 1$ . Such algorithm would sequentially

- i. select a maximal sequence  $(e_1, \dots, e_q)$ ,  $q \geq 1$ , of requests such that the drop-off station of request  $e_i$  matches the pick-up station of request  $e_{i+1}$ , that is,  $d_{e_i} = o_{e_{i+1}}$ , for  $i = 1, \dots, q - 1$ ,
- ii. assign all the  $q$  demands of the sequence to an available vehicle, and
- iii. remove all the  $q$  demands of the sequence from  $E$ .

The total number of stops of that greedy solution clearly equals

$$\sum_{v \in V} \max\{|\delta^+(v)|, |\delta^-(v)|\},$$

and matches lower bound (4). The greedy solution therefore is optimal and can be found in  $O(m)$ .  $\square$

We now present some polynomially-solvable U-SNP cases when the instance graph fulfills some structural properties.

**Proposition 4.** *If  $C = 2$  and graph  $G = (V, E)$  is complete then U-SNP is polynomially solvable.*

*Proof.* When  $G$  is a complete graph, PTGP and U-SNP are equivalent for no parallel requests exist. The algorithm proposed by Bermond et al. (2007) for solving PTGP when the grooming ratio  $g = 2$  thus applies for solving U-SNP with  $C = 2$ .  $\square$

We now present a polynomially-solvable version of U-SNP that does not rely on PTGP.

**Proposition 5.** *If graph  $G = (V, E)$  is a star then U-SNP is polynomially solvable.*

*Proof.* Let  $\{E_1, \dots, E_{p^*}\}$  be a partition of  $E$  corresponding to an optimal solution to instance  $(V, E, 1, p)$ , where  $p^* \leq p$  denotes the number of vehicles serving at least one request. Such a solution may be obtained in polynomial time using the greedy algorithm of Proposition 3 and yields  $n - 1 + \max\{|\delta^+(v_c)|, |\delta^-(v_c)|\}$  stops, where  $v_c$  denotes the central node of  $G$  (i.e., the node that is adjacent to all the other nodes). Note that  $1 \leq |E_i| \leq 2$  for all  $i \in \{1, \dots, p^*\}$ . Moreover if  $E_i$  is composed of two requests  $e_1 = (o_{e_1}, d_{e_1})$  and  $e_2 = (o_{e_2}, d_{e_2})$ , we then must have

$$o_{e_1} < d_{e_1} = v_c = o_{e_2} < d_{e_2}.$$

Let  $\{E'_1, \dots, E'_{q^*}\}$ ,  $q^* = \lceil \frac{p^*}{C} \rceil$ , be the partition of  $E$  obtained from partition  $\{E_1, \dots, E_{p^*}\}$  as follows

$$\begin{aligned} E'_i &= \bigcup_{j=C(i-1)+1}^{iC} E_j & \text{for } i = 1, \dots, q^* - 1, \\ E'_{q^*} &= \bigcup_{j=C(q^*-1)+1}^{p^*} E_j. \end{aligned}$$

Partition  $\{E'_1, \dots, E'_{q^*}\}$  clearly defines a feasible solution to  $(V, E, C, p)$  and yields  $n - 1 + \max\{\frac{|\delta^+(v_c)|, |\delta^-(v_c)|}{C}\}$  stops. This total number of stops matches the lower bound (4). Therefore  $\{E'_1, \dots, E'_{q^*}\}$  is an optimal solution to  $(V, E, C, p)$ .  $\square$

#### 4. Computational Complexity of U-SNP

In Section 2.2 we pointed out the equivalence between U-SNP and PTGP. Since PTGP has been proved to be NP-hard for any fixed grooming factor  $g \geq 2$  by Amini et al. (2009), so is U-SNP. The conjecture of Pimenta et al. (2017), mentioned in Section 2.2, thus is proved. To prove the NP-hardness of PTGP, Amini et al. (2009) use a reduction from the NP-Hard problem of finding the maximum number of edge-disjoint triangles (i.e., cycles of length

3) in a tripartite graph. This proof however cannot be directly used to show the NP-hardness of PTGP and U-SNP on bipartite graphs for those graphs do not contain any odd cycles. In this section, we prove that U-SNP (and thus PTGP) is NP-hard for any fixed capacity  $C \geq 2$  on planar bipartite graphs, and thus improve the known computational-complexity results for U-SNP and PTGP.

