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# CONTINUITY OF THE DUAL HAAR MEASURE

JEAN N. RENAULT

ABSTRACT. Given a locally compact group bundle, we show that the system of the Plancherel weights of their  $C^*$ -algebras is lower semi-continuous. As a corollary, we obtain that the dual Haar sytem of a continuous Haar system of a locally compact abelian group bundle is also continuous.

Let  $G$  be a locally compact abelian group with Haar measure  $\lambda$ . The dual Haar measure  $\hat{\lambda}$  on the dual group  $\hat{G}$  is the Haar measure on  $\hat{G}$  which makes the Fourier transform defined by

$$\mathcal{F}(f)(\chi) = \int f(\gamma) \overline{\chi(\gamma)} d\lambda(\gamma)$$

for  $f$  in the space  $C_c(G)$  of complex-valued continuous functions with compact support on  $G$ , an isometry from  $L^2(G, \lambda)$  to  $L^2(\hat{G}, \hat{\lambda})$ . Suppose now that  $p : G \rightarrow X$  is a locally compact abelian group bundle. Here, we mean that  $G$  and  $X$  are locally compact Hausdorff spaces,  $p$  is a continuous surjection and the fibres  $G_x = p^{-1}(x)$  are abelian groups. We also require that the multiplication, as a map from  $G \times_X G$  to  $G$ , and the inverse, as a map from  $G$  to  $G$  are continuous. A Haar system for  $G$  is a family of measures  $(\lambda^x)_{x \in X}$ , where for all  $x \in X$ ,  $\lambda^x$  is a Haar measure of  $G_x$ ; it is said to be continuous if for all  $f \in C_c(G)$ , the function  $x \mapsto \int f d\lambda^x$  is continuous. Then we can form the dual group bundle  $\hat{p} : \hat{G} \rightarrow X$ . As a set,  $\hat{G}$  is the disjoint union of the dual groups  $\hat{G}_x$ . Moreover, it can be identified with the spectrum of the commutative  $C^*$ -algebra  $C^*(G)$ , where  $G$  is viewed as a locally compact groupoid with Haar system  $(\lambda^x)_{x \in X}$ . Hence it is endowed with a locally compact topology. It can be checked ([12, Corollary 3.4]) that  $\hat{p} : \hat{G} \rightarrow X$  is indeed a locally compact abelian group bundle in the above sense. Of course, one expects that the dual Haar system  $(\hat{\lambda}^x)_{x \in X}$  is continuous. This is stated as Proposition 3.6 of [12]. However it was recently pointed to the authors by Henrik Kreidler that their proof is defective. This note corrects this and gives a more general result, based on the fact that the dual Haar measure  $\hat{\lambda}$  on the dual group  $\hat{G}$  of a locally compact abelian group  $G$ , viewed as its Haar weight and defined as the canonical weight of  $L^\infty(\hat{G})$  acting on  $L^2(\hat{G}, \hat{\lambda})$ , corresponds under the Fourier transform to the Plancherel weight of  $G$ , defined as the canonical weight of the von Neumann algebra  $VN(G)$  of  $G$  acting on  $L^2(G, \lambda)$ . Therefore one can consider a locally compact group bundle  $p : G \rightarrow X$  where the fibres  $G_x$  are no longer abelian. Our main result is Proposition 2.3 which says that the Plancherel weight of  $G_x$  varies continuously in a suitable sense. This lead us to the definition of a lower semi-continuous  $C_0(X)$ -weight on a  $C_0(X)$ - $C^*$ -algebra which we illustrate by three examples.

The first section recalls the construction of the canonical weight of a left Hilbert algebra and the properties which are needed to prove the crucial Corollary 1.6. In the second section, we consider the case of a locally compact group bundle and prove our main result. This example motivates the definition of a  $C_0(X)$ -weight of a  $C_0(X)$ - $C^*$ -algebra, given in the third section. The fact that the Plancherel  $C_0(X)$ -weight of the  $C^*$ -algebra of a group bundle is densely defined and lower semi-continuous gives Proposition 3.6 of [12].

