



HAL
open science

C *-algebras of extensions of groupoids by group bundles

Marius Ionescu, Alex Kumjian, Jean N Renault, Aidan Sims, Dana P Williams

► **To cite this version:**

Marius Ionescu, Alex Kumjian, Jean N Renault, Aidan Sims, Dana P Williams. C *-algebras of extensions of groupoids by group bundles. *Journal of Functional Analysis*, In press, 280 (5), pp.108892. 10.1016/j.jfa.2020.108892 . hal-03116987

HAL Id: hal-03116987

<https://hal.science/hal-03116987v1>

Submitted on 20 Jan 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

C^* -ALGEBRAS OF EXTENSIONS OF GROUPOIDS BY GROUP BUNDLES

MARIUS IONESCU, ALEX KUMJIAN, JEAN N. RENAULT, AIDAN SIMS, AND DANA P. WILLIAMS

ABSTRACT. Given a normal subgroup bundle \mathcal{A} of the isotropy bundle of a groupoid Σ , we obtain a twisted action of the quotient groupoid Σ/\mathcal{A} on the bundle of group C^* -algebras determined by \mathcal{A} whose twisted crossed product recovers the groupoid C^* -algebra $C^*(\Sigma)$. Restricting to the case where \mathcal{A} is abelian, we describe $C^*(\Sigma)$ as the C^* -algebra associated to a \mathbf{T} -groupoid over the transformation groupoid obtained from the canonical action of Σ/\mathcal{A} on the Pontryagin dual space of \mathcal{A} . We give some illustrative examples of this result.

INTRODUCTION

The objective of this paper is to develop tools for analyzing C^* -algebras of very general groupoids via a version of the Mackey machine, and also to construct Cartan subalgebras in C^* -algebras of large classes of groupoids that are not necessarily étale. Specifically, one of the fundamental components of the modern Mackey Subgroup Analysis for group representations comes from Green’s [Gre78, Proposition 1]: if H is a normal subgroup of a locally compact group G , then $C^*(G)$ is a twisted crossed product $C^*(G, C^*(H), \kappa)$ of $C^*(H)$ by G . The idea is then to describe the representation theory and the structure of $C^*(G)$ in terms of representations of H and the action of G on the spectrum of $C^*(H)$ induced by conjugation. Variants of this approach are sometimes called “Mackey’s Little Group Method”.

Our main theorem generalizes Green’s theorem to the situation of a locally compact Hausdorff groupoid Σ and a normal subgroup bundle \mathcal{A} of the isotropy groupoid of Σ . Our result says that the full C^* -algebra of Σ coincides with a twisted crossed product of $C^*(\mathcal{A})$ by Σ . If \mathcal{A} is amenable, this descends to an isomorphism of reduced C^* -algebras. We pay particular attention to the situation when \mathcal{A} is abelian, and hence has a Pontryagin dual space $\hat{\mathcal{A}}$. The quotient $\mathcal{G} = \Sigma/\mathcal{A}$ is a topological groupoid, and we use our main theorem to prove that $C^*(\Sigma)$ is isomorphic to the restricted C^* -algebra $C^*(\hat{\mathcal{A}} \rtimes \mathcal{G}; \hat{\Sigma})$ of a \mathbf{T} -groupoid $\hat{\Sigma}$ over a transformation groupoid $\hat{\mathcal{A}} \rtimes \mathcal{G}$. As the theory of \mathbf{T} -groupoids—also called twists—is well studied, this provides powerful tools for studying $C^*(\Sigma)$. This was the approach for the main result in [MRW96], and in fact, our isomorphism result is a significant generalization

Date: 6 November 2020.

This research was supported by Australian Research Council grant DP180100595. This work was also partially supported by Simons Foundation Collaboration grants #209277 (MI), #353626 (AK) and #507798 (DPW), and by Grant N0017318WR00251 from the Office of Naval Research and by the Naval Research Laboratory. This work was also facilitated by visits of the second and fifth author to the University of Wollongong as well as a visit by the second author to the Université d’Orléans. We thank our respective hosts and their institutions for their hospitality and support. We thank the anonymous referee of an earlier version of this paper for very thorough and detailed report that allowed us to improve the exposition.

of, and is motivated by, [MRW96, Proposition 4.5]. As an illustrative example, we discuss the special case of a locally compact Hausdorff group Σ with a closed normal abelian subgroup \mathcal{A} ; even in this special case, our results have something new to say.

Accordingly, in Section 2, we consider a second countable locally compact Hausdorff groupoid and subgroup bundle \mathcal{A} of the isotropy groupoid of Σ whose unit space coincides with that of Σ . Then \mathcal{A} acts on the left and right of Σ and we say that \mathcal{A} is normal if the orbits $\sigma\mathcal{A}$ and $\mathcal{A}\sigma$ coincide for all $\sigma \in \Sigma$. If \mathcal{A} is normal, then the quotient $\mathcal{G} = \Sigma/\mathcal{A}$ is a groupoid and we can view Σ as an extension of \mathcal{A} by \mathcal{G} . Assuming that both Σ and \mathcal{A} have Haar systems, Σ acts by automorphisms on $C^*(\mathcal{A})$. We show that there is a Green–Renault twisting map κ for this action. This allows us to form a twisted crossed product of which Green’s twisted crossed products are a special case. When \mathcal{A} is an abelian group bundle these twisted crossed products are the same as those in [Ren91, Ren87]. Our first main result (Theorem 2.11) extends Green’s result by proving that $C^*(\Sigma)$ is isomorphic to the twisted crossed product; it also establishes a similar result for the reduced norm when \mathcal{A} is amenable. There is overlap with a result in [BM16]; we discuss this in detail in Remark 2.13.

In Section 3 we restrict to the case that \mathcal{A} is an abelian group bundle. Then the Gelfand dual, $\hat{\mathcal{A}}$, of $C^*(\mathcal{A})$ is a locally compact right \mathcal{G} -space and we can form the transformation groupoid $\hat{\mathcal{A}} \rtimes \mathcal{G}$. We can build a natural \mathbf{T} -groupoid, $\hat{\Sigma}$, over $\hat{\mathcal{A}} \rtimes \mathcal{G}$. Our second main result (3.3) is that $C^*(\Sigma)$ is isomorphic to the restricted groupoid C^* -algebra $C^*(\hat{\mathcal{A}} \rtimes \mathcal{G}; \hat{\Sigma})$ of this \mathbf{T} -groupoid with a similar assertion for the reduced norms. It follows from [BL17] that $C^*(\Sigma)$ belongs to the UCT class if and only if it is nuclear.

In Section 4, we give a number of applications and examples of our results. We first consider in Section 4.1 the situation of a closed normal abelian subgroup H of a locally compact group G . The quotient G/H acts on H by conjugation, and hence also on the Pontryagin dual \hat{H} viewed as a topological space. Our construction yields a \mathbf{T} -groupoid $\hat{\Sigma}$ over $\hat{H} \rtimes (G/H)$. Our main theorem identifies the group C^* -algebra $C^*(G)$ with the C^* -algebra $C^*(\hat{H} \rtimes (G/H); \hat{\Sigma})$ of this \mathbf{T} -groupoid, and similarly at the level of reduced C^* -algebras (cf., [ZM68]). As specific examples of this result we demonstrate how to recover two standard descriptions of the C^* -algebra of the integer Heisenberg group (see Example 4.2). Our second class of examples is that of extensions of effective étale groupoids \mathcal{G} by bundles \mathcal{A} of abelian groups. In this situation, the semidirect product $\hat{\mathcal{A}} \rtimes \mathcal{G}$ is also étale, and we prove in Theorem 4.6 that $C_0(\hat{\mathcal{A}})$ embeds as a Cartan subalgebra of $C_r^*(\Sigma)$. This generalizes a result in [BNR⁺16]. We conclude Section 4 with a few examples including applications to higher-rank Deaconu–Renault groupoids and, in particular, groupoids associated to row-finite higher rank graphs with no sources.

This paper is a revised and shortened version of [IKR⁺20a].

1. PRELIMINARIES

Our results require that we work with Green–Renault twisted crossed products of C^* -algebras by locally compact Hausdorff groupoids, and in particular, the theory of \mathbf{T} -groupoids and their restricted groupoid C^* -algebras. In order to take a unified approach, we use the theory of Fell bundles over groupoids and their associated C^* -algebras. For convenience, we give a brief overview of the required background with selected references.

1.1. Open maps. Open maps are ubiquitous when working with groupoids and bundles. An important tool we refer to frequently is known as *Fell's Criterion*. For convenience we recall the formal statement from [FD88a, Proposition II.13.2].

Lemma 1.1 (Fell's Criterion). *A surjection $f : X \rightarrow Y$ is an open map if and only if for every $x \in X$ and every net $\{y_\alpha\}$ converging to $f(x)$ in Y there is a subnet $\{y_{\alpha_j}\}$ and a net $\{x_{\alpha_j}\}$ in X such that $f(x_{\alpha_j}) = y_{\alpha_j}$ for all j , and $x_{\alpha_j} \rightarrow x$.*

1.2. Upper-semicontinuous Banach bundles. A (upper-semicontinuous) Banach bundle over a space X is a topological space \mathcal{E} together with a continuous open surjection $p : \mathcal{E} \rightarrow X$ and complex Banach space structures on each fibre $E(x) := p^{-1}(x)$ such that

- (a) $a \mapsto \|a\|$ is upper-semicontinuous from \mathcal{E} to \mathbf{R}^+ ,
- (b) $(a, b) \mapsto a + b$ is continuous from $\mathcal{E} * \mathcal{E} = \{(a, b) \in \mathcal{E} \times \mathcal{E} : p(a) = p(b)\}$ to \mathcal{E} ,
- (c) $(\lambda, a) \mapsto \lambda a$ is continuous from $\mathbf{C} \times \mathcal{E}$ to \mathcal{E} , and
- (d) If $\{a_i\}$ is a net in \mathcal{E} such that $p(a_i) \rightarrow x$ and $\|a_i\| \rightarrow 0$, then $a_i \rightarrow 0_x$ where 0_x is the zero element in $E(x)$.

If in addition each fibre $E(x)$ is a C^* -algebra and $(a, b) \mapsto ab$ and $a \mapsto a^*$ are continuous on $\mathcal{E} * \mathcal{E}$ and \mathcal{E} , respectively, then we call \mathcal{E} a (upper-semicontinuous) C^* -bundle. If axiom (a) is replaced by “ $a \mapsto \|a\|$ is continuous”, then we call \mathcal{E} either a continuous Banach bundle or a continuous C^* -bundle. We normally drop the adjective “upper-semicontinuous” and add “continuous” only when we specialize to that case.

An excellent reference for continuous Banach bundles is §§13–14 of [FD88a, Chap. II]. For more details in the general case see [MW08a, Appendix A] and, for the C^* -bundle case, [Wil07, Appendix C]. If $p : \mathcal{E} \rightarrow X$ is a Banach bundle, we will write $\Gamma_0(X; \mathcal{E})$ for the continuous sections of \mathcal{E} which vanish at infinity. Lazar's [Laz18, Theorem 2.9] shows that every Banach bundle $p : \mathcal{E} \rightarrow X$ is guaranteed to have sufficiently many sections in the sense that given $a \in E(x)$ there is a $f \in \Gamma_0(X; \mathcal{E})$ such that $f(x) = a$. Furthermore, $\Gamma_0(X; \mathcal{E})$ is a Banach space with respect to the supremum norm. A primary motivation for working with upper-semicontinuous bundles rather than restricting to continuous ones is that a $C_0(X)$ -algebra A is always $C_0(X)$ -isomorphic to $\Gamma_0(X, \mathcal{A})$ for an appropriate upper-semicontinuous C^* -bundle \mathcal{A} [Wil07, Theorem C.26].

In our constructions below, we need to work with Banach bundles that arise as quotients of Banach bundles by isometric groupoid actions as in [KMRW98, §2] (where these constructions were developed for continuous bundles). Specifically, we let G be a second countable locally compact Hausdorff groupoid with a Haar system λ . We suppose that $p : \mathcal{E} \rightarrow G^{(0)}$ is a Banach bundle admitting a continuous G -action $\gamma \cdot a = \alpha_\gamma(a)$ where $\alpha_\gamma : E(s(\gamma)) \rightarrow E(r(\gamma))$ is an *isometric* Banach space isomorphism of the fibres of \mathcal{E} .

Now let X be a *free* and proper left G -space. We can form the pullback $X * \mathcal{E} = \{(x, a) \in X \times \mathcal{E} : r(x) = p(a)\}$. Then $X * \mathcal{E}$ is a Banach bundle over X which is a continuous Banach bundle if \mathcal{E} is. We get a left G -action on $X * \mathcal{E}$ given by

$$\gamma \cdot (x, a) = (\gamma \cdot x, \alpha_\gamma(a)).$$

We will write \mathcal{E}^X for the orbit space $G \backslash (X * \mathcal{E})$ with its quotient topology, and write $[x, a]$ for the orbit of (x, a) .

Lemma 1.2. *The map $p^X : \mathcal{E}^X \rightarrow G \backslash X$ given by $p^X([x, a]) = G \cdot x$ is a continuous open surjection.*

Proof. Continuity is clear. To see that p^X is open, we use Fell's criterion (see Lemma 1.1). Suppose that $G \cdot x_i \rightarrow G \cdot x = p^X([x, a])$. It suffices to lift a subnet to \mathcal{E}^X . After passing to a subnet, and relabeling, we can assume that $x_i \rightarrow x$. Since $r(x_i) \rightarrow r(x) = p(a)$ in $G^{(0)}$ and since p is open, we can pass to another subnet, relabel, and assume that there are $a_i \rightarrow a$ in \mathcal{E} such that $p(a_i) = r(x_i)$. But then $[x_i, a_i] \rightarrow [x, a]$ as required. \square

To see that \mathcal{E}^X is an Banach bundle, we have to equip the fibres $(p^X)^{-1}(G \cdot x)$ with Banach space structures. The map $a \mapsto [x, a]$ is a bijection of $E(r(x))$ onto $(p^X)^{-1}(G \cdot x)$. Thus we can define $\|[x, a]\| = \|a\|$, $[x, a] + [x, b] = [x, a + b]$, and $\lambda[x, a] = [x, \lambda a]$. These structures are independent of our choice of representative $x \in G \cdot x$ and makes $(p^X)^{-1}(G \cdot x)$ into a Banach space. Note that $E^X(G \cdot x) := (p^X)^{-1}(G \cdot x)$ is isomorphic to $E(r(x))$; however, this identification is non-canonical as it depends on the choice of representative for $G \cdot x$.

Lemma 1.3. *Suppose that $p : \mathcal{E} \rightarrow G^{(0)}$ is a (continuous) Banach bundle on which G acts by isometric isomorphisms $\alpha_\gamma : E(s(\gamma)) \rightarrow E(r(\gamma))$. Let $p^X : \mathcal{E}^X \rightarrow G \backslash X$ be the quotient bundle described above with the given Banach space structure on the fibres $E^X(G \cdot x)$. Then \mathcal{E}^X is a (continuous) Banach bundle over $G \backslash X$. In particular, if \mathcal{E} is a C^* -bundle and G acts by $*$ -isomorphisms, then \mathcal{E}^X is a C^* -bundle.*

Proof. The proof proceeds by checking that the axioms for a Banach bundle hold just as in [KMRW98, Proposition 2.15]. The only ‘‘upgrade’’ from [KMRW98] is to include the possibility that p is merely upper-semicontinuous. But if $[x_i, a_i] \rightarrow [x, a]$ in \mathcal{E}^X with $\|a_i\| \geq \epsilon > 0$ for all i , then we can pass to a subnet, relabel, and assume that $(\gamma_i x_i, \alpha_{\gamma_i}(a_i)) \rightarrow (x, a)$. Since each α_{γ_i} is isometric, $\|a\| \geq \epsilon$. \square

It will be convenient to describe the sections of a quotient bundle in terms of the sections of the original bundle. If G acts isometrically on \mathcal{E} as above, then given $\check{f} \in \Gamma(G \backslash X; \mathcal{E}^X)$, since the G -action on X is free, there is a function $f : X \rightarrow \mathcal{E}$ such that $f(x) \in E(r(x))$ and

$$\check{f}(G \cdot x) = [x, f(x)].$$

Furthermore,

$$f(\gamma \cdot x) = \alpha_\gamma(f(x)).$$

Now suppose that $x_i \rightarrow x$. Then

$$\check{f}(G \cdot x_i) = [x, f(x_i)] \rightarrow \check{f}(G \cdot x) = [x, f(x)].$$

Since p^X is open, we can pass to a subnet, relabel, and assume that there are γ_i such that

$$(\gamma_i \cdot x_i, \alpha_{\gamma_i}(f(x_i))) \rightarrow (x, f(x)) \quad \text{in } X * \mathcal{E}.$$

Since the G -action on X is free and proper, we can assume that $\gamma_i \rightarrow s(x)$. But then since the G action on $X * \mathcal{E}$ is continuous, $(x_i, f(x_i)) \rightarrow (x, f(x))$, and in particular, $f(x_i) \rightarrow f(x)$. Since we can repeat this argument for any subnet of $\{f(x_i)\}$ it follows that the original net converges to $f(x)$ and $f : X \rightarrow \mathcal{E}$ is continuous. As a consequence, we have the following proposition where, as is standard, we have identified sections $f \in \Gamma(X; X * \mathcal{E})$ of the pullback with continuous functions $f : X \rightarrow \mathcal{E}$ such that $p(f(x)) = r(x)$.

