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# The spectrum of the Dirac operator on $SU_2/Q_8$

Nicolas Ginoux\*

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**Abstract.** We compute the fundamental Dirac operator for the three-parameter-family of homogeneous Riemannian metrics and the four different spin structures on  $SU_2/Q_8$ , where  $Q_8$  denotes the group of quaternions. We deduce its spectrum for the Berger metrics and show the sharpness of Christian Bär's upper bound for the smallest Dirac eigenvalue in the particular case where  $SU_2/Q_8$  is a homogeneous minimal hypersurface of  $S^4$ .

*Mathematics Subject Classification:* 53C27, 53C30, 58C40

*Key words:* Spin geometry, homogeneous manifolds, spectral theory

Throughout this paper and unless explicitly mentioned we denote by  $M$  the quotient of  $SU_2$  by the right-action of the group of quaternions  $Q_8$ , i.e., the group with 8 elements defined by  $\{\pm I_2, \pm A_1, \pm A_2, \pm A_3\}$  with  $A_1 := \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$ ,  $A_2 := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$  and  $A_3 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ . The manifold  $M$  is a 3-dimensional compact connected spin homogeneous space and at the same time the simplest example of homogeneous hypersurface in the round sphere with 3 different principal curvatures, see e.g. [6] and end of Section 2. Using classical techniques (see e.g. [2]) we first compute the Dirac operator of  $M$  for any homogeneous metric and any spin structure:

## Theorem 0.1

- i) The manifold  $M$  carries a 3-parameter family of homogeneous Riemannian metrics which are given by the orthonormal bases  $\{X_1 := a_1 A_1, X_2 := a_2 A_2, X_3 := a_3 A_3\}$  of  $\mathfrak{su}(2)$ , where  $a_1, a_2, a_3 \in \mathbb{R}^*$ . Conversely, every homogeneous metric on  $M$  is of that form.*
- ii) The isotropy representation  $\alpha$  of  $M$  is given in the basis  $(X_1, X_2, X_3)$  of  $\mathfrak{su}(2)$  by*

$$\begin{aligned} \alpha(\pm I_2) &= I_3 & \alpha(\pm A_1) &= \text{diag}(1, -1, -1) \\ \alpha(\pm A_2) &= \text{diag}(-1, 1, -1) & \alpha(\pm A_3) &= \text{diag}(-1, -1, 1). \end{aligned}$$

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In particular the manifold  $M$  is orientable.

iii) The manifold  $M$  is spin and carries exactly 4 spin structures, each one corresponding to one of the following group homomorphisms  $\mathbb{Q}_8 \xrightarrow{\varepsilon_j} \{-1, 1\}$ :  $\varepsilon_0 \equiv 1$  and  $\text{Ker}(\varepsilon_j) = \{\pm I_2, \pm A_j\}$  for  $j \in \{1, 2, 3\}$ .

iv) The finite dimensional Dirac operator  $D_n$  corresponding to the irreducible representation of  $\text{SU}_2$  on the space  $V_n$  of homogeneous polynomials of degree  $n$  in two variables is non-trivial only if  $n$  is odd. In that situation

$$D_n = D'_n - \frac{a_1^2 a_2^2 + a_2^2 a_3^2 + a_1^2 a_3^2}{2a_1 a_2 a_3} \text{Id}$$

where  $D'_n$  is described by a  $\frac{n+1}{2} \times \frac{n+1}{2}$  tridiagonal matrix. More precisely, there exists a basis  $(v_0, \dots, v_{\frac{n-1}{2}})$  in which  $D'_n$  can be expressed as

0) in case  $M$  carries the spin structure given by  $\varepsilon_0$ ,

$$\begin{aligned} D'_n(v_k) &= (-1)^k a_1 (n-2k)v_k + (k+1)(a_2 + (-1)^k a_3)v_{k+1} \\ &\quad + (n-k+1)(a_2 - (-1)^k a_3)v_{k-1}, \quad 0 \leq k < \frac{n-1}{2} \\ D'_n(v_{\frac{n-1}{2}}) &= \left(a_1 + \frac{n+1}{2}(a_2 + a_3)\right)v_{\frac{n-1}{2}} + \frac{n+3}{2}(a_2 - a_3)v_{\frac{n-3}{2}} \end{aligned}$$

if  $n \equiv 1 \pmod{4}$  and

$$\begin{aligned} D'_n(v_k) &= -(-1)^k a_1 (n-2k)v_k + (k+1)(a_2 - (-1)^k a_3)v_{k+1} \\ &\quad + (n-k+1)(a_2 + (-1)^k a_3)v_{k-1}, \quad 0 \leq k < \frac{n-1}{2} \\ D'_n(v_{\frac{n-1}{2}}) &= \left(a_1 - \frac{n+1}{2}(a_2 + a_3)\right)v_{\frac{n-1}{2}} + \frac{n+3}{2}(a_2 - a_3)v_{\frac{n-3}{2}} \end{aligned}$$

if  $n \equiv 3 \pmod{4}$ .

1) in case  $M$  carries the spin structure given by  $\varepsilon_1$ ,

$$\begin{aligned} D'_n(v_k) &= (-1)^k a_1 (n-2k)v_k + (k+1)(a_2 + (-1)^k a_3)v_{k+1} \\ &\quad + (n-k+1)(a_2 - (-1)^k a_3)v_{k-1}, \quad 0 \leq k < \frac{n-1}{2} \\ D'_n(v_{\frac{n-1}{2}}) &= \left(a_1 - \frac{n+1}{2}(a_2 + a_3)\right)v_{\frac{n-1}{2}} + \frac{n+3}{2}(a_2 - a_3)v_{\frac{n-3}{2}} \end{aligned}$$

if  $n \equiv 1 \pmod{4}$  and

$$\begin{aligned} D'_n(v_k) &= -(-1)^k a_1 (n-2k)v_k + (k+1)(a_2 - (-1)^k a_3)v_{k+1} \\ &\quad + (n-k+1)(a_2 + (-1)^k a_3)v_{k-1}, \quad 0 \leq k < \frac{n-1}{2} \\ D'_n(v_{\frac{n-1}{2}}) &= \left(a_1 + \frac{n+1}{2}(a_2 + a_3)\right)v_{\frac{n-1}{2}} + \frac{n+3}{2}(a_2 - a_3)v_{\frac{n-3}{2}} \end{aligned}$$

if  $n \equiv 3 \pmod{4}$ .

2) in case  $M$  carries the spin structure given by  $\varepsilon_2$ ,

$$\begin{aligned} D'_n(v_k) &= -(-1)^k a_1(n-2k)v_k + (k+1)(a_2 - (-1)^k a_3)v_{k+1} \\ &\quad + (n-k+1)(a_2 + (-1)^k a_3)v_{k-1}, \quad 0 \leq k < \frac{n-1}{2} \end{aligned}$$

$$D'_n(v_{\frac{n-1}{2}}) = \left(-a_1 + \frac{n+1}{2}(a_2 - a_3)\right)v_{\frac{n-1}{2}} + \frac{n+3}{2}(a_2 + a_3)v_{\frac{n-3}{2}}$$

if  $n \equiv 1 \pmod{4}$  and

$$\begin{aligned} D'_n(v_k) &= (-1)^k a_1(n-2k)v_k + (k+1)(a_2 + (-1)^k a_3)v_{k+1} \\ &\quad + (n-k+1)(a_2 - (-1)^k a_3)v_{k-1}, \quad 0 \leq k < \frac{n-1}{2} \end{aligned}$$

$$D'_n(v_{\frac{n-1}{2}}) = \left(-a_1 - \frac{n+1}{2}(a_2 - a_3)\right)v_{\frac{n-1}{2}} + \frac{n+3}{2}(a_2 + a_3)v_{\frac{n-3}{2}}$$

if  $n \equiv 3 \pmod{4}$ .

3) in case  $M$  carries the spin structure given by  $\varepsilon_3$ ,

$$\begin{aligned} D'_n(v_k) &= -(-1)^k a_1(n-2k)v_k + (k+1)(a_2 - (-1)^k a_3)v_{k+1} \\ &\quad + (n-k+1)(a_2 + (-1)^k a_3)v_{k-1}, \quad 0 \leq k < \frac{n-1}{2} \end{aligned}$$

$$D'_n(v_{\frac{n-1}{2}}) = \left(-a_1 - \frac{n+1}{2}(a_2 - a_3)\right)v_{\frac{n-1}{2}} + \frac{n+3}{2}(a_2 + a_3)v_{\frac{n-3}{2}}$$

if  $n \equiv 1 \pmod{4}$  and

$$\begin{aligned} D'_n(v_k) &= (-1)^k a_1(n-2k)v_k + (k+1)(a_2 + (-1)^k a_3)v_{k+1} \\ &\quad + (n-k+1)(a_2 - (-1)^k a_3)v_{k-1}, \quad 0 \leq k < \frac{n-1}{2} \end{aligned}$$

$$D'_n(v_{\frac{n-1}{2}}) = \left(-a_1 + \frac{n+1}{2}(a_2 - a_3)\right)v_{\frac{n-1}{2}} + \frac{n+3}{2}(a_2 + a_3)v_{\frac{n-3}{2}}$$

if  $n \equiv 3 \pmod{4}$ .

We deduce the spectrum of the Dirac operator  $D$  of  $M$  for the so-called Berger metrics, which form a 2-parameter subfamily of homogeneous metrics:

**Corollary 0.2** *With the notations of Theorem 0.1, assume furthermore that  $a_2 = a_3$ . Then the spectrum of the operator  $D + \frac{2a_1^2 + a_2^2}{2a_1} \text{Id}$  on  $M$  for the metric induced by  $a_1, a_2$  and the spin structure given by  $\varepsilon_j$  ( $j \in \{0, 1, 2, 3\}$ ) consists of the following family of eigenvalues:*

0. for  $j = 0$ ,

$$\begin{aligned} &\bigcup_{\substack{n \in \mathbb{N} \\ n \equiv 1 \pmod{4}}} \left\{ a_1 \pm \sqrt{(n-2k-1)^2 a_1^2 + 4(n-k)(k+1)a_2^2} \right. \\ &\quad \left. \mid k \in \left\{ 0, \dots, \frac{n-5}{2} \right\} \text{ even, } a_1 + (n+1)a_2 \right\} \end{aligned}$$

$$\bigcup_{\substack{n \in \mathbb{N} \\ n \equiv 3 \pmod{4}}} \left\{ a_1 \pm \sqrt{(n-2k-1)^2 a_1^2 + 4(n-k)(k+1)a_2^2} \right. \\ \left. | k \in \left\{ 1, \dots, \frac{n-5}{2} \right\} \text{ odd}, a_1 - (n+1)a_2, -na_1 \right\},$$

each eigenvalue having multiplicity  $n+1$  for the corresponding  $n$ .

1. for  $j = 1$ ,

$$\bigcup_{\substack{n \in \mathbb{N} \\ n \equiv 1 \pmod{4}}} \left\{ a_1 \pm \sqrt{(n-2k-1)^2 a_1^2 + 4(n-k)(k+1)a_2^2} \right. \\ \left. | k \in \left\{ 0, \dots, \frac{n-5}{2} \right\} \text{ even}, a_1 - (n+1)a_2 \right\} \\ \bigcup_{\substack{n \in \mathbb{N} \\ n \equiv 3 \pmod{4}}} \left\{ a_1 \pm \sqrt{(n-2k-1)^2 a_1^2 + 4(n-k)(k+1)a_2^2} \right. \\ \left. | k \in \left\{ 1, \dots, \frac{n-5}{2} \right\} \text{ odd}, a_1 + (n+1)a_2, -na_1 \right\},$$

each eigenvalue having multiplicity  $n+1$  for the corresponding  $n$ .

