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CONTINUITY OF THE DUAL HAAR MEASURE

JEAN N. RENAULT

ABSTRACT. Given a locally compact group bundle, we show that the system of the Plancherel weights of their C^* -algebras is lower semi-continuous. As a corollary, we obtain that the dual Haar system of a continuous Haar system of a locally compact abelian group bundle is also continuous.

Let G be a locally compact abelian group with Haar measure λ . The dual Haar measure $\hat{\lambda}$ on the dual group \hat{G} is the Haar measure on \hat{G} which makes the Fourier transform defined by

$$\mathcal{F}(f)(\chi) = \int f(\gamma) \overline{\chi(\gamma)} d\lambda(\gamma)$$

for f in the space $C_c(G)$ of complex-valued continuous functions with compact support on G , an isometry from $L^2(G, \lambda)$ to $L^2(\hat{G}, \hat{\lambda})$. Suppose now that $p : G \rightarrow X$ is a locally compact abelian group bundle. Here, we mean that G and X are locally compact Hausdorff spaces, p is a continuous surjection and the fibres $G_x = p^{-1}(x)$ are abelian groups. We also require that the multiplication, as a map from $G \times_X G$ to G , and the inverse, as a map from G to G are continuous. A Haar system for G is a family of measures $(\lambda^x)_{x \in X}$, where for all $x \in X$, λ^x is a Haar measure of G_x ; it is said to be continuous if for all $f \in C_c(G)$, the function $x \mapsto \int f d\lambda^x$ is continuous. Then we can form the dual group bundle $\hat{p} : \hat{G} \rightarrow X$. As a set, \hat{G} is the disjoint union of the dual groups \hat{G}_x . Moreover, it can be identified with the spectrum of the commutative C^* -algebra $C^*(G)$, where G is viewed as a locally compact groupoid with Haar system $(\lambda^x)_{x \in X}$. Hence it is endowed with a locally compact topology. It can be checked ([12, Corollary 3.4]) that $\hat{p} : \hat{G} \rightarrow X$ is indeed a locally compact abelian group bundle in the above sense. Of course, one expects that the dual Haar system $(\hat{\lambda}^x)_{x \in X}$ is continuous. This is stated as Proposition 3.6 of [12]. However it was recently pointed to the authors by Henrik Kreidler that their proof is defective. This note corrects this and gives a more general result, based on the fact that the dual Haar measure $\hat{\lambda}$ on the dual group \hat{G} of a locally compact abelian group G , viewed as its Haar weight and defined as the canonical weight of $L^\infty(\hat{G})$ acting on $L^2(\hat{G}, \hat{\lambda})$, corresponds under the Fourier transform to the Plancherel weight of G , defined as the canonical weight of the von Neumann algebra $VN(G)$ of G acting on $L^2(G, \lambda)$. Therefore one can consider a locally compact group bundle $p : G \rightarrow X$ where the fibres G_x are no longer abelian. Our main result is Proposition 2.3 which says that the Plancherel weight of G_x varies continuously in a suitable sense. This lead us to the definition of a lower semi-continuous $C_0(X)$ -weight on a $C_0(X)$ - C^* -algebra which we illustrate by three examples.

The first section recalls the construction of the canonical weight of a left Hilbert algebra and the properties which are needed to prove the crucial Corollary 1.6. In the second section, we consider the case of a locally compact group bundle and prove our main result. This example motivates the definition of a $C_0(X)$ -weight of a $C_0(X)$ - C^* -algebra, given in the third section. The fact that the Plancherel $C_0(X)$ -weight of the C^* -algebra of a group bundle is densely defined and lower semi-continuous gives Proposition 3.6 of [12].