Proposition 1.4. *The sections in $\check{f} \in \Gamma(G \backslash X; \mathcal{E}^X)$ are in one-to-one correspondence with sections $f \in \Gamma(X; X * \mathcal{E})$ such that*

$$(1.1) \quad f(\gamma \cdot x) = \alpha_\gamma(f(x)) \quad \text{for all } (\gamma, x) \in G * X.$$

We have

$$\check{f}(G \cdot x) = [x, f(x)].$$

In particular, we can identify $\Gamma_c(G \backslash X; \mathcal{E}^X)$ with the space $C_c(G \backslash X, X, \mathcal{E}, \alpha)$ of continuous functions $f : X \rightarrow \mathcal{E}$ which transform according to (1.1), and such that the support of f has compact image in $G \backslash X$.

We say a net $\{\check{f}_i\}$ in $\Gamma_c(G \backslash X; \mathcal{E}^X)$ converges to \check{f} in the inductive-limit topology if $\check{f}_i \rightarrow \check{f}$ uniformly and the supports of the \check{f}_i are all contained in a fixed compact set. This is equivalent to saying that the corresponding functions $f_i \in C_c(G \backslash X, X, \mathcal{E}, \alpha)$ converge uniformly to f with supports all contained in a set with compact image in $G \backslash X$.

1.3. Fell bundles and their C*-algebras. Fell bundles over groupoids are a natural generalization of Fell's Banach *-algebraic bundles from [FD88b, Chapter VIII]. They were introduced in [Yam87]. For more details, see [Kum98, MW08a]. Roughly speaking a Fell bundle \mathcal{B} over a locally compact Hausdorff groupoid \mathcal{G} is an upper-semicontinuous Banach bundle $p : \mathcal{B} \rightarrow \mathcal{G}$ endowed with a continuous involution $b \mapsto b^*$ and a continuous multiplication $(a, b) \mapsto ab$ from $\mathcal{B}^{(2)} = \{(b, b') : (p(b), p(b')) \in \mathcal{G}^{(2)}\}$ to \mathcal{B} such that—with respect to the operations, actions, and inner products induced by the involution and multiplication—the fibres $B(x) = p^{-1}(x)$ over units $x \in \mathcal{G}^{(0)}$ are C*-algebras and such that each fibre $B(\gamma)$ is a $B(r(\gamma))$ - $B(s(\gamma))$ -imprimitivity bimodule. We write $\Gamma_c(\mathcal{G}; \mathcal{B})$ for the *-algebra of continuous compactly supported sections of \mathcal{B} under convolution and involution.

Each Fell bundle has both a full and a reduced C*-algebra. The full C*-algebra $C^*(\mathcal{G}, \mathcal{B})$ is universal for representations of the bundle that are continuous in the inductive-limit topology. The reduced C*-algebra is obtained from a representation of $C_c(\mathcal{G}; \mathcal{B})$ on the Hilbert module $\mathcal{H}(\mathcal{B})$ defined as follows (see [Kum98, MT11, SW13, Hol17]).

The restriction of \mathcal{B} to a bundle over the unit space is an upper-semicontinuous C*-bundle, and we write $\Gamma_0(\mathcal{G}^{(0)}; \mathcal{B})$ for the C*-algebra of C_0 -sections of this bundle. Define $\langle \cdot, \cdot \rangle_{C_0(\mathcal{G}^{(0)}, \mathcal{B})}$ on $\Gamma_c(\mathcal{G}; \mathcal{B})$ by $\langle f, g \rangle_{C_0(\mathcal{G}^{(0)}, \mathcal{B})} = (f^* * g)|_{\mathcal{G}^{(0)}}$. This is a pre-inner product, and the completion

$$\mathcal{H}(\mathcal{B}) := \overline{\Gamma_c(\mathcal{G}; \mathcal{B})}^{\|\cdot\|_{C_0(\mathcal{G}^{(0)}, \mathcal{B})}}$$

is a Hilbert module. Left multiplication in $\Gamma_c(\mathcal{G}; \mathcal{B})$ extends to a left action $\phi : \Gamma_c(\mathcal{G}; \mathcal{B}) \rightarrow \mathcal{L}(\mathcal{H}(\mathcal{B}))$ by adjointable operators, and the reduced norm on $\Gamma_c(\mathcal{G}; \mathcal{B})$ is given by $\|f\|_r = \|\phi(f)\|_{\mathcal{L}(\mathcal{H}(\mathcal{B}))}$.

1.4. Twisted groupoid crossed products. There are a number of treatments of groupoid crossed products and twisted crossed products in the literature. Here we use a Fell bundle approach as in [MW08a, §2]. Thus a groupoid dynamical system $(\mathcal{E}, \Sigma, \vartheta)$ consists of a C*-bundle $p : \mathcal{E} \rightarrow \Sigma^{(0)}$, a locally compact Hausdorff groupoid Σ , and isomorphisms $\vartheta_\sigma : E(s(\sigma)) \rightarrow E(r(\sigma))$ such that $\vartheta_{\sigma\tau} = \vartheta_\sigma \circ \vartheta_\tau$ and such that $\sigma \cdot e := \vartheta_\sigma(e)$ is a continuous action of Σ on \mathcal{E} . Often in the notation the bundle \mathcal{E} is replaced by the corresponding

$C_0(\Sigma^{(0)})$ -algebra $A := \Gamma_0(\Sigma^{(0)}; \mathcal{E})$, and we still call (A, Σ, ϑ) a groupoid dynamical system. As in [MW08a, Example 2.1], we can form a Fell bundle $\mathcal{B} = r^* \mathcal{E} = \{(a, \sigma) : p(a) = r(\sigma)\}$ over Σ with multiplication and involution given by

$$(a, \sigma)(b, \tau) = (a\vartheta_\sigma(b), \sigma\tau) \quad \text{and} \quad (a, \sigma)^* = (\vartheta_\sigma^{-1}(a), \sigma^{-1}).$$

Then, if Σ has a Haar system, the crossed product $C^*(\mathcal{E}, \Sigma, \vartheta)$ is the Fell bundle C^* -algebra $C^*(\Sigma; \mathcal{B})$ built out of $\Gamma_c(\Sigma; \mathcal{B})$.

To get a twisted crossed product, we first require that Σ is a unit space fixing extension of a subgroupoid group bundle \mathcal{A} . That is, we have

$$(1.2) \quad \begin{array}{ccccc} \mathcal{A} & \xrightarrow{\iota} & \Sigma & \xrightarrow{p} & \mathcal{G} \\ & \searrow & \Downarrow & \swarrow & \\ & & \Sigma^{(0)} & & \end{array}$$

where ι is the inclusion map, \mathcal{G} is a locally compact groupoid, and p is a continuous open surjection with kernel $\iota(\mathcal{A})$ restricting to a homeomorphism of $\Sigma^{(0)}$ and $\mathcal{G}^{(0)}$. Notice that we can identify \mathcal{G} with either of the orbit spaces $\mathcal{A} \backslash \Sigma$ or Σ / \mathcal{A} where the \mathcal{A} -action is given by multiplication in Σ .

Let $U(E(u)) \subset M(E(u))$ be the unitary group of $E(u)$, and let $\coprod_{u \in \Sigma^{(0)}} U(E(u))$ be the corresponding (algebraic) group bundle over $\Sigma^{(0)}$. A (Green–Renault) *twisting map* for ϑ is a unit-space fixing homomorphism $\kappa : \mathcal{A} \rightarrow \coprod_{u \in \Sigma^{(0)}} U(E(u))$ that induces an action by isometric Banach space isomorphisms of \mathcal{A} on \mathcal{E} so that $(a, e) \mapsto a \cdot e := \kappa(a)e$ is continuous from $\mathcal{A} * \mathcal{E} \rightarrow \mathcal{E}$ such that

$$(1.3) \quad \begin{aligned} \vartheta_a(e) &= \kappa(a)e\kappa(a)^* \quad \text{for all } (a, e) \in \mathcal{A} * \mathcal{E}, \text{ and} \\ \kappa(\sigma a \sigma^{-1}) &= \bar{\vartheta}_\sigma(\kappa(a)) \quad \text{for all } (\sigma, a) \in \Sigma * \mathcal{A}. \end{aligned}$$

Since $e \mapsto e^*$ is continuous, so is $(a, e) \mapsto e\kappa(a)^* = (\kappa(a)e^*)^*$. Then we call $(\mathcal{G}, \Sigma, \mathcal{E}, \vartheta, \kappa)$ a Green–Renault twisted groupoid dynamical system.

Given $(\mathcal{G}, \Sigma, \mathcal{E}, \vartheta, \kappa)$, we can form a Fell bundle $\mathcal{B} = \mathcal{B}(\mathcal{E}, \vartheta, \kappa)$ over \mathcal{G} as in [MW08a, Example 2.5] as follows. Define a left \mathcal{A} -action by isometric isomorphisms on $r^* \mathcal{E} = \Sigma * \mathcal{E} = \{(\sigma, e) \in \Sigma \times \mathcal{E} : r(\sigma) = p_\mathcal{E}(e)\}$ by

$$a \cdot (\sigma, e) = (a\sigma, e\kappa(a)^*).$$

Then form a the Banach bundle quotient $\mathcal{B} = \mathcal{B}(\mathcal{E}, \vartheta, \kappa) = \mathcal{E}^\Sigma$ over $\mathcal{G} = \mathcal{A} \backslash \Sigma$ by forming the quotient $\mathcal{A} \backslash r^* \mathcal{E}$. We will write $[\sigma, e]$ for the orbit of (σ, e) in \mathcal{B} . Thus $p_\mathcal{B} : \mathcal{B} \rightarrow \mathcal{G}$ is given by $p_\mathcal{B}([\sigma, e]) = \dot{\sigma}$.

If $(\dot{\sigma}, \dot{\tau}) \in \mathcal{G}^{(2)}$, then we can define

$$(1.4) \quad (\sigma, e)(\tau, f) = (\sigma\tau, e\vartheta_\sigma(f)),$$

and compute that

$$(a\sigma, e\kappa(a)^*)(b\tau, f\kappa(b)^*) = (a\sigma b\sigma^{-1}) \cdot (\sigma\tau, e\vartheta_\sigma(f)).$$

Hence (1.4) is well defined on elements of \mathcal{B} and defines a “multiplication map” on $\mathcal{B}^{(2)} = \{([\sigma, e], [\tau, f]) \in \mathcal{B} \times \mathcal{B} : (\dot{\sigma}, \dot{\tau}) \in \mathcal{G}^{(2)}\}$ given by

$$(1.5) \quad [\sigma, e][\tau, f] := [\sigma\tau, e\vartheta_\sigma(f)].$$

Similarly, we get a well-defined involution on \mathcal{B} defined by

$$(1.6) \quad [\sigma, e]^* := [\sigma^{-1}, \vartheta_\sigma^{-1}(e^*)].$$

Lemma 1.5. *With multiplication and involution defined by (1.5) and (1.6), respectively, $\mathcal{B}(\mathcal{E}, \vartheta, \kappa)$ is a Fell bundle over \mathcal{G} .*

Proof. The continuity of multiplication and the involution follows from the openness of the projection $p_{\mathcal{B}} : \mathcal{B} \rightarrow \mathcal{G}$ and the continuity of the action of Σ on \mathcal{E} . The algebraic properties of multiplication are routine to verify as are axioms (a), (b), and (c) of [MW08a, Definition 1.1].

For (d) and (e), recall that the fibre $B(\dot{\sigma})$ of \mathcal{B} over $\dot{\sigma} \in \mathcal{G}$ is $\{[\sigma, e]\}$ equipped with the Banach-space structure induced by the map $e \mapsto [\sigma, e]$ for $e \in E(r(\sigma))$. So for $u \in \Sigma^{(0)}$, the space $B(u)$ is a C^* -algebra isomorphic to $E(u)$ with the induced multiplication and involution coming from \mathcal{B} . This is (d).

For (e), we must show that $B(\dot{\sigma})$ is a $B(r(\sigma)) - B(s(\sigma))$ -imprimitivity bimodule under the actions and inner products induced by \mathcal{B} . These actions and inner products are exactly those coming from viewing $E(r(\sigma))$ as a $E(r(\sigma)) - E(s(\sigma))$ -imprimitivity bimodule using the isomorphism $\vartheta_\sigma^{-1} : E(r(\sigma)) \rightarrow E(s(\sigma))$ (see [MW08a, Example 2.1]). \square

If \mathcal{G} has a Haar system, then the twisted groupoid crossed product $C^*(\mathcal{G}, \Sigma, \mathcal{E}, \kappa)$ is the Fell bundle C^* -algebra $C^*(\mathcal{G}; \mathcal{B})$ where $\mathcal{B} = \mathcal{B}(\mathcal{E}, \vartheta, \kappa)$ as above. Recall that sections $\check{f} \in \Gamma(\mathcal{G}; \mathcal{B})$ are determined by continuous functions $f : \Sigma \rightarrow \mathcal{E}$ such that

$$(1.7) \quad f(a\sigma) = f(\sigma)\kappa(a)^* \quad \text{for all } (a, \sigma) \in \mathcal{A} * \mathcal{E}.$$

Then

$$\check{f}(\dot{\sigma}) = [\sigma, f(\sigma)].$$

Thus $\Gamma_c(\mathcal{G}; \mathcal{B})$ is isomorphic to the space $C_c(\mathcal{G}, \Sigma, \mathcal{E}, \kappa)$ of continuous functions $f : \Sigma \rightarrow \mathcal{E}$ that satisfy (1.7) and whose support has compact image in \mathcal{G} , with operations

$$f * g(\sigma) = \int_{\mathcal{G}} f(\tau)\vartheta_\tau(g(\tau^{-1}\sigma)) d\alpha^{r(\sigma)}(\dot{\tau}) \quad \text{and} \quad f^*(\sigma) = \vartheta_\sigma(f(\sigma^{-1})^*).$$

For example,

$$\check{f} * \check{g}(\dot{\sigma}) = \int_{\mathcal{G}} \check{f}(\dot{\tau})\check{g}(\dot{\tau}^{-1}\dot{\sigma}) d\alpha^{r(\sigma)}(\dot{\tau}) = \int_{\mathcal{G}} [\sigma, f(\tau)\vartheta_\tau(g(\tau^{-1}\sigma))] d\alpha^{r(\sigma)}(\dot{\tau}),$$

and since $(\sigma, a) \mapsto [\sigma, a]$ is an isomorphism of Banach spaces, this gives

$$\check{f} * \check{g}(\dot{\sigma}) = \left[\sigma, \int_{\mathcal{G}} f(\tau)\vartheta_\tau(g(\tau^{-1}\sigma)) d\alpha^{r(\sigma)}(\dot{\tau}) \right].$$

The I -norm on $C_c(\mathcal{G}, \Sigma, \mathcal{E}, \kappa)$ is given by

$$\|f\|_I = \max \left\{ \sup_{u \in \Sigma^{(0)}} \int_{\mathcal{G}} \|f(\sigma)\| d\alpha^u(\dot{\sigma}), \sup_{u \in \Sigma^{(0)}} \int_{\mathcal{G}} \|f(\sigma)\| d\alpha_u(\dot{\sigma}) \right\}.$$

Therefore the Fell bundle C^* -algebra $C^*(\mathcal{G}; \mathcal{B})$ is the completion of $C_c(\mathcal{G}, \Sigma, \mathcal{E}, \kappa)$ universal for I -norm decreasing representations of $C_c(\mathcal{G}, \Sigma, \mathcal{E}, \kappa)$.

Remark 1.6. If Σ is a group G and \mathcal{A} a normal subgroup N , then κ is a twisting map for (N, G, ϑ) as in [Gre78]. The extension of twists to groupoid dynamical systems comes from [Ren87, §3] where it was assumed that \mathcal{A} was an abelian group bundle and that \mathcal{E} was continuous.

For our isomorphism result, we will need the special case of the twisted crossed product where $\mathcal{A} = \Sigma^{(0)} \times \mathbf{T}$. Such groupoids are called either \mathbf{T} -groupoids or twists. As in [MW08a, Example 2.3], the associated full and reduced C^* -algebras, $C^*(\mathcal{G}; \Sigma)$ and $C_r^*(\mathcal{G}; \Sigma)$ coincide with the full and reduced C^* -algebras of the Fell line-bundle \mathcal{B} over \mathcal{G} . In particular, both these C^* -algebras are completions of $C_c(\mathcal{G}; \Sigma)$ consisting of $f \in C_c(\Sigma)$ such that $f(z\sigma) = zf(\sigma)$ for all $z \in \mathbf{T}$ and $\sigma \in \Sigma$ (note that the complex conjugate appearing in equation (2.3) following [MW08a, Example 2.3] is a typo).

Hence the reduced norm on $C_c(\mathcal{G}; \Sigma)$ is realized by the representation of $C_c(\mathcal{G}; \Sigma)$ on the Hilbert module $\mathcal{H}(\mathcal{G}; \Sigma)$ given by taking the completion of $C_c(\mathcal{G}; \Sigma)$ with respect to $\langle f, g \rangle := (f^* * g)|_{\mathcal{G}^{(0)}}$. Restricting to the special case where the twist Σ is the trivial twist $\mathcal{G} \times \mathbf{T}$, we see that the reduced norm on $C_c(\mathcal{G})$ is realized by the representation of $C_c(\mathcal{G})$ on the Hilbert module $\mathcal{H}(\mathcal{G})$ obtained as the completion of $C_c(\mathcal{G})$ with respect to $\langle f, g \rangle = (f^* * g)|_{\mathcal{G}^{(0)}}$.