2. for  $j = 2$  and  $j = 3$ ,

$$\bigcup_{\substack{n \in \mathbb{N} \\ n \equiv 1 \pmod{4}}} \left\{ a_1 \pm \sqrt{(n-2k-1)^2 a_1^2 + 4(n-k)(k+1)a_2^2} \right. \\ \left. | k \in \left\{ 1, \dots, \frac{n-3}{2} \right\} \text{ odd}, -na_1 \right\} \\ \bigcup_{\substack{n \in \mathbb{N} \\ n \equiv 3 \pmod{4}}} \left\{ a_1 \pm \sqrt{(n-2k-1)^2 a_1^2 + 4(n-k)(k+1)a_2^2} \right. \\ \left. | k \in \left\{ 0, \dots, \frac{n-3}{2} \right\} \text{ even} \right\},$$

each eigenvalue having multiplicity  $n+1$  for the corresponding  $n$ .

In the case where  $a_1 = a_2 = a_3$ , i.e.,  $M$  is a space-form with positive curvature, we reobtain the Dirac spectrum computed by Christian Bär in [3, Thm. 2], see Corollary 3.2.

On the other hand, considering  $M$  as embedded homogeneous hypersurface in the 4-dimensional round sphere  $S^4$  one could ask if the following inequality due to Christian Bär [5, Cor. 4.3] is an equality:

$$\lambda_1(D^2) \leq \frac{9}{4}(\mathcal{H}^2 + 1), \quad (1)$$

where  $\lambda_1(D^2)$  is the smallest eigenvalue of the Dirac Laplacian on  $M$  (for the induced metric and spin structure) and  $\mathcal{H}$  is the mean curvature of  $M$  in  $S^4$ . This question takes its origin in the study of the equality case in Christian Bär's

estimate [5, Cor. 4.3] for the smallest eigenvalue  $\lambda_1(D^2)$  of the Dirac Laplacian. If this inequality is an equality, then the mean curvature of the hypersurface has to be constant, nevertheless the reverse statement has up to now neither been proved nor been contradicted. We give a partial answer to that question for  $M$ :

**Corollary 0.3** *With the notations of Theorem 0.1, assume furthermore that  $M$  carries a homogeneous metric coming from a minimal embedding in  $S^4$  and the spin structure described by  $\varepsilon_0$ . Then (1) is an equality.*

The paper is organized as follows. In the first section we describe the metrics and spin structures on  $M$  and thus prove Theorem 0.1 *i)–iii)*. In the second one we compute the Dirac operator of  $M$  (Theorem 0.1 *iv)*) and the eigenvalue of  $D_1$  (Corollary 2.9), which in the case where  $M$  is a hypersurface of  $S^4$  turns out to coincide with the upper bound in (1), see Corollary 2.11. In the third section we prove Corollary 0.2 and derive the Dirac spectrum of  $M$  in case its metric either is of constant sectional curvature or comes from a minimal embedding in  $S^4$ , see Corollary 3.2. We deduce in Corollary 3.3 the existence of non-zero real Killing spinors in the first case and Corollary 0.3 in the other one.

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## 1 Metrics and spin structures on $M$

The Lie-algebra of  $Q_8$  being trivial the adjoint representation  $\alpha$  of the homogeneous space  $M$  is nothing but the restriction of the adjoint map  $SU_2 \rightarrow \text{Aut}(\mathfrak{su}(2))$  to  $Q_8$ , where  $\mathfrak{su}(2)$  denotes the Lie-algebra of  $SU_2$ . We define the scalar product  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{su}(2)$  by declaring the following basis to be orthonormal:

$$\begin{aligned} X_1 &:= a_1 A_1 \\ X_2 &:= a_2 A_2 \\ X_3 &:= a_3 A_3, \end{aligned}$$

where  $a_1, a_2, a_3 \in \mathbb{R}^*$  are fixed parameters. The map  $\alpha$  is given in the basis  $(X_1, X_2, X_3)$  of  $\mathfrak{su}(2)$  by

$$\begin{aligned} \alpha(\pm I_2) &= I_3 \\ \alpha(\pm A_1) &= \text{diag}(1, -1, -1) \\ \alpha(\pm A_2) &= \text{diag}(-1, 1, -1) \\ \alpha(\pm A_3) &= \text{diag}(-1, -1, 1), \end{aligned}$$

therefore it obviously preserves  $\langle \cdot, \cdot \rangle$  which hence induces a homogeneous metric on  $M$ . Using the form of  $\alpha$  in the basis  $(A_1, A_2, A_3)$  computed above it is easy to prove that every homogeneous metric on  $M$  comes from such a scalar product on  $\mathfrak{su}(2)$ , i.e., it admits  $\{a_1 A_1, a_2 A_2, a_3 A_3\}$  as orthonormal basis for suitable

$a_1, a_2, a_3 \in \mathbb{R}^*$ . Note also that  $\alpha$  preserves the orientation of  $\mathfrak{su}(2)$ , so that if we choose  $(X_1, X_2, X_3)$  as positively-oriented orthonormal basis of  $\mathfrak{su}(2)$  then  $\alpha$  is expressed in that basis by a map  $\mathbb{Q}_8 \xrightarrow{\alpha} \mathrm{SO}_3$ .

We now examine the spin structures on  $M$  considering the metric and the orientation given by  $(X_1, X_2, X_3)$ . From [2, Lemma 3] the manifold  $M$  is spin if and only if its isotropy representation  $\alpha$  lifts to  $\mathrm{Spin}_3$  through the non-trivial two-fold covering  $\mathrm{Spin}_3 \xrightarrow{\xi} \mathrm{SO}_3$ , and in that case spin structures on  $M$  are in one-to-one correspondence with those lifts, each one of those being uniquely determined by a group homomorphism  $\mathbb{Q}_8 \xrightarrow{\varepsilon} \{-1, 1\}$ . Here  $\mathbb{Q}_8$  already lies in  $\mathrm{SU}_2 \cong \mathrm{Spin}_3$  so that  $M$  is obviously spin. Denoting by  $\widehat{\alpha}$  the inclusion  $\mathbb{Q}_8 \subset \mathrm{SU}_2$ , every spin structure on  $M$  is uniquely described by a map  $\widetilde{\alpha} : \mathbb{Q}_8 \rightarrow \mathrm{SU}_2$  of the form  $\widetilde{\alpha}(h) = \varepsilon(h)\widehat{\alpha}(h)$  for every  $h \in \mathbb{Q}_8$ , where  $\varepsilon : \mathbb{Q}_8 \rightarrow \{-1, 1\}$  is a group homomorphism. But there are exactly 4 such homomorphisms: the trivial one  $\varepsilon_0 \equiv 1$  and the  $\varepsilon_j$ 's,  $j = 1, 2, 3$ , with  $\mathrm{Ker}(\varepsilon_j) = \{\pm I_2, \pm A_j\}$ . This proves Theorem 0.1 *i) – iii)*.

In the following we shall call the spin structure corresponding to  $\varepsilon_j \cdot \widehat{\alpha}$  the  $\varepsilon_j$ -spin structure on  $M$ .

## 2 The Dirac operator on $M$

Let us denote by  $\mathrm{Spin}_n \xrightarrow{\delta_n} \mathrm{Aut}(\Sigma_n)$  the spinor representation in dimension  $n$ . We recall the following theorem allowing the representation-theoretical computation of the fundamental Dirac operator on a homogeneous space, see e.g. [2, Thm. 2 & Prop. 1]:

**Theorem 2.1** *Let  $M := G/H$  be an  $n$ -dimensional Riemannian homogeneous spin manifold with  $G$  compact and simply-connected. Let  $\mathfrak{p}$  be a supplementary subspace of  $\mathfrak{h}$  in  $\mathfrak{g}$ . Fix a p.o.n.b  $(X_1, \dots, X_n)$  of  $\mathfrak{p}$  and let  $\alpha : H \rightarrow \mathrm{SO}_n$  be the isotropy representation of  $M$  expressed in the basis  $(X_1, \dots, X_n)$ . Let  $\widetilde{\alpha} : H \rightarrow \mathrm{Spin}_n$  be the lift of  $\alpha$  to  $\mathrm{Spin}_n$  induced by the given spin structure of  $M$  and  $\Sigma_{\widetilde{\alpha}}M \rightarrow M$  be the spinor bundle of  $M$  associated with  $\widetilde{\alpha}$ . Let  $\widehat{G}$  be the set of equivalence classes of irreducible unitary representations of  $G$  (in the following we shall always identify an element of  $\widehat{G}$  with one of its representants).*

*i) The space  $L^2(M, \Sigma_{\widetilde{\alpha}}M)$  splits under the unitary left action of  $G$  into a direct Hilbert sum*

$$\bigoplus_{\gamma \in \widehat{G}} V_\gamma \otimes \overline{\mathrm{Hom}_H(V_\gamma, \Sigma_n)} \quad (2)$$

*where  $V_\gamma$  is the space of the representation  $\gamma$  (i.e.,  $\gamma : G \rightarrow \mathrm{U}(V_\gamma)$ ) and*

$$\mathrm{Hom}_H(V_\gamma, \Sigma_n) := \left\{ f \in \mathrm{Hom}(V_\gamma, \Sigma_n) \text{ s.t.} \right. \\ \left. \forall h \in H, f \circ \gamma(h) = (\delta_n \circ \widetilde{\alpha})(h) \circ f \right\}.$$

*ii) The Dirac operator  $D$  of  $M$  preserves each summand of (2); more precisely, if  $(e_1, \dots, e_n)$  denotes the canonical basis of  $\mathbb{R}^n$ , then for every*

$\gamma \in \widehat{G}$ , the restriction of  $D$  to  $V_\gamma \otimes \text{Hom}_H(V_\gamma, \Sigma_n)$  is given by  $\text{Id} \otimes D_\gamma$ , where, for every  $A \in \text{Hom}_H(V_\gamma, \Sigma_n)$ ,

$$D_\gamma(A) := - \sum_{k=1}^n e_k \cdot A \circ T_e \gamma(X_k) + \left( \sum_{i=1}^n \beta_i e_i + \sum_{i < j < k} \alpha_{ijk} e_i \cdot e_j \cdot e_k \right) \cdot A, \quad (3)$$

and

$$\begin{aligned} \beta_i &:= \frac{1}{2} \sum_{j=1}^n \langle [X_j, X_i]_{\mathfrak{p}}, X_j \rangle \\ \alpha_{ijk} &:= \frac{1}{4} (\langle [X_i, X_j]_{\mathfrak{p}}, X_k \rangle + \langle [X_j, X_k]_{\mathfrak{p}}, X_i \rangle + \langle [X_k, X_i]_{\mathfrak{p}}, X_j \rangle) \end{aligned}$$

(here and henceforth  $X_{\mathfrak{p}}$  will denote the image of  $X \in \mathfrak{g}$  under the projection  $\mathfrak{g} \rightarrow \mathfrak{p}$  with kernel  $\mathfrak{h}$ ).

The following statement will be useful for taking the symmetries of  $M$  into account, see Examples 2.4 below.

**Lemma 2.2** *Under the hypotheses of Theorem 2.1 let  $\langle \cdot, \cdot \rangle'$  be a further homogeneous metric on  $M$  and  $f : G \rightarrow G$  be a Lie-group-homomorphism such that  $f(H) \subset H$  and  $f_* := [T_e f]$  is an orientation-preserving isometry  $(T_{[e]}M, \langle \cdot, \cdot \rangle) \rightarrow (T_{[e]}M, \langle \cdot, \cdot \rangle')$ .*

*Then the pull-back spin structure  $f^* \text{Spin}_{\tilde{\alpha}}(TM)$  is described by*

$$\begin{aligned} H &\longrightarrow \text{Spin}_n \\ h &\longmapsto \widehat{f}^{-1} \cdot \tilde{\alpha} \circ f(h) \cdot \widehat{f} \end{aligned}$$

where  $\widehat{f} \in \text{Spin}_n$  satisfies  $\xi(\widehat{f}) = f_*$ .

*Proof:* The proof relies on the identity  $f_* \circ \text{Ad}(g) = \text{Ad}(f(g)) \circ f_*$  for every  $g \in G$ , which implies in particular

$$\alpha(h) = f_*^{-1} \circ \alpha(f(h)) \circ f_*$$

for every  $h \in H$ . □

### Notes 2.3

1. Of course the homomorphism describing the pull-back spin structure in Lemma 2.2 is well-defined since  $\widehat{f}$  is uniquely determined up to a sign.
2. One should pay attention that Lemma 2.2 can only be applied once p.o.n.b.  $(X_1, \dots, X_n)$  and  $(X'_1, \dots, X'_n)$  of  $\mathfrak{p}$  w.r.t.  $\langle \cdot, \cdot \rangle$  and  $\langle \cdot, \cdot \rangle'$  respectively have been chosen. Then all the objects above should be expressed in those bases, see Examples 2.4 below.