Inspired by the work of Dyer & Frieze (1985), we propose polynomial reductions from the well-known *3-Dimensional Matching Problem* (3DMP). This problem is one of the 21 classic NP-Complete problems proposed by Karp (1972) and can be stated as follows. Given three disjoint sets  $X$ ,  $Y$ , and  $Z$  with equal cardinality  $q$ , and a set of triples  $T \subseteq X \times Y \times Z$ , a *3-dimensional matching* is a subset  $M$  of  $T$  such that for any two distinct triples  $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2)$  of  $M$ ,  $x_1 \neq x_2$ ,  $y_1 \neq y_2$ , and  $z_1 \neq z_2$ . 3DMP consists of deciding whether or not there exists a 3-dimensional matching of cardinality  $q$ . Dyer & Frieze (1986) proved that 3DMP is NP-Complete even when the associated bipartite graph  $H = (T, S, E')$  is restricted to be planar, where  $S = X \cup Y \cup Z$  and

$$E' = \bigcup_{t=(x,y,z) \in T} \{(t, x), (t, y), (t, z)\}.$$

Figure 6 represents such graph  $H$ . The restriction of 3DMP to planar bipartite graphs is referred to as *planar 3DMP*.

**Theorem 1.** *U-SNP is NP-Hard even when restricted to the case where  $C = 2$  and  $G$  is a planar bipartite graph.*

*Proof.* Consider an instance of 3DMP as described above such that graph  $H = (T, S, E')$  is planar bipartite. Let  $G = (V, E)$  be the graph obtained from  $H$  as follows. (See Figure 7.) Besides the sets  $X$  and  $T$ , node set  $V$  is also composed of the sets

- $X' = \{x'_t : t \in T\}$ ,
- $X^i = \{x_k^i : x \in X, k = 1, \dots, \deg_H(x) - 1\}$  for  $i = 1, 2$ ,
- $W = \{w_v : v \in Y \cup Z\}$ , and

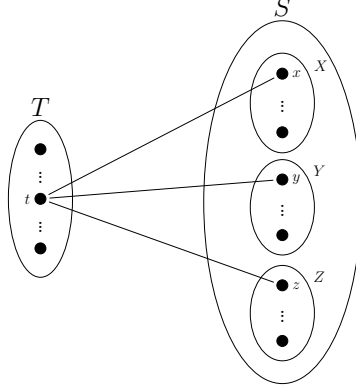


Figure 6: 3DMP bipartite graph  $H = (T, S, E')$

- $W^j = \{w_k^j : w \in W, k = 1, \dots, \deg_H(w) - 1\}$  for  $j = 1, 2, 3$ ,

where  $\deg_H(s)$  denotes the degree of node  $s \in S$  in graph  $H$ . Since  $|X^i| = |T| - q$  for  $i = 1, 2$  and  $|W^j| = 2|T| - 2q$  for  $j = 1, 2, 3$ , the cardinality of  $V$  is then equal to  $10|T| - 5q$ .

The edge set  $E$  of  $G$  is composed of

- $E_t = \{(x'_t, t), (x'_t, x), (t, w_y), (t, w_z)\}$  for  $t = (x, y, z) \in T$ ,
- $E_x = \{(x, x_k^1), (x_k^1, x_k^2) : k = 1, \dots, \deg_H(x) - 1\}$  for  $x \in X$ , and
- $E_w = \{(w, w_k^1), (w_k^1, w_k^2), (w_k^2, w_k^3) : k = 1, \dots, \deg_H(w) - 1\}$  for  $w \in W$ .

Notice that sets  $E_x$ ,  $x \in X$ , and  $E_w$ ,  $w \in W$ , are composed of paths of length 2 and 3 that only share nodes in  $X$  and  $W$ , respectively. The cardinality of  $E$  clearly is equal to  $12|T| - 8q$ . Notice that graph  $G$  is planar bipartite because  $H$  is.