## 1. THE PLANCHEREL WEIGHT OF A LOCALLY COMPACT GROUP

We recall first some elements of Tomita-Takesaki's theory, using the standard notation from [16]. Given a left Hilbert algebra  $\mathcal{A}$  where the product, the involution and the scalar product are respectively denoted by  $ab$ ,  $a^\sharp$  and  $(a|b)$ , we denote by  $\mathcal{H}$  the Hilbert space completion of  $\mathcal{A}$  and by  $\pi : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{H})$  the left representation. We denote by  $\mathcal{M} = \pi(\mathcal{A})''$  the left von Neumann algebra of  $\mathcal{A}$ . We denote by  $S$  the closure of the involution  $a \mapsto a^\sharp$  and by  $F$  its adjoint. The domain of  $S$  [resp.  $F$ ] is denoted by  $\mathcal{D}^\sharp$  [resp.  $\mathcal{D}^\flat$ ] and one writes  $\xi^\sharp = S\xi$  [resp.  $\eta^\flat = F\eta$ ] for  $\xi \in \mathcal{D}^\sharp$  [resp. for  $\eta \in \mathcal{D}^\flat$ ]. An element  $\eta \in \mathcal{H}$  is called right bounded if there exists a bounded operator  $\pi'(\eta)$  on  $\mathcal{H}$  such that  $\pi'(\eta)a = \pi(a)\eta$  for all  $a \in \mathcal{A}$ . One then writes  $\xi\eta = \pi'(\eta)\xi$  for all  $\xi \in \mathcal{H}$ . The set of right bounded elements is denoted by  $\mathcal{B}'$ . One shows that  $\mathcal{A}' = \mathcal{B}' \cap \mathcal{D}^\flat$  with involution  $\xi^\flat$  is a right Hilbert algebra in the same Hilbert space  $\mathcal{H}$ . An element  $\xi \in \mathcal{H}$  is called left bounded if there exists a bounded operator  $\pi(\xi)$  on  $\mathcal{H}$  such that  $\pi(\xi)\eta = \pi'(\eta)\xi$  for all  $\eta \in \mathcal{A}'$ . One then writes  $\xi\eta = \pi(\xi)\eta$  for all  $\eta \in \mathcal{H}$ . If  $\xi$  is left bounded and  $\eta$  is right bounded, the notation is consistent:  $\pi(\xi)\eta = \xi\eta = \pi'(\eta)\xi$ . The set of left bounded elements is denoted by  $\mathcal{B}$ . We shall need the following well-known lemmas.

**Lemma 1.1.** *Let  $\mathcal{B}$  be the set of left bounded elements of  $\mathcal{H}$ . Then,*

- (i)  $\mathcal{B}$  is a linear subspace of  $\mathcal{H}$  containing  $\mathcal{A}$ .
- (ii)  $\pi(\mathcal{B})$  is contained in the left von Neumann algebra  $\mathcal{M}$ .
- (iii)  $\mathcal{B}$  is stable under  $\mathcal{M}$ . More precisely, for  $T \in \mathcal{M}$  and  $\xi \in \mathcal{B}$ ,  $\pi(T\xi) = T\pi(\xi)$ .

*Proof.* See [2, Section 2]. □

**Lemma 1.2.** *Let  $\xi, \xi' \in \mathcal{B}$  such that  $\pi(\xi') = \pi(\xi)^*$ . Then  $\xi$  belongs to  $\mathcal{D}^\sharp$  and  $\xi' = \xi^\sharp$ .*

*Proof.* For all  $\eta_1, \eta_2 \in \mathcal{A}'$ , one has:

$$\begin{aligned}
(\xi | \eta_1 \eta_2^b) &= (\xi | \pi'(\eta_2^b) \eta_1) \\
&= (\xi | \pi'(\eta_2)^* \eta_1) \\
&= (\pi'(\eta_2) \xi | \eta_1) \\
&= (\pi(\xi) \eta_2 | \eta_1) \\
&= (\pi(\xi')^* \eta_2 | \eta_1) \\
&= (\eta_2 | \pi(\xi') \eta_1) \\
&= (\eta_2 | \pi'(\eta_1) \xi') \\
&= (\pi'(\eta_1)^* \eta_2 | \xi') \\
&= (\pi'(\eta_1^b) \eta_2 | \xi') \\
&= (\eta_2 \eta_1^b | \xi')
\end{aligned}$$