**Examples 2.4** Consider again  $M := \text{SU}_2/\text{Q}_8$ , fix  $a_1, a_2, a_3 \in \mathbb{R}^*$  and as above set  $X_k := a_k A_k$  for  $k \in \{1, 2, 3\}$ . We write  $(M, \langle \cdot, \cdot \rangle_{a_1, a_2, a_3}, \varepsilon_j)$  for  $M$  endowed with the metric and the orientation given by  $(X_1, X_2, X_3)$  and the  $\varepsilon_j$ -spin structure ( $j \in \{0, 1, 2, 3\}$ ).

1. Set  $X'_1 := X_1$ ,  $X'_2 := -X_2$  and  $X'_3 := -X_3$ . Let  $f(A_1) := A_1$ ,  $f(A_2) := -A_2$  and  $f(A_3) := -A_3$ . Setting  $f(I_2) := I_2$  and extending  $f$  linearly one obtains a Lie-group-homomorphism  $\text{SU}_2 \rightarrow \text{SU}_2$  inducing an orientation-preserving isometry  $(M, \langle \cdot, \cdot \rangle_{a_1, a_2, a_3}) \rightarrow (M, \langle \cdot, \cdot \rangle_{a_1, -a_2, -a_3})$ . The matrix of  $f_* = f$  in the bases  $(X_1, X_2, X_3)$  and  $(X'_1, X'_2, X'_3)$  respectively is the identity so that  $\widehat{f} = 1$  can be chosen. Applying Lemma 2.2 the pull-back of the  $\varepsilon_j$ -spin structure by  $f$  is then described by

$$\text{Q}_8 \rightarrow \text{SU}_2, \quad h \mapsto \varepsilon_j(h)f(h)$$

(remember that  $-I_2 \in \text{Ker}(\varepsilon_j)$ ), i.e., the pull-back of the  $\varepsilon_0$ - (resp.  $\varepsilon_2$ -) spin structure is the  $\varepsilon_1$ - (resp.  $\varepsilon_3$ -) one. In other words, changing the sign of both  $a_2$  and  $a_3$  changes neither the metric nor the orientation, however it permutes the  $\varepsilon_0$ - (resp.  $\varepsilon_2$ -) spin structure with the  $\varepsilon_1$ - (resp.  $\varepsilon_3$ -) one. In particular the Dirac operator on e.g.  $(M, \langle \cdot, \cdot \rangle_{a_1, a_2, a_3}, \varepsilon_0)$  coincides with that of  $(M, \langle \cdot, \cdot \rangle_{a_1, -a_2, -a_3}, \varepsilon_1)$ .

2. Let  $\sigma$  be a permutation of  $\{0, 1, 2, 3\}$  with  $\sigma(0) = 0$  and set  $X'_k := a_{\sigma(k)} A_k$  for  $k \in \{1, 2, 3\}$ . Let  $f(A_1) := A_{\sigma^{-1}(1)}$ ,  $f(A_2) := A_{\sigma^{-1}(2)}$  and  $f(A_3) := \varepsilon(\sigma) A_{\sigma^{-1}(3)}$  where  $\varepsilon(\sigma) \in \{-1, 1\}$  is the signature of  $\sigma$ . Setting in the same way as just above  $f(I_2) := I_2$  and extending  $f$  linearly one obtains a Lie-group-homomorphism  $\text{SU}_2 \rightarrow \text{SU}_2$  inducing an orientation-preserving isometry  $(M, \langle \cdot, \cdot \rangle_{a_1, a_2, a_3}) \rightarrow (M, \langle \cdot, \cdot \rangle_{a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}})$ . This time the matrix of  $f_* = f$  in the bases  $(X_1, X_2, X_3)$  and  $(X'_1, X'_2, X'_3)$  respectively is not the identity, however it coincides with the matrix of  $f$  in the basis  $(A_1, A_2, A_3)$  so that, per definition of the universal 2-fold covering map,

$$\widehat{f}^{-1} \cdot f(h) \cdot \widehat{f} = h$$

for any lift  $\widehat{f}$  of  $f$  to  $\text{SU}_2$  and every  $h \in \text{Q}_8$ . The pull-back through  $f$  of the  $\varepsilon_j$ -spin structure is therefore the  $(\varepsilon_j \circ f)$ -one, that is, the  $\varepsilon_{\sigma(j)}$ -one. In other words, permuting the coefficients  $a_1, a_2, a_3$  induces an orientation-preserving isometry permuting the spin structure in the reverse way, the  $\varepsilon_0$ -one staying unchanged under that transformation. In particular the Dirac operator on  $(M, \langle \cdot, \cdot \rangle_{a_1, a_2, a_3}, \varepsilon_j)$  coincides with that of  $(M, \langle \cdot, \cdot \rangle_{a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}}, \varepsilon_{\sigma^{-1}(j)})$ .

3. It is well-known that, for any fixed metric and spin structure on  $M$ , the Dirac operators for the two different orientations are just opposite from one another (this is always the case in odd dimensions). For example, if one turns  $a_1$  into  $-a_1$  and lets  $a_2$  and  $a_3$  unchanged, then the Dirac operator on e.g.  $(M, \langle \cdot, \cdot \rangle_{-a_1, a_2, a_3}, \varepsilon_0)$  coincides with minus that of  $(M, \langle \cdot, \cdot \rangle_{a_1, -a_2, -a_3}, \varepsilon_0)$ , i.e., with minus that of  $(M, \langle \cdot, \cdot \rangle_{a_1, a_2, a_3}, \varepsilon_1)$ .

Note that Examples 2.4 essentially exhausts all possible isometric transformations of  $M$  since the only Lie-group-automorphisms  $f$  of  $SU_2$  preserving  $Q_8$  are characterized by  $f(A_k) = \epsilon(k)A_{\sigma(k)}$  for some permutation  $\sigma$  of  $\{1, 2, 3\}$  and  $\epsilon(k) \in \{-1, 1\}$ .

We come now to the computation of the Dirac operator on  $M = SU_2/Q_8$ . We begin with the part of the Dirac operator that does not depend on the representation  $\gamma$  of  $SU_2$ . Note also that this part only depends on the metric chosen on  $M$  and not on its spin structure.

**Proposition 2.5** *For the metric on  $M$  given by  $a_1, a_2, a_3$  we have  $\beta_j = 0$  for every  $j \in \{1, 2, 3\}$  and  $\alpha_{123} = \frac{a_1^2 a_2^2 + a_2^2 a_3^2 + a_1^2 a_3^2}{2a_1 a_2 a_3}$ . In particular*

$$\sum_{j=1}^3 \beta_j e_j \cdot + \alpha_{123} e_1 \cdot e_2 \cdot e_3 \cdot = -\frac{a_1^2 a_2^2 + a_2^2 a_3^2 + a_1^2 a_3^2}{2a_1 a_2 a_3} \text{Id.}$$

*Proof:* We compute the Lie-brackets  $[X_j, X_k]$  for all  $1 \leq j < k \leq 3$ . Since  $A_1 A_2 = -A_2 A_1 = A_3$  we have

$$\begin{aligned} [X_1, X_2] &= a_1 a_2 [A_1, A_2] \\ &= 2a_1 a_2 A_3 \\ &= \frac{2a_1 a_2}{a_3} X_3, \end{aligned}$$

and analogously  $[X_2, X_3] = \frac{2a_2 a_3}{a_1} X_1$ ,  $[X_3, X_1] = \frac{2a_1 a_3}{a_2} X_2$ . We straightforwardly deduce that  $\beta_1 = \beta_2 = \beta_3 = 0$ . Furthermore,

$$\begin{aligned} \alpha_{123} &= \frac{1}{4} (\langle [X_1, X_2], X_3 \rangle + \langle [X_2, X_3], X_1 \rangle + \langle [X_3, X_1], X_2 \rangle) \\ &= \frac{1}{4} \left( \frac{2a_1 a_2}{a_3} + \frac{2a_2 a_3}{a_1} + \frac{2a_1 a_3}{a_2} \right) \\ &= \frac{a_1^2 a_2^2 + a_2^2 a_3^2 + a_1^2 a_3^2}{2a_1 a_2 a_3}. \end{aligned}$$

It remains to notice that, by convention, the complex volume form  $i^{\lfloor \frac{3+1}{2} \rfloor} e_1 \cdot e_2 \cdot e_3 = -e_1 \cdot e_2 \cdot e_3$  acts by the identity on  $\Sigma_3$ . This concludes the proof.  $\square$

We next determine the space of equivariant homomorphisms for each  $\gamma \in \widehat{SU}_2$  and each  $\varepsilon_j$ -spin structure on  $M$ . First recall that the irreducible unitary representations of  $SU_2$  are given by its natural action on the  $n+1$ -dimensional vector spaces of all  $n$ -graded homogeneous complex polynomials in two variables: set, for any  $n \in \mathbb{N}$  (we include  $n = 0$ )

$$V_n := \{P \in \mathbb{C}[z_1, z_2], \quad P = 0 \text{ or } P \text{ homogeneous and } d^c P = n\}.$$

Then  $SU_2$  acts on  $V_n$  through

$$\begin{aligned}\pi_n : SU_2 &\longrightarrow \text{Aut}(V_n) \\ A &\longmapsto (\pi_n(A) : P \mapsto P \circ R_A),\end{aligned}$$

where  $P \circ R_A(z) := P(zA)$  for every  $z = (z_1 \ z_2) \in \mathbb{C}^2$ . From now on we shall always work with the following basis of  $V_n$ :

$$(P_k(z_1, z_2) := z_1^{n-k} z_2^k, \quad 0 \leq k \leq n).$$

Identifying  $\text{Spin}_3$  to  $SU_2$  the spinor representation  $\text{Spin}_3 \xrightarrow{\delta_3} \text{Aut}(\Sigma_3)$  is equivalent to the standard representation  $SU_2 \longrightarrow \text{Aut}(\mathbb{C}^2)$ . For every lift  $\varepsilon_j \cdot \hat{\alpha}$  of the isotropy representation  $\alpha$  of  $M$  the space of equivariant homomorphisms for  $\pi_n$  and for the  $\varepsilon_j$ -spin structure - that we shall denote by  $\text{Hom}_{\mathbb{Q}_8, \varepsilon_j}(V_n, \mathbb{C}^2)$  - is then given by

$$\text{Hom}_{\mathbb{Q}_8, \varepsilon_j}(V_n, \mathbb{C}^2) = \{f \in \text{Hom}(V_n, \mathbb{C}^2) \text{ s.t. } f \circ \pi_n(h) = \varepsilon_j(h)h \circ f \quad \forall h \in \mathbb{Q}_8\}.$$

We fix the following basis  $(F_0, \dots, F_n, G_0, \dots, G_n)$  of  $\text{Hom}(V_n, \mathbb{C}^2)$  (which is that of [2, p.73]): set, for every  $k \in \{0, \dots, n\}$ ,

$$F_k(P_l) := \begin{cases} (1 \ 0) & \text{if } l = k \text{ and } k \text{ even} \\ (0 \ 1) & \text{if } l = k \text{ and } k \text{ odd} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$G_k(P_l) := \begin{cases} (0 \ 1) & \text{if } l = k \text{ and } k \text{ even} \\ (1 \ 0) & \text{if } l = k \text{ and } k \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

W.r.t. the bases  $(P_0, \dots, P_n)$  and  $((1 \ 0), (0 \ 1))$  of  $V_n$  and  $\mathbb{C}^2$  respectively the elements  $F_k$  and  $G_k$  are described by matrices of the form:

$$F_k = \begin{pmatrix} 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad G_k = \begin{pmatrix} 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & 0 & \dots & 0 \end{pmatrix}$$

if  $k$  is even and

$$F_k = \begin{pmatrix} 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & 0 & \dots & 0 \end{pmatrix}, \quad G_k = \begin{pmatrix} 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

if  $k$  is odd, where the “1” always stands in the  $(k+1)^{\text{st}}$  column.