Consider any numbering of the nodes of  $V$  (i.e., the stations with respect to U-SNP) that respects the following precedence order

$$W_2 \prec X' \prec W_1 \prec X \prec W_3 \prec T \prec X_2 \prec W \prec X_1, \quad (5)$$

that is, as they appear in Figure 7. Notice that sets  $X$ ,  $W$ ,  $T$ ,  $X'$ ,  $X^1$ ,  $X^2$ ,  $W^1$ ,  $W^2$ , and  $W^3$  correspond to stable sets of  $G$ . The 4-tuple  $\mathcal{I} = (V, E, 2, p)$ ,



$p = 12|T| - 8q$ , is a U-SNP instance whose construction can be done in polynomial time.

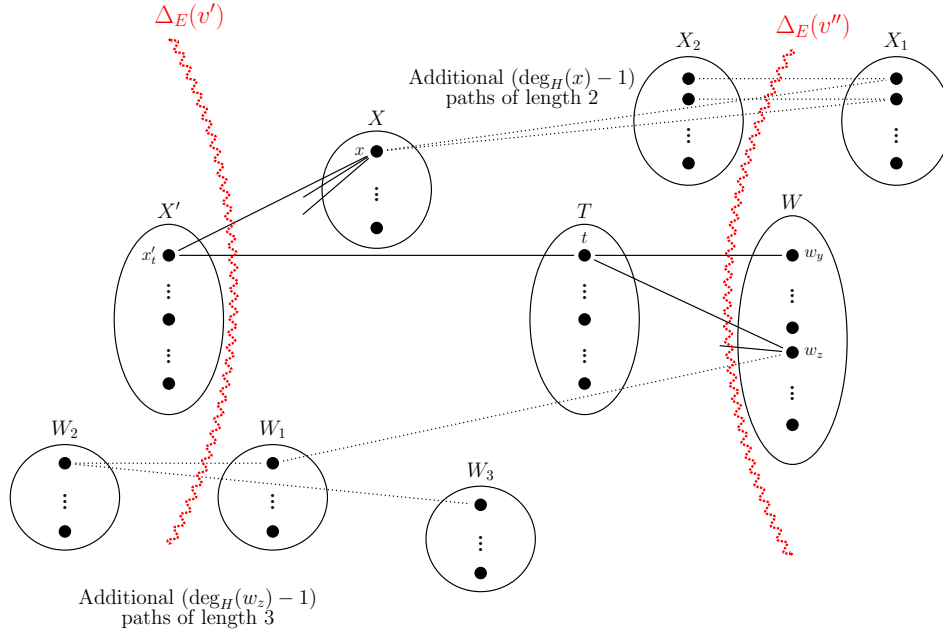


Figure 7: Construction of graph  $G$

We now claim that the instance of planar 3DMP has a 3-dimensional matching of cardinality  $q$  if and only if  $\mathcal{I}$  has a solution  $\{E_1, \dots, E_p\}$  with  $|E| + \frac{|E|}{4}$  stops.

Let  $v'$  and  $v''$  be the nodes in  $X'$  and  $X^2$  with the largest number, respectively. For (5) we have  $\Delta_E(v') \cup \Delta_E(v'') = E$ . (See Figure (7).) To respect its capacity limit of 2 requests at a time, any vehicle  $i \in \{1, \dots, p\}$  may serve at most 4 requests, that is,

$$|E_i| \leq 4 \quad \text{for } i = 1, \dots, p. \quad (6)$$

Moreover from the construction of graph  $G$ , any cycle in  $G$  must contain at least three edges of either  $\Delta_E(v')$  or  $\Delta_E(v'')$ . The subgraph  $G_i = (V(E_i), E_i)$ , if nonempty, clearly is a forest and vehicle  $i$  thus makes  $|E_i| + c(G_i)$  stops where  $c(G_i)$  denotes the number of connected components of

$G_i$  for  $i = 1, \dots, p$ . Suppose without loss of generality that only the first  $p'$  vehicles serve a request, that is,  $E_i \neq \emptyset$  for  $i = 1, \dots, p'$  and  $E_i = \emptyset$  for  $i = p' + 1, \dots, p$ . The total number of stops of solution  $\{E_1, \dots, E_p\}$  therefore equals

$$\sum_{i=1}^{p'} |V[E_i]| = \sum_{i=1}^{p'} (|E_i| + c(G_i)) = |E| + \sum_{i=1}^{p'} c(G_i).$$

Using (3) and  $\Delta_E(v') \cup \Delta_E(v'')$ , we obtain

$$\sum_{i=1}^{p'} c(G_i) \geq p' \geq p_{\min} \geq \frac{|E|}{4}$$

and  $|E| + \frac{|E|}{4}$  is a lower bound on the total number of stops of  $\{E_1, \dots, E_p\}$ .