2. EXTENSIONS OF GROUP BUNDLES

Let Σ be a second-countable locally compact Hausdorff groupoid with a Haar system $\lambda = \{\lambda^u\}_{u \in \Sigma^{(0)}}$. Let \mathcal{A} be a closed subgroupoid of the the isotropy group bundle $\Sigma' := \{\sigma \in \Sigma : s(\sigma) = r(\sigma)\}$ such that $\mathcal{A}^{(0)} = \Sigma^{(0)}$. We summarize this by saying that \mathcal{A} is a *wide subgroup bundle* of Σ . We let $p_{\mathcal{A}} = r|_{\mathcal{A}} = s|_{\mathcal{A}}$ be the projection from \mathcal{A} onto $\Sigma^{(0)}$ and write $A(u)$ for the fibre of \mathcal{A} over u .

We assume throughout that \mathcal{A} has a Haar system $\{\beta^u\}_{u \in \Sigma^{(0)}}$. Even though Σ is assumed to have a Haar system, \mathcal{A} has a Haar system only if the fibres of \mathcal{A} are well-behaved as described in the following result taken from [Ren91, §1] (see also [Wil19, Theorem 6.12]).

Lemma 2.1. *Let \mathcal{A} be a closed subgroup bundle of Σ . Then the following are equivalent.*

- (a) *There is a Haar system $\beta = \{\beta^u\}_{u \in \Sigma^{(0)}}$ for \mathcal{A} .*
- (b) *The projection map $p_{\mathcal{A}} : \mathcal{A} \rightarrow \Sigma^{(0)}$ is open.*
- (c) *The map $u \mapsto A(u)$ is continuous from $\Sigma^{(0)}$ into the space of closed subgroups of Σ in the Fell topology.*

Note that \mathcal{A} acts freely and properly on the left as well as on the right of Σ . Since $p_{\mathcal{A}}$ is open, the orbit maps for \mathcal{A} -actions are open [Wil19, Proposition 2.12], so both orbit spaces $\mathcal{A} \backslash \Sigma$ and Σ / \mathcal{A} are locally compact Hausdorff in our situation. We say that \mathcal{A} is *normal* if the orbits $\mathcal{A}\sigma$ and $\sigma\mathcal{A}$ coincide for all $\sigma \in \Sigma$. Note that \mathcal{A} is normal if and only if Σ acts on \mathcal{A} by conjugation.

If \mathcal{A} is normal, then as subsets of Σ , $(\sigma_1\mathcal{A})(\sigma_2\mathcal{A}) = \sigma_1\sigma_2\mathcal{A}$ whenever $s(\sigma_1) = r(\sigma_2)$. This allows us to impose a groupoid structure on $\mathcal{G} = \Sigma / \mathcal{A}$ in the obvious way. Conversely, if $p : \Sigma \rightarrow \mathcal{G}$ is a continuous, open groupoid homomorphism which induces a homeomorphism of $\Sigma^{(0)}$ onto $\mathcal{G}^{(0)}$, then the kernel is a wide normal subgroup bundle and we can identify $\Sigma^{(0)}$ and $\mathcal{G}^{(0)}$. We can summarize these observations as follows where as usual we write $\tilde{\sigma}$ in place of $p(\sigma) = \sigma\mathcal{A}$.

Lemma 2.2. *Suppose that \mathcal{A} is a wide normal subgroup bundle of Σ with a Haar system. Consider the orbit space $\mathcal{G} = \mathcal{A} \backslash \Sigma = \Sigma / \mathcal{A}$. Define $\mathcal{G}^{(2)} = \{(\dot{\sigma}_1, \dot{\sigma}_2) : s(\sigma_1) = r(\sigma_2)\}$, and define a multiplication map $\mathcal{G}^{(2)} \rightarrow \mathcal{G}$ by $(\dot{\sigma}_1)(\dot{\sigma}_2) = \sigma_1 \sigma_2 \mathcal{A}$. Then \mathcal{G} is a locally compact groupoid, and $\mathcal{G}^{(0)} \cong \Sigma^{(0)}$ via $u\mathcal{A} \mapsto u$. Equivalently, \mathcal{A} is a wide normal subgroup bundle of Σ if and only if we have a unit space fixing groupoid extension (1.2) where ι is the inclusion map, p is the orbit map, and we identify $\Sigma^{(0)}$ and $\mathcal{G}^{(0)}$.*

Remark 2.3. The isotropy bundle $\mathcal{A} = \Sigma'$ is always a normal subgroup bundle. However, Lemma 2.1 implies that Σ' has a Haar system only when the isotropy map $u \mapsto \Sigma(u) := \Sigma'_u$ is continuous. Nevertheless, there are interesting examples where $\mathcal{A} \subsetneq \Sigma'$ has a Haar system even when Σ' does not. A number of such instances are given in Section 4.2.

For the remainder of this section, Σ will be a second countable locally compact Hausdorff groupoid with a Haar system λ , and \mathcal{A} will be a wide normal subgroup bundle with a Haar system β . We will write \mathcal{G} for the groupoid Σ / \mathcal{A} as in Lemma 2.1 and $p : \Sigma \rightarrow \mathcal{G}$ will be the quotient map as in (1.2).

The following can be proved using the uniqueness of Haar measure on the $A(u)$ exactly as in [MRW96, Lemma 4.1].

Lemma 2.4. *Let Σ and \mathcal{A} be as above. Then there is a continuous homomorphism $\delta : \Sigma \rightarrow \mathbf{R}^+$, called the modular map for the extension (1.2), such that*

$$\int_{\mathcal{A}} f(\sigma a \sigma^{-1}) d\beta^{s(\sigma)}(a) = \delta(\sigma) \int_{\mathcal{A}} f(a) d\beta^{r(\sigma)}(a).$$

Note that if $a \in A(u)$, then $\delta(a) = \Delta_{A(u)}(a)$.

Remark 2.5. Note that $\delta(\sigma)$ is the inverse of the similar constant $\omega(\sigma)$ used in [MRW96], and δ has the advantage that it restricts to the modular function on each $A(u)$.

We let $C_{c,p}(\Sigma)$ be the collection of continuous functions on Σ such that $\text{supp } b \cap p^{-1}(K)$ is compact for all $K \subset \mathcal{G}$ compact.

Lemma 2.6. *Suppose that Σ , \mathcal{A} , \mathcal{G} , and p are as above.*

(a) *There is a $b \in C_{c,p}(\Sigma)$, called a Bruhat section, such that*

$$(2.1) \quad \int_{\mathcal{A}} b(\sigma a) d\beta^{s(\sigma)}(a) = 1 \quad \text{for all } \sigma \in \Sigma.$$

(b) *There is a surjection $Q : C_c(\Sigma) \rightarrow C_c(\mathcal{G})$ given by*

$$Q(f)(\dot{\sigma}) = \int_{\mathcal{A}} f(\sigma a) d\beta^{s(\sigma)}(a).$$

(c) *There is a Haar system $\alpha = \{\alpha_u\}_{u \in \Sigma^{(0)}}$ on \mathcal{G} such that for all $f \in C_c(\Sigma)$ and $u \in \Sigma^{(0)}$ we have*

$$(2.2) \quad \int_{\Sigma} f(\sigma) d\lambda^u(\sigma) = \int_{\mathcal{G}} \int_{\mathcal{A}} f(\sigma a) d\beta^{s(\sigma)}(a) d\alpha^u(\dot{\sigma}).$$

Proof. By left invariance, if $f \in C_c(\Sigma)$ and $\dot{\sigma} = \dot{\tau} \in \mathcal{G}$ then

$$\int_{\mathcal{A}} f(\sigma a) d\beta^{s(\sigma)}(a) = \int_{\mathcal{A}} f(\tau a) d\beta^{s(\tau)}(a).$$

Hence for $f \in C_c(\Sigma)$ there is a well-defined map $\gamma \mapsto m^\gamma(f)$ from \mathcal{G} to \mathbf{C} such that

$$m^{\dot{\sigma}}(f) = \int_{\mathcal{A}} f(\sigma a) d\beta^{s(\sigma)}(a) \quad \text{for all } \sigma \in \Sigma.$$

The collection $m = \{m^\gamma\}_{\gamma \in \mathcal{G}}$ is a p -system of measures by [Wil19, Lemma 2.21(b)]. The existence of b follows from the paracompactness of \mathcal{G} as in [Wil19, Proposition 3.18]. This proves (a) and that $Q(f) \in C_c(\mathcal{G})$. Surjectivity follows from (a) and we have established (b).

(c) To show that a Radon measure α^u exists satisfying (2.2), it will suffice to see that whenever

$$(2.3) \quad \int_{\mathcal{A}} f(\sigma a) d\beta^{s(\sigma)}(a) = 0 \quad \text{for all } \sigma \in r^{-1}(u)$$

it follows that

$$\int_{\Sigma} f(\sigma) d\lambda^u(\sigma) = 0.$$

(Then we can define a linear functional on $C_c(\mathcal{G})$ by $\alpha^u(Q(f)) = \lambda^u(f)$.) But if (2.3) holds, then for any $h \in C_c(\Sigma)$,

$$(2.4) \quad \begin{aligned} \int_{\mathcal{A}} f * h(a) d\beta^u(a) &= \int_{\mathcal{A}} \int_{\Sigma} f(a\sigma) h(\sigma^{-1}) d\lambda^u(\sigma) d\beta^u(a) \\ &= \int_{\Sigma} \left(\delta(\sigma)^{-1} \int_{\mathcal{A}} f(\sigma a) d\beta^{s(\sigma)}(a) \right) h(\sigma^{-1}) d\lambda^u(\sigma) = 0. \end{aligned}$$

Taking h to be a multiple of b in (2.1) by an appropriate function in $C_c(\Sigma)$, we can assume that

$$\int_{\mathcal{A}} h(\sigma^{-1}a) d\beta^u(a) = 1$$

for all $\sigma \in \text{supp } f$. Then the left-hand side of (2.4) is exactly

$$\int_{\Sigma} f(\sigma) d\lambda^u(\sigma).$$

If $F \in C_c(\mathcal{G})$, then

$$\int_{\mathcal{G}} F(\gamma) d\alpha^u(\gamma) = \int_{\Sigma} F(\dot{\sigma}) b(\sigma) d\lambda^u(\sigma).$$

Hence $u \mapsto \alpha^u(F)$ is continuous. For left-invariance, suppose that $\eta = \dot{\tau}$. Then

$$\begin{aligned} \int_{\mathcal{G}} F(\eta\gamma) d\alpha^{s(\eta)}(\gamma) &= \int_{\Sigma} F(p(\tau\sigma)) b(\sigma) d\lambda^{s(\tau)}(\sigma) \\ &= \int_{\Sigma} F(p(\sigma)) b(\tau^{-1}\sigma) d\lambda^{r(\tau)}(\sigma) \\ &= \int_{\mathcal{G}} F(\gamma) d\alpha^{r(\eta)}(\gamma). \end{aligned} \quad \square$$

Since \mathcal{A} has a Haar system, we can form the C^* -algebra $C^*(\mathcal{A})$. Then it is not hard to verify that $C^*(\mathcal{A})$ is a $C_0(\Sigma^{(0)})$ -algebra with fibres $C^*(\mathcal{A})(u)$ identified with the group C^* -algebras $C^*(A(u))$. (Since we are working in a groupoid context, we treat $A(u)$ as a groupoid with a single unit. This means that the modular function $\Delta_{A(u)}$ does not appear in the formula for the involution on $C_c(A(u))$.) To form a groupoid dynamical system we need to exhibit $C^*(\mathcal{A})$ as the section algebra of a C^* -bundle.

Using [Wil07, Theorem C.26], we can equip

$$\mathcal{E} := \coprod_{u \in \Sigma^{(0)}} C^*(A(u))$$

with a topology such that it becomes an upper-semicontinuous C^* -bundle over $\Sigma^{(0)}$ such that each $f \in C_c(\mathcal{A})$ defines a continuous section $\check{f} \in \Gamma_c(\Sigma^{(0)}; \mathcal{E})$ given by $\check{f}(u)(a) = f(a)$. Then $C^*(\mathcal{A})$ can be identified with $\Gamma_0(\Sigma^{(0)}; \mathcal{E})$.

For each $\sigma \in \Sigma$, we get an isomorphism

$$\vartheta_\sigma : C^*(A(s(\sigma))) \rightarrow C^*(A(r(\sigma)))$$

given on $h \in C_c(A(s(\sigma)))$ by

$$\vartheta_\sigma(h)(a) = \delta(\sigma)h(\sigma^{-1}a\sigma).$$

Proposition 2.7 (cf., [Goe12, Proposition 2.6]). *With the notation just established, and writing $\vartheta = \{\vartheta_\sigma\}$, the triple $(\mathcal{E}, \Sigma, \vartheta)$ is a groupoid dynamical system.*

Proof. We have already established that each ϑ_σ is an isomorphism, and it is easy to check that $\vartheta_{\sigma\tau} = \vartheta_\sigma \circ \vartheta_\tau$. It only remains to check the continuity of $(\sigma, s) \mapsto \vartheta_\sigma(s)$ from $\Sigma * \mathcal{E}$ to \mathcal{E} . We establish this using [MW08b, Lemma 4.3] (see also [KMRW98, Lemma 2.13]).

We must consider the pullbacks

$$r^*(C^*(\mathcal{A})) = \Gamma_0(\Sigma; r^*\mathcal{E}) \quad \text{and} \quad s^*(C^*(\mathcal{A})) = \Gamma_0(\Sigma; s^*\mathcal{E}).$$

But we can clearly identify sections in $\Gamma(\Sigma; r^*\mathcal{E})$ with continuous functions $f : \Sigma \rightarrow \mathcal{E}$ such that $p_{\mathcal{E}}(f(\sigma)) = r(\sigma)$, and similarly for sections of $s^*\mathcal{E}$. Hence we can define $\vartheta : \Gamma_0(\Sigma; r^*\mathcal{E}) \rightarrow \Gamma_0(\Sigma; s^*\mathcal{E})$ by $\vartheta(f)(\sigma) = \vartheta_\sigma(f(\sigma))$. Then ϑ is an isomorphism, and the continuity of the action follows from [MW08b, Lemma 4.3]. \square

If $t \in A(u)$, then we can define $\kappa(t) : C_c(A(u)) \rightarrow C_c(A(u))$ by

$$\kappa(t)h(a) = \delta(t)^{\frac{1}{2}}h(t^{-1}a) = \Delta_{A(u)}(t)^{\frac{1}{2}}h(t^{-1}a).$$

Since $(\kappa(t)h)^* * (\kappa(t)k) = h^* * k$, it follows that $\kappa(t)$ extends to a unitary in the multiplier algebra $M(C^*(A(u)))$. It is straightforward to check that $\kappa(tt') = \kappa(t)\kappa(t')$.

Lemma 2.8. *The action of \mathcal{A} on \mathcal{E} is continuous. That is, the map $(t, s) \mapsto \kappa(t)s$ is continuous from $\mathcal{A} * \mathcal{E} \rightarrow \mathcal{E}$.*

Proof. Suppose that we have a net $(t_i, s_i) \rightarrow (t_0, s_0)$ in $\mathcal{A} * \mathcal{E}$. We need to show that $\kappa(t_i)s_i \rightarrow \kappa(t_0)s_0$ in \mathcal{E} . Let $u_i = p_{\mathcal{A}}(t_i) = p_{\mathcal{E}}(s_i)$. Using [Wil07, Proposition C.20], it will suffice to show that given $\epsilon > 0$, there are $s'_i \rightarrow s'_0$ in \mathcal{E} such that $p_{\mathcal{E}}(s'_i) = u_i$ and that we eventually have $\|s'_i - \kappa(t_i)s_i\| < \epsilon$.

Let $f \in C_c(\mathcal{A})$ be such that $\|\check{f}(u_0) - s_0\| < \epsilon$. Then we eventually have $\|\check{f}(u_i) - s_i\| < \epsilon$. Since \mathcal{A} acts by unitaries, we eventually have $\|\kappa(t_i)\check{f}(u_i) - \kappa(t_i)s_i\| < \epsilon$.

Therefore it will suffice to show that $s'_i := \kappa(t_i)\check{f}(u_i) \rightarrow s'_0 := \kappa(t_0)\check{f}(u_0)$ in \mathcal{E} . For this, we form the pullback bundle $p_{\mathcal{A}}^*\mathcal{E} = \{(t, s) \in \mathcal{A} \times \mathcal{E} : p_{\mathcal{A}}(t) = p_{\mathcal{E}}(s)\}$. If $F \in C_c(\mathcal{A} * \mathcal{A})$, then we get a section of $p_{\mathcal{A}}^*\mathcal{E}$ given by $\check{F}(t) = (t, F(t, \cdot)) \in C_c(A(p_{\mathcal{A}}(t)))$. If $F(t, a) = f(t)g(a)$ for $f, g \in C_c(\mathcal{A})$, then we clearly have $\check{F} \in \Gamma(\mathcal{A}; p_{\mathcal{A}}^*\mathcal{E})$. Since finite sums of such functions are dense in the inductive-limit topology on $C_c(\mathcal{A} * \mathcal{A})$, we have $\check{F} \in \Gamma(\mathcal{A}; p_{\mathcal{A}}^*\mathcal{E})$ for all $F \in C_c(\mathcal{A} * \mathcal{A})$.