**Lemma 2.6** *Let  $M$  carry the  $\varepsilon_j$ -spin structure for  $j \in \{0, 1, 2, 3\}$ . Then  $\text{Hom}_{\mathbb{Q}_8, \varepsilon_j}(V_n, \mathbb{C}^2) = \{0\}$  if  $n$  is even. Moreover*

0. *for  $j = 0$  we have*

$$\text{Hom}_{\mathbb{Q}_8, \varepsilon_0}(V_n, \mathbb{C}^2) = \begin{cases} \bigoplus_{k=0}^{\frac{n-1}{2}} \mathbb{C}(F_k + F_{n-k}) & \text{if } n \equiv 1 \pmod{4} \\ \bigoplus_{k=0}^{\frac{n-1}{2}} \mathbb{C}(G_k - G_{n-k}) & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

1. for  $j = 1$  we have

$$\text{Hom}_{\mathbb{Q}_8, \varepsilon_1}(V_n, \mathbb{C}^2) = \begin{cases} \bigoplus_{k=0}^{\frac{n-1}{2}} \mathbb{C}(F_k - F_{n-k}) & \text{if } n \equiv 1 \pmod{4} \\ \bigoplus_{k=0}^{\frac{n-1}{2}} \mathbb{C}(G_k + G_{n-k}) & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

2. for  $j = 2$  we have

$$\text{Hom}_{\mathbb{Q}_8, \varepsilon_2}(V_n, \mathbb{C}^2) = \begin{cases} \bigoplus_{k=0}^{\frac{n-1}{2}} \mathbb{C}(G_k + G_{n-k}) & \text{if } n \equiv 1 \pmod{4} \\ \bigoplus_{k=0}^{\frac{n-1}{2}} \mathbb{C}(F_k - F_{n-k}) & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

3. for  $j = 3$  we have

$$\text{Hom}_{\mathbb{Q}_8, \varepsilon_3}(V_n, \mathbb{C}^2) = \begin{cases} \bigoplus_{k=0}^{\frac{n-1}{2}} \mathbb{C}(G_k - G_{n-k}) & \text{if } n \equiv 1 \pmod{4} \\ \bigoplus_{k=0}^{\frac{n-1}{2}} \mathbb{C}(F_k + F_{n-k}) & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

*Proof:* Since  $-\mathbf{I}_2 \in \text{Ker}(\varepsilon_j)$  any element  $f \in \text{Hom}_{\mathbb{Q}_8, \varepsilon_j}(V_n, \mathbb{C}^2)$  must satisfy  $f \circ \pi_n(-\mathbf{I}_2) = -f$ , with  $\pi_n(-\mathbf{I}_2) = (-1)^n \text{Id}_{V_n}$ , so that the condition reads

$$(-1)^n f = -f,$$

which requires  $f = 0$  as soon as  $n$  is even.

From now on, we assume that  $n$  is odd. We compute  $\pi_n(A_j)$  for  $j = 1, 2$  (remember that  $A_1$  and  $A_2$  generate  $\mathbb{Q}_8$ ): for every  $k \in \{0, \dots, n\}$  and  $z \in \mathbb{C}^2$ ,

$$\begin{aligned} \{\pi_n(A_1)\}(P_k)(z) &= P_k \left( (z_1 \ z_2) \cdot \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \right) \\ &= P_k(-iz_1, iz_2) \\ &= (-iz_1)^{n-k} (iz_2)^k \\ &= (-1)^{n-k} i^n z_1^{n-k} z_2^k, \end{aligned}$$

i.e.,  $\{\pi_n(A_1)\}(P_k) = (-1)^{n-k} i^n P_k$ . Analogously,

$$\begin{aligned} \{\pi_n(A_2)\}(P_k)(z) &= P_k \left( (z_1 \ z_2) \cdot \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right) \\ &= P_k(iz_2, iz_1) \\ &= (iz_2)^{n-k} (iz_1)^k, \end{aligned}$$

i.e.,  $\{\pi_n(A_2)\}(P_k) = i^n P_{n-k}$ . The conditions  $f \circ \pi_n(A_l) = \varepsilon_j(A_l) A_l \circ f$  for  $l = 1, 2$  then read

$$\begin{cases} f(P_k) &= (-1)^{k+\frac{n-1}{2}} i \varepsilon_j(A_1) (A_1 \circ f)(P_k) \\ f(P_{n-k}) &= (-1)^{\frac{n+1}{2}} i \varepsilon_j(A_2) (A_2 \circ f)(P_k) \end{cases} \quad (4)$$

for every  $k \in \{0, 1, \dots, n\}$ . From now on we denote by  $\begin{pmatrix} f_{1k} \\ f_{2k} \end{pmatrix} := f(P_k) \in \mathbb{C}^2$ .

We examine each case separately.

• *Case  $j = 0$ :* In that case the conditions (4) are equivalent to

$$\begin{cases} f(P_k) &= (-1)^{k+\frac{n-1}{2}} i(A_1 \circ f)(P_k) \\ f(P_{n-k}) &= (-1)^{\frac{n+1}{2}} i(A_2 \circ f)(P_k), \end{cases}$$

that is,

$$\begin{cases} f_{1k} &= (-1)^{k+\frac{n-1}{2}} f_{1k} \\ f_{2k} &= (-1)^{k+\frac{n+1}{2}} f_{2k} \\ f_{1n-k} &= (-1)^{\frac{n-1}{2}} f_{2k} \\ f_{2n-k} &= (-1)^{\frac{n-1}{2}} f_{1k}. \end{cases}$$

If  $n \equiv 1$  (4) then those identities become

$$\begin{cases} f_{1k} &= (-1)^k f_{1k} \\ f_{2k} &= -(-1)^k f_{2k} \\ f_{1n-k} &= f_{2k} \\ f_{2n-k} &= f_{1k}, \end{cases}$$

hence  $f_{1k} = 0$  if  $k$  is odd (resp.  $f_{2k} = 0$  if  $k$  is even) and  $(f_{1n-k}, f_{2n-k}) = (f_{2k}, f_{1k})$  for every  $0 \leq k \leq \frac{n-1}{2}$ . We deduce that

$$f = f_{10}(F_0 + F_n) + f_{21}(F_1 + F_{n-1}) + \dots + f_{1\frac{n-1}{2}}(F_{\frac{n-1}{2}} + F_{\frac{n+1}{2}})$$

and the result in that case.

If  $n \equiv 3$  (4) then those identities become

$$\begin{cases} f_{1k} &= -(-1)^k f_{1k} \\ f_{2k} &= (-1)^k f_{2k} \\ f_{1n-k} &= -f_{2k} \\ f_{2n-k} &= -f_{1k}, \end{cases}$$

hence  $f_{1k} = 0$  if  $k$  is even (resp.  $f_{2k} = 0$  if  $k$  is odd) and  $(f_{1n-k}, f_{2n-k}) = (-f_{2k}, -f_{1k})$  for every  $0 \leq k \leq \frac{n-1}{2}$ . We deduce that

$$f = f_{20}(G_0 - G_n) + f_{11}(G_1 - G_{n-1}) + \dots + f_{1\frac{n-1}{2}}(G_{\frac{n-1}{2}} - G_{\frac{n+1}{2}})$$

and the result in that case.

• *Case  $j = 1$ :* In that case the conditions (4) are equivalent to

$$\begin{cases} f(P_k) &= (-1)^{k+\frac{n-1}{2}} i(A_1 \circ f)(P_k) \\ f(P_{n-k}) &= (-1)^{\frac{n-1}{2}} i(A_2 \circ f)(P_k), \end{cases}$$

that is,

$$\begin{cases} f_{1k} &= (-1)^{k+\frac{n-1}{2}} f_{1k} \\ f_{2k} &= (-1)^{k+\frac{n+1}{2}} f_{2k} \\ f_{1n-k} &= (-1)^{\frac{n+1}{2}} f_{2k} \\ f_{2n-k} &= (-1)^{\frac{n+1}{2}} f_{1k}. \end{cases}$$

If  $n \equiv 1 \pmod{4}$  then those identities become

$$\begin{cases} f_{1k} &= (-1)^k f_{1k} \\ f_{2k} &= -(-1)^k f_{2k} \\ f_{1n-k} &= -f_{2k} \\ f_{2n-k} &= -f_{1k}, \end{cases}$$

hence  $f_{1k} = 0$  if  $k$  is odd (resp.  $f_{2k} = 0$  if  $k$  is even) and  $(f_{1n-k}, f_{2n-k}) = (-f_{2k}, -f_{1k})$  for every  $0 \leq k \leq \frac{n-1}{2}$ . We deduce that

$$f = f_{10}(F_0 - F_n) + f_{21}(F_1 - F_{n-1}) + \dots + f_{1\frac{n-1}{2}}(F_{\frac{n-1}{2}} - F_{\frac{n+1}{2}})$$

and the result in that case.

If  $n \equiv 3 \pmod{4}$  then those identities become

$$\begin{cases} f_{1k} &= -(-1)^k f_{1k} \\ f_{2k} &= (-1)^k f_{2k} \\ f_{1n-k} &= f_{2k} \\ f_{2n-k} &= f_{1k}, \end{cases}$$

hence  $f_{1k} = 0$  if  $k$  is even (resp.  $f_{2k} = 0$  if  $k$  is odd) and  $(f_{1n-k}, f_{2n-k}) = (f_{2k}, f_{1k})$  for every  $0 \leq k \leq \frac{n-1}{2}$ . We deduce that

$$f = f_{20}(G_0 + G_n) + f_{11}(G_1 + G_{n-1}) + \dots + f_{1\frac{n-1}{2}}(G_{\frac{n-1}{2}} + G_{\frac{n+1}{2}})$$

and the result in that case.

• *Case  $j = 2$ :* In that case the conditions (4) are equivalent to

$$\begin{cases} f(P_k) &= (-1)^{k+\frac{n+1}{2}} i(A_1 \circ f)(P_k) \\ f(P_{n-k}) &= (-1)^{\frac{n+1}{2}} i(A_2 \circ f)(P_k), \end{cases}$$

that is,

$$\begin{cases} f_{1k} &= (-1)^{k+\frac{n+1}{2}} f_{1k} \\ f_{2k} &= (-1)^{k+\frac{n-1}{2}} f_{2k} \\ f_{1n-k} &= (-1)^{\frac{n-1}{2}} f_{2k} \\ f_{2n-k} &= (-1)^{\frac{n-1}{2}} f_{1k}. \end{cases}$$

If  $n \equiv 1 \pmod{4}$  then those identities become

$$\begin{cases} f_{1k} &= -(-1)^k f_{1k} \\ f_{2k} &= (-1)^k f_{2k} \\ f_{1n-k} &= f_{2k} \\ f_{2n-k} &= f_{1k}, \end{cases}$$

hence  $f_{1k} = 0$  if  $k$  is even (resp.  $f_{2k} = 0$  if  $k$  is odd) and  $(f_{1n-k}, f_{2n-k}) = (f_{2k}, f_{1k})$  for every  $0 \leq k \leq \frac{n-1}{2}$ . We deduce that

$$f = f_{20}(G_0 + G_n) + f_{11}(G_1 + G_{n-1}) + \dots + f_{2\frac{n-1}{2}}(G_{\frac{n-1}{2}} + G_{\frac{n+1}{2}})$$

and the result in that case.

If  $n \equiv 3 \pmod{4}$  then those identities become

$$\begin{cases} f_{1k} &= (-1)^k f_{1k} \\ f_{2k} &= -(-1)^k f_{2k} \\ f_{1n-k} &= -f_{2k} \\ f_{2n-k} &= -f_{1k}, \end{cases}$$

hence  $f_{1k} = 0$  if  $k$  is odd (resp.  $f_{2k} = 0$  if  $k$  is even) and  $(f_{1n-k}, f_{2n-k}) = (-f_{2k}, -f_{1k})$  for every  $0 \leq k \leq \frac{n-1}{2}$ . We deduce that

$$f = f_{10}(F_0 - F_n) + f_{21}(F_1 - F_{n-1}) + \dots + f_{2\frac{n-1}{2}}(F_{\frac{n-1}{2}} - F_{\frac{n+1}{2}})$$

and the result in that case.