1. THE PLANCHEREL WEIGHT OF A LOCALLY COMPACT GROUP

We recall first some elements of Tomita-Takesaki's theory, using the standard notation from [16]. Given a left Hilbert algebra \mathcal{A} where the product, the involution and the scalar product are respectively denoted by ab , a^\sharp and $(a|b)$, we denote by \mathcal{H} the Hilbert space completion of \mathcal{A} and by $\pi : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{H})$ the left representation. We denote by $\mathcal{M} = \pi(\mathcal{A})''$ the left von Neumann algebra of \mathcal{A} . We denote by S the closure of the involution $a \mapsto a^\sharp$ and by F its adjoint. The domain of S [resp. F] is denoted by \mathcal{D}^\sharp [resp. \mathcal{D}^b] and one writes $\xi^\sharp = S\xi$ [resp. $\eta^b = F\eta$] for $\xi \in \mathcal{D}^\sharp$ [resp. for $\eta \in \mathcal{D}^b$]. An element $\eta \in \mathcal{H}$ is called right bounded if there exists a bounded operator $\pi'(\eta)$ on \mathcal{H} such that $\pi'(\eta)a = \pi(a)\eta$ for all $a \in \mathcal{A}$. One then writes $\xi\eta = \pi'(\eta)\xi$ for all $\xi \in \mathcal{H}$. The set of right bounded elements is denoted by \mathcal{B}' . One shows that $\mathcal{A}' = \mathcal{B}' \cap \mathcal{D}^b$ with involution ξ^b is a right Hilbert algebra in the same Hilbert space \mathcal{H} . An element $\xi \in \mathcal{H}$ is called left bounded if there exists a bounded operator $\pi(\xi)$ on \mathcal{H} such that $\pi(\xi)\eta = \pi'(\eta)\xi$ for all $\eta \in \mathcal{A}'$. One then writes $\xi\eta = \pi(\xi)\eta$ for all $\eta \in \mathcal{H}$. If ξ is left bounded and η is right bounded, the notation is consistent: $\pi(\xi)\eta = \xi\eta = \pi'(\eta)\xi$. The set of left bounded elements is denoted by \mathcal{B} . We shall need the following well-known lemmas.

Lemma 1.1. *Let \mathcal{B} be the set of left bounded elements of \mathcal{H} . Then,*

- (i) \mathcal{B} is a linear subspace of \mathcal{H} containing \mathcal{A} .
- (ii) $\pi(\mathcal{B})$ is contained in the left von Neumann algebra \mathcal{M} .
- (iii) \mathcal{B} is stable under \mathcal{M} . More precisely, for $T \in \mathcal{M}$ and $\xi \in \mathcal{B}$, $\pi(T\xi) = T\pi(\xi)$.

Proof. See [2, Section 2]. □

Lemma 1.2. *Let $\xi, \xi' \in \mathcal{B}$ such that $\pi(\xi') = \pi(\xi)^*$. Then ξ belongs to \mathcal{D}^\sharp and $\xi' = \xi^\sharp$.*

Proof. For all $\eta_1, \eta_2 \in \mathcal{A}'$, one has:

$$\begin{aligned}
(\xi | \eta_1 \eta_2^b) &= (\xi | \pi'(\eta_2^b) \eta_1) \\
&= (\xi | \pi'(\eta_2)^* \eta_1) \\
&= (\pi'(\eta_2) \xi | \eta_1) \\
&= (\pi(\xi) \eta_2 | \eta_1) \\
&= (\pi(\xi')^* \eta_2 | \eta_1) \\
&= (\eta_2 | \pi(\xi') \eta_1) \\
&= (\eta_2 | \pi'(\eta_1) \xi') \\
&= (\pi'(\eta_1)^* \eta_2 | \xi') \\
&= (\pi'(\eta_1^b) \eta_2 | \xi') \\
&= (\eta_2 \eta_1^b | \xi')
\end{aligned}$$

According to [16, Lemma 3.4] and the comments which follow, this suffices to conclude. \square

Definition 1.3. The canonical weight τ of the left Hilbert algebra \mathcal{A} is the map $\tau : \mathcal{M}_+ \rightarrow [0, \infty]$ defined by

$$\tau(T) = \begin{cases} \|\xi\|^2 & \text{if } \exists \xi \in \mathcal{B} : T = \pi(\xi)^* \pi(\xi) \\ \infty & \text{otherwise} \end{cases}$$

Lemma 1.4. [2, Théorème 2.11] *The above canonical weight τ is well-defined and is a faithful, semi-finite, σ -weakly lower semi-continuous weight on \mathcal{M}_+ .*

The following lemma is certainly well-known but I did not find a reference for it.