We can assume that all the t_i are in a compact neighborhood V of t_0 . Then if $\phi \in C_c^+(\mathcal{A})$ is such that $\phi \equiv 1$ on V , then $F(t, a) = \phi(t)\kappa(t)f(a)$ defines an element of $C_c(\mathcal{A} * \mathcal{A})$ and $\check{F}(t_i) = (t_i, \kappa(t_i)\check{f}(u_i))$. Since projection on the second factor is continuous from $p_{\mathcal{A}}^*\mathcal{E}$ to \mathcal{E} , the result follows. \square

Since $h\kappa(t)^*(a) = \delta(t)^{\frac{1}{2}}h(at)$, it is routine to check that $\vartheta_t(s) = \kappa(t)s\kappa(t)^*$ for all $(t, s) \in \mathcal{A} * \mathcal{E}$, and that $\kappa(\sigma t \sigma^{-1}) = \overline{\vartheta}_{\sigma}(\kappa(t))$ for all $(\sigma, t) \in \Sigma * \mathcal{A}$. It follows from Lemma 2.8 that κ is a twisting map for $(\mathcal{E}, \Sigma, \vartheta)$. Then we can form the twisted groupoid crossed product $C^*(\mathcal{G}, \Sigma, \mathcal{E}, \kappa)$. Recall that as described in Section 1.4, the latter is the Fell bundle C^* -algebra for the Fell bundle $\mathcal{B}(\mathcal{E}, \vartheta, \kappa) = \mathcal{E}^{\Sigma}$ associated to the twist. Hence $\mathcal{B}(\mathcal{E}, \vartheta, \kappa)$ is the quotient of $r^*\mathcal{E} = \{(\sigma, a) \in \Sigma \times \mathcal{A} : r(\sigma) = p_{\mathcal{E}}(a)\}$ by the \mathcal{A} -action $t \cdot (\sigma, a) = (t\sigma, a\kappa(t)^*)$.

We can identify $\Gamma_c(\mathcal{G}; \mathcal{B}(\mathcal{E}, \vartheta, \kappa))$ with the collection $C_c(\mathcal{G}, \Sigma, \mathcal{E}, \kappa)$ of continuous functions $f : \Sigma \rightarrow \mathcal{E}$ such that $p_{\mathcal{E}}(f(\sigma)) = r(\sigma)$ and

$$(2.5) \quad f(a\sigma) = f(\sigma)\kappa(a)^*$$

whose support has compact image in \mathcal{G} . The $*$ -algebra structure on $C_c(\mathcal{G}, \Sigma, \mathcal{E}, \kappa)$ is given by

$$(2.6) \quad f * g(\sigma) = \int_{\mathcal{G}} f(\tau)\vartheta_{\tau}(g(\tau^{-1}\sigma))d\alpha^{r(\sigma)}(\dot{\tau}) \quad \text{and} \quad f^*(\sigma) = \vartheta_{\sigma}(f(\sigma^{-1})^*).$$

If $f \in C_c(\Sigma)$ and $\sigma \in \Sigma$, we let $j(f)(\sigma)$ be the element of $C_c(A(r(\sigma)))$ given by $a \mapsto \delta(\sigma)^{\frac{1}{2}}f(a\sigma)$. In particular, $j(f) \in \Gamma_c(\Sigma; r^*\mathcal{E})$ and a quick computation verifies that it satisfies (2.5) and is an element of $C_c(\mathcal{G}, \Sigma, \mathcal{E}, \kappa)$.

Remark 2.9. Let $\{g_i\}$ be a net in $C_c(\mathcal{G}, \Sigma, \mathcal{E}, \kappa)$. We say that $g_i \rightarrow g$ in the inductive-limit topology if $g_i \rightarrow g$ uniformly and the supports of the g_i are all contained in some B such that $p(B)$ is compact in \mathcal{G} . This implies that $\check{g}_i \rightarrow \check{g}$ in the inductive-limit topology on $\Gamma_c(\mathcal{G}; \mathcal{B}(\mathcal{E}, \vartheta, \kappa))$. Now suppose that $f_i \rightarrow f$ in the inductive-limit topology on Σ so that there is a compact set $K \subset \Sigma$ such that $\text{supp } f_i \subset K$ for all i and $f_i \rightarrow f$ uniformly. We claim that $j(f_i) \rightarrow j(f)$ in the inductive-limit topology on $C_c(\mathcal{G}, \Sigma, \mathcal{E}, \kappa)$. Certainly we have the supports of the $j(f_i)$ all contained in the image of K . Moreover if $\sigma \in K$, then $\text{supp } j(f_i)(\sigma) \subset K^{-1}K \cap \mathcal{A}$. Since $u \mapsto \beta^u(K^{-1}K \cap \mathcal{A})$ is bounded and δ is bounded on K , it follows that $j(f_i) \rightarrow j(f)$ uniformly on K . Since $\|j(f_i)(\sigma) - j(f)(\sigma)\|$ depends only on $\dot{\sigma}$, the claim follows.

Lemma 2.10. *The map $f \mapsto j(f)$ is a $*$ -homomorphism of $C_c(\Sigma)$ into $C_c(\mathcal{G}, \Sigma, \mathcal{E}, \kappa)$ and the range of j is dense in $C_c(\mathcal{G}, \Sigma, \mathcal{E}, \kappa)$ in the inductive limit topology.*

Proof. Using (2.6) we have

$$\begin{aligned}
j(f * g)(\sigma)(a') &= \delta(\sigma)^{\frac{1}{2}} f * g(a' \sigma) \\
&= \delta(\sigma)^{\frac{1}{2}} \int_{\Sigma} f(\tau) g(\tau^{-1} a' \sigma) d\lambda^{r(\sigma)}(\tau) \\
&= \delta(\sigma)^{\frac{1}{2}} \int_{\mathcal{G}} \int_{\mathcal{A}} f(\tau a) g(a^{-1} \tau^{-1} a' \sigma) \beta^{s(\tau)}(a) d\alpha^{r(\sigma)}(\dot{\tau}) \\
&= \delta(\sigma)^{\frac{1}{2}} \int_{\mathcal{G}} \delta(\tau) \int_{\mathcal{A}} f(a\tau) g(\tau^{-1} a^{-1} a' \sigma) d\beta^{r(\tau)}(a) d\alpha^{r(\sigma)}(\dot{\tau}) \\
&= \int_{\mathcal{G}} \int_{\mathcal{A}} \delta(\tau)^{\frac{1}{2}} f(a\tau) \delta(\tau) \delta(\tau^{-1} \sigma)^{\frac{1}{2}} g(\tau^{-1} a^{-1} a' \sigma) d\beta^{r(\tau)}(a) d\alpha^{r(\sigma)}(\dot{\tau}) \\
&= \int_{\mathcal{G}} \int_{\mathcal{A}} j(f)(\tau)(a) \vartheta_{\tau}(j(g)(\tau^{-1} \sigma))(a^{-1} a') \beta^{r(\tau)}(a) d\alpha^{r(\sigma)}(\dot{\tau}) \\
&= \int_{\mathcal{G}} j(f)(\tau) * \vartheta_{\tau}(j(g)(\tau^{-1} \sigma))(a') d\alpha^{r(\sigma)}(\dot{\tau})
\end{aligned}$$

which, arguing as in [Wil07, Lemma 1.108], is

$$= \left(\int_{\mathcal{G}} j(f)(\tau) * \vartheta_{\tau}(j(g)(\tau^{-1} \sigma)) d\alpha^{r(\sigma)}(\dot{\tau}) \right)(a').$$

Thus $j(f * g)(\sigma) = j(f) * j(g)(\sigma)$ as required.

Similarly,

$$\begin{aligned}
j(f^*)(\sigma)(a) &= \delta(\sigma)^{\frac{1}{2}} f^*(a\sigma) = \delta(\sigma)^{\frac{1}{2}} \overline{f(\sigma^{-1} a^{-1} \sigma \sigma^{-1})} \\
&= \overline{\vartheta_{\sigma}(j(f)(\sigma^{-1}))(a^{-1})} = (\vartheta_{\sigma}(j(f)(\sigma^{-1})))^*(a).
\end{aligned}$$

Thus $j(f^*) = j(f)^*$, and we have shown that j is a *-homomorphism.

To see that the range is dense, let $F \in C_c(\mathcal{G}, \Sigma, \mathcal{E}, \kappa)$. Let K be a compact neighborhood of $p(\text{supp } F)$. Given $\epsilon > 0$ it will suffice to find $f \in C_c(\Sigma)$ such that $p(\text{supp } j(f)) \subset K$ and $\|j(f) - F\|_{\infty} < \epsilon$.

Given $\sigma \in \Sigma$, there exists $f_{\sigma} \in C_c(\Sigma)$ such that

$$\|j(f_{\sigma})(\sigma) - F(\sigma)\| < \epsilon.$$

Since $r^* \mathcal{E}$ is an upper-semicontinuous Banach bundle and κ is unitary-valued, there is an open neighborhood, $V_{\dot{\sigma}}$ of $\dot{\sigma}$ in \mathcal{G} such that

$$\|j(f_{\sigma})(\tau) - F(\tau)\| < \epsilon \quad \text{for all } \dot{\tau} \in V_{\dot{\sigma}}.$$

Fix $\sigma_1, \dots, \sigma_n \in \Sigma$ such that the $V_{\dot{\sigma}_k}$ cover $p(\text{supp } F)$. Let $\{\phi_k\} \subset C_c^+(\mathcal{G})$ be such that $\text{supp } \phi_k \subset V_{\dot{\sigma}_k} \cap K$ and

$$\sum_{k=1}^n \phi_k(\dot{\tau}) = 1$$

if $\tau \in p(\text{supp } F)$ and bounded by 1 otherwise. Then

$$f(\sigma) = \sum_{k=1}^n \phi_k(\dot{\sigma}) f_{\sigma_k}(\sigma)$$

belongs to $C_c(\Sigma)$, $p(\text{supp } j(f)) \subset K$, and

$$\|j(f)(\sigma) - F(\sigma)\| \leq \sum_{k=1}^n \|\phi_k(\dot{\sigma})(j(f_{\sigma_k})(\sigma) - F(\sigma))\| < \epsilon \sum_{k=1}^n \phi_k(\dot{\sigma}) \leq \epsilon.$$

This suffices. □

Theorem 2.11. *We let Σ , \mathcal{A} , \mathcal{G} , \mathcal{E} , and ϑ be as above with twisting map κ . The $*$ -homomorphism $j : C_c(\Sigma) \rightarrow C_c(\mathcal{G}, \Sigma, \mathcal{E}, \kappa)$ defined by $j(f)(\sigma)(a) = \delta(\sigma)^{\frac{1}{2}} f(a\sigma)$ in Lemma 2.10 is isometric for the (universal) C^* -norm and therefore extends to an isomorphism $j : C^*(\Sigma) \rightarrow C^*(\mathcal{G}, \Sigma, \mathcal{E}, \kappa)$. If \mathcal{A} is amenable, then j is also isometric for reduced norms and extends to an isomorphism $j_r : C_r^*(\Sigma) \rightarrow C_r^*(\mathcal{G}, \Sigma, \mathcal{E}, \kappa)$.*

Remark 2.12. If \mathcal{A} is not amenable, the situation is complicated. One might expect to replace $C^*(\mathcal{A})$ with $C_r^*(\mathcal{A})$ in the construction of \mathcal{E} to obtain a bundle \mathcal{E}_r and an isomorphism $C_r^*(\Sigma) \cong C_r^*(\mathcal{G}, \Sigma, \mathcal{E}_r, \kappa_r)$. However, as shown in [Wil15, Arm19], while $C_r^*(\mathcal{A})$ is a $C_0(\mathcal{G}^{(0)})$ -algebra, its fibre over u need not be isomorphic to $C_r^*(A(u))$. We have not pursued this subtlety further.

Remark 2.13. With some effort, the assertion that $C^*(\Sigma)$ and $C^*(\mathcal{G}, \Sigma, \mathcal{E}, \kappa)$ are isomorphic can be derived from a general result in a 2016 preprint due to Buss and Meyer [BM16]. The map $p : \Sigma \rightarrow \mathcal{G}$ from (1.2) is an example of what is called a ‘‘groupoid fibration with fibre \mathcal{A} ’’ in [BM16]. The Fell bundle constructed in [BM16, §6] can be shown to be isomorphic (as Fell bundles) to the Fell bundle $\mathcal{B}(\mathcal{E}, \vartheta, \kappa) = \mathcal{E}^\Sigma$ corresponding to $C^*(\mathcal{G}, \Sigma, \mathcal{E}, \kappa)$ constructed above. Then one can show that j is the composition of the isomorphism from [BM16, Theorem 6.2] and the isomorphism induced by the Fell bundle isomorphism above. Since our situation is considerably more concrete, we are able to explicitly describe both the bundle and the isomorphism. This will be crucial in the next section where we specialize to the case \mathcal{A} is an abelian group bundle. Furthermore, the case of the reduced norm is not considered in [BM16].

To prove Theorem 2.11 we will need the following technical result from [Ren91, Corollary 1.8]. We have included the details for completeness.

Lemma 2.14. *If μ is a quasi-invariant measure on $\Sigma^{(0)}$ with respect to Σ , then μ is also quasi-invariant with respect to \mathcal{G} . If $\underline{\Delta}$ is a modular function on \mathcal{G} for μ , then $\Delta(\sigma) = \delta(\sigma)\underline{\Delta}(\dot{\sigma})$ is a modular function on Σ for μ . In particular, we can assume both $\underline{\Delta}$ and Δ are homomorphisms into \mathbf{R}^+ .*

Proof. Let $b \in C_{c,p}^+(\Sigma)$ be a Bruhat section as in Lemma 2.6(a). Suppose $f \in C_c(\mathcal{G})$. Then

$$\begin{aligned} \nu_{\mathcal{G}}(f) &= \int_{\Sigma^{(0)}} \int_{\mathcal{G}} f(\lambda) d\alpha^u(\lambda) d\mu(u) \\ &= \int_{\Sigma^{(0)}} \int_{\mathcal{G}} \int_{\mathcal{A}} f(\dot{\sigma}) b(\sigma a) d\beta^{s(\sigma)}(a) d\alpha^u(\dot{\sigma}) d\mu(u) \\ &= \int_{\Sigma^{(0)}} \int_{\Sigma} f(\dot{\sigma}) b(\sigma) d\lambda^u(\sigma) d\mu(u). \end{aligned}$$

So for any modular function Δ for μ on Σ , we obtain

$$\begin{aligned} \nu_{\mathcal{G}}(f) &= \int_{\Sigma^{(0)}} \int_{\Sigma} f(\dot{\sigma}^{-1}) b(\sigma^{-1}) \Delta(\sigma^{-1}) d\lambda^u(\sigma) d\mu(u) \\ (2.7) \quad &= \int_{\Sigma^{(0)}} \int_{\mathcal{G}} f(\dot{\sigma}^{-1}) \int_{\mathcal{A}} b(a^{-1}\sigma^{-1}) \Delta(a^{-1}\sigma^{-1}) d\beta^{s(\sigma)}(a) d\alpha^u(\dot{\sigma}) d\mu(u). \end{aligned}$$

Define $B : \mathcal{G} \rightarrow \mathbf{C}$ by

$$B(\dot{\sigma}) = \int_{\mathcal{A}} b(a^{-1}\sigma^{-1}) \Delta(a^{-1}\sigma^{-1}) d\beta^{s(\sigma)}(a).$$

Then (2.7) gives

$$\nu_{\mathcal{G}}(f) = \int_{\Sigma^{(0)}} \int_{\mathcal{G}} f(\gamma^{-1}) B(\gamma) d\sigma^u(\gamma) d\mu(u) = \nu_{\mathcal{G}}^{-1}(fB^*).$$

Since δ agrees with $\Delta_{\mathcal{A}(u)}$, we have

$$\begin{aligned} B(\dot{\sigma}) &= \delta(\sigma)^{-1} \int_{\mathcal{A}} b(\sigma^{-1}a^{-1}) \Delta(\sigma^{-1}a^{-1}) d\beta^{r(\sigma)}(a) \\ &= \int_{\mathcal{A}} b(\sigma^{-1}a) \Delta(\sigma^{-1}a) \delta(\sigma^{-1}a^{-1}) d\beta^{r(\sigma)}(a). \end{aligned}$$

Since Δ and δ never vanish, it follows from (2.1) that B never vanishes. Hence $\nu_{\mathcal{G}}$ and $\nu_{\mathcal{G}}^{-1}$ are equivalent. Thus, by definition, μ is \mathcal{G} -quasi-invariant.