• *Case  $j = 3$* : In that case the conditions (4) are equivalent to

$$\begin{cases} f(P_k) &= (-1)^{k+\frac{n+1}{2}} i(A_1 \circ f)(P_k) \\ f(P_{n-k}) &= (-1)^{\frac{n-1}{2}} i(A_2 \circ f)(P_k), \end{cases}$$

that is,

$$\begin{cases} f_{1k} &= (-1)^{k+\frac{n+1}{2}} f_{1k} \\ f_{2k} &= (-1)^{k+\frac{n-1}{2}} f_{2k} \\ f_{1n-k} &= (-1)^{\frac{n+1}{2}} f_{2k} \\ f_{2n-k} &= (-1)^{\frac{n+1}{2}} f_{1k}. \end{cases}$$

If  $n \equiv 1 \pmod{4}$  then those identities become

$$\begin{cases} f_{1k} &= -(-1)^k f_{1k} \\ f_{2k} &= (-1)^k f_{2k} \\ f_{1n-k} &= -f_{2k} \\ f_{2n-k} &= -f_{1k}, \end{cases}$$

hence  $f_{1k} = 0$  if  $k$  is even (resp.  $f_{2k} = 0$  if  $k$  is odd) and  $(f_{1n-k}, f_{2n-k}) = (-f_{2k}, -f_{1k})$  for every  $0 \leq k \leq \frac{n-1}{2}$ . We deduce that

$$f = f_{20}(G_0 - G_n) + f_{11}(G_1 - G_{n-1}) + \dots + f_{2\frac{n-1}{2}}(G_{\frac{n-1}{2}} - G_{\frac{n+1}{2}})$$

and the result in that case.

If  $n \equiv 3 \pmod{4}$  then those identities become

$$\begin{cases} f_{1k} &= (-1)^k f_{1k} \\ f_{2k} &= -(-1)^k f_{2k} \\ f_{1n-k} &= f_{2k} \\ f_{2n-k} &= f_{1k}, \end{cases}$$

hence  $f_{1k} = 0$  if  $k$  is odd (resp.  $f_{2k} = 0$  if  $k$  is even) and  $(f_{1n-k}, f_{2n-k}) = (f_{2k}, f_{1k})$  for every  $0 \leq k \leq \frac{n-1}{2}$ . We deduce that

$$f = f_{10}(F_0 + F_n) + f_{21}(F_1 + F_{n-1}) + \dots + f_{2\frac{n-1}{2}}(F_{\frac{n-1}{2}} + F_{\frac{n+1}{2}})$$

and the result in that case. This concludes the proof.  $\square$

It remains to compute the map  $T_{1_2}\pi_n$  for every (odd)  $n$ .

**Lemma 2.7** *The endomorphisms  $T_{1_2}\pi_n(X_j)$ ,  $1 \leq j \leq 3$ , are given in the basis  $(P_0, \dots, P_n)$  of  $V_n$  by:*

$$\begin{aligned} \{T_{1_2}\pi_n(X_1)\}(P_k) &= -ia_1(n-2k)P_k \\ \{T_{1_2}\pi_n(X_2)\}(P_k) &= ia_2((n-k)P_{k+1} + kP_{k-1}) \\ \{T_{1_2}\pi_n(X_3)\}(P_k) &= a_3(-(n-k)P_{k+1} + kP_{k-1}) \end{aligned}$$

for every  $k \in \{0, \dots, n\}$ , with the convention  $P_{-1} = P_{n+1} = 0$ .

*Proof:* For every  $X \in \mathfrak{su}_2$ ,  $P \in V_n$  and  $z \in \mathbb{C}^2$ , we have

$$\begin{aligned}
(\{T_{I_2}\pi_n(X)\}(P))(z) &= \frac{d}{dt}\Big|_{t=0} (P \circ R_{\exp(tX)})(z) \\
&= \frac{d}{dt}\Big|_{t=0} (P \circ R_{\exp(tX)}(z)) \\
&= \frac{d}{dt}\Big|_{t=0} (P(z \exp(tX))) \\
&= d_z P(zX) \\
&= \frac{\partial P}{\partial z_1}(z)(zX)_1 + \frac{\partial P}{\partial z_2}(z)(zX)_2.
\end{aligned}$$

Since  $zA_1 = (-iz_1 \ iz_2)$ ,  $zA_2 = (iz_2 \ iz_1)$  and  $zA_3 = (-z_2 \ z_1)$  we have, for every  $k \in \{0, \dots, n\}$

$$\begin{aligned}
\{T_{I_2}\pi_n(X_1)\}(P_k) &= a_1\{T_{I_2}\pi_n(A_1)\}(P_k) \\
&= a_1 \left( -iz_1 \frac{\partial P_k}{\partial z_1}(z) + iz_2 \frac{\partial P_k}{\partial z_2}(z) \right) \\
&= -ia_1 \left( (n-k)z_1 z_1^{n-k-1} z_2^k - kz_2 z_1^{n-k} z_2^{k-1} \right) \\
&= -ia_1 \left( (n-k)z_1^{n-k} z_2^k - kz_1^{n-k} z_2^k \right) \\
&= -ia_1(n-2k)P_k.
\end{aligned}$$

For  $X_2$  we have

$$\begin{aligned}
\{T_{I_2}\pi_n(X_2)\}(P_k) &= a_2\{T_{I_2}\pi_n(A_2)\}(P_k) \\
&= a_2 \left( iz_2 \frac{\partial P_k}{\partial z_1}(z) + iz_1 \frac{\partial P_k}{\partial z_2}(z) \right) \\
&= ia_2 \left( (n-k)z_1^{n-k-1} z_2^{k+1} + kz_1^{n-k+1} z_2^{k-1} \right) \\
&= ia_2 \left( (n-k)P_{k+1} + kP_{k-1} \right),
\end{aligned}$$

and for  $X_3$  we obtain

$$\begin{aligned}
\{T_{I_2}\pi_n(X_3)\}(P_k) &= a_3\{T_{I_2}\pi_n(A_3)\}(P_k) \\
&= a_3 \left( -z_2 \frac{\partial P_k}{\partial z_1}(z) + z_1 \frac{\partial P_k}{\partial z_2}(z) \right) \\
&= a_3 \left( -(n-k)z_1^{n-k-1} z_2^{k+1} + kz_1^{n-k+1} z_2^{k-1} \right) \\
&= a_3 \left( -(n-k)P_{k+1} + kP_{k-1} \right).
\end{aligned}$$

Note that the above expressions for  $\{T_{I_2}\pi_n(X_2)\}(P_k)$  and  $\{T_{I_2}\pi_n(X_3)\}(P_k)$  are also valid for  $k = 0$  or  $k = n$  with the convention  $P_{-1} = P_{n+1} = 0$ . The result follows.  $\square$

We now compute the component  $D_n$  of the Dirac operator of  $M$  acting on  $\text{Hom}_{\mathbb{Q}_8, \varepsilon_j}(V_n, \mathbb{C}^2)$ , see (3). We adopt henceforth the following convention:  $F_k := G_k := 0$  as soon as  $k \notin \{0, \dots, n\}$ .

The fix part of  $D_n$  has already been computed in Proposition 2.5, so that only the endomorphism  $D'_n$  of  $\text{Hom}_{\mathbb{Q}_8, \varepsilon_j}(V_n, \mathbb{C}^2)$  given by

$$D'_n A = - \sum_{j=1}^3 e_j \cdot A \circ T_{I_2} \pi_n(X_j)$$

for every  $A \in \text{Hom}_{\mathbb{Q}_8, \varepsilon_j}(V_n, \mathbb{C}^2)$ , remains to be made explicit.

First note that the Clifford product by  $e_j$  can be identified with the matrix multiplication by  $A_j$  for  $j \in \{1, 2, 3\}$ .

Furthermore, it is straightforward to show using Lemma 2.7 that, for every  $k \in \{0, 1, \dots, n\}$ ,

$$\begin{aligned} F_k \circ T_{I_2} \pi_n(X_1) &= -ia_1(n-2k)F_k \\ F_k \circ T_{I_2} \pi_n(X_2) &= ia_2((n-k+1)G_{k-1} + (k+1)G_{k+1}) \\ F_k \circ T_{I_2} \pi_n(X_3) &= a_3(-(n-k+1)G_{k-1} + (k+1)G_{k+1}). \end{aligned}$$

Those identities still hold for  $k=0$  or  $n$  using our convention above on the  $F_k$ 's and  $G_k$ 's. To obtain the corresponding identities on the  $G_k$ 's one just has to exchange the roles of  $F_l$  and  $G_l$  for every  $l$ :

$$\begin{aligned} G_k \circ T_{I_2} \pi_n(X_1) &= -ia_1(n-2k)G_k \\ G_k \circ T_{I_2} \pi_n(X_2) &= ia_2((n-k+1)F_{k-1} + (k+1)F_{k+1}) \\ G_k \circ T_{I_2} \pi_n(X_3) &= a_3(-(n-k+1)F_{k-1} + (k+1)F_{k+1}). \end{aligned}$$

We deduce the following set of identities:

$$\begin{aligned} (F_k \pm F_{n-k}) \circ T_{I_2} \pi_n(X_1) &= -ia_1(n-2k)(F_k \mp F_{n-k}) \\ (F_k \pm F_{n-k}) \circ T_{I_2} \pi_n(X_2) &= ia_2\left((k+1)(G_{k+1} \pm G_{n-k-1}) \right. \\ &\quad \left. + (n-k+1)(G_{k-1} \pm G_{n-k+1})\right) \\ (F_k \pm F_{n-k}) \circ T_{I_2} \pi_n(X_3) &= a_3\left((k+1)(G_{k+1} \mp G_{n-k-1}) \right. \\ &\quad \left. - (n-k+1)(G_{k-1} \mp G_{n-k+1})\right) \\ (G_k \pm G_{n-k}) \circ T_{I_2} \pi_n(X_1) &= -ia_1(n-2k)(G_k \mp G_{n-k}) \\ (G_k \pm G_{n-k}) \circ T_{I_2} \pi_n(X_2) &= ia_2\left((k+1)(F_{k+1} \pm F_{n-k-1}) \right. \\ &\quad \left. + (n-k+1)(F_{k-1} \pm F_{n-k+1})\right) \\ (G_k \pm G_{n-k}) \circ T_{I_2} \pi_n(X_3) &= a_3\left((k+1)(F_{k+1} \mp F_{n-k-1}) \right. \\ &\quad \left. - (n-k+1)(F_{k-1} \mp F_{n-k+1})\right). \end{aligned} \tag{5}$$

On the other hand, it is also a short calculation to show

$$\begin{aligned} A_1 \cdot (F_k \pm F_{n-k}) &= (-1)^{k+1}i(F_k \mp F_{n-k}) \\ A_2 \cdot (F_k \pm F_{n-k}) &= i(G_k \pm G_{n-k}) \\ A_3 \cdot (F_k \pm F_{n-k}) &= (-1)^{k+1}(G_k \mp G_{n-k}) \\ A_1 \cdot (G_k \pm G_{n-k}) &= (-1)^k i(G_k \mp G_{n-k}) \\ A_2 \cdot (G_k \pm G_{n-k}) &= i(F_k \pm F_{n-k}) \\ A_3 \cdot (G_k \pm G_{n-k}) &= (-1)^k(F_k \mp F_{n-k}). \end{aligned} \tag{6}$$