According to [16, Lemma 3.4] and the comments which follow, this suffices to conclude.  $\square$

**Definition 1.3.** The canonical weight  $\tau$  of the left Hilbert algebra  $\mathcal{A}$  is the map  $\tau : \mathcal{M}_+ \rightarrow [0, \infty]$  defined by

$$\tau(T) = \begin{cases} \|\xi\|^2 & \text{if } \exists \xi \in \mathcal{B} : T = \pi(\xi)^* \pi(\xi) \\ \infty & \text{otherwise} \end{cases}$$

**Lemma 1.4.** [2, Théorème 2.11] *The above canonical weight  $\tau$  is well-defined and is a faithful, semi-finite,  $\sigma$ -weakly lower semi-continuous weight on  $\mathcal{M}_+$ .*

The following lemma is certainly well-known but I did not find a reference for it.

**Lemma 1.5.** *For all  $\xi \in \mathcal{B}$ , one has*

$$\tau(\pi(\xi)\pi(\xi)^*) = \begin{cases} \|\xi^\sharp\|^2 & \text{if } \xi \in \mathcal{D}^\sharp \\ \infty & \text{otherwise} \end{cases}$$

*Proof.* Suppose that  $\xi \in \mathcal{D}^\sharp$ . Then  $\xi$  belongs to the full Hilbert algebra  $\mathcal{A}'' = \mathcal{B} \cap \mathcal{D}^\sharp$  of  $\mathcal{A}$ . Therefore,  $\xi^\sharp$  is also left bounded and  $\pi(\xi)^* = \pi(\xi^\sharp)$ . Thus,

$$\tau(\pi(\xi)\pi(\xi)^*) = \tau(\pi(\xi^\sharp)^* \pi(\xi^\sharp)) = \|\xi^\sharp\|^2$$

Suppose now that the left handside is finite. Then there exists  $\eta \in \mathcal{B}$  such that  $\pi(\xi)\pi(\xi)^* = \pi(\eta)^* \pi(\eta)$ . Let  $\pi(\xi) = U|\pi(\xi)|$  [resp.  $\pi(\eta) = V|\pi(\eta)|$ ] be the polar decomposition of  $\pi(\xi)$  [resp.  $\pi(\eta)$ ]. All these operators belong to the left von Neumann algebra  $\mathcal{M}$ . Our assumption implies that  $|\pi(\eta)| = U|\pi(\xi)|U^*$ . Therefore,

$$\begin{aligned}
(U^*V^*\pi(\eta))^* &= \pi(\eta)^*VU \\
&= |\pi(\eta)|V^*VU \\
&= |\pi(\eta)|U \\
&= U|\pi(\xi)| \\
&= \pi(\xi)
\end{aligned}$$

Moreover, according to Lemma 1.1,  $U^*V^*\pi(\eta) = \pi(U^*V^*\eta)$  and  $\zeta = U^*V^*\eta$  is left bounded. Since  $\pi(\xi) = \pi(\zeta)^*$  we deduce from Lemma 1.2 that  $\xi \in \mathcal{D}^\sharp$ .  $\square$

We consider now the left Hilbert algebra associated with the left regular representation of a locally compact group  $G$  endowed with a left Haar measure  $\lambda$ . Using the framework of [14, pages 56-58], we choose the left Hilbert algebra  $\mathcal{A} = C_c(G)$ , where the product is the usual convolution product, the involution, denoted by  $f^*$  instead of  $f^\sharp$ , is  $f^*(\gamma) = \overline{f(\gamma^{-1})}$  and the scalar product is