Let $\underline{\Delta}$ be a modular function for \mathcal{G} (with respect to μ). Then

$$\begin{aligned} &\int_{\Sigma^{(0)}} \int_{\Sigma} f(\sigma^{-1}) \underline{\Delta}(\dot{\sigma}^{-1}) \delta(\sigma)^{-1} d\lambda^u(\sigma) d\mu(u) \\ &= \int_{\Sigma^{(0)}} \int_{\mathcal{G}} \underline{\Delta}(\dot{\sigma}^{-1}) \delta(\sigma)^{-1} \int_{\mathcal{A}} f(a^{-1}\sigma^{-1}) \delta(a)^{-1} d\beta^{s(\sigma)}(a) d\alpha^u(\dot{\sigma}) d\mu(u) \\ &= \int_{\Sigma^{(0)}} \int_{\mathcal{G}} Q(f)(\dot{\sigma}^{-1}) \underline{\Delta}(\dot{\sigma}^{-1}) d\alpha^u(\dot{\sigma}) d\mu(u) \\ &= \int_{\Sigma^{(0)}} \int_{\mathcal{G}} Q(f)(\dot{\sigma}) d\alpha^u(\dot{\sigma}) d\mu(u) \\ &= \int_{\Sigma^{(0)}} \int_{\Sigma} f(\sigma) d\lambda^u(\sigma) d\mu(u). \end{aligned}$$

Hence $\underline{\Delta}(\dot{\sigma})\delta(\sigma)$ is a modular function for Σ . Work of Ramsay—see [Wil19, Proposition 7.6]—implies we can take $\underline{\Delta}$ to be a homomorphism. Then we can let $\Delta(\sigma) = \underline{\Delta}(\dot{\sigma})\delta(\sigma)$. This completes the proof. \square

Proof of Theorem 2.11. Suppose that $f_i \rightarrow f$ in the inductive-limit topology on $C_c(\Sigma)$. Then as in Remark 2.9, $j(f_i) \rightarrow j(f)$ in the inductive-limit topology on $C_c(\mathcal{G}, \Sigma, \mathcal{E}, \kappa)$. Thus if \underline{L} is a nondegenerate representation of $\Gamma_c(\mathcal{G}; \mathcal{B}(\mathcal{E}, \vartheta, \kappa))$, then $\underline{L} \circ j$ is continuous in the inductive-limit topology on $C_c(\Sigma)$ and therefore bounded with respect to the C^* -norm:

$$\|\underline{L}(j(f))\| \leq \|f\| \quad \text{for all } f \in C_c(\Sigma).$$

Since \underline{L} is arbitrary, j extends to a homomorphism $j : C^*(\Sigma) \rightarrow C^*(\mathcal{G}, \Sigma, \mathcal{E}, \kappa)$.

Proving that j is isometric for universal norms requires considerably more work. The idea is straightforward. Given a nondengenerate representation L of $C_c(\Sigma)$, it will suffice to produce a representation \underline{L} of $\Gamma_c(\mathcal{G}; \mathcal{B}(\mathcal{E}, \vartheta, \kappa))$ such that $L(f) = \underline{L}(j(f))$. Applying this to a faithful L will show that j is isometric.

Using Renault's Disintegration Theorem (cf., e.g., [Wil19, Theorem 8.2]), we can assume that L is the integrated form of a unitary representation $(\mu, \Sigma^{(0)} * \mathcal{H}, \hat{L})$ where μ is a quasi-invariant measure on $\Sigma^{(0)}$, $\Sigma^{(0)} * \mathcal{H}$ is a Borel Hilbert bundle over $\Sigma^{(0)}$, and \hat{L} is a groupoid homomorphism of Σ into $\text{Iso}(\Sigma^{(0)} * \mathcal{H})$ of the form $\hat{L}(\sigma) = (r(\sigma), L_\sigma, s(\sigma))$. By Lemma 2.14, μ is quasi-invariant with respect to \mathcal{G} . Furthermore, we may assume that Δ is given by $\Delta(\sigma) = \delta(\sigma)\underline{\Delta}(\dot{\sigma})$ where δ is given by Lemma 2.4, and $\underline{\Delta}$ is a modular function for \mathcal{G} . We can also assume that Δ and $\underline{\Delta}$ are homomorphisms.

We are going to realize \underline{L} as the integrated form of a Borel $*$ -functor $\hat{\pi} : \mathcal{B}(\mathcal{E}, \vartheta, \kappa) \rightarrow \text{End}(\Sigma^{(0)} * \mathcal{H})$ as in [MW08a, Definition 4.5 and Proposition 4.10]. To start, consider $\sigma \in \Sigma$ and $h \in C_c(A(r(\sigma)))$. Define $\bar{\pi}_\sigma(h) : \mathcal{H}(s(\sigma)) \rightarrow \mathcal{H}(r(\sigma))$ by

$$\bar{\pi}_\sigma(h)\xi = \int_{\mathcal{A}} h(a)L_{a\sigma}(\xi)\delta(a)^{-\frac{1}{2}} d\beta^{r(\sigma)}(a).$$

It is routine to check that

$$(2.8) \quad \bar{\pi}_{t\sigma}(h) = \bar{\pi}_\sigma(h\kappa(t)) \quad \text{for } (t, \sigma) \in \mathcal{A} * \Sigma.$$

Since

$$(\bar{\pi}_\sigma(h)\xi \mid \eta) = \int_{\mathcal{A}} h(a)\delta(a)^{-\frac{1}{2}}(L_{a\sigma}(\xi) \mid \eta) d\beta^{r(\sigma)},$$

it follows from the usual Cauchy–Schwarz estimate that

$$\begin{aligned} |(\bar{\pi}_\sigma(h)\xi \mid \eta)|^2 &\leq \|\xi\|^2 \|\eta\|^2 \left(\int_{\mathcal{A}} |h(a)|\delta(a)^{-\frac{1}{2}} d\beta^{r(\sigma)}(a) \right)^2 \\ &\leq \|\xi\|^2 \|\eta\|^2 \|h\|_{I,r} \|h\|_{I,s} \leq \|\xi\|^2 \|\eta\|^2 \|h\|_I^2. \end{aligned}$$

Therefore $\bar{\pi}_\sigma$ extends to all of $C^*(A(r(\sigma)))$ and still satisfies (2.8). Hence we get a map $\pi : \mathcal{B}(\mathcal{E}, \vartheta, \kappa) \rightarrow \text{End}(\Sigma^{(0)} * \mathcal{H})$ defined by

$$\pi([\sigma, s]) = \bar{\pi}_\sigma(s).$$

It is not hard to check that π determines a $*$ -functor $\hat{\pi}(b) = (r(b), \pi(b), s(b))$. To see that $\hat{\pi}$ is Borel we need to show that

$$\dot{\sigma} \mapsto (\pi(\check{f}(\dot{\sigma}))(h(s(\sigma))) \mid k(r(\sigma)))$$

is Borel for all $\check{f} \in \Gamma(\mathcal{G}; \mathcal{B}(\mathcal{E}, \vartheta, \kappa))$ and all Borel sections h and k of $\Sigma^{(0)} * \mathcal{H}$. By Lemma 2.10, we can assume that \check{f} is defined by $f \in C_c(\mathcal{G}, \Sigma, \mathcal{E}, \kappa)$ in the range of j . Thus $f(\sigma) \in C_c(A(r(\sigma)))$ for all $\sigma \in \Sigma$. Then

$$\begin{aligned} (\pi(\check{f}(\dot{\sigma}))(h(s(\sigma))) | k(r(\sigma))) &= (\pi_\sigma(f(\sigma))(h(s(\sigma))) | k(r(\sigma))) \\ &= \int_{\mathcal{A}} f(\sigma)(a) (L_{a\sigma}(h(s(\sigma))) | k(r(\sigma))) \delta(a)^{-\frac{1}{2}} d\beta^{r(\sigma)}(a), \end{aligned}$$

which is Borel since L is. Let \underline{L} be the integrated form of $\hat{\pi}$.

Take $f \in C_c(\Sigma)$ and $\xi \in L^2(\Sigma^{(0)} * \mathcal{H}, \mu)$. Then

$$\begin{aligned} L(f)\xi(u) &= \int_{\Sigma} f(\sigma) L_\sigma(\xi(s(\sigma))) \Delta(\sigma)^{-\frac{1}{2}} d\lambda^u(\sigma) \\ &= \int_{\mathcal{G}} \int_{\mathcal{A}} f(\sigma a) L_{\sigma a}(\xi(s(\sigma))) \Delta(\sigma a)^{-\frac{1}{2}} d\beta^{s(\sigma)}(a) d\alpha^u(\dot{\sigma}) \\ &= \int_{\mathcal{G}} \delta(\sigma) \int_{\mathcal{A}} f(a\sigma) L_{a\sigma}(\xi(s(\sigma))) \Delta(a\sigma)^{-\frac{1}{2}} d\beta^u(a) d\alpha^u(\dot{\sigma}) \\ &= \int_{\mathcal{G}} \delta(\sigma)^{\frac{1}{2}} \int_{\mathcal{A}} j(f)(\sigma)(a) L_{a\sigma}(\xi(s(\sigma))) \underline{\Delta}(\dot{\sigma})^{-\frac{1}{2}} \delta(a\sigma)^{-\frac{1}{2}} d\beta^u(a) d\alpha^u(\dot{\sigma}) \\ &= \int_{\mathcal{G}} \pi(j(f(\gamma)))(h(s(\gamma))) \underline{\Delta}(\gamma)^{-\frac{1}{2}}(\gamma) \\ &= \underline{L}(j(f))\xi(u). \end{aligned}$$

This shows that j is isometric.

The surjectivity of j follows from Lemma 2.10. This finishes the proof that j extends to an isomorphism $j : C^*(\Sigma) \rightarrow C^*(\mathcal{G}, \Sigma, \mathcal{E}, \kappa)$.

It remains to prove that j is isometric for reduced norms when \mathcal{A} is amenable. First, note that by [Hol17, Example 3.14] the space $C_c(\Sigma)$ completes to a $C^*(\Sigma)$ - $C^*(\mathcal{A})$ correspondence $X(\Sigma)$ with actions given by convolution and inner product $\langle f, g \rangle_{C^*(\mathcal{A})} = (f^* * g)|_{\mathcal{A}}$. Let \mathcal{B} denote the Fell bundle $\mathcal{B}(\mathcal{E}, \vartheta, \kappa)$ described above so that $C^*(\mathcal{G}, \Sigma, \mathcal{E}, \kappa)$ is the C^* -algebra $C^*(\mathcal{G}, \mathcal{B})$ of this bundle, and similarly for reduced C^* -algebras. Let $\mathcal{H}(\mathcal{B})$ be the right-Hilbert $C^*(\mathcal{A})$ -module of Section 1.3, so that the left action of $C_c(\mathcal{G}; \mathcal{B})$ on $\mathcal{H}(\mathcal{B})$ determined by multiplication in $C_c(\mathcal{G}; \mathcal{B})$ is isometric for the reduced norm. By construction of these maps, the map $j : C_c(\Sigma) \rightarrow C^*(\mathcal{G}, \Sigma, \mathcal{E}, \kappa)$ extends to a right- $C^*(\mathcal{A})$ -module isomorphism $\rho : X(\Sigma) \rightarrow \mathcal{H}(\mathcal{B})$, which satisfies

$$(2.9) \quad \rho(f \cdot \xi) = j(f) \cdot \rho(\xi) \quad \text{for } f \in C_c(\Sigma) \text{ and } \xi \in X(\Sigma).$$

Let $\mathcal{H}(\Sigma)$ and $\mathcal{H}(\mathcal{A})$ be the Hilbert modules described in Section 1.3 that carry faithful representations of $C_r^*(\Sigma)$ and $C_r^*(\mathcal{A})$ respectively. We can form the modules $X(\Sigma) \otimes_{C^*(\mathcal{A})} \mathcal{H}(\mathcal{A})$ and $\mathcal{H}(\mathcal{B}) \otimes_{C^*(\mathcal{A})} \mathcal{H}(\mathcal{A})$, and the isomorphism $\rho : X(\Sigma) \rightarrow \mathcal{H}(\mathcal{B})$ defined above determines an isomorphism $\rho \otimes \text{id} : X(\Sigma) \otimes_{C^*(\mathcal{A})} \mathcal{H}(\mathcal{A}) \rightarrow \mathcal{H}(\mathcal{B}) \otimes_{C^*(\mathcal{A})} \mathcal{H}(\mathcal{A})$ which again intertwines the left actions similarly to (2.9). Since the bundle \mathcal{A} is an amenable groupoid, the left action of $C^*(\mathcal{A}) = C_r^*(\mathcal{A})$ on $\mathcal{H}(\mathcal{A})$ is faithful, and it follows that the map $T \mapsto T \otimes 1$ from $\mathcal{L}(\mathcal{H}(\mathcal{B}))$ to $\mathcal{L}(\mathcal{H}(\mathcal{B}) \otimes_{C^*(\mathcal{A})} \mathcal{H}(\mathcal{A}))$ is isometric. So for $f \in C_c(\Sigma)$, we have

$$\|\rho(f) \otimes 1\|_{\mathcal{L}(\mathcal{H}(\mathcal{B}) \otimes_{C^*(\mathcal{A})} \mathcal{H}(\mathcal{A}))} = \|\rho(f)\|_{C_r^*(\mathcal{G}, \Sigma, \mathcal{E}, \kappa)}.$$

We will show that the map $f \otimes a \mapsto f \cdot a$ extends to an isomorphism $X(\Sigma) \otimes_{C^*(\mathcal{A})} \mathcal{H}(A)$ to $\mathcal{H}(\Sigma)$. This will complete the proof since then

$$\|f\|_{C_r^*(\Sigma)} = \|f \otimes 1\|_{\mathcal{L}(X(\Sigma) \otimes_{C^*(\mathcal{A})} \mathcal{H}(A))} = \|\rho(f) \otimes 1\|_{\mathcal{L}(\mathcal{H}(\mathcal{B}) \otimes_{C^*(\mathcal{A})} \mathcal{H}(A))} = \|\rho(f)\|_{C_r^*(\mathcal{G}, \Sigma, \mathcal{E}, \kappa)}$$

for all $f \in C_c(\Sigma)$.

To see that $f \otimes a \mapsto f \cdot a$ extends to the desired isomorphism, we fix $f, g \in C_c(\Sigma)$ and $a, b \in C_c(\mathcal{A})$, and calculate:

$$\begin{aligned} \langle f \otimes a, g \otimes b \rangle_{C_0(\Sigma^{(0)})} &= (a^* * (f^* * g)|_{\mathcal{A}} * b)|_{C_0(\Sigma^{(0)})} \quad \text{and} \\ \langle f \cdot a, g \cdot b \rangle_{C_0(\Sigma^{(0)})} &= (a^* * (f^* * g) * b)|_{C_0(\Sigma^{(0)})}. \end{aligned}$$

For any $h \in C_c(\Sigma)$, and any $x \in \Sigma^{(0)}$, we have

$$(a^* * h * b)(x) = \int_{\Sigma} \int_{\Sigma} a^*(\beta) h(\gamma) b((\beta\gamma)^{-1}x) d\lambda^x(\beta) d\lambda^{s(\beta)}(\gamma).$$

Since $a, b \in C_c(\mathcal{A})$, the integrand is nonzero only when $\beta, \beta^{-1}\gamma \in \mathcal{A}_x$, and this forces $\gamma \in \mathcal{A}_x$. So

$$(a^* * (f^* * g) * b)|_{\Sigma^{(0)}} = (a^* * (f^* * g)|_{\mathcal{A}} * b)|_{\Sigma^{(0)}}.$$

This completes the proof of Theorem 2.11. \square

3. THE ABELIAN CASE

In this section, we specialize to the case where \mathcal{A} is a normal *abelian* subgroup bundle of Σ with a Haar system β as above. Then $C^*(\mathcal{A})$ is a commutative C^* -algebra. Let $\hat{\mathcal{A}}$ be the Gelfand dual space of nonzero complex homomorphisms $\chi : C^*(\mathcal{A}) \rightarrow \mathbf{C}$, then the Gelfand transform \mathcal{F} is an isomorphism of $C^*(\mathcal{A})$ onto $C_0(\hat{\mathcal{A}})$. As usual, we write $\hat{f} = \mathcal{F}(f)$ for all $f \in C^*(\mathcal{A})$. As shown in [MRW96, Corollary 3.4], the Gelfand dual $\hat{\mathcal{A}}$ is an abelian group bundle $\hat{\pi} : \hat{\mathcal{A}} \rightarrow \Sigma^{(0)}$ with fibres $\hat{\mathcal{A}}(u) := (A(u))^\wedge$. If $\chi \in \hat{\mathcal{A}}$, the corresponding complex homomorphism on $C_c(\mathcal{A})$ is given by

$$(3.1) \quad \chi(f) = \int_{A(\hat{\pi}(\chi))} f(a) \overline{\chi(a)} d\beta^{\hat{\pi}(\chi)}(a) = \hat{f}(\chi).$$

(The complex conjugate appearing on the right-hand side of (3.1) is included to match up with our prejudice for the form of the Fourier transform.)