Bringing (5) and (6) together we deduce that

$$\begin{aligned}
D'_n(F_k \pm F_{n-k}) &= - \sum_{j=1}^3 e_j \cdot (F_k \pm F_{n-k}) \circ T_{I_2} \pi_n(X_j) \\
&= - \sum_{j=1}^3 A_j \cdot (F_k \pm F_{n-k}) \circ T_{I_2} \pi_n(X_j) \\
&\stackrel{(5)}{=} ia_1(n-2k)A_1 \cdot (F_k \mp F_{n-k}) \\
&\quad - ia_2A_2 \cdot \left( (k+1)(G_{k+1} \pm G_{n-k-1}) + (n-k+1)(G_{k-1} \pm G_{n-k+1}) \right) \\
&\quad - a_3A_3 \cdot \left( (k+1)(G_{k+1} \mp G_{n-k-1}) - (n-k+1)(G_{k-1} \mp G_{n-k+1}) \right) \\
&\stackrel{(6)}{=} (-1)^k a_1(n-2k)(F_k \pm F_{n-k}) \\
&\quad + a_2 \left( (k+1)(F_{k+1} \pm F_{n-k-1}) + (n-k+1)(F_{k-1} \pm F_{n-k+1}) \right) \\
&\quad + (-1)^k a_3 \left( (k+1)(F_{k+1} \pm F_{n-k-1}) - (n-k+1)(F_{k-1} \pm F_{n-k+1}) \right) \\
&= (-1)^k a_1(n-2k)(F_k \pm F_{n-k}) \\
&\quad + (k+1)(a_2 + (-1)^k a_3)(F_{k+1} \pm F_{n-k-1}) \\
&\quad + (n-k+1)(a_2 - (-1)^k a_3)(F_{k-1} \pm F_{n-k+1}).
\end{aligned}$$

Similarly,

$$\begin{aligned}
D'_n(G_k \pm G_{n-k}) &= - \sum_{j=1}^3 A_j \cdot (G_k \pm G_{n-k}) \circ T_{I_2} \pi_n(X_j) \\
&\stackrel{(5)}{=} ia_1(n-2k)A_1 \cdot (G_k \mp G_{n-k}) \\
&\quad - ia_2A_2 \cdot \left( (k+1)(F_{k+1} \pm F_{n-k-1}) + (n-k+1)(F_{k-1} \pm F_{n-k+1}) \right) \\
&\quad - a_3A_3 \cdot \left( (k+1)(F_{k+1} \mp F_{n-k-1}) - (n-k+1)(F_{k-1} \mp F_{n-k+1}) \right) \\
&\stackrel{(6)}{=} -(-1)^k a_1(n-2k)(G_k \pm G_{n-k}) \\
&\quad + a_2 \left( (k+1)(G_{k+1} \pm G_{n-k-1}) + (n-k+1)(G_{k-1} \pm G_{n-k+1}) \right) \\
&\quad - (-1)^k a_3 \left( (k+1)(G_{k+1} \pm G_{n-k-1}) - (n-k+1)(G_{k-1} \pm G_{n-k+1}) \right) \\
&= -(-1)^k a_1(n-2k)(G_k \pm G_{n-k}) \\
&\quad + (k+1)(a_2 - (-1)^k a_3)(G_{k+1} \pm G_{n-k-1}) \\
&\quad + (n-k+1)(a_2 + (-1)^k a_3)(G_{k-1} \pm G_{n-k+1}).
\end{aligned}$$

Note that, for  $k = \frac{n-1}{2}$ ,  $F_{k+1} \pm F_{n-k-1} = \pm(F_k \pm F_{n-k})$  and the same holds for the  $G_k$ 's, so that

$$\begin{aligned}
&D'_n(F_{\frac{n-1}{2}} \pm F_{\frac{n+1}{2}}) \\
&= (-1)^{\frac{n-1}{2}} a_1(F_{\frac{n-1}{2}} \pm F_{\frac{n+1}{2}}) \\
&\quad + \frac{n+1}{2} (a_2 + (-1)^{\frac{n-1}{2}} a_3)(F_{\frac{n+1}{2}} \pm F_{\frac{n-1}{2}})
\end{aligned}$$

$$\begin{aligned}
& + \frac{n+3}{2}(a_2 - (-1)^{\frac{n-1}{2}}a_3)(F_{\frac{n-3}{2}} \pm F_{\frac{n+3}{2}}) \\
= & \left( (-1)^{\frac{n-1}{2}}a_1 \pm \frac{n+1}{2}(a_2 + (-1)^{\frac{n-1}{2}}a_3) \right) (F_{\frac{n-1}{2}} \pm F_{\frac{n+1}{2}}) \\
& + \frac{n+3}{2}(a_2 - (-1)^{\frac{n-1}{2}}a_3)(F_{\frac{n-3}{2}} \pm F_{\frac{n+3}{2}})
\end{aligned}$$

and in the same way

$$\begin{aligned}
& D'_n(G_{\frac{n-1}{2}} \pm G_{\frac{n+1}{2}}) \\
= & -(-1)^{\frac{n-1}{2}}a_1(G_{\frac{n-1}{2}} \pm G_{\frac{n+1}{2}}) \\
& + \frac{n+1}{2}(a_2 - (-1)^{\frac{n-1}{2}}a_3)(G_{\frac{n+1}{2}} \pm G_{\frac{n-1}{2}}) \\
& + \frac{n+3}{2}(a_2 + (-1)^{\frac{n-1}{2}}a_3)(G_{\frac{n-3}{2}} \pm G_{\frac{n+3}{2}}) \\
= & \left( -(-1)^{\frac{n-1}{2}}a_1 \pm \frac{n+1}{2}(a_2 - (-1)^{\frac{n-1}{2}}a_3) \right) (G_{\frac{n-1}{2}} \pm G_{\frac{n+1}{2}}) \\
& + \frac{n+3}{2}(a_2 + (-1)^{\frac{n-1}{2}}a_3)(G_{\frac{n-3}{2}} \pm G_{\frac{n+3}{2}}).
\end{aligned}$$

Denoting by  $(v_0, \dots, v_{\frac{n-1}{2}})$  the basis of  $\text{Hom}_{\mathbb{Q}_8, \varepsilon_j}(V_n, \mathbb{C}^2)$  computed in Lemma 2.6 we conclude the proof of Theorem 0.1 *iv*).

**Note 2.8** From Theorem 0.1 *iv*) the matrix representing the operator  $D_n$  in the basis  $(v_0, \dots, v_{\frac{n-1}{2}})$  is not symmetric. Beware however that this basis does not take  $A_1, A_2, A_3$  into account the same way and turns out not to be orthonormal.

We now make the eigenvalue of  $D_1$  explicit:

**Corollary 2.9** Fix  $j \in \{0, 1, 2, 3\}$  and let  $\epsilon_1, \epsilon_2, \epsilon_3 \in \{-1, 1\}$  be defined by  $\epsilon_l := -(-1)^{\delta_{j0} + \delta_{jl}}$  for  $l \in \{1, 2, 3\}$ . Then under the assumptions of Theorem 0.1 the following number is an eigenvalue of the Dirac operator of  $M$  for the spin structure given by  $\varepsilon_j$  and the metric induced by  $a_1, a_2, a_3$ :

$$\frac{-(\epsilon_2 a_2 - \epsilon_3 a_3)^2 a_1^2 + 2a_2 a_3 (\epsilon_2 a_2 + \epsilon_3 a_3) a_1 - a_2^2 a_3^2}{2a_1 a_2 a_3}.$$

If in particular  $\epsilon_2 \epsilon_3 a_2 a_3 > 0$  then there exists  $a_1 \in \mathbb{R}^*$  such that for the corresponding metric the Dirac operator of  $M$  has a non-zero kernel.

*Proof:* For  $n = 1$  the operator  $D'_n$  can be expressed from Theorem 0.1 as

$$D'_1 = (\epsilon_1 a_1 + \epsilon_2 a_2 + \epsilon_3 a_3) \text{Id}$$

for the  $\epsilon_l$ 's defined above (beware that they depend on  $j$ ). Therefore the corresponding Dirac operator  $D_n$  is given by

$$\begin{aligned}
D_1 & = \left( \epsilon_1 a_1 + \epsilon_2 a_2 + \epsilon_3 a_3 - \frac{a_1^2 a_2^2 + a_2^2 a_3^2 + a_1^2 a_3^2}{2a_1 a_2 a_3} \right) \text{Id} \\
& = \frac{-(\epsilon_2 a_2 - \epsilon_3 a_3)^2 a_1^2 + 2a_2 a_3 (\epsilon_2 a_2 + \epsilon_3 a_3) a_1 - a_2^2 a_3^2}{2a_1 a_2 a_3} \text{Id},
\end{aligned}$$

from which the first statement follows.

An elementary computation shows that, if  $\epsilon_2\epsilon_3a_2a_3 > 0$ , then the numerator of the eigenvalue vanishes for

$$a_1 = \frac{a_2a_3(\epsilon_2a_2 + \epsilon_3a_3) \pm 2(\epsilon_2\epsilon_3a_2a_3)^{\frac{3}{2}}}{(\epsilon_2a_2 - \epsilon_3a_3)^2}$$

in the case  $\epsilon_2a_2 \neq \epsilon_3a_3$  and

$$a_1 = \frac{\epsilon_2a_3}{4}$$

if  $\epsilon_2a_2 = \epsilon_3a_3$ . Note that none of those numbers can vanish because of  $a_2a_3 \neq 0$ . This concludes the proof. □

### Notes 2.10

1. It follows from Corollary 2.9 that, for any given spin structure on  $M$ , there exists a 2-parameter-family of Riemannian metrics for which  $M$  admits non-zero harmonic spinors. This is not a surprise since the existence of such metrics already follows from a purely theoretical result by Christian Bär [4]. However we can make some of those metrics explicit here.
2. There may exist non-zero harmonic spinors for other metrics on  $M$  and possibly without needing the condition  $\epsilon_2\epsilon_3a_2a_3 > 0$  from Corollary 2.9, since we have up to now only studied the eigenvalue corresponding to one particular representation.
3. In the same way the eigenvalue computed in Corollary 2.9 is not necessarily the smallest one in absolute value. Choose for example the  $\epsilon_0$ -spin structure,  $a_2 = a_3 < 0$  and  $a_1 \in ]-\frac{a_2}{8}, -\frac{a_2}{2}[$ . Then  $\frac{4a_1a_2 - a_2^2}{2a_1}$  and  $-\frac{8a_1a_2 + a_2^2}{2a_1}$  are eigenvalues of the Dirac operator of  $M$ , the first one corresponding to  $n = 1$  (i.e., to the one computed in Corollary 2.9) and the second one to  $n = 3$ , see Corollary 0.2. However one has from the assumptions on  $a_1, a_2, a_3$  that  $|\frac{8a_1a_2 + a_2^2}{2a_1}| < |\frac{4a_1a_2 - a_2^2}{2a_1}|$ .

We end this section with an important remark which actually constitutes the main motivation for this work. The manifold  $M$  can be seen as hypersurface of the 4-dimensional round sphere  $S^4$  (with sectional curvature 1): consider the manifold  $\{A \in M_{3 \times 3}(\mathbb{R}), {}^tA = A, \text{tr}(A) = 0 \text{ and } \text{tr}(A^2) = 2\} \cong S^4$  with metric  $(A, B) \mapsto \langle A, B \rangle := \frac{1}{2}\text{tr}(AB)$ . Let  $B := \text{diag}(\lambda, -\lambda - \mu, \mu) \in S^4$  where  $\lambda, \mu \in \mathbb{R}$  satisfy  $\lambda + 2\mu \neq 0, \lambda \neq \mu, \mu + 2\lambda \neq 0$  and  $\lambda^2 + (\lambda + \mu)^2 + \mu^2 = 2$ . Set

$$N := \{\pi(P) \cdot B \cdot \pi(P)^{-1}, P \in \text{SU}_2\} \subset S^4,$$

where  $\text{SU}_2 \xrightarrow{\pi} \text{SO}_3$  is the universal 2-fold covering map. Then it is an elementary exercise to show that  $N$  is a hypersurface of  $S^4$  which is diffeomorphic to  $\text{SU}_2/\text{Q}_8$ , that the homogeneous metric induced by the inclusion map  $N \subset S^4$

is given by  $a_1 := -\frac{1}{2(\lambda+2\mu)}$ ,  $a_2 := \frac{1}{2(\mu-\lambda)}$ ,  $a_3 := \frac{1}{2(\mu+2\lambda)}$  and that choosing  $\nu_B := \frac{1}{\sqrt{3}}\text{diag}(2\mu+\lambda, \lambda-\mu, -2\lambda-\mu) \in T_B S^4$  as unit normal vector field the induced spin structure on  $N$  is the  $\varepsilon_0$ -one. Here beware that the metrics obtained form a one-parameter strict subfamily of that of all homogeneous metrics on  $M$ . Furthermore, the Weingarten endomorphism-field of  $N$  w.r.t.  $\nu_B$  - seen as endomorphism of  $\mathfrak{su}(2)$  - is given in the basis  $(X_1, X_2, X_3)$  of  $\mathfrak{su}(2)$  by

$$\text{Mat}(\mathcal{A}) = \sqrt{3} \cdot \text{diag}\left(\frac{\lambda}{2\mu+\lambda}, \frac{\mu+\lambda}{\mu-\lambda}, -\frac{\mu}{2\lambda+\mu}\right).$$