If the total number of stops is exactly  $|E| + \frac{|E|}{4}$ , then each vehicle serves either no requests or 4 requests that induce a connected subgraph of  $G$  for a total of 5 stops. Each set of requests defining a path of length 3 in  $E_w$ ,  $w \in W$ , thus must be served by a different vehicle. For each of these vehicles serves 4 connected requests, they also must serve as many requests between  $T$  and  $W$  as paths of length 3. Each node in  $W$  then is incident with exactly one unassigned request whose pick-up station belongs to  $T$ .

Similarly each set of requests defining a path of length 2 in  $E_x$ ,  $x \in X$ , must be served by a different vehicle that also needs to serve two additional requests, one from  $X'$  to  $X$  and one from  $X'$  to  $T$ . Notice that these two additional requests could not both have been from  $X'$  to  $X$  for they would have left a connected component composed of 3 unassigned requests (i.e., a star whose central node is in  $T$ ). Each node in  $X$  then is incident with exactly one unassigned request whose pick-up station belongs to  $X'$ , the latter being incident with exactly one unassigned request whose drop-off belongs to  $T$ .

The  $q$  remaining nonempty vehicles therefore serve sets of 4 requests that induce trees with two internal nodes, one in  $X'$  and one in  $T$ , and one leaf in each of the sets  $X$ ,  $Y$ , and  $Z$ . The loads of these vehicles thus correspond

to a 3-dimensional matching of cardinality  $q$  in  $H$ .

Conversely from any 3-dimensional matching of cardinality  $q$  in  $H$ , a solution to  $\mathcal{I}$  with  $|E| + \frac{|E|}{4}$  stops can easily be obtained by following the foregoing process in a reverse order.  $\square$

The proof of the NP-Hardness of U-SNP for even values of  $C$  on planar bipartite graphs is similar to the one of Theorem 1. Every edge in the constructed graph is indeed replaced by a path of length  $\frac{C}{2}$ . (The constructed instance's graph for  $C = 4$  is depicted in Figure 8.) The next theorem therefore is stated with no proof. (A detailed proof can be found in Colares (2019).)