$$(f | g) = \int f \bar{g} d\lambda^{-1}.$$

The left representation is denoted by  $L$  instead of  $\pi$ . It acts on the Hilbert space  $\mathcal{H} = L^2(G, \lambda^{-1})$  by  $L(f)g = f * g$  for  $f, g \in C_c(G)$ . The left von Neumann algebra  $\mathcal{M}$  is the group von Neumann algebra  $VN(G)$ . The canonical weight  $\tau$  of this left Hilbert algebra is also called the Plancherel weight of the group  $G$  ([15, 17]). We also refer the reader to [6, Section 2] for a deep study of its properties. It satisfies

$$\tau(L(f^* * f)) = \|f\|^2 = \int |f|^2 d\lambda^{-1} = (f^* * f)(e)$$

for  $f \in C_c(G)$ . When  $G$  is abelian, we can use the Fourier transform to compute the Plancherel weight (see [17, VII.3]). The canonical weight  $\hat{\tau}$  of the commutative Hilbert algebra  $L^2(\hat{G}, \hat{\lambda}) \cap L^\infty(\hat{G})$  is the integral with respect to the dual Haar measure  $\hat{\lambda}$  on  $L^\infty(\hat{G})_+$ . This weight is called the Haar weight of the dual group  $\hat{G}$ . The Fourier transform  $\mathcal{F}$  implements an isomorphism between the full left Hilbert algebra of the regular representation of  $G$  and the Hilbert algebra  $L^2(\hat{G}, \hat{\lambda}) \cap L^\infty(\hat{G})$ . Therefore,  $\tau(T) = \hat{\tau}(\mathcal{F} \circ T \circ \mathcal{F}^{-1})$  for  $T \in VN(G)$ . One has in particular

$$\tau(L(a)) = \int \mathcal{G}(a) d\hat{\lambda},$$

for  $a \in C^*(G)_+$  where  $\mathcal{G} : C^*(G) \rightarrow C_0(\hat{G})$  is the Gelfand transform.

Let us express Lemma 1.5 in the case of the left regular representation of a locally compact group  $G$  with a left Haar measure  $\lambda$ .

**Corollary 1.6.** *For all left bounded element  $\xi$  of  $L^2(G, \lambda^{-1})$ , one has*

$$\tau(L(\xi)L(\xi)^*) = \int |\xi|^2 d\lambda.$$

*Remark 1.7.* One can give a more direct proof of this result since the equality  $L(\xi)^*\eta = \xi^* * \eta$  can be established by usual integration techniques and [5, Lemme 3.1]. However, its natural framework is Tomita-Takesaki's theory.

## 2. THE PLANCHEREL WEIGHT OF A GROUP BUNDLE

We consider now the case of a locally compact group bundle  $p : G \rightarrow X$ . We assume that the groups  $G_x$  have a left Haar measure  $\lambda^x$  such that  $x \mapsto \lambda^x$  is continuous. This is the particular case of a locally compact groupoid with Haar system where the range and source maps coincide. Thus, we can construct its  $C^*$ -algebra  $C^*(G)$  as usual. If the groups  $G_x$  are abelian, it is a commutative  $C^*$ -algebra. Then the Gelfand transform identifies it with  $C_0(\hat{G})$ . The space  $\hat{G}$

is the total space of the bundle  $\hat{p} : \hat{G} \rightarrow X$ , where the fibre  $\hat{G}_x$  is the dual group of  $G_x$ . For  $a \in C^*(G)_+$  and  $x \in X$ , we define

$$\mathcal{T}(a)(x) \stackrel{\text{def}}{=} \int \mathcal{G}(a) d\widehat{\lambda}^x = \int \mathcal{G}_x(a_x) d\widehat{\lambda}^x = \tau_x \circ L_x(a_x),$$

where  $L_x$  is the left regular representation of  $G_x$  and  $\tau_x$  is the Plancherel weight of  $G_x$ . The last expression is defined when the groups are non-abelian and we turn now to this case.

**Lemma 2.1.** *Let  $G \rightarrow X$  be a locally compact group bundle with Haar system  $\lambda = (\lambda^x)_{x \in X}$ . Then,*

- (i)  $x \mapsto C^*(G_x)$  is an upper semi-continuous field of  $C^*$ -algebras;
- (ii) its sectional algebra is  $C^*(G)$ .