In particular, the mean curvature  $\mathcal{H} := \frac{1}{3}\text{tr}(\mathcal{A})$  of  $N$  in  $S^4$  w.r.t.  $\nu_B$  is

$$\mathcal{H} = \frac{3\sqrt{3} \cdot \lambda\mu(\lambda+\mu)}{(2\mu+\lambda)(\mu-\lambda)(2\lambda+\mu)}.$$

**Corollary 2.11** *Under the hypotheses of Theorem 0.1 assume furthermore that  $M$  sits in  $S^4$ , i.e., that  $a_1 = -\frac{1}{2(\lambda+2\mu)}$ ,  $a_2 = \frac{1}{2(\mu-\lambda)}$ ,  $a_3 = \frac{1}{2(\mu+2\lambda)}$  for some  $\lambda, \mu \in \mathbb{R}$  satisfying  $\lambda+2\mu \neq 0$ ,  $\lambda \neq \mu$ ,  $\mu+2\lambda \neq 0$  and  $\lambda^2 + (\lambda+\mu)^2 + \mu^2 = 2$ . Then  $\frac{9}{4}(\mathcal{H}^2 + 1)$  is an eigenvalue of the Dirac Laplacian of  $M$  for the induced  $(\varepsilon_0)$ -spin structure.*

*Proof:* The result follows straightforward from Corollary 2.9 in the case  $j = 0$  and from an elementary computation giving

$$\begin{aligned} \frac{9}{4}(\mathcal{H}^2 + 1) &= \frac{9}{(\lambda+2\mu)^2(\mu-\lambda)^2(\mu+2\lambda)^2} \\ &= \left( \frac{-(a_2 - a_3)^2 a_1^2 + 2a_2 a_3 (a_2 + a_3) a_1 - a_2^2 a_3^2}{2a_1 a_2 a_3} \right)^2. \end{aligned}$$

□

Corollary 2.11 confirms what had been already noticed since Christian Bär's work [5] on extrinsic upper eigenvalue bounds for the lower part of the Dirac spectrum: for any compact orientable hypersurface  $\overline{M}^m$  with constant mean curvature  $\mathcal{H}$  (and carrying the induced metric and spin structure) in the  $(m+1)$ -dimensional round sphere the number  $\frac{m^2}{4}(\mathcal{H}^2 + 1)$  is an eigenvalue of its Dirac Laplacian. However the question still remains open whether this eigenvalue should be the smallest one or not.

### 3 Computation of the spectrum of the Dirac operator on $M$ for particular metrics

Although the matrices representing the Dirac operator  $D$  of  $M$  have a "simple" shape (they are tridiagonal, see Theorem 0.1), their spectrum is still hard to compute explicitly since there does not exist any general formula giving the

eigenvalues of such matrices. It is however possible to compute them for particular values of the parameters  $a_1, a_2, a_3 \in \mathbb{R}^*$ , i.e., for particular metrics on  $M$ . In Corollary 0.2 we do it for the so-called Berger metrics on  $M$  (compare with [2, p.71] where the author chooses  $a_2 = 1 = -a_3$  and  $a_1 = -\frac{1}{T}$  with  $T > 0$ ). Namely, if we assume that  $a_2 = a_3$  then the identities for  $D'_n(F_k \pm F_{n-k})$  and  $D'_n(G_k \pm G_{n-k})$  become

$$\begin{aligned} D'_n(F_k \pm F_{n-k}) &= (-1)^k a_1 (n-2k)(F_k \pm F_{n-k}) \\ &\quad + (k+1)(1 + (-1)^k) a_2 (F_{k+1} \pm F_{n-k-1}) \\ &\quad + (n-k+1)(1 - (-1)^k) a_2 (F_{k-1} \pm F_{n-k+1}) \end{aligned}$$

and

$$\begin{aligned} D'_n(G_k \pm G_{n-k}) &= -(-1)^k a_1 (n-2k)(G_k \pm G_{n-k}) \\ &\quad + (k+1)(1 - (-1)^k) a_2 (G_{k+1} \pm G_{n-k-1}) \\ &\quad + (n-k+1)(1 + (-1)^k) a_2 (G_{k-1} \pm G_{n-k+1}) \end{aligned}$$

for every  $k \in \{0, \dots, \frac{n-1}{2}\}$ . In particular, if  $k$  is even, then

$$\begin{aligned} D'_n(F_k \pm F_{n-k}) &= a_1 (n-2k)(F_k \pm F_{n-k}) \\ &\quad + 2(k+1) a_2 (F_{k+1} \pm F_{n-k-1}) \end{aligned}$$

and

$$\begin{aligned} D'_n(G_k \pm G_{n-k}) &= -a_1 (n-2k)(G_k \pm G_{n-k}) \\ &\quad + 2(n-k+1) a_2 (G_{k-1} \pm G_{n-k+1}). \end{aligned}$$

If  $k$  is odd then

$$\begin{aligned} D'_n(F_k \pm F_{n-k}) &= -a_1 (n-2k)(F_k \pm F_{n-k}) \\ &\quad + 2(n-k+1) a_2 (F_{k-1} \pm F_{n-k+1}) \end{aligned}$$

and

$$\begin{aligned} D'_n(G_k \pm G_{n-k}) &= a_1 (n-2k)(G_k \pm G_{n-k}) \\ &\quad + 2(k+1) a_2 (G_{k+1} \pm G_{n-k-1}). \end{aligned}$$

We now consider each case separately. Remember that from Theorem 2.1 the Dirac operator  $D$  restricted to  $V_n \otimes \text{Hom}_{Q_s, \varepsilon_j}(V_n, \mathbb{C}^2)$  is given by  $\text{Id} \otimes D_n$  where  $D_n = D'_n - (\frac{a_1^2 a_2^2 + a_1^2 a_3^2 + a_2^2 a_3^2}{2a_1 a_2 a_3}) \text{Id}$ . In particular the multiplicity of each eigenvalue of  $D_n$  should be counted  $n+1$  times for the spectrum of  $D$ .

• *Case  $j = 0$ :*

\* If  $n \equiv 1 \pmod{4}$ : It follows from the identities just above and from Lemma 2.6 that the matrix of  $D'_n$  consists of  $\frac{n-1}{4}$  blocks on the diagonal of the form

$$\begin{pmatrix} (n-2k)a_1 & 2(n-k)a_2 \\ 2(k+1)a_2 & -(n-2(k+1))a_1 \end{pmatrix}$$

where  $k \in \{0, \dots, \frac{n-5}{2}\}$  is even and of the isolated eigenvalue  $a_1 + (n+1)a_2$  (corresponding to  $k = \frac{n-1}{2}$ ). The eigenvalues of each such  $2 \times 2$ -matrix are simple and given by

$$a_1 \pm \sqrt{((n-2k)(n-2(k+1)) + 1)a_1^2 + 4(n-k)(k+1)a_2^2}$$

with  $((n-2k)(n-2(k+1)) + 1) = (n-2k-1)^2$ .

\* If  $n \equiv 3 \pmod{4}$ : It follows from the identities just above and from Lemma 2.6 that the matrix of  $D'_n$  consists of  $\frac{n-3}{4}$  blocks on the diagonal of the form

$$\begin{pmatrix} (n-2k)a_1 & 2(n-k)a_2 \\ 2(k+1)a_2 & -(n-2(k+1))a_1 \end{pmatrix}$$

where  $k \in \{1, \dots, \frac{n-5}{2}\}$  is odd and of the isolated eigenvalues  $-na_1$  (corresponding to  $k=0$ ) and  $a_1 - (n+1)a_2$  (corresponding to  $k = \frac{n-1}{2}$ ).

This shows 0.

• *Case  $j = 1$ :*

\* If  $n \equiv 1 \pmod{4}$ : It follows from the identities just above and from Lemma 2.6 that the matrix of  $D'_n$  consists of  $\frac{n-1}{4}$  blocks on the diagonal of the form

$$\begin{pmatrix} (n-2k)a_1 & 2(n-k)a_2 \\ 2(k+1)a_2 & -(n-2(k+1))a_1 \end{pmatrix}$$

where  $k \in \{0, \dots, \frac{n-5}{2}\}$  is even and of the isolated eigenvalue  $a_1 - (n+1)a_2$  (corresponding to  $k = \frac{n-1}{2}$ ). The eigenvalues of each such  $2 \times 2$ -matrix have already been computed in the case  $j = 0$  above.

\* If  $n \equiv 3 \pmod{4}$ : It follows from the identities just above and from Lemma 2.6 that the matrix of  $D'_n$  consists of  $\frac{n-3}{4}$  blocks on the diagonal of the form

$$\begin{pmatrix} (n-2k)a_1 & 2(n-k)a_2 \\ 2(k+1)a_2 & -(n-2(k+1))a_1 \end{pmatrix}$$

where  $k \in \{1, \dots, \frac{n-5}{2}\}$  is odd and of the isolated eigenvalues  $-na_1$  (corresponding to  $k=0$ ) and  $a_1 + (n+1)a_2$  (corresponding to  $k = \frac{n-1}{2}$ ).

This shows 1.

• *Case  $j = 2$  or  $j = 3$ :* Since  $a_2 = a_3$  the Dirac spectra for the  $\varepsilon_2$ - and  $\varepsilon_3$ - spin structures coincide, see Examples 2.4.2 with  $\sigma = (2 \ 3)$ .

\* If  $n \equiv 1 \pmod{4}$ : It follows from the identities just above and from Lemma 2.6 that the matrix of  $D'_n$  consists of  $\frac{n-1}{4}$  blocks on the diagonal of the form

$$\begin{pmatrix} (n-2k)a_1 & 2(n-k)a_2 \\ 2(k+1)a_2 & -(n-2(k+1))a_1 \end{pmatrix}$$

where  $k \in \{1, \dots, \frac{n-3}{2}\}$  is odd and of the isolated eigenvalue  $-na_1$  (corresponding to  $k=0$ ).

\* If  $n \equiv 3 \pmod{4}$ : It follows from the identities just above and from Lemma 2.6 that the matrix of  $D'_n$  consists of  $\frac{n+1}{4}$  blocks on the diagonal of the form

$$\begin{pmatrix} (n-2k)a_1 & 2(n-k)a_2 \\ 2(k+1)a_2 & -(n-2(k+1))a_1 \end{pmatrix}$$

where  $k \in \{0, \dots, \frac{n-3}{2}\}$  is even.

This shows 2. and concludes the proof of Corollary 0.2.

**Note 3.1** Of course one should understand each upper bound (e.g.  $\frac{n-5}{2}$ ) for the possible values of  $k$  in Corollary 0.2 as follows: if for a given  $n$  it is negative then the corresponding eigenvalues do not appear. For example if  $M$  carries the  $\varepsilon_0$ -spin structure and  $n = 1$  then  $D_n + \frac{2a_1^2 + a_2^2}{2a_1} \text{Id}$  has only one eigenvalue, namely  $a_1 + 2a_2$  (with multiplicity 2). Similarly, if  $j = 2, 3$  and  $n = 1$ , then only  $-a_1$  appears with multiplicity 2.

One could in a similar way compute the spectrum of the Dirac operator for  $a_2 = -a_3$ , in which case the spectra would coincide for the  $\varepsilon_0$ - and the  $\varepsilon_1$ -spin structure on  $M$  (use Examples 2.4).

