Exact eddy-viscosity equation for turbulent wall flows

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Abstract

One major obstacle of turbulence modeling for CFD is the large uncertainty of the scale-determining equation in the existing popular RANS models like the k - ϵ or k - ω models. This equation has been usually constructed on a phenomenological basis and suffers from a lack of theory. This does not only affect RANS, but also hybrid methods, since most of these are designed in the RANS framework. A focus on models that contain a transport equation for the eddy viscosity ν_t may be an optimal point of attack because it enables the derivation of exact conclusions. Indeed, an analysis that focusses on three canonical turbulent wall flows: channel and pipe flows, together with zero-pressure gradient boundary layers over a flat plate, has been presented recently [1, 2]. It yields a determination of the von Karman constant, $\kappa = 0.40$, and analytical formulas for the mean strain rate $S = \partial U/\partial y$, the Reynolds shear stress τ_{xy} , hence ν_t , with U the mean flow, x (resp. y) the streamwise (resp. wall-normal) coordinate. Tests on various DNS and experimental data suggest that these formulas are valid for $Re_\tau \to \infty$. For such flows, we use the ν_t formula of [2], valid for $Re_\tau \gtrsim 500$, to derive an exact ν_t - equation of the standard form

$$\sigma \frac{\partial \nu_t}{\partial t} \;=\; \frac{\partial}{\partial y}\Big(\nu_t \frac{\partial \nu_t}{\partial y}\Big) \;+\; P_\nu \;-\; D_\nu \;=\; 0 \qquad (1)$$

with $P_\nu > 0$ the production, $D_\nu > 0$ the dissipation term. Denoting T_ν the turbulent diffusion term, the eddy-viscosity budget thus reads $T_\nu \;+\; P_\nu \;-\; D_\nu \;=\; 0$, where all terms are analytic. We use this to derive relevant theoretical conclusions about the length-scales implied, and about production and dissipation for high Re_τ and $Re_\tau = \infty$. Moreover, the exact ν_t - equation (1) may serve as a test bench of existing RANS models with a ν_t - equation [3, 4, 5, 6]. We thus explain some deficiencies of [3, 4] and propose a modification of the most promising model [5, 6]. Our results should be brought to the attention of the community since they may open the way to improved RANS and 'hybrid' or 'sensitive' simulations.

To illustrate our approach, we display the eddy-viscosity budget at large and infinite Re_τ , in the inner and outer regions, together with a validation, in the figures 1 and 2. In figure 1 the abscissa $y^+ = yu_\tau/\nu$ with u_τ the friction velocity, ν the fluid viscosity. In figure 2 the abscissa implies the macroscopic length scale δ which is the half-channel height, pipe radius, or 99% boundary layer thickness with respect to channel flow, pipe flow, and boundary layer, respectively. Of course $Re_\tau = \delta^+$. The fact that dissipation dominates in the near-wall region (figure 1) proves that dissipation of ν_t is mainly due to universal near-wall motions. The fact that production dominates in an intermediate region (figure 2) proves that production of ν_t is due to large-scale outer motions. We will complete this with a detailed analysis of the models [3, 4, 5, 6] using the channel flow DNS database [7]. The analysis of [5, 6] reveals that the 'scale adaptive' term of their ν_t - equation is responsible for discrepancies with the exact model. This may be fixed by replacing this term by a term similar to our D_ν term: $D_{\nu N} \;=\; (\kappa\nu_t/(\ell_{vK} f))^2$ with $\ell_{vK} = \kappa|S/(\partial S/\partial y)|$ the von Karman length scale, limited from below by the turbulent length scale ℓ, and $f(y^+)$ an analytic damping function. This modified model, which is in principle much more general (i.e., not only applicable to turbulent wall flows, but also applicable to more complex flows), may be as 'sensitive' or 'scale-adaptive' as the original one, but more accurate.