*Proof.* See for example [10, Section 5]. □

Equivalently, this lemma says that  $C^*(G)$  is a  $C_0(X)$ - $C^*$ -algebra. We shall view an element  $a$  of  $C^*(G)$  as a continuous field  $x \mapsto a_x$ , where  $a_x \in C^*(G_x)$ . For  $a \in C^*(G)_+$ , we define as in the commutative case the function

$$\mathcal{T}(a) : x \mapsto \tau_x(L_x(a_x)),$$

where  $L_x$  is the left regular representation of  $C^*(G_x)$  and  $\tau_x$  is the Plancherel weight of  $G_x$ . Our main result will be that the function  $x \mapsto \mathcal{T}(a)(x)$  is lower semi-continuous. We consider now weights on  $C^*$ -algebras rather than von Neumann algebras and refer the reader to [1] for the main definitions and results.

**Lemma 2.2.** *Let  $\varphi : A_+ \rightarrow [0, \infty]$  be a weight on a  $C^*$ -algebra  $A$ . Assume that  $(e_i)_{i \in I}$  is an approximate identity for  $A$  such that  $\|e_i\| \leq 1$  for all  $i \in I$ . Let  $a \in A_+$ . Then,*

- (i)  $\varphi(a^{1/2}e_i a^{1/2}) \leq \varphi(a)$ ;
- (ii) if  $\varphi$  is lower semi-continuous, then  $\varphi(a) = \sup_i \varphi(a^{1/2}e_i a^{1/2})$ .

*Proof.* For (i), we have  $a^{1/2}e_i a^{1/2} \leq \|e_i\|a \leq a$ . Since  $\varphi$  is increasing, we obtain the desired inequality. For (ii), since  $a^{1/2}e_i a^{1/2}$  converges to  $a$ , we have

$$\varphi(a) \leq \liminf \varphi(a^{1/2}e_i a^{1/2}) \leq \sup \varphi(a^{1/2}e_i a^{1/2}) \leq \varphi(a)$$

□

**Proposition 2.3.** *Let  $G \rightarrow X$  be a locally compact group bundle with Haar system  $\lambda = (\lambda^x)_{x \in X}$ . Given  $a \in C^*(G)_+$ , the function*

$$\mathcal{T}(a) : x \mapsto \tau_x(L_x(a_x))$$

*is lower semi-continuous.*

*Proof.* Proposition 2.10 of [11] gives the existence of an approximate unit  $(e_i)$  of  $C^*(G)$  where each  $e_i$  is a finite sum of elements of the form  $f * f^*$  with  $f \in C_c(G)$ ; moreover, according to its construction,  $\|e_i\|_I$  tends to 1, where  $\|f\|_I = \max(\sup_x \int |f| d\lambda^x, \sup_x \int |f| d\lambda_x)$  and  $\lambda_x = (\lambda^x)^{-1}$ . Replacing  $e_i$  by  $(1/\|e_i\|_I)e_i$ , we can assume that  $\|e_i\|_I = 1$  for all  $i \in I$ . We then have  $\|e_i\| \leq \|e_i\|_I \leq 1$ . For

all  $x \in X$ , the image  $(e_i(x))$  of  $(e_i)$  in  $C^*(G_x)$  satisfies the same assumptions. Since the Plancherel weight  $\tau_x$  is  $\sigma$ -weakly lower semi-continuous, we have

$$\tau_x(L_x(a_x)) = \sup_i \tau_x(L_x(a_x^{1/2} e_i(x) a_x^{1/2})).$$

We will show that the function  $x \mapsto \tau_x(L_x(a_x^{1/2} e_i(x) a_x^{1/2}))$  is lower semi-continuous. This will imply that the function  $x \mapsto \tau_x(L_x(a_x))$  is lower semi-continuous as lower upper bound of a family of lower semi-continuous functions. It suffices to show that  $x \mapsto \tau_x(L_x(a_x^{1/2} * (f * f^*)_x * a_x^{1/2}))$  is lower semi-continuous for all  $f \in C_c(G)$ . Let us fix  $x \in X$ . According to Lemma 1.1,  $f_x$  and  $g_x = L_x(a_x^{1/2})f_x$  are left bounded elements of  $L^2(G_x, \lambda_x)$ . Thus we have according to Lemma 1.5 :

$$\tau_x(L_x(a_x^{1/2} * (f * f^*)_x * a_x^{1/2})) = \tau_x(L_x(g_x)L_x(g_x)^*) = \int |g_x(\gamma)|^2 d\lambda^x(\gamma).$$