Lemma 1.5. *For all $\xi \in \mathcal{B}$, one has*

$$\tau(\pi(\xi)\pi(\xi)^*) = \begin{cases} \|\xi^\sharp\|^2 & \text{if } \xi \in \mathcal{D}^\sharp \\ \infty & \text{otherwise} \end{cases}$$

Proof. Suppose that $\xi \in \mathcal{D}^\sharp$. Then ξ belongs to the full Hilbert algebra $\mathcal{A}'' = \mathcal{B} \cap \mathcal{D}^\sharp$ of \mathcal{A} . Therefore, ξ^\sharp is also left bounded and $\pi(\xi)^* = \pi(\xi^\sharp)$. Thus,

$$\tau(\pi(\xi)\pi(\xi)^*) = \tau(\pi(\xi^\sharp)^* \pi(\xi^\sharp)) = \|\xi^\sharp\|^2$$

Suppose now that the left handside is finite. Then there exists $\eta \in \mathcal{B}$ such that $\pi(\xi)\pi(\xi)^* = \pi(\eta)^* \pi(\eta)$. Let $\pi(\xi) = U|\pi(\xi)|$ [resp. $\pi(\eta) = V|\pi(\eta)|$] be the polar decomposition of $\pi(\xi)$ [resp. $\pi(\eta)$]. All these operators belong to the left von Neumann algebra \mathcal{M} . Our assumption implies that $|\pi(\eta)| = U|\pi(\xi)|U^*$. Therefore,

$$\begin{aligned}
(U^*V^*\pi(\eta))^* &= \pi(\eta)^*VU \\
&= |\pi(\eta)|V^*VU \\
&= |\pi(\eta)|U \\
&= U|\pi(\xi)| \\
&= \pi(\xi)
\end{aligned}$$

Moreover, according to Lemma 1.1, $U^*V^*\pi(\eta) = \pi(U^*V^*\eta)$ and $\zeta = U^*V^*\eta$ is left bounded. Since $\pi(\xi) = \pi(\zeta)^*$ we deduce from Lemma 1.2 that $\xi \in \mathcal{D}^\sharp$. \square

We consider now the left Hilbert algebra associated with the left regular representation of a locally compact group G endowed with a left Haar measure λ . Using the framework of [14, pages 56-58], we choose the left Hilbert algebra $\mathcal{A} = C_c(G)$, where the product is the usual convolution product, the involution, denoted by f^* instead of f^\sharp , is $f^*(\gamma) = \overline{f(\gamma^{-1})}$ and the scalar product is

$$(f | g) = \int f \bar{g} d\lambda^{-1}.$$

The left representation is denoted by L instead of π . It acts on the Hilbert space $\mathcal{H} = L^2(G, \lambda^{-1})$ by $L(f)g = f * g$ for $f, g \in C_c(G)$. The left von Neumann algebra \mathcal{M} is the group von Neumann algebra $VN(G)$. The canonical weight τ of this left Hilbert algebra is also called the Plancherel weight of the group G ([15, 17]). We also refer the reader to [6, Section 2] for a deep study of its properties. It satisfies

$$\tau(L(f^* * f)) = \|f\|^2 = \int |f|^2 d\lambda^{-1} = (f^* * f)(e)$$

for $f \in C_c(G)$. When G is abelian, we can use the Fourier transform to compute the Plancherel weight (see [17, VII.3]). The canonical weight $\hat{\tau}$ of the commutative Hilbert algebra $L^2(\hat{G}, \hat{\lambda}) \cap L^\infty(\hat{G})$ is the integral with respect to the dual Haar measure $\hat{\lambda}$ on $L^\infty(\hat{G})_+$. This weight is called the Haar weight of the dual group \hat{G} . The Fourier transform \mathcal{F} implements an isomorphism between the full left Hilbert algebra of the regular representation of G and the Hilbert algebra $L^2(\hat{G}, \hat{\lambda}) \cap L^\infty(\hat{G})$. Therefore, $\tau(T) = \hat{\tau}(\mathcal{F} \circ T \circ \mathcal{F}^{-1})$ for $T \in VN(G)$. One has in particular

$$\tau(L(a)) = \int \mathcal{G}(a) d\hat{\lambda},$$

for $a \in C^*(G)_+$ where $\mathcal{G} : C^*(G) \rightarrow C_0(\hat{G})$ is the Gelfand transform.