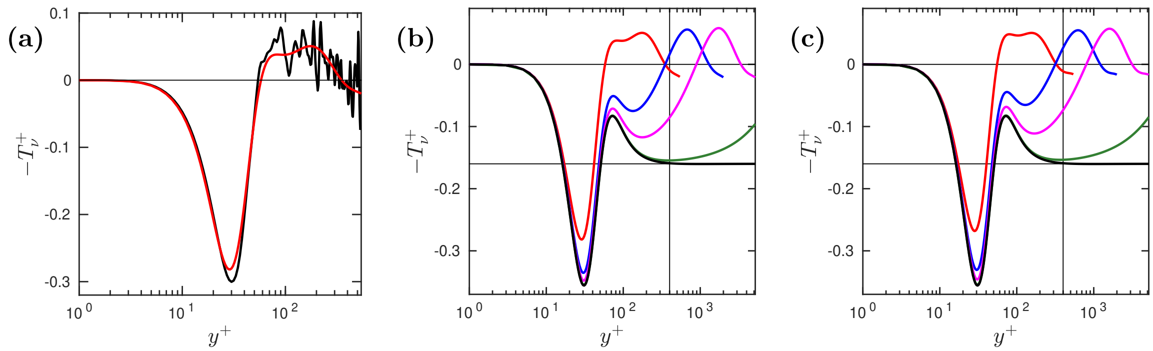

Fig. 1 : Eddy-viscosity budget in the inner region for channel and pipe flows. **(a)** Validation of our model for channel flow at $Re_\tau = 543$: the black (resp. red) curve shows $-T_\nu^+ = -T_\nu/u_\tau^2$ computed with finite differences from the DNS data of [7] (resp. with our analytical model). According to our model, for **(b)** channel flow, **(c)** pipe flow, the black curve shows $-D_\infty^+ = \lim_{Re_\tau\to\infty}(-D_\nu^+)$ at fixed y^+, the (red, blue, magenta, green) curves show $-T_\nu^+$ for $Re_\tau = (543, 1995, 5186, 80000)$. The function $D_\infty^+(y^+)$ is universal in that it does not depend on the flow case. The vertical line at $y^+ = 400$ and the horizontal line at $-T_\nu^+ = -\kappa^2$ mark the beginning of the log layer where ν_t approaches $\nu\kappa y$.

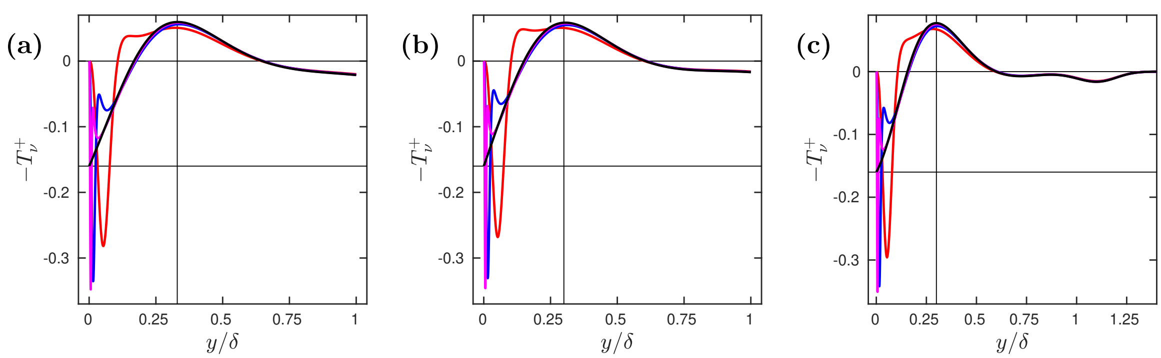

Fig. 2 : Exact eddy-viscosity budget in the outer region for **(a)** channel flow, **(b)** pipe flow, **(c)** boundary layer. The black curve shows $-T_\infty^+ = \lim_{Re_\tau\to\infty}(-T_\nu^+)$ at fixed y/δ, the (red, blue, magenta) curves show $-T_\nu^+$ for $Re_\tau = (543, 1995, 5186)$. The function $T_\infty^+(y^+)$ depends on the flow case. The horizontal line at $-T_\nu^+ = -\kappa^2$ marks the end of the log layer. The vertical lines at **(a)** $y/\delta = 0.33$, **(b,c)** $y/\delta = 0.3$ mark the production peak.

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