Note that  $g : x \mapsto g_x$  is an element of the  $C_0(X)$ -Hilbert module  $L^2(G, \lambda^{-1})$  (see [7, Section 2]). If the groups  $G_x$  are unimodular, then

$$\int |g(\gamma)|^2 d\lambda^x(\gamma) = \int |g(\gamma)|^2 d\lambda_x(\gamma)$$

depends continuously on  $x$ . If not, we first observe that for all  $\rho \in C_c(G)$ ,  $\rho g$  belongs to  $L^2(G, \lambda^{-1})$ . Let  $D : G \rightarrow \mathbf{R}_+^*$  be the modular function of the group bundle  $G$ . It is a continuous homomorphism such that  $\lambda^x = D\lambda_x$  for all  $x \in X$  (see [8, Lemma 2.4]). There exists an increasing sequence  $(D_n)$  of continuous non-negative functions with compact support which converges pointwise to  $D$ . Then, for all  $x \in X$ ,  $\int D_n(\gamma)|g(\gamma)|^2 d\lambda_x(\gamma)$  is an increasing sequence which converges to

$$\int D(\gamma)|g(\gamma)|^2 d\lambda_x(\gamma) = \int |g(\gamma)|^2 d\lambda^x(\gamma).$$

Therefore, the function  $x \mapsto \int |g(\gamma)|^2 d\lambda^x(\gamma)$  is lower semi-continuous as a limit of an increasing sequence of continuous functions.  $\square$

### 3. $C_0(X)$ -WEIGHTS ON $C_0(X)$ - $C^*$ -ALGEBRAS

In this section, given a topological space  $X$ ,  $LSC(X)_+$  denotes the convex cone of lower semi-continuous functions  $f : X \rightarrow [0, \infty]$ .

**Definition 3.1.** Let  $X$  be a locally compact Hausdorff space. A  $C_0(X)$ -weight on a  $C_0(X)$ - $C^*$ -algebra  $A$  is a map

$$\Phi : A_+ \rightarrow LSC(X)_+$$

such that

- (i)  $\Phi(a + b) = \Phi(a) + \Phi(b)$  for all  $a, b \in A_+$ ;
- (ii)  $\Phi(ha) = h\Phi(a)$  for all  $h \in C_0(X)_+$  and  $a \in A_+$ .

It is called lower semi-continuous if  $a_n \rightarrow a$  implies  $\Phi(a) \leq \liminf \Phi(a_n)$  and densely defined if its domain, defined as

$$P = \{a \in A_+ : \Phi(a) \text{ is finite and continuous}\}$$

is dense in  $A_+$ .

In order to include the first example, we need to modify slightly this definition. Given a topological space  $X$ , we denote by  $C_b(X)$  the space of complex-valued bounded continuous functions on  $X$ . We say that a  $C^*$ -algebra  $A$  is a  $C_b(X)$ -algebra if it is endowed with a nondegenerate morphism of  $C_b(X)$  into the centre of the multiplier algebra of  $A$ . Then we define a  $C_b(X)$ -weight by replacing  $C_0(X)$  by  $C_b(X)$  in the above definition.

*Remark 3.2.* The above definition of a  $C_0(X)$ -weight on a  $C_0(X)$ - $C^*$ -algebra is motivated by the examples below; it is different from the definition of a  $C^*$ -valued weight given in [9].