Let us express Lemma 1.5 in the case of the left regular representation of a locally compact group G with a left Haar measure λ .

Corollary 1.6. *For all left bounded element ξ of $L^2(G, \lambda^{-1})$, one has*

$$\tau(L(\xi)L(\xi)^*) = \int |\xi|^2 d\lambda.$$

Remark 1.7. One can give a more direct proof of this result since the equality $L(\xi)^*\eta = \xi^* * \eta$ can be established by usual integration techniques and [5, Lemme 3.1]. However, its natural framework is Tomita-Takesaki's theory.

2. THE PLANCHEREL WEIGHT OF A GROUP BUNDLE

We consider now the case of a locally compact group bundle $p : G \rightarrow X$. We assume that the groups G_x have a left Haar measure λ^x such that $x \mapsto \lambda^x$ is continuous. This is the particular case of a locally compact groupoid with Haar system where the range and source maps coincide. Thus, we can construct its C^* -algebra $C^*(G)$ as usual. If the groups G_x are abelian, it is a commutative C^* -algebra. Then the Gelfand transform identifies it with $C_0(\hat{G})$. The space \hat{G}

is the total space of the bundle $\hat{p} : \hat{G} \rightarrow X$, where the fibre \hat{G}_x is the dual group of G_x . For $a \in C^*(G)_+$ and $x \in X$, we define

$$\mathcal{T}(a)(x) \stackrel{\text{def}}{=} \int \mathcal{G}(a) d\widehat{\lambda}^x = \int \mathcal{G}_x(a_x) d\widehat{\lambda}^x = \tau_x \circ L_x(a_x),$$

where L_x is the left regular representation of G_x and τ_x is the Plancherel weight of G_x . The last expression is defined when the groups are non-abelian and we turn now to this case.

Lemma 2.1. *Let $G \rightarrow X$ be a locally compact group bundle with Haar system $\lambda = (\lambda^x)_{x \in X}$. Then,*

- (i) $x \mapsto C^*(G_x)$ is an upper semi-continuous field of C^* -algebras;
- (ii) its sectional algebra is $C^*(G)$.

Proof. See for example [10, Section 5]. □

Equivalently, this lemma says that $C^*(G)$ is a $C_0(X)$ - C^* -algebra. We shall view an element a of $C^*(G)$ as a continuous field $x \mapsto a_x$, where $a_x \in C^*(G_x)$. For $a \in C^*(G)_+$, we define as in the commutative case the function

$$\mathcal{T}(a) : x \mapsto \tau_x(L_x(a_x)),$$

where L_x is the left regular representation of $C^*(G_x)$ and τ_x is the Plancherel weight of G_x . Our main result will be that the function $x \mapsto \mathcal{T}(a)(x)$ is lower semi-continuous. We consider now weights on C^* -algebras rather than von Neumann algebras and refer the reader to [1] for the main definitions and results.

Lemma 2.2. *Let $\varphi : A_+ \rightarrow [0, \infty]$ be a weight on a C^* -algebra A . Assume that $(e_i)_{i \in I}$ is an approximate identity for A such that $\|e_i\| \leq 1$ for all $i \in I$. Let $a \in A_+$. Then,*

- (i) $\varphi(a^{1/2}e_i a^{1/2}) \leq \varphi(a)$;
- (ii) if φ is lower semi-continuous, then $\varphi(a) = \sup_i \varphi(a^{1/2}e_i a^{1/2})$.