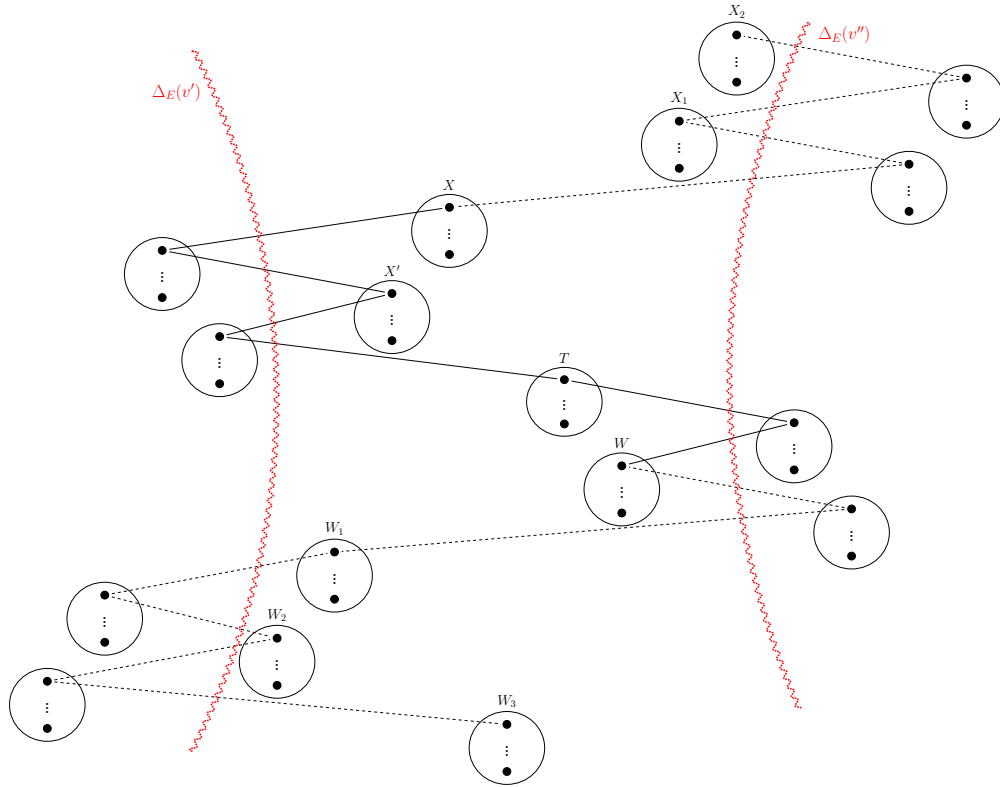


Figure 8: Construction of graph  $G$  for  $C = 4$

**Theorem 2.** *The U-SNP is NP-Hard even when restricted to the case where  $C = 2k$ , for any  $k \in \mathbb{Z}^+$ , and  $G$  is a planar bipartite graph.  $\square$*

Our proof of Theorem 1 relies on the ability to enforce that no more than  $2C$  requests are assigned to each vehicle by constructing an instance with two stations (namely  $v'$  and  $v''$ ) covering the whole set of requests. Each nonempty vehicle then serves  $C$  requests crossing  $v'$  and  $C$  requests crossing  $v''$ . For even values of  $C$ , substituting each edge in the construction of graph  $G$  for a path of length  $\frac{C}{2}$  preserves the validity of that argument. However for odd values of  $C$ , such subdivision-based approach cannot be directly applied. We therefore provide a slightly different reduction from 3DMP to prove the NP-Hardness of U-SNP with odd values of  $C$ .

**Theorem 3.** *The U-SNP is NP-Hard even when restricted to the case where  $C = 2k + 1$ , for  $k \in \mathbb{Z}^+$ , and  $G$  is a planar bipartite graph.*

*Proof.* Consider an instance of 3DMP with  $H = (T, S, E')$  planar bipartite. (See Figure 6.) Let  $k$  be a positive integer. Following a similar approach as in Theorem 1, we construct graph  $G = (V, E)$  of an U-SNP instance with vehicles' capacity  $C = 2k + 1$ . Starting from  $H$ , we apply the next three operations

- i. replace the edges between  $T$  and  $Y \cup Z$  with paths of length  $k$ ,
- ii. attach  $\deg_H(i) - 1$  disjoint paths of length  $k + 1$  to each node  $i \in Y \cup Z$ ,  
and
- iii. attach  $\deg_H(i) - 1$  disjoint paths of length  $2k$  to each node  $i \in X$ .

Figure 9 depicts graph  $G$  for  $k = 2$ .