Since \mathcal{A} is abelian in this section, $\delta(\sigma)$ depends only on $\dot{\sigma}$ and each β^u is bi-invariant. The right action of Σ on $\hat{\mathcal{A}}$ given by

$$\chi \cdot \sigma(a) := \chi(\sigma a \sigma^{-1})$$

factors through a right action of \mathcal{G} on $\hat{\mathcal{A}}$. So we may form the transformation groupoid $\hat{\mathcal{A}} \rtimes \mathcal{G}$ for the action of \mathcal{G} on the space $\hat{\mathcal{A}}$. Recall that we identify the unit space $\hat{\mathcal{A}} \rtimes \mathcal{G}$ with $\hat{\mathcal{A}}$. Thus $r(\chi, \gamma) = \chi$, $s(\chi, \sigma) = \chi \cdot \sigma$, and $(\chi, \gamma)(\chi \cdot \gamma, \eta) = (\chi, \gamma\eta)$. We can equip $\hat{\mathcal{A}} \rtimes \mathcal{G}$ with the Haar system $\underline{\alpha} = \{\underline{\alpha}^\chi\}_{\chi \in \hat{\mathcal{A}}} = \{\delta_\chi \times \alpha^{\hat{\pi}(\chi)}\}$.

3.1. The associated \mathbf{T} -groupoid. We want to build a \mathbf{T} -groupoid associated to Σ just as in [MRW96, §4] except that there \mathcal{G} was assumed to be principal. We start by defining

$$\mathcal{D} = \{ (\chi, z, \sigma) \in \hat{\mathcal{A}} \times \mathbf{T} \times \Sigma : \hat{\pi}(\chi) = r(\sigma) \}.$$

We can make \mathcal{D} into a locally compact Hausdorff groupoid by identifying it with $\hat{\mathcal{A}} \rtimes \Sigma \times \mathbf{T}$. Thus,

$$(\chi, z_1, \sigma_1)(\chi \cdot \sigma_1, z_2, \sigma_2) = (\chi, z_1 z_2, \sigma_1 \sigma_2) \quad \text{and} \quad (\chi, z, \sigma)^{-1} = (\chi \cdot \sigma, \bar{z}, \sigma^{-1}).$$

We can identify $\mathcal{D}^{(0)}$ with $\hat{\mathcal{A}}$, and then

$$r(\chi, z, \sigma) = \chi \quad \text{and} \quad s(\chi, z, \sigma) = \chi \cdot \sigma.$$

Let H be the subgroupoid of \mathcal{D} consisting of triples of the form $(\chi, \overline{\chi(a)}, a)$ for $a \in A(\hat{\pi}(\chi))$. Note that if $(\chi_n, \overline{\chi_n(a_n)}, a_n) \rightarrow (\chi, z, \sigma)$ in \mathcal{D} , then $a_n \rightarrow \sigma$ and $\sigma = a \in A(\hat{\pi}(\chi))$ since \mathcal{A} is closed in Σ . Then $\chi_n(a_n) \rightarrow \chi(a)$ by [MRW96, Proposition 3.3]. Hence H is closed in \mathcal{D} with $H^{(0)} = \mathcal{D}^{(0)}$. To see that H has an open range map, we use Fell's Criterion (Lemma 1.1).

So suppose that $\chi_n \rightarrow \chi = r(\chi, \overline{\chi(a)}, a)$ in $\hat{\mathcal{A}} \times \Sigma^{(0)}$. Since $p : \mathcal{A} \rightarrow \Sigma^{(0)}$ is open, we can pass to a subnet, relabel, and assume that there are $a_n \rightarrow a$ in \mathcal{A} such that $p(a_n) = \hat{\pi}(\chi_n)$. Then $(\chi_n, \overline{\chi_n(a_n)}, a_n) \rightarrow (\chi, \overline{\chi(a)}, a)$ in \mathcal{D} .

We have now showed that H has open range and source maps. Hence the quotient map $q : \mathcal{D} \rightarrow \mathcal{D}/H$ is open. Furthermore, if $d \in \mathcal{D}$, then it is not hard to see that $dH = Hd$. Thus, as in Lemma 2.2, we can form the locally compact Hausdorff groupoid $\hat{\Sigma} := \mathcal{D}/H$, and the elements of $\hat{\Sigma}$ are given by triples $[\chi, z, \sigma]$ where $[\chi, z, a\sigma] = [\chi, \chi(a)z, \sigma]$ for all $a \in \mathcal{A}$. Thus we can define maps $i : (\hat{\mathcal{A}} \times \mathbf{T}) \rightarrow \hat{\Sigma}$ and $j : \hat{\Sigma} \rightarrow \hat{\mathcal{A}} \rtimes \mathcal{G}$ by

$$i(\chi, z) = [\chi, z, \hat{\pi}(\chi)] \quad \text{and} \quad j([\chi, z, \sigma]) = (\chi, \dot{\sigma}).$$

Proposition 3.1. *With respect to the maps i and j above, $\hat{\Sigma}$ is a \mathbf{T} -groupoid over $\hat{\mathcal{A}} \rtimes \mathcal{G}$:*

$$(3.2) \quad \begin{array}{ccccc} \hat{\mathcal{A}} \times \mathbf{T} & \xrightarrow{i} & \hat{\Sigma} & \xrightarrow{j} & \hat{\mathcal{A}} \rtimes \mathcal{G} \\ & \searrow & \Downarrow & \swarrow & \\ & & \hat{\mathcal{A}} & & \end{array}$$

Proof. The map i is clearly injective. Suppose that $i(\chi_n, z_n) = [\chi_n, z_n, \hat{\pi}(\chi_n)] \rightarrow [\chi, z, \sigma]$. Since $q : \mathcal{D} \rightarrow \hat{\Sigma}$ is open, we may assume that there exist $a_n \rightarrow a$ in \mathcal{A} such that $(\chi_n, \overline{\chi_n(a_n)z_n}, a_n) \rightarrow (\chi, z, \sigma)$. But then $\sigma = a$ and $z_n \rightarrow \chi(a)z$. Thus $[\chi, z, \sigma] = [\chi, z, a] = [\chi, \chi(a)z, \hat{\pi}(\chi)] \in i(\hat{\mathcal{A}} \times \mathbf{T})$ and i has closed range. Replacing σ by $\hat{\pi}(\chi)$ in the above shows that $z_n \rightarrow z$ and it follows that i is a homeomorphism onto its range as required.

If $j([\chi, z, \sigma])$ is a unit, then $\dot{\sigma} = \hat{\pi}(\chi)$ and $\sigma = a \in A(\hat{\pi}(\chi))$. But then

$$[\chi, z, a] = [\chi, \chi(a)z, \hat{\pi}(\chi)] \in i(\hat{\mathcal{A}} \times \mathbf{T}).$$

In other words, (3.2) is “exact” at $\hat{\Sigma}$.

We still need to check that j is open. Again, we employ Fell's criterion (see Lemma 1.1). Suppose that $(\chi_n, \gamma_n) \rightarrow (\chi, \dot{\sigma}) = [\chi, z, \sigma]$ in $\hat{\mathcal{A}} \rtimes \mathcal{G}$. Since p is open, we can pass to

a subnet, relabel, and assume that there are $\sigma_n \rightarrow \sigma$ such that $p(\sigma_n) = \gamma_n$. But then $(\chi_n, z, \sigma_n) \rightarrow (\chi, z, \sigma)$. This suffices. \square

Given our Haar system on $\hat{\mathcal{A}} \rtimes \mathcal{G}$, we can build the restricted C^* -algebra $C^*(\hat{\mathcal{A}} \rtimes \mathcal{G}; \hat{\Sigma})$. Recall that this C^* -algebra is built from functions $\tilde{F} \in C_c(\hat{\Sigma})$ such that

$$z' \tilde{F}([\chi, z, \sigma]) = \tilde{F}([\chi, z'z, \sigma]) \quad \text{for all } z' \in \mathbf{T}.$$

Also recall that we write $C_c(\hat{\mathcal{A}} \rtimes \mathcal{G}; \hat{\Sigma})$ for the space of all such functions. To make the notation easier to work with, notice that any $\tilde{F} \in C_c(\hat{\mathcal{A}} \rtimes \mathcal{G}; \hat{\Sigma})$ is determined by its values on classes of the form $[\chi, 1, \sigma]$. Hence we identify $C_c(\hat{\mathcal{A}} \rtimes \mathcal{G}; \hat{\Sigma})$ with the collection $C_c^{\hat{\Sigma}}(\hat{\mathcal{A}} \rtimes \Sigma)$ of continuous functions F on $\hat{\mathcal{A}} \rtimes \Sigma$ such that

$$F(\chi, a\sigma) = \chi(a)F(\chi, \sigma) \quad \text{for all } a \in A(r(\sigma))$$

and such that the support of F has compact image in $\hat{\mathcal{A}} \rtimes \mathcal{G}$.

If $F, G \in C_c^{\hat{\Sigma}}(\hat{\mathcal{A}} \rtimes \Sigma)$, then

$$F * G(\chi, \sigma) = \int_{\mathcal{G}} F(\chi, \tau) G(\chi \cdot \tau, \tau^{-1}\sigma) d\alpha^{r(\sigma)}(p(\tau))$$

and

$$F^*(\chi, \sigma) = \overline{F(\chi \cdot \sigma, \sigma^{-1})}.$$

3.2. The isomorphism. The Gelfand transform gives us an isomorphism of $C^*(\mathcal{A}) = \Gamma_0(\Sigma^{(0)}; \mathcal{E})$ onto $C_0(\hat{\mathcal{A}}) = \Gamma_0(\Sigma^{(0)}; \hat{\mathcal{E}})$ where $\hat{\mathcal{E}} = \coprod C_0(\hat{A}(u))$ is the bundle described just before (3.1).

Our constructions in the previous section give us a dynamical system $(\hat{\mathcal{E}}, \Sigma, \hat{\vartheta})$ where $\hat{\vartheta}_\sigma : C_0(\hat{A}(s(\sigma))) \rightarrow C_0(\hat{A}(r(\sigma)))$ is given by

$$\hat{\vartheta}_\sigma(\hat{h})(\chi) = \hat{h}(\chi \cdot \sigma).$$

The corresponding left-action of \mathcal{A} on $r^*\hat{\mathcal{E}}$ is determined on $\hat{h} \in C_0(\hat{A}(u))$ by

$$\hat{\kappa}(t)\hat{h}(\chi) = \overline{\chi(t)\hat{h}(\chi)}.$$

We form the Fell bundle $\hat{\mathcal{B}}(\hat{\mathcal{E}}, \hat{\vartheta}, \hat{\kappa}) = \mathcal{A} \setminus r^*\hat{\mathcal{E}}$ for the twist $\hat{\kappa}$ on $(\hat{\mathcal{E}}, \Sigma, \hat{\vartheta})$. Sections $\check{g} \in \Gamma_c(\mathcal{G}; \hat{\mathcal{B}}(\hat{\mathcal{E}}, \hat{\vartheta}, \hat{\kappa}))$ are determined by $g \in C_c(\mathcal{G}, \Sigma, \hat{\mathcal{E}}, \hat{\kappa})$ where $g : \Sigma \rightarrow \hat{\mathcal{E}}$ is continuous, and satisfies

$$g(t\sigma)(\chi) = \chi(t)g(\sigma)(\chi)$$

for all $(t, \sigma) \in \mathcal{A} * \Sigma$, and has support with compact image in \mathcal{G} .

From Theorem 2.11, we get isomorphisms $C^*(\Sigma) \rightarrow C^*(\mathcal{G}; \hat{\mathcal{B}}(\hat{\mathcal{E}}, \hat{\vartheta}, \hat{\kappa}))$ and $C_r^*(\Sigma) \rightarrow C_r^*(\mathcal{G}; \hat{\mathcal{B}}(\hat{\mathcal{E}}, \hat{\vartheta}, \hat{\kappa}))$ that send $f \in C_c(\Sigma)$ to the section $\hat{j}(f)$ given by $\hat{j}(\sigma) = (j(f)(\sigma))^\wedge$. Hence

$$\hat{j}(f)(\sigma)(\chi) = \delta(\sigma)^{\frac{1}{2}} \int_{\mathcal{A}} f(a\sigma) \overline{\chi(a)} d\beta^{r(\sigma)}(a).$$

Let $C_\infty^{\widehat{\Sigma}}(\widehat{\mathcal{A}} \rtimes \Sigma)$ be the collection of all $f \in C_0(\widehat{\mathcal{A}} *_r \Sigma)$ such that there is a compact set $K \subset \mathcal{G}$ such that $f(\chi, \sigma) = 0$ if $\dot{\sigma} \notin K$. Since $r^*(C_0(\widehat{\mathcal{A}})) \cong C_0(\Sigma *_r \widehat{\mathcal{A}})$, we obtain a one-to-one correspondence between $C_\infty^{\widehat{\Sigma}}(\widehat{\mathcal{A}} \rtimes \Sigma)$ and $C_c(\mathcal{G}, \Sigma, \widehat{\mathcal{E}}, \widehat{\kappa})$ that carries $f \in C_\infty^{\widehat{\Sigma}}(\widehat{\mathcal{A}} \rtimes \Sigma)$ to the element $F_f \in C_c(\mathcal{G}, \Sigma, \widehat{\mathcal{E}}, \widehat{\kappa})$ given by

$$F_f(\sigma)(\chi) = f(\chi, \sigma).$$

Proposition 3.2. *The map $f \mapsto F_f$ is a *-isomorphism of $C_\infty^{\widehat{\Sigma}}(\widehat{\mathcal{A}} \rtimes \Sigma)$ into $C_c(\mathcal{G}, \Sigma, \widehat{\mathcal{E}}, \widehat{\kappa})$ which extends to isomorphisms*

$$C^*(\widehat{\mathcal{A}} \rtimes \mathcal{G}; \widehat{\Sigma}) \cong C^*(\mathcal{G}, \Sigma, \widehat{\mathcal{E}}, \widehat{\kappa}) \quad \text{and} \quad C_r^*(\widehat{\mathcal{A}} \rtimes \mathcal{G}; \widehat{\Sigma}) \cong C_r^*(\mathcal{G}, \Sigma, \widehat{\mathcal{E}}, \widehat{\kappa}).$$

Proof. We first prove the isomorphism of full C*-algebras. If $\chi \in \widehat{\mathcal{A}}(r(\sigma))$, then since evaluation at χ passes through the integral,

$$\begin{aligned} F_f * F_g(\sigma)(\chi) &= \int_{\mathcal{G}} F_f(\tau)(\chi) \hat{\nu}_\tau(F_g(\tau^{-1}\sigma)(\chi)) d\alpha^{r(\sigma)}(\dot{\tau}) \\ &= \int_{\mathcal{G}} F_f(\tau)(\chi) F_g(\tau^{-1}\sigma)(\chi \cdot \tau) d\alpha^{r(\sigma)}(\dot{\tau}) \\ &= \int_{\mathcal{G}} f(\chi, \tau) g(\chi \cdot \tau, \tau^{-1}\sigma) d\alpha^{r(\sigma)}(\dot{\tau}). \end{aligned}$$

Hence $F_{f*g} = F_f * F_g$. A similar computation shows that $F_{f*} = F_f^*$. Thus $f \mapsto F_f$ is a *-isomorphism onto its range.

The $\|\cdot\|_I$ -norm on $C_c(\mathcal{G}, \Sigma, \widehat{\mathcal{E}}, \widehat{\kappa})$ is given by

$$\|F_f\|_I = \max \{ \|F_f\|_{I,r}, \|F_f\|_{I,s} \},$$

where

$$\|F_f\|_{I,r} = \sup_{u \in \Sigma^{(0)}} \int_{\mathcal{G}} \|F_f(\sigma)\|_\infty d\alpha^u(\dot{\sigma}) \quad \text{and} \quad \|F_f\|_{I,s} = \sup_{u \in \Sigma^{(0)}} \int_{\mathcal{G}} \|F_f(\sigma)\|_\infty d\alpha_u(\dot{\sigma}).$$

The set $\{F_f : f \in C_\infty^{\widehat{\Sigma}}(\widehat{\mathcal{A}} \rtimes \Sigma)\}$ is clearly dense in $C_c(\mathcal{G}, \Sigma, \widehat{\mathcal{E}}, \widehat{\kappa})$ in this $\|\cdot\|_I$ -norm. There exists $\chi_\sigma \in \widehat{\mathcal{A}}(r(\sigma))$ such that $\|F_f(\sigma)\|_\infty = |f(\chi_\sigma, \sigma)|$. Thus

$$\begin{aligned} \|F_f\|_{I,r} &= \sup_{u \in \Sigma^{(0)}} \int_{\mathcal{G}} |f(\chi_\sigma, \sigma)| d\alpha^u(\dot{\sigma}) \\ &= \sup_{\substack{u \in \Sigma^{(0)} \\ \hat{\pi}(\chi) = u}} \int_{\mathcal{G}} |f(\chi, \sigma)| d\alpha^u(\dot{\sigma}) \\ &= \sup_{\chi \in \widehat{\mathcal{A}}} \int_{\mathcal{G}} |f(\chi, \sigma)| d\alpha^{\hat{\pi}(\chi)}(\dot{\sigma}) \\ &= \|f\|_{I,r}. \end{aligned}$$

Similarly, $\|F_f\|_{I,s} = \|f\|_{I,s}$, and $f \mapsto F_f$ is isometric for the respective I -norms. The isomorphism of full C*-algebras follows.