**Lemma 3.3.** *The domain  $P$  of a  $C_b(X)$ -weight or of  $C_0(X)$ -weight is hereditary: if  $0 \leq b \leq a$  and  $a \in P$ , then  $b \in P$ .*

*Proof.* See [4, Lemme 4.4.2.i]. □

**Example 3.4.** *The canonical center-valued trace* (cf [3, Section 3.4]). Let  $A$  be a  $C^*$ -algebra and let  $X = \text{Prim}(A)$  its primitive ideal space endowed with the Jacobson topology. Through the Dauns-Hofmann theorem, we view  $A$  as a  $C_b(X)$ -algebra. Given  $a \in A_+$ , according to [13, Proposition 4.4.9], one can define  $\Phi(a)(P) = \text{Trace}(\pi(a))$  where  $P$  is a primitive ideal and  $\pi$  is any irreducible representation admitting  $P$  as kernel, and  $\Phi(a)$  is lower semi-continuous on  $X$ . Condition (i) of the definition results from the additivity of the trace and condition (ii) from the irreducibility of  $\pi$ . Thus  $\Phi : A_+ \rightarrow LSC(X)_+$  is a  $C_b(X)$ -weight. Since the usual Trace is  $\sigma$ -weakly lower semi-continuous,  $\Phi$  is lower semi-continuous. By definition,  $\Phi$  is densely defined if and only if  $A$  is a continuous-trace  $C^*$ -algebra.

**Example 3.5.** *The Plancherel  $C_0(X)$ -weight of a group bundle.* This is the example described in the previous section:

**Theorem 3.6.** *Let  $p : G \rightarrow X$  be a locally compact group bundle endowed with a continuous Haar system. Then the map*

$$\mathcal{T} : C^*(G)_+ \rightarrow LSC(X)_+$$

*such that  $\mathcal{T}(a)(x) = \tau_x(L_x(a_x))$  for  $a \in C^*(G)_+$  and where  $\tau_x$  is the Plancherel weight of  $G_x$  and  $L_x$  the left regular representation of  $G_x$  is a densely defined lower semi-continuous  $C_0(X)$ -weight, which we call the Plancherel  $C_0(X)$ -weight of the group bundle.*

*Proof.* The main point was to show that the range of this map is contained in  $LSC(X)_+$ , which was done in Proposition 2.3. The conditions (i) and (ii) of the definition are clear. It is densely defined since its domain contains the elements  $f^* * f$  where  $f \in C_c(G)$ , whose linear span is dense in  $C^*(G)$ . The lower semi-continuity of  $\tau_x \circ L_x$  for all  $x \in X$  gives the lower semi-continuity of  $\mathcal{T}$ . □

**Example 3.7.** *The commutative case.* Let  $\pi : Y \rightarrow X$  be a continuous, open and surjective map, where  $Y$  and  $X$  are locally compact Hausdorff spaces. For  $x \in X$ , let  $Y_x = \pi^{-1}(x)$  be the fibre over  $x$ . Endowed with the fundamental family



of continuous sections  $C_c(Y)$ ,  $x \mapsto C_0(Y_x)$  is a continuous field of  $C^*$ -algebras. Its  $C^*$ -algebra of continuous sections is identified to  $C_0(Y)$ . Thus  $C_0(Y)$  is a  $C_0(X)$ - $C^*$ -algebra. The following proposition gives a description of the densely defined lower semi-continuous  $C_0(X)$ -weights of  $C_0(Y)$ .

**Proposition 3.8.** *In the above situation, we have natural one-to-one correspondence between*

- (i) *the continuous  $\pi$ -systems of measures, by which we mean families of Radon measures  $\alpha = (\alpha_x)_{x \in X}$  on  $Y$ , where  $\alpha_x$  is supported on  $\pi^{-1}(x)$ , and such that for all  $f \in C_c(Y)$ , the function  $x \mapsto \int f d\alpha_x$  is continuous;*
- (ii) *the densely defined and lower semi-continuous  $C_0(X)$ -weights of  $C_0(Y)$ .*