Proof. For (i), we have $a^{1/2}e_i a^{1/2} \leq \|e_i\|a \leq a$. Since φ is increasing, we obtain the desired inequality. For (ii), since $a^{1/2}e_i a^{1/2}$ converges to a , we have

$$\varphi(a) \leq \liminf \varphi(a^{1/2}e_i a^{1/2}) \leq \sup \varphi(a^{1/2}e_i a^{1/2}) \leq \varphi(a)$$

□

Proposition 2.3. *Let $G \rightarrow X$ be a locally compact group bundle with Haar system $\lambda = (\lambda^x)_{x \in X}$. Given $a \in C^*(G)_+$, the function*

$$\mathcal{T}(a) : x \mapsto \tau_x(L_x(a_x))$$

is lower semi-continuous.

Proof. Proposition 2.10 of [11] gives the existence of an approximate unit (e_i) of $C^*(G)$ where each e_i is a finite sum of elements of the form $f * f^*$ with $f \in C_c(G)$; moreover, according to its construction, $\|e_i\|_I$ tends to 1, where $\|f\|_I = \max(\sup_x \int |f| d\lambda^x, \sup_x \int |f| d\lambda_x)$ and $\lambda_x = (\lambda^x)^{-1}$. Replacing e_i by $(1/\|e_i\|_I)e_i$, we can assume that $\|e_i\|_I = 1$ for all $i \in I$. We then have $\|e_i\| \leq \|e_i\|_I \leq 1$. For

all $x \in X$, the image $(e_i(x))$ of (e_i) in $C^*(G_x)$ satisfies the same assumptions. Since the Plancherel weight τ_x is σ -weakly lower semi-continuous, we have

$$\tau_x(L_x(a_x)) = \sup_i \tau_x(L_x(a_x^{1/2} e_i(x) a_x^{1/2})).$$

We will show that the function $x \mapsto \tau_x(L_x(a_x^{1/2} e_i(x) a_x^{1/2}))$ is lower semi-continuous. This will imply that the function $x \mapsto \tau_x(L_x(a_x))$ is lower semi-continuous as lower upper bound of a family of lower semi-continuous functions. It suffices to show that $x \mapsto \tau_x(L_x(a_x^{1/2} * (f * f^*)_x * a_x^{1/2}))$ is lower semi-continuous for all $f \in C_c(G)$. Let us fix $x \in X$. According to Lemma 1.1, f_x and $g_x = L_x(a_x^{1/2})f_x$ are left bounded elements of $L^2(G_x, \lambda_x)$. Thus we have according to Lemma 1.5 :

$$\tau_x(L_x(a_x^{1/2} * (f * f^*)_x * a_x^{1/2})) = \tau_x(L_x(g_x)L_x(g_x)^*) = \int |g_x(\gamma)|^2 d\lambda^x(\gamma).$$

Note that $g : x \mapsto g_x$ is an element of the $C_0(X)$ -Hilbert module $L^2(G, \lambda^{-1})$ (see [7, Section 2]). If the groups G_x are unimodular, then

$$\int |g(\gamma)|^2 d\lambda^x(\gamma) = \int |g(\gamma)|^2 d\lambda_x(\gamma)$$

depends continuously on x . If not, we first observe that for all $\rho \in C_c(G)$, ρg belongs to $L^2(G, \lambda^{-1})$. Let $D : G \rightarrow \mathbf{R}_+^*$ be the modular function of the group bundle G . It is a continuous homomorphism such that $\lambda^x = D\lambda_x$ for all $x \in X$ (see [8, Lemma 2.4]). There exists an increasing sequence (D_n) of continuous non-negative functions with compact support which converges pointwise to D . Then, for all $x \in X$, $\int D_n(\gamma)|g(\gamma)|^2 d\lambda_x(\gamma)$ is an increasing sequence which converges to

$$\int D(\gamma)|g(\gamma)|^2 d\lambda_x(\gamma) = \int |g(\gamma)|^2 d\lambda^x(\gamma).$$

Therefore, the function $x \mapsto \int |g(\gamma)|^2 d\lambda^x(\gamma)$ is lower semi-continuous as a limit of an increasing sequence of continuous functions. \square

3. $C_0(X)$ -WEIGHTS ON $C_0(X)$ - C^* -ALGEBRAS

In this section, given a topological space X , $LSC(X)_+$ denotes the convex cone of lower semi-continuous functions $f : X \rightarrow [0, \infty]$.