For the isomorphism of reduced C*-algebras, let $\mathcal{H}(\widehat{\mathcal{B}})$ be the right-Hilbert $C_0(\widehat{\mathcal{A}})$ -module described in Section 1.3 for the Fell bundle $\widehat{\mathcal{B}}$. Regard $\mathcal{H}(\widehat{\mathcal{B}})$ as a right $C_0(\widehat{\mathcal{A}})$ -module

by identifying $C_0(\mathcal{G}^{(0)}; \hat{\mathcal{E}})$ with $C_0(\hat{\mathcal{A}})$ via $f \mapsto (\chi \mapsto f(p(\chi))(\chi))$. Then $C_c(\mathcal{G}, \hat{\mathcal{B}})$ acts on the left of $\mathcal{H}(\hat{\mathcal{B}})$ by convolution, and the map implementing this action is isometric for the reduced norm on $C_c(\mathcal{G}, \hat{\mathcal{B}})$. Now consider the Hilbert module $\mathcal{H}(\hat{\mathcal{A}} \rtimes \mathcal{G}; \hat{\Sigma})_{C_0(\hat{\mathcal{A}})}$ obtained from the twist $\hat{\Sigma}$ as described in Section 1.3, so that the left action of $C_c(\hat{\mathcal{A}} \rtimes \mathcal{G}; \hat{\Sigma})$ on $\mathcal{H}(\hat{\mathcal{A}} \rtimes \mathcal{G}; \hat{\Sigma})$ by convolution is isometric for the reduced norm on $C_c(\hat{\mathcal{A}} \rtimes \mathcal{G}; \hat{\Sigma})$. A straightforward calculation shows that $\langle f, g \rangle_{C_0(\hat{\mathcal{A}})} = \langle F_f, F_g \rangle_{C_0(\hat{\mathcal{A}})}$ and so $f \mapsto F_f$ extends to an isomorphism of Hilbert modules $\mathcal{H}(\hat{\mathcal{A}} \rtimes \mathcal{G}; \hat{\Sigma}) \cong \mathcal{H}(\hat{\mathcal{B}})$, which intertwines the left actions because $f \mapsto F_f$ is a homomorphism. Hence

$$\|f\|_{C_r^*(\hat{\mathcal{A}} \rtimes \mathcal{G}; \hat{\Sigma})} = \|f\|_{\mathcal{L}(\mathcal{H}(\hat{\mathcal{A}} \rtimes \mathcal{G}; \hat{\Sigma}))} = \|F_f\|_{\mathcal{L}(\mathcal{H}(\hat{\mathcal{B}}))} = \|F_f\|_{C_r^*(\mathcal{G}, \Sigma, \hat{\mathcal{E}}, \hat{\kappa})}. \quad \square$$

Since we can identify $C_\infty^{\hat{\Sigma}}(\hat{\mathcal{A}} \rtimes \Sigma)$ and $C_c(\mathcal{G}, \Sigma, \hat{\mathcal{E}}, \hat{\kappa})$, we can view $C_\infty^{\hat{\Sigma}}(\hat{\mathcal{A}} \rtimes \Sigma)$ as a dense subalgebra of $C^*(\hat{\mathcal{A}} \rtimes \mathcal{G}; \hat{\Sigma})$. Furthermore, if $f \in C_c(\Sigma)$, then

$$(3.3) \quad \Phi(f)(\chi, \sigma) = \delta(\sigma)^{\frac{1}{2}} \int_{\mathcal{A}} f(a\sigma) \overline{\chi(a)} d\beta^r(\sigma)(a)$$

defines an element of $C_\infty^{\hat{\Sigma}}(\hat{\mathcal{A}} \rtimes \Sigma)$. Noticing that $\Phi(f)(\chi, \sigma) = j(f)(\sigma)(\chi)$, we can combine Theorem 2.11 and Proposition 3.2 to obtain the main result in this section.

Theorem 3.3. *Let Σ be a second countable locally compact groupoid with a Haar system λ . Suppose that \mathcal{A} is a closed abelian normal subgroup bundle with a Haar system β . Then the map $\Phi : C_c(\Sigma) \rightarrow C_\infty^{\hat{\Sigma}}(\hat{\mathcal{A}} \rtimes \Sigma)$ given in (3.3) extends to an isomorphism of $C^*(\Sigma)$ onto the restricted C^* -algebra $C^*(\hat{\mathcal{A}} \rtimes \mathcal{G}; \hat{\Sigma})$. Moreover, this isomorphism descends to an isomorphism $C_r^*(\Sigma) \cong C_r^*(\hat{\mathcal{A}} \rtimes \mathcal{G}; \hat{\Sigma})$.*

Remark 3.4. Theorem 3.3 generalizes [MRW96, Proposition 4.5] where it was assumed that $\mathcal{A} = \Sigma'$, so that \mathcal{G} is principal, and that $C^*(\Sigma)$ is CCR.

4. EXAMPLES AND APPLICATIONS

4.1. Closed normal abelian subgroups. Let G be a locally compact group, and let $H \leq G$ be a closed normal abelian subgroup. Putting $\mathcal{A} := H$, $\Sigma := G$ and $\mathcal{G} := G/H$, we obtain an instance of the situation of Section 3, in which the groupoids involved have a single unit.

The group G acts on the left of H by conjugation, this descends to an action of G/H because H is abelian, and these left actions induce right actions of G and G/H on the space \hat{H} . So we obtain an extension of groupoids

$$\hat{H} \times H \longrightarrow \hat{H} \rtimes G \longrightarrow \hat{H} \rtimes (G/H)$$

with common unit space \hat{H} (regarded as a topological space).

The groupoid \mathcal{D} of Section 3.1 is the cartesian product $(\hat{H} \rtimes G) \times \mathbf{T}$ of the transformation groupoid for the action of G on \hat{H} , with the circle group. The closed subgroupoid $\iota(\hat{H} \times H)$ is the set $\{((\chi, h), \overline{\chi(h)}) : (\chi, h) \in \hat{H} \times H\}$. So the associated \mathbf{T} -groupoid of Section 3.1 is the quotient

$$\hat{\Sigma} := \{[(\chi, g), z] : (\chi, g) \in \hat{H} \rtimes G, z \in \mathbf{T}\},$$

in which $[(\chi, g), z] = [(\chi, hg), \overline{\chi(h)}z]$ for any $h \in H$.

The inclusion $\iota : (\chi, h) \mapsto [(\chi, h), \overline{\chi(h)}]$ and the groupoid homomorphism $\pi : [(\chi, g), z] \mapsto (\chi, gH)$ yield the twist

$$\widehat{H} \times \mathbf{T} \xrightarrow{\iota} \widehat{\Sigma} \xrightarrow{\pi} \widehat{H} \rtimes (G/H),$$

and Theorem 3.3 yields an isomorphism $C^*(G) \cong C^*(\widehat{H} \rtimes (G/H); \widehat{\Sigma})$.

Remark 4.1. An interesting special case of the construction described above occurs when the closed normal abelian subgroup H is a clopen subgroup as studied by Zeller-Meier [ZM68, 2.31]. This implies first that the quotient group G/H is a discrete group, and second that the group C^* -algebra $C^*(H) \cong C_0(\widehat{H})$ is a subalgebra of $C^*(G)$ that contains an approximate identity for $C^*(G)$. Since G/H is discrete, there exists a continuous section $\widehat{H} \rtimes (G/H) \rightarrow G$ for the quotient map. Therefore the twist $\widehat{\Sigma}$ is topologically trivial, in the sense that it is determined by a continuous \mathbf{T} -valued 2-cocycle on $\widehat{H} \rtimes (G/H)$.

Example 4.2. As an illustrative example, consider the integer Heisenberg group

$$G = \langle a, b, c \mid ab = cba, ca = ac, cb = bc \rangle.$$

(A similar analysis applies for the Heisenberg group of upper-triangular 3×3 real matrices with diagonal entries equal to 1, but the integer example is slightly easier to describe.) We consider two natural (clopen) normal abelian subgroups of G in the context of our result.

- (a) First consider the subgroup $H_c \cong \mathbf{Z}$ generated by the element c . This H_c is precisely the center $\mathcal{Z}(G)$, and so the action of $G/H_c \cong \mathbf{Z}^2$ on $\widehat{H}_c \cong \mathbf{T}$ is trivial. Hence the semidirect product $\widehat{H}_c \rtimes (G/H_c)$ is just the product $\widehat{H}_c \times (G/H_c) \cong \mathbf{T} \times \mathbf{Z}^2$ regarded as group bundle. As discussed in Remark 4.1, $C^*(H_c) \cong C(\mathbf{T})$ embeds as a unital subalgebra of $C^*(G)$ which is central because H_c is central, and therefore makes $C^*(G)$ into a $C(\mathbf{T})$ -algebra. For $z \in \mathbf{T}$, the fibre of $\widehat{\Sigma}$ corresponding to the character $\chi_z : c \mapsto z$ of H_c is the extension of \mathbf{Z}^2 corresponding to the 2-cocycle $((m_1, m_2), (n_1, n_2)) \mapsto z^{m_2 n_1}$. The corresponding fibre of $C^*(H)$ is isomorphic to the rotation algebra A_θ where $z = e^{2\pi i \theta}$. So we recover, in this instance, Anderson and Paschke's description [AP89] of $C^*(G)$ as the section algebra of a field of rotation algebras.
- (b) Now consider the subgroup $H_{b,c} \cong \mathbf{Z}^2$ of G generated by b and c , so that $\widehat{H}_{b,c} \cong \mathbf{T}^2$. The quotient $G/H_{b,c}$ is isomorphic to \mathbf{Z} via $aH_{b,c} \mapsto 1$, and acts on \mathbf{T}^2 by $1 \cdot (w, z) = (zw, z)$, inducing an action α of \mathbf{Z} on $C(\mathbf{T}^2)$. The twist $\widehat{\Sigma}$ is the trivial twist over $\mathbf{T}^2 \rtimes \mathbf{Z}$, and so we recover the well-known description $C^*(G) \cong C^*(\mathbf{T}^2 \rtimes \mathbf{Z}; \widehat{\Sigma}) \cong C^*(\mathbf{T}^2 \rtimes \mathbf{Z})$ as the crossed-product algebra $C(\mathbf{T}^2) \rtimes_\alpha \mathbf{Z}$.

Remark 4.3. Williams proved in [Wil81, pp 357–358] that when G/H is abelian and G is a semidirect product, $G \cong H \rtimes (G/H)$, then $C^*(G) \cong C^*(\widehat{H} \rtimes (G/H))$. Note that the extension $\widehat{\Sigma}$ is trivial.

Remark 4.4. The situation described in Example 4.2(a) generalises as follows. If H is any closed subgroup of $\mathcal{Z}(G)$, then $C^*(H)$ embeds in the centre of the multiplier algebra of $C^*(G)$, making $C^*(G)$ into a $C_0(\widehat{H})$ -algebra. The transformation group $\widehat{H} \rtimes (G/H)$

coincides with the group bundle $\widehat{H} \times (G/H)$. Each $\chi \in \widehat{H}$ determines a central extension $\mathbf{T} \rightarrow \widehat{\Sigma}_\chi \rightarrow G/H$ in which $\widehat{\Sigma}_\chi$ coincides with the quotient group $(G \times \mathbf{T})/\{(h, \overline{\chi(h)}) : h \in H\}$. The fibre of $C^*(G)$ corresponding to a character χ of H is the twisted group C^* -algebra of this extension.

4.2. Extensions of effective étale groupoids. For the convenience of the reader we recall our standing hypotheses and the extension (1.2) where Σ is a unit space fixing extension of a subgroupoid group bundle \mathcal{A} . That is, we have

$$(4.1) \quad \begin{array}{ccccc} \mathcal{A} & \xrightarrow{\iota} & \Sigma & \xrightarrow{p} & \mathcal{G} \\ & \searrow & \Downarrow & \swarrow & \\ & & \Sigma^{(0)} & & \end{array}$$

where ι is the inclusion map and p is a continuous open surjection restricting to a homeomorphism of $\Sigma^{(0)}$ and $\mathcal{G}^{(0)}$. We require that all groupoids in the extension are second countable locally compact, Hausdorff, both Σ and \mathcal{A} have Haar systems and \mathcal{A} is an abelian group bundle.

In this section, we consider the situation where the groupoid \mathcal{G} in the extension (4.1) is an effective étale groupoid; that is, $r, s : \mathcal{G} \rightarrow \mathcal{G}^{(0)}$ are local homeomorphisms, and the topological interior of the isotropy bundle of \mathcal{G} is $\mathcal{G}^{(0)}$. Since our standing hypotheses include that \mathcal{G} is second-countable and Hausdorff, the latter is equivalent to the condition that \mathcal{G} is topologically principal in the sense that the set $\{x \in \mathcal{G}^{(0)} : \mathcal{G}_x^x = \{x\}\}$ is dense in $\mathcal{G}^{(0)}$ [Ren08, Proposition 3.6].

It follows that the transformation groupoid $\widehat{\mathcal{A}} \rtimes \mathcal{G}$ is also étale and effective; this is certainly well known but also easy to prove directly:

Lemma 4.5. *Let \mathcal{G} be a locally compact, Hausdorff groupoid acting on the right of a locally compact Hausdorff space X . If \mathcal{G} is étale (resp., effective) then so is $X \rtimes \mathcal{G}$.*

Proof. Suppose that \mathcal{G} is étale and fix $(x, \gamma) \in X \rtimes \mathcal{G}$. Fix an open bisection neighbourhood U of γ in \mathcal{G} and an open neighbourhood W of $x \in X$. Then $(W \times U) \cap (X \rtimes \mathcal{G}) = \{(y, \eta) \in W \times U \mid s(y) = r(\eta)\}$ is an open bisection neighbourhood of (x, γ) . Hence $X \rtimes \mathcal{G}$ is étale.

Now suppose that \mathcal{G} is effective. Suppose that $U \subseteq X \rtimes \mathcal{G}$ is open and consists entirely of isotropy. Then it is a union of sets of the form $W * V$ where $W \subseteq X$ is open, and $V \subseteq \mathcal{G}$ is an open bisection consisting of isotropy. For any such set, since \mathcal{G} is effective, we have $V \subseteq \mathcal{G}^{(0)}$, and so $W * V \subseteq X * \mathcal{G}^{(0)} = (X \rtimes \mathcal{G})^{(0)}$. \square

It follows from our main theorem that $C_r^*(\Sigma)$ is equal to the reduced C^* -algebra of a twist $\widehat{\Sigma}$ over the étale effective groupoid $\widehat{\mathcal{A}} \rtimes \mathcal{G}$, and so Renault's theory of Cartan subalgebras of C^* -algebras applies. In the case where Σ is étale the first assertion in the following theorem may be deduced from [BNR⁺16, Corollary 4.5]. In addition, we provide an explicit construction of the Weyl twist which is not included in [BNR⁺16]. Our result is also related to Theorem 5.8 of [DGN⁺20]: in their case Σ is étale but they also consider a continuous \mathbf{T} -valued 2-cocycle.

Theorem 4.6. *Suppose that Σ is an extension of a wide normal abelian subgroup bundle with a Haar system as in (4.1). Further assume that \mathcal{G} is étale and effective. Then $C^*(\mathcal{A}) \cong$*

$C_0(\hat{\mathcal{A}})$ is a Cartan subalgebra of $C_r^*(\Sigma)$. The Weyl twist of the Cartan pair $(C_r^*(\Sigma), C^*(\mathcal{A}))$ is $(\hat{\mathcal{A}} \rtimes \mathcal{G}, \widehat{\Sigma})$.

Proof. We apply [Ren08, Theorem 5.2] to the twist $\hat{\mathcal{A}} \times \mathbf{T} \longrightarrow \widehat{\Sigma} \longrightarrow \hat{\mathcal{A}} \rtimes \mathcal{G}$ to see that the pair $(C_r^*(\hat{\mathcal{A}} \rtimes \mathcal{G}; \widehat{\Sigma}), C_0(\hat{\mathcal{A}}))$ is a Cartan pair with Weyl twist $(\hat{\mathcal{A}} \rtimes \mathcal{G}; \widehat{\Sigma})$. The result then follows from the final statement of Theorem 3.3. \square

Many of our examples do not require the full power of our theorem. Note, though, that in the next example Σ is not étale and so [BNR⁺16, Corollary 4.5] cannot be used.

Example 4.7. Let (A, B) be a Cartan pair and let (\mathcal{G}, Σ) be the associated Weyl twist. Then Σ is an extension of \mathcal{G} by the wide normal abelian subgroup bundle $\mathcal{A} \cong \mathbf{T} \times \mathcal{G}^{(0)}$. Moreover, since (A, B) is a Cartan pair, \mathcal{G} is étale and effective. Hence, by Theorem 4.6, $C^*(\mathcal{A}) \cong C_0(\mathbf{Z} \times \mathcal{G}^{(0)})$ is a Cartan subalgebra of $C_r^*(\Sigma)$.