*Proof.* When  $X$  is reduced to a point, this is a well-known result given for example in the introduction of [1]. A crucial point which we shall use again below is that a densely defined weight on  $C_0(Y)$  is necessarily finite on  $C_c(Y)_+$  because the linear span of its domain of definition is a dense ideal, hence it contains the minimal dense ideal, called the Pedersen ideal, which in our case is  $C_c(Y)$  ([13, 5.6.3.]). The proof of the general case is essentially the same. Suppose that  $\alpha = (\alpha_x)_{x \in X}$  is a continuous  $\pi$ -system. For  $f \in C_0(Y)_+$ , we define  $\Phi(f)(x) = \int f d\alpha_x$ . There exists an increasing sequence  $(f_n)$  in  $C_c(Y)_+$  converging uniformly to  $f$ . Therefore,  $\Phi(f) = \sup_n \Phi(f_n)$  is lower semi-continuous. It is clear that the other properties of a  $C_0(X)$ -weight are satisfied. Since its domain contains  $C_c(Y)$ ,  $\Phi$  is densely defined. The lower semi-continuity of the Radon measures gives the lower semi-continuity of  $\Phi$ . Conversely, let  $\Phi$  be a densely defined and lower semi-continuous  $C_0(X)$  weight on  $C_0(Y)$ . For all  $x \in X$ , we can define  $\varphi_x : C_0(Y_x)_+ \rightarrow [0, \infty]$  such that  $\Phi(f)(x) = \varphi_x(f|_{Y_x})$  for all  $f$ . Then  $\varphi_x$  is a densely defined and lower semi-continuous weight on  $C_0(Y_x)$ . As we recalled at the beginning, there exists a unique Radon measure  $\alpha_x$  such that  $\varphi_x(f) = \int f d\alpha_x$  for all  $f \in C_0(Y_x)$ . By assumption, the linear span of the domain of definition  $P$  of  $\Phi$  is dense. Since it is also an ideal, it contains the Pedersen ideal  $C_c(Y)$  of  $C_0(Y)$ . This shows that  $\alpha = (\alpha_x)_{x \in X}$  is a continuous  $\pi$ -system of measures.  $\square$

**Example 3.9.** *The Haar  $C_0(X)$ -weight of a groupoid.* This example is a particular case of the previous example. Let  $G$  be a locally compact groupoid. We use the range map  $r : G \rightarrow G^{(0)}$  to turn the commutative  $C^*$ -algebra  $C_0(G)$  into a  $C_0(X)$ -algebra, where  $X = G^{(0)}$ . Then a continuous Haar system  $(\lambda^x)_{x \in X}$  defines a densely defined and lower semi-continuous  $C_0(X)$ -weight of  $C_0(G)$ , which in accordance with the group case, can be called the Haar  $C_0(X)$ -weight of  $G$ . On the other hand, we cannot define a Plancherel  $C_0(X)$ -weight for an arbitrary groupoid  $G$  since  $C^*(G)$  is usually not a  $C_0(X)$ -algebra.

We now have all the elements to prove the Proposition 3.6 in [12].

**Corollary 3.10.** *Let  $p : G \rightarrow X$  a locally compact bundle of abelian groups, equipped with a continuous Haar system  $\lambda = (\lambda^x)_{x \in X}$ . Then the family of dual Haar measures  $\hat{\lambda} = (\hat{\lambda}^x)_{x \in X}$  is a continuous Haar system for  $\hat{p} : \hat{G} \rightarrow X$ .*

*Proof.* From Theorem 3.6, we know that the Plancherel  $C_0(X)$ -weight  $\mathcal{T}$  on  $C^*(G)$  is lower semi-continuous and densely defined. Therefore  $\Phi = \mathcal{T} \circ \mathcal{G}^{-1}$ , where

$\mathcal{G} : C^*(G) \rightarrow C_0(\hat{G})$  is the Gelfand transform, is a lower semi-continuous and densely defined  $C_0(X)$ -weight on  $C_0(\hat{G})$ . Moreover, for all  $x \in X$ , the Plancherel weight  $\tau_x$  of  $G_x$  correspond to the Haar weight of  $\hat{G}_x$ , which is given by the Radon measure  $\hat{\lambda}^x$ . From Proposition 3.8, the  $\hat{p}$ -system  $\hat{\lambda} = (\hat{\lambda}^x)_{x \in X}$  is continuous.  $\square$

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