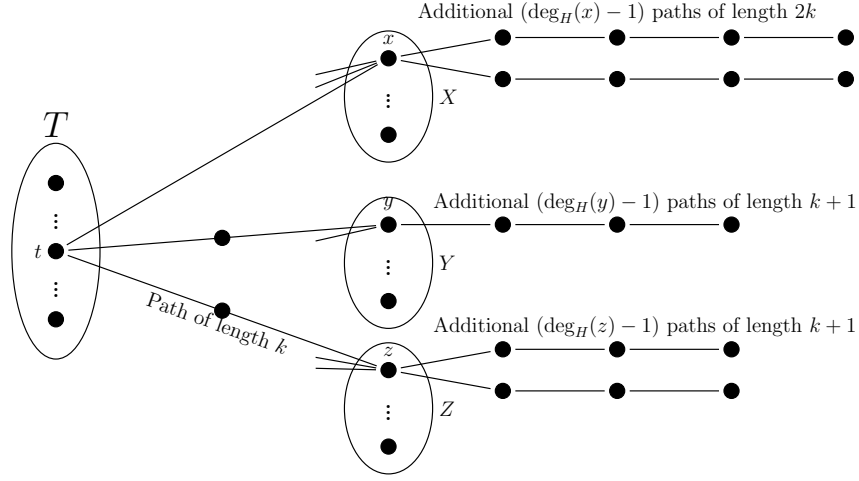


Figure 9: Construction of graph  $G$  for Theorem 3, with  $k = 2$

The number of nodes in  $G$  clearly equals  $(6k + 1)|T| - (4k - 1)q$  and its number of edges  $(2k + 1)(3|T| - 2q)$ . Graph  $G$  can thus be constructed in polynomial time. Notice that for  $H$  is planar bipartite, so is graph  $G$ . The node set of  $G$  can therefore be partitioned into two disjoint and independent sets  $A$  and  $B$ . Consider any numbering of the nodes of  $V$  that respects the precedence order  $A \prec B$ . Let  $\mathcal{I} = (V, E, 2k+1, p)$ ,  $p = (6k+3)|T| - (4k+2)q$ , be the thus obtained U-SNP instance.

We claim that the instance of 3DMP has a 3-dimensional matching if and only if  $\mathcal{I}$  has a solution  $\{E_1, \dots, E_p\}$  with  $|E| + \frac{|E|}{2k+1}$  stops. Let  $v'$  denote the node in  $A$  having the largest number. From the construction of  $G$ , we clearly have  $\Delta_E(v') = E$ . Therefore each vehicle may take at most  $C = 2k + 1$  demands. Moreover from the construction of  $G$ , no cycle in  $G$  contains less than  $2k + 2$  (i.e.,  $C + 1$ ) edges. The subgraph  $G_i = (V(E_i), E_i)$  associated with vehicle  $i = 1, \dots, p$ , if nonempty, clearly is a forest. The argument, similar to the one used in the proof of Theorem 1, consisting of

- i. proving that  $G_i$  is connected with  $|E_i| = 2k + 1$ , if nonempty, for  $i = 1, \dots, p$ ,
- ii. assigning the requests of any additional path of length  $2k$  together with one of the requests between  $T$  and  $X$  to a same vehicle, and

- iii. assigning the requests of any additional path of length  $k + 1$  together with the  $k$  requests of one of the paths between  $T$  and  $Y \cup Z$  to a same vehicle

to end up with each node in  $S$  being incident with exactly one unassigned request whose other station belongs to  $T$  would hence complete the proof.

□

We summarize the three foregoing theorems into the following corollary that validates the conjecture of Pimenta et al. (2017) even on planar bipartite graphs.

**Corollary 1.** *The U-SNP is NP-Hard for any fixed capacity  $C \geq 2$ , even when restricted to the case where  $G$  is a planar bipartite graph.* □

Notice that all the optimal solutions to the U-SNP instances constructed in proofs of Theorems 1, 2, and 3 need exactly  $p_{\min}$  nonempty vehicles. We can therefore state the following.

**Corollary 2.** *The U-SNP is NP-Hard for any fixed capacity  $C \geq 2$  and  $p = p_{\min}$ , even when restricted to the case where  $G$  is a planar bipartite graph.* □.

U-SNP graphs  $G$ , constructed in the proofs of Theorems 1, 2, and 3, clearly have no parallel edges. For U-SNP and PTGP are equivalent as long as U-SNP does not have any parallel requests, Corollary 1 naturally extends to the path traffic grooming problem, improving the state-of-the-art computational-complexity results for this problem.

**Corollary 3.** *The PTGP is NP-Hard for any fixed grooming ratio  $g \geq 2$ , even when restricted to the case where the request graph  $H$  is a planar bipartite graph.* □

## 5. Conclusion

In this paper, we have studied the computational complexity of the unit stop number problem (U-SNP), a problem that arises in the management

of a dial-and-ride system with identical autonomous vehicles along a closed circuit. Besides providing polynomial-time algorithms to solve special cases of U-SNP, we have proved the NP-Hardness of U-SNP on planar bipartite graphs, and hence have positively proved a conjecture of Pimenta et al. (2017) for any vehicle’s capacity greater than or equal to 2. An equivalence between U-SNP and the path traffic grooming problem (PTGP) yields an improvement of the computational-complexity results known in the literature.

As a recent optimization problem, U-SNP needs to be addressed from various angles. It would be worthwhile to study the computational complexity of U-SNP for instances whose associated graph fulfills some specific properties. The structural properties of the optimal solutions to U-SNP, introduced in this paper, could represent valuable insights when designing heuristics or approximation algorithms, or when developing mixed-integer programming models for this problem or its variants.

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