Example 4.8. Let X be a locally compact Hausdorff space and let $n \mapsto \sigma^n$ be an action of \mathbf{N}^k by local homeomorphisms of X . Let Σ be the associated Deaconu–Renault groupoid $\{(x, m - n, y) : \sigma^m(x) = \sigma^n(y)\}$. Let \mathcal{A} denote the interior of the isotropy in Σ , and suppose that \mathcal{A} is a clopen subset of Σ (this may seem like a restrictive hypothesis but it is automatic if Σ is minimal by [KPS16, Proposition 2.1] applied to the cocycle $c(x, m, y) = m$, in which case it is given by $\mathcal{A} = \{(x, m, x) : (y, m, y) \in \Sigma \text{ for all } y \in X\}$ [SW16, Proposition 3.10]). Then \mathcal{A} is an abelian group bundle. Specifically, for each $x \in X$, the set

$$\text{Stab}_x^{\text{ess}} \{p - q : p, q \in \mathbf{Z}^k \text{ and } \sigma^p = \sigma^q \text{ on a neighbourhood of } x\}$$

is a subgroup of \mathbf{Z}^k , we have $\text{Stab}_x^{\text{ess}} = \text{Stab}_y^{\text{ess}}$ whenever $\Sigma_y^x \neq \emptyset$, and as a set,

$$\mathcal{A} = \bigsqcup_{x \in X} \{x\} \times \text{Stab}_x^{\text{ess}}.$$

Since \mathcal{A} is clopen, we can form the quotient groupoid $\mathcal{G} = \Sigma/\mathcal{A}$, which is étale and effective. A quick calculation shows that the action of \mathcal{G} on \mathcal{A} is given by $\gamma \cdot (s(\gamma), m) = (r(\gamma), m)$, and so the induced action of \mathcal{G} on $\hat{\mathcal{A}}$ is given by $(r(\gamma), \chi) \cdot \gamma = (s(\gamma), \chi)$ for $\chi \in (\text{Stab}_{r(\gamma)}^\wedge)$. So, as a groupoid, $\hat{\mathcal{A}} \rtimes \mathcal{G}$ is just the fibred product $\hat{\mathcal{A}} * \mathcal{G}$ —for example if \mathcal{G} is minimal, then $\hat{\mathcal{A}} \rtimes \mathcal{G} \cong \hat{\mathcal{A}}_x \times \mathcal{G}$ for any $x \in X$. Since \mathcal{A} is open in Σ , the map $p_{\mathcal{A}}$ is open and Theorem 3.3 applies. Combining with Theorem 4.6, this implies that $(C^*(\hat{\mathcal{A}} \rtimes \mathcal{G}; \widehat{\Sigma}), C_0(\hat{\mathcal{A}}))$ is a Cartan pair with Weyl twist $(\hat{\mathcal{A}} \rtimes \mathcal{G}; \widehat{\Sigma})$.

Example 4.9. As a particular case of the preceding example, let Λ be a row-finite k -graph with no sources, and let \mathcal{G}_Λ be the associated infinite-path groupoid, which is the Deaconu–Renault groupoid for the action of \mathbf{N}^k on Λ^∞ by shift maps [KP00]. Suppose that the interior \mathcal{I} of the isotropy in \mathcal{G}_Λ is closed—for example, this is automatic if Λ is cofinal, in which case $\mathcal{I} \cong \Lambda^\infty \times \text{Per}(\Lambda)$ [KPS16, Corollary 2.2]. (See [BLY17, Theorem 4.4] for a characterisation of when \mathcal{I} is closed in \mathcal{G}_Λ .) Then our main theorem implies that $C^*(\mathcal{I}) \cong C_0(\hat{\mathcal{I}})$ is a Cartan subalgebra of $C^*(\Lambda)$, recovering (c) \implies (b) of [BNR⁺16, Corollary 4.6], and also gives a recipe for describing the associated Weyl twist.

4.3. Transformation groupoids. Assume that Γ is a locally compact Hausdorff groupoid with Haar system which acts on the right on a locally compact space X . We write $s_X : X \rightarrow \Gamma^{(0)}$ for the moment map and we let $\Sigma := X \rtimes \Gamma$ be the transformation groupoid. Recall that $X \rtimes \Gamma = \{(x, \gamma) \in X \times \Gamma : s_X(x) = r(\gamma)\}$ with the topology inherited from the product topology and the range and source maps given by $r(x, \gamma) = (x, r(\gamma))$ and $s(x, \gamma) = (x \cdot \gamma, s(\gamma))$. Then $(x, \gamma)(x \cdot \gamma, \eta) = (x, \gamma\eta)$ while $(x, \gamma)^{-1} = (x \cdot \gamma, \gamma^{-1})$. We identify the unit space of $X \rtimes \Gamma$ with X as usual via the map $(x, r(\gamma)) \rightarrow x$. Assume that \mathcal{N} is a wide normal subgroup bundle of Γ which acts trivially on X : $x \cdot a = x$ for all $x \in X$ and $a \in \mathcal{N}$. Then the action of Γ on X descends to an action of Γ/\mathcal{N} on X . Therefore, if we let $\mathcal{A} := X * \mathcal{N}$ be the pull-back bundle and let $\mathcal{G} := X \rtimes \Gamma / \mathcal{A}$, we obtain a groupoid extension as in (1.2) with $\Sigma^{(0)} = X$. If \mathcal{N} is abelian, then $\mathcal{A} = X * \mathcal{N}$ is also abelian. Then the \mathbf{T} -groupoid $\widehat{\Sigma}$ constructed in Section 3.1 can be viewed as equivalence classes with $x \in X$, $\chi \in \widehat{N}_{s(x)}$, $\gamma \in \Gamma$ and $z \in \mathbf{T}$ such that $[(\chi, z, x, a\gamma)] = [(\chi, \chi(a)z, x, \gamma)]$. The transformation groupoid $\widehat{\mathcal{A}} \rtimes \mathcal{G}$ is represented by triples $\{(\chi, x, \dot{\gamma}) : \chi \in \widehat{N}_{s(x)} \text{ and } r(\gamma) = s(x)\}$. Then Theorem 3.3 implies that the transformation groupoid C^* -algebra $C^*(\Sigma)$ is isomorphic to the restricted groupoid C^* -algebra $C^*(\widehat{\mathcal{A}} \rtimes \mathcal{G}; \widehat{\Sigma})$ and similarly for the reduced algebras.

Example 4.10 (Rational Rotation Algebra). We consider the group $\Gamma = \mathbf{Z}$ acting on \mathbf{T} via rotation by $\alpha \in \mathbf{Q}$: $z \cdot k := ze^{2\pi i k \alpha}$. If $\alpha = m/n$ with m and n relatively prime, then $n\mathbf{Z}$ fixes the action. Hence we get an instance of the above situation with \mathcal{N} the group $n\mathbf{Z}$, $\Gamma = \mathbf{Z}$ and $\mathcal{G} = \mathbf{Z}_n$. Therefore the groupoid extension is

$$\mathbf{T} \times n\mathbf{Z} \xhookrightarrow{i} \mathbf{T} \rtimes \mathbf{Z} \xrightarrow{p} \mathbf{T} \rtimes \mathbf{Z}_n.$$

The C^* -algebra $C^*(\mathbf{T} \rtimes \mathbf{Z})$ is the rational rotation C^* -algebra \mathcal{A}_α (see, for example, [DB84]). If we let $\mathbf{T}_n = \mathbf{T}/\mathbf{Z}_n$ be the dual of $n\mathbf{Z}$, then $\widehat{\Sigma}$ consists of equivalence classes of elements of $\mathbf{T}_n \times \mathbf{T}^2 \times \mathbf{Z}$ where $[\chi, w, z, nl + k] = [\chi^{nl}, w, z, k]$. In particular, we realize the rotation algebra \mathcal{A}_α as the appropriate completion of continuous functions F on $\mathbf{T}_n \times \mathbf{T} \times \mathbf{Z}$ such that $F(\chi, w, nl + k) = \chi^{nl} F(\chi, w, k)$.

Remark 4.11. The arguments from the above example can be easily extended to actions of \mathbf{Z} on compact spaces by periodic homeomorphisms. These ideas can be sharpened by viewing the construction of $\widehat{\Sigma}$ in Section 3.1 as an instance of general ‘‘pushout’’ construction that will pursue in a future project [IKR⁺20b].

Example 4.12. A large class of examples is given by twisted groupoid crossed products $(\mathcal{G}, \Sigma, \mathcal{E}, \vartheta, \kappa)$ as defined in Section 1.4 under the assumption that the C^* -bundle \mathcal{E} consists of abelian C^* -algebras. Then the $C_0(\Sigma^{(0)})$ -algebra $A = \Gamma_0(\Sigma^{(0)}; \mathcal{E})$ is an abelian C^* -algebra and, thus, isomorphic to $C_0(X)$, where X is the Gelfand spectrum of A . Since A is a $C_0(\Sigma^{(0)})$ -algebra the bundle map $s : X \rightarrow \Sigma^{(0)}$ is continuous. Moreover, $X = \bigsqcup X_u$, where X_u is the spectrum of the fibre $E(u)$. The action of Σ on \mathcal{E} induces a right action on X via $\vartheta(f)(x) = f(x \cdot \sigma)$ for all $f \in C_0(X_{s(\sigma)})$. Since the action of \mathcal{A} on \mathcal{E} is unitarily implemented as in (1.3) and since $E(u)$ is abelian, \mathcal{A} acts trivially on X . Thus we obtain an extension as above with $\mathcal{N} := \mathcal{A}$ and $\Gamma := \Sigma$. Moreover, $C^*(\mathcal{G}, \Sigma, \mathcal{E}, \kappa)$ is isomorphic to $C^*(X \rtimes \Sigma)$. Indeed one can prove that the map that sends $f \in C_c(X \rtimes \Sigma)$ to $\hat{f} \in C_c(\mathcal{G}, \Sigma, \mathcal{E}, \kappa)$ defined

via

$$\hat{f}(\sigma)(x) := \delta(\sigma)^{1/2} \int_{\mathcal{A}} f(x, a\sigma) \overline{\kappa(a)} d\beta^{r(\sigma)}(a)$$

extends to an isomorphism of C^* -algebras. In the above equation $\{\beta^u\}$ is the Haar system on \mathcal{A} and δ is the modular map defined in Lemma 2.4. Theorem 3.3 implies that $C^*(\mathcal{G}, \Sigma, \mathcal{E}, \kappa)$ is isomorphic to the restricted C^* -algebra $C^*(\hat{\mathcal{A}} \rtimes \mathcal{G}; \hat{\Sigma})$, where $\hat{\Sigma}$ and $\hat{\mathcal{A}} \rtimes \mathcal{G}$ are described above. Moreover, the isomorphism descends to the level of reduced C^* -algebras.

REFERENCES

- [AP89] Joel Anderson and William Paschke, *The rotation algebra*, Houston J. Math. **15** (1989), 1–26.
- [Arm19] Becky Armstrong, *Simplicity of C^* -algebras of topological higher-rank graphs*, Ph.D. Dissertation, 2019.
- [BL17] Selçuk Barlak and Xin Li, *Cartan subalgebras and the UCT problem*, Adv. Math. **316** (2017), 748–769.
- [BLY17] Jonathan Brown, Hui Li, and Dilian Yang, *Cartan subalgebras of topological graph algebras and k -graph C^* -algebras*, Münster J. Math. **10** (2017), 367–381.
- [BM16] A. Buss and R. Meyer, *Iterated crossed products for groupoid fibrations*, preprint, 2016. (arXiv:math.OA.1604.02015v1).
- [BNR⁺16] Jonathan H. Brown, Gabriel Nagy, Sarah Reznikoff, Aidan Sims, and Dana P. Williams, *Cartan subalgebras in C^* -algebras of Hausdorff étale groupoids*, Integral Equations Operator Theory **85** (2016), 109–126.
- [DB84] Marc De Brabanter, *The classification of rational rotation C^* -algebras*, Arch. Math. (Basel) **43** (1984), 79–83.
- [DGN⁺20] A. Duwenig, E. Gillaspy, R. Norton, S. Reznikoff, and S. Wright, *Cartan subalgebras for non-principal twisted groupoid C^* -algebras*, J. Funct. Anal. **279** (2020), 108611, 40.
- [FD88a] James M. G. Fell and Robert S. Doran, *Representations of $*$ -algebras, locally compact groups, and Banach $*$ -algebraic bundles. Vol. 1*, Pure and Applied Mathematics, vol. 125, Academic Press Inc., Boston, MA, 1988. Basic representation theory of groups and algebras.
- [FD88b] ———, *Representations of $*$ -algebras, locally compact groups, and Banach $*$ -algebraic bundles. Vol. 2*, Pure and Applied Mathematics, vol. 126, Academic Press Inc., Boston, MA, 1988. Banach $*$ -algebraic bundles, induced representations, and the generalized Mackey analysis.
- [Goe12] Geoff Goehle, *The Mackey machine for crossed products by regular groupoids. II*, Rocky Mountain J. Math. **42** (2012), 873–900.
- [Gre78] Philip Green, *The local structure of twisted covariance algebras*, Acta Math. **140** (1978), 191–250.
- [Hol17] Rohit Dilip Holkar, *Topological construction of C^* -correspondences for groupoid C^* -algebras*, J. Operator Theory **77** (2017), 217–241.
- [IKR⁺20a] Marius Ionescu, Alex Kumjian, Jean N. Renault, Aidan Sims, and Dana P. Williams, *A Mackey normal subgroup analysis for groupoids*, 2020. (Unpublished arXiv:2001.01312v2).
- [IKR⁺20b] ———, *Pushouts of group bundles*, 2020. In preparation.
- [KMRW98] Alexander Kumjian, Paul S. Muhly, Jean N. Renault, and Dana P. Williams, *The Brauer group of a locally compact groupoid*, Amer. J. Math. **120** (1998), 901–954.
- [KP00] Alex Kumjian and David Pask, *Higher rank graph C^* -algebras*, New York J. Math. **6** (2000), 1–20.
- [KPS16] Alex Kumjian, David Pask, and Aidan Sims, *Simplicity of twisted C^* -algebras of higher-rank graphs and crossed products by quasifree actions*, J. Noncommut. Geom. **10** (2016), 515–549.
- [Kum98] Alex Kumjian, *Fell bundles over groupoids*, Proc. Amer. Math. Soc. **126** (1998), 1115–1125.
- [Laz18] Aldo J. Lazar, *A selection theorem for Banach bundles and applications*, J. Math. Anal. Appl. **462** (2018), 448–470.
- [MRW96] Paul S. Muhly, Jean N. Renault, and Dana P. Williams, *Continuous-trace groupoid C^* -algebras. III*, Trans. Amer. Math. Soc. **348** (1996), 3621–3641.

- [MT11] El-Kaïoum M. Moutuou and Jean-Louis Tu, *Equivalence of Fell systems and their reduced C^* -algebras*, preprint, 2011. (arXiv:math.OA.1101.1235v1).
- [MW08a] Paul S. Muhly and Dana P. Williams, *Equivalence and disintegration theorems for Fell bundles and their C^* -algebras*, *Dissertationes Math. (Rozprawy Mat.)* **456** (2008), 1–57.
- [MW08b] ———, *Renault’s equivalence theorem for groupoid crossed products*, NYJM Monographs, vol. 3, State University of New York University at Albany, Albany, NY, 2008. Available at <http://nyjm.albany.edu:8000/m/2008/3.htm>.
- [Ren08] Jean N. Renault, *Cartan subalgebras in C^* -algebras*, *Irish Math. Soc. Bull.* **61** (2008), 29–63.
- [Ren87] ———, *Représentation des produits croisés d’algèbres de groupoïdes*, *J. Operator Theory* **18** (1987), 67–97.
- [Ren91] ———, *The ideal structure of groupoid crossed product C^* -algebras*, *J. Operator Theory* **25** (1991), 3–36.
- [SW13] Aidan Sims and Dana P. Williams, *An equivalence theorem for reduced Fell bundle C^* -algebras*, *New York J. Math.* **19** (2013), 159–178.
- [SW16] ———, *The primitive ideals of some étale groupoid C^* -algebras*, *Algebr. Represent. Theory* **19** (2016), 255–276.
- [Wil07] Dana P. Williams, *Crossed products of C^* -algebras*, *Mathematical Surveys and Monographs*, vol. 134, American Mathematical Society, Providence, RI, 2007.
- [Wil15] Rufus Willett, *A non-amenable groupoid whose maximal and reduced C^* -algebras are the same*, *Münster J. Math.* **8** (2015), 241–252.
- [Wil19] Dana P. Williams, *A tool kit for groupoid C^* -algebras*, *Mathematical Surveys and Monographs*, vol. 241, American Mathematical Society, Providence, RI, 2019.
- [Wil81] ———, *The topology on the primitive ideal space of transformation group C^* -algebras and $C.C.R.$ transformation group C^* -algebras*, *Trans. Amer. Math. Soc.* **266** (1981), 335–359.
- [Yam87] Shigeru Yamagami, *On the ideal structure of C^* -algebras over locally compact groupoids*, 1987. (Unpublished manuscript).
- [ZM68] Georges Zeller-Meier, *Produits croisés d’une C^* -algèbre par un groupe d’automorphismes*, *J. Math. Pures Appl.* (9) **47** (1968), 101–239.

DEPARTMENT OF MATHEMATICS, UNITED STATES NAVAL ACADEMY, ANNAPOLIS, MD 21402 USA
Email address: ionescu@usna.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NEVADA, RENO NV 89557 USA
Email address: alex@unr.edu

INSTITUT DENIS POISSON (UMR 7013), UNIVERSITÉ D’ORLÉANS ET CNRS, 45067 ORLÉANS CEDEX
 2, FRANCE
Email address: jean.renault@univ-orleans.fr

SCHOOL OF MATHEMATICS AND APPLIED STATISTICS, UNIVERSITY OF WOLLONGONG, NSW 2522, AUS-
 TRALIA
Email address: asims@uow.edu.au

DEPARTMENT OF MATHEMATICS, DARTMOUTH COLLEGE, HANOVER, NH 03755-3551 USA
Email address: dana.williams@Dartmouth.edu