We end this section with deriving from Corollary 0.2 the spectrum of the Dirac operator on  $M$  for any of the 4 spin structures and the following metrics: for one of the metrics with constant sectional curvature and for one of the 6 metrics induced by minimal isometric embeddings into  $S^4$  (i.e., for  $(\lambda = 0, \mu = \pm 1)$ ,  $(\lambda = \pm 1, \mu = 0)$  or  $(\lambda, \mu) = \pm(1, -1)$ , see end of Section 2). In the first case the spectrum has already been computed by Christian Bär in [3, Thm. 2] and it can be easily checked that his results coincide with ours.

**Corollary 3.2** *Under the hypotheses of Theorem 0.1, assume furthermore that*

*i)  $a_1 = a_2 = a_3 = 1$ . Then the spectrum of the Dirac operator of  $M$  w.r.t. the  $\varepsilon_0$ -spin structure consists of the family*

$$\left| \begin{array}{ll} \frac{3}{2} + 4k & \text{with multiplicity } 2(k+1)(2k+1) \\ \frac{3}{2} + 4k + 2 & \text{with multiplicity } 4k(k+1) \\ -\frac{3}{2} - 4k - 1 & \text{with multiplicity } 2k(2k+1) \\ -\frac{3}{2} - 4k - 3 & \text{with multiplicity } 4(k+1)(k+2) \end{array} \right.$$

*where  $k$  runs over  $\mathbb{N}$  and w.r.t. any of the other spin structures  $\varepsilon_j$  of the family*

$$\left| \begin{array}{ll} \frac{3}{2} + 4k & \text{with multiplicity } 2k(2k+1) \\ \frac{3}{2} + 4k + 2 & \text{with multiplicity } 4(k+1)^2 \\ -\frac{3}{2} - 4k - 1 & \text{with multiplicity } 2(k+1)(2k+1) \\ -\frac{3}{2} - 4k - 3 & \text{with multiplicity } 4(k+1)^2 \end{array} \right.$$

*where  $k$  runs over  $\mathbb{N}$ .*

*ii)  $a_1 = -\frac{1}{4}, a_2 = a_3 = \frac{1}{2}$ . Then the spectrum of the Dirac operator of  $M$*

\* w.r.t. the  $\varepsilon_0$ -spin structure is given by

$$\begin{aligned} & \bigcup_{\substack{n \in \mathbb{N} \\ n \equiv 1 \pmod{4}}} \left\{ \frac{1}{2} \pm \frac{1}{4} \sqrt{(n-2k-1)^2 + 16(n-k)(k+1)} \right. \\ & \qquad \left. | k \in \left\{ 0, \dots, \frac{n-5}{2} \right\} \text{ even}, \frac{n}{2} + 1 \right\} \\ & \bigcup \bigcup_{\substack{n \in \mathbb{N} \\ n \equiv 3 \pmod{4}}} \left\{ \frac{1}{2} \pm \frac{1}{4} \sqrt{(n-2k-1)^2 + 16(n-k)(k+1)} \right. \\ & \qquad \left. | k \in \left\{ 1, \dots, \frac{n-5}{2} \right\} \text{ odd}, -\frac{n}{2}, \frac{n+3}{4} \right\}, \end{aligned}$$

each eigenvalue having multiplicity  $n+1$  for the corresponding  $n$ .

\* w.r.t. the  $\varepsilon_1$ -spin structure is given by

$$\begin{aligned} & \bigcup_{\substack{n \in \mathbb{N} \\ n \equiv 1 \pmod{4}}} \left\{ \frac{1}{2} \pm \frac{1}{4} \sqrt{(n-2k-1)^2 + 16(n-k)(k+1)} \right. \\ & \qquad \left. | k \in \left\{ 0, \dots, \frac{n-5}{2} \right\} \text{ even}, -\frac{n}{2} \right\} \\ & \bigcup \bigcup_{\substack{n \in \mathbb{N} \\ n \equiv 3 \pmod{4}}} \left\{ \frac{1}{2} \pm \frac{1}{4} \sqrt{(n-2k-1)^2 + 16(n-k)(k+1)} \right. \\ & \qquad \left. | k \in \left\{ 1, \dots, \frac{n-5}{2} \right\} \text{ odd}, \frac{n}{2} + 1, \frac{n+3}{4} \right\}, \end{aligned}$$

each eigenvalue having multiplicity  $n+1$  for the corresponding  $n$ .

\* w.r.t. the  $\varepsilon_2$ - or  $\varepsilon_3$ -spin structure is given by

$$\begin{aligned} & \bigcup_{\substack{n \in \mathbb{N} \\ n \equiv 1 \pmod{4}}} \left\{ \frac{1}{2} \pm \frac{1}{4} \sqrt{(n-2k-1)^2 + 16(n-k)(k+1)} \right. \\ & \qquad \left. | k \in \left\{ 1, \dots, \frac{n-3}{2} \right\} \text{ odd}, \frac{n+3}{4} \right\} \\ & \bigcup \bigcup_{\substack{n \in \mathbb{N} \\ n \equiv 3 \pmod{4}}} \left\{ \frac{1}{2} \pm \frac{1}{4} \sqrt{(n-2k-1)^2 + 16(n-k)(k+1)} \right. \\ & \qquad \left. | k \in \left\{ 0, \dots, \frac{n-3}{2} \right\} \text{ even} \right\}, \end{aligned}$$

each eigenvalue having multiplicity  $n+1$  for the corresponding  $n$ .

*Proof:* In case  $a_1 = a_2 = a_3 = 1$  one has on the one hand

$$(n-2k-1)^2 a_1^2 + 4(n-k)(k+1) a_2^2 = (n+1)^2$$

for every possible  $k$  and on the other hand  $\frac{2a_1^2+a_2^2}{2a_1} = \frac{3}{2}$ . The result in *i*) straightforward follows using Corollary 0.2 and Examples 2.4. Assuming now  $a_1 = -\frac{1}{4}$  and  $a_2 = a_3 = \frac{1}{2}$ , one has

$$\begin{aligned} a_1 \pm \sqrt{(n-2k-1)^2 a_1^2 + 4(n-k)(k+1)a_2^2} \\ = -\frac{1}{4} \pm \frac{\sqrt{(n-2k-1)^2 + 16(n-k)(k+1)}}{4} \end{aligned}$$

and  $\frac{2a_1^2+a_2^2}{2a_1} = -\frac{3}{4}$ . This concludes the proof.  $\square$

One can deduce from Corollary 3.2 and Examples 2.4 the spectrum of the Dirac operator of  $M$  for any spin structure and any metric induced by  $(a_1, a_1, a_1)$  with  $a_1 \in \mathbb{R}^*$  or any metric induced by a minimal embedding into  $S^4$ : in the first case rescale by  $a_1$ , in the second one exchange the roles of  $a_1, a_2, a_3$  and possibly multiply all of them by  $-1$ .

For the next corollary recall that, for a given  $\beta \in \mathbb{C}$ , a  $\beta$ -Killing spinor on a spin manifold  $N$  is a smooth section  $\psi$  of the spinor bundle of  $N$  such that  $\nabla_X \psi = \beta X \cdot \psi$  for every  $X \in TN$ .

**Corollary 3.3** *Under the hypotheses of Theorem 0.1 the following holds:*

- i) If  $a_1 = a_2 = a_3 = 1$  then the  $\varepsilon_0$ -spin structure is the only one for which  $M$  admits a non-zero space of Killing spinors, which is then 2-dimensional and associated to the constant  $\beta = -\frac{1}{2}$ . In particular  $\frac{3}{2}$  is in absolute value the smallest eigenvalue of the Dirac operator of  $M$  for the  $\varepsilon_0$ -spin structure.*
- ii) If  $a_1 = -\frac{1}{4}, a_2 = a_3 = \frac{1}{2}$  and  $M$  carries the  $\varepsilon_0$ -spin structure then  $\frac{3}{2}$  is in absolute value the smallest eigenvalue of the Dirac operator of  $M$ . In particular inequality (1) is an equality on  $M$  for the induced metric and spin structure.*

*Proof:* If  $a_1 = a_2 = a_3 = 1$  then on the one hand the metric induced on  $M$  has constant sectional curvature 1; on the other hand Corollary 3.2 *i*) implies that the smallest eigenvalue in absolute value of the Dirac operator of  $M$  is  $\frac{3}{2}$  with multiplicity 2 w.r.t. the  $\varepsilon_0$ -spin structure and  $-\frac{5}{2}$  with multiplicity 2 w.r.t. any of the other spin structures (both obtained for  $n = 1$ , i.e., they are the eigenvalues computed in Corollary 2.9). Now  $M$  carries a non-trivial Killing spinor if and only if the smallest eigenvalue of its Dirac Laplacian coincides with T. Friedrich's lower bound  $\frac{3}{4(3-1)} \inf_M(\text{Scal}_M)$  in terms of the scalar curvature of  $M$ , see [7]. Here  $\frac{3}{4(3-1)} \text{Scal}_M = \frac{9}{4}$  so that  $M$  carries a 2-dimensional space of non-zero Killing spinors only for the  $\varepsilon_0$ -spin structure; in that case the corresponding constant  $\beta$  should obviously be  $-\frac{1}{2}$ . This shows *i*)

If  $a_1 = -\frac{1}{4}, a_2 = a_3 = \frac{1}{2}$  and  $M$  carries the  $\varepsilon_0$ -spin structure then from Corollary 3.2 *ii*) the eigenvalues corresponding to  $n = 1$  and  $n = 3$  are  $\frac{3}{2}$  and  $-\frac{3}{2}, \frac{3}{2}$  with

multiplicities 2, 4 and 4 respectively. Next we show that all eigenvalues corresponding to  $n \geq 5$  are greater than  $\frac{3}{2}$  in absolute value. Since this is obviously the case for  $\frac{n}{2} + 1$ ,  $-\frac{n}{2}$  and  $\frac{n+3}{4}$  we just have to deal with the eigenvalues  $\frac{1}{5} \pm \frac{1}{4} \sqrt{(n-2k-1)^2 + 16(n-k)(k+1)}$ , of which absolute value is greater than  $\frac{3}{2}$  if and only if

$$(n-2k-1)^2 + 16(n-k)(k+1) - 64 > 0 \quad (7)$$

for every  $k \in \{0, \dots, \frac{n-5}{2}\}$ . The l.h.s. of (7) is a trinomial in  $k$  with negative dominant coefficient and of which roots are given by  $\frac{n-1}{2} \pm \sqrt{\frac{(n-3)(n+5)}{3}}$ . If  $n \geq 5$  then  $\frac{n-1}{2} - \sqrt{\frac{(n-3)(n+5)}{3}} < 0 < \frac{n-1}{2} < \frac{n-1}{2} + \sqrt{\frac{(n-3)(n+5)}{3}}$ , which shows that (7) is satisfied. Hence  $\frac{3}{2}$  is in absolute value the smallest eigenvalue of the Dirac operator. Apply Corollary 2.11 to the case  $\lambda = 0$  and  $\mu = 1$  to conclude.  $\square$

That  $M$  admits a 2-dimensional space of Killing spinors w.r.t. its  $\varepsilon_0$ -spin structure and any normal metric is also not a surprise, see [1, Cor. 5.2.5 (1b)]. Moreover, following the symmetry arguments already used above (see Examples 2.4 Corollary 3.3 *ii*) actually holds for any of the metrics induced by a minimal embedding into  $S^4$ . This proves Corollary 0.3.

Corollary 0.3 provides a further example (after geodesic spheres [5] and generalized Clifford tori [8]) of homogeneous hypersurface of the round sphere for which Christian Bär's inequality [5, Cor. 4.3] is an equality for the smallest Dirac eigenvalue. Here it should furthermore be noticed that, still under the assumptions of Corollary 0.3, the multiplicity of the smallest eigenvalue of the Dirac Laplacian on  $M$  is greater than the corresponding one on the 3-dimensional round sphere. This shows an analogy with the generalized Clifford tori tested in [8], on which the multiplicity of the smallest eigenvalue of the Dirac Laplacian is also greater than or equal to the corresponding one on the round sphere of same dimension.

We conjecture that the inequality in [5, Cor. 4.3] for the smallest Dirac eigenvalue is an equality for every homogeneous hypersurface in the round sphere. We refer to [9] for further work in this direction.

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