Definition 3.1. Let X be a locally compact Hausdorff space. A $C_0(X)$ -weight on a $C_0(X)$ - C^* -algebra A is a map

$$\Phi : A_+ \rightarrow LSC(X)_+$$

such that

- (i) $\Phi(a + b) = \Phi(a) + \Phi(b)$ for all $a, b \in A_+$;
- (ii) $\Phi(ha) = h\Phi(a)$ for all $h \in C_0(X)_+$ and $a \in A_+$.

It is called lower semi-continuous if $a_n \rightarrow a$ implies $\Phi(a) \leq \liminf \Phi(a_n)$ and densely defined if its domain, defined as

$$P = \{a \in A_+ : \Phi(a) \text{ is finite and continuous}\}$$

is dense in A_+ .

In order to include the first example, we need to modify slightly this definition. Given a topological space X , we denote by $C_b(X)$ the space of complex-valued bounded continuous functions on X . We say that a C^* -algebra A is a $C_b(X)$ -algebra if it is endowed with a nondegenerate morphism of $C_b(X)$ into the centre of the multiplier algebra of A . Then we define a $C_b(X)$ -weight by replacing $C_0(X)$ by $C_b(X)$ in the above definition.

Remark 3.2. The above definition of a $C_0(X)$ -weight on a $C_0(X)$ - C^* -algebra is motivated by the examples below; it is different from the definition of a C^* -valued weight given in [9].

Lemma 3.3. *The domain P of a $C_b(X)$ -weight or of $C_0(X)$ -weight is hereditary: if $0 \leq b \leq a$ and $a \in P$, then $b \in P$.*

Proof. See [4, Lemme 4.4.2.i]. □

Example 3.4. *The canonical center-valued trace* (cf [3, Section 3.4]). Let A be a C^* -algebra and let $X = \text{Prim}(A)$ its primitive ideal space endowed with the Jacobson topology. Through the Dauns-Hofmann theorem, we view A as a $C_b(X)$ -algebra. Given $a \in A_+$, according to [13, Proposition 4.4.9], one can define $\Phi(a)(P) = \text{Trace}(\pi(a))$ where P is a primitive ideal and π is any irreducible representation admitting P as kernel, and $\Phi(a)$ is lower semi-continuous on X . Condition (i) of the definition results from the additivity of the trace and condition (ii) from the irreducibility of π . Thus $\Phi : A_+ \rightarrow LSC(X)_+$ is a $C_b(X)$ -weight. Since the usual Trace is σ -weakly lower semi-continuous, Φ is lower semi-continuous. By definition, Φ is densely defined if and only if A is a continuous-trace C^* -algebra.

Example 3.5. *The Plancherel $C_0(X)$ -weight of a group bundle.* This is the example described in the previous section:

Theorem 3.6. *Let $p : G \rightarrow X$ be a locally compact group bundle endowed with a continuous Haar system. Then the map*

$$\mathcal{T} : C^*(G)_+ \rightarrow LSC(X)_+$$

such that $\mathcal{T}(a)(x) = \tau_x(L_x(a_x))$ for $a \in C^(G)_+$ and where τ_x is the Plancherel weight of G_x and L_x the left regular representation of G_x is a densely defined lower semi-continuous $C_0(X)$ -weight, which we call the Plancherel $C_0(X)$ -weight of the group bundle.*

Proof. The main point was to show that the range of this map is contained in $LSC(X)_+$, which was done in Proposition 2.3. The conditions (i) and (ii) of the definition are clear. It is densely defined since its domain contains the elements $f^* * f$ where $f \in C_c(G)$, whose linear span is dense in $C^*(G)$. The lower semi-continuity of $\tau_x \circ L_x$ for all $x \in X$ gives the lower semi-continuity of \mathcal{T} . □

Example 3.7. *The commutative case.* Let $\pi : Y \rightarrow X$ be a continuous, open and surjective map, where Y and X are locally compact Hausdorff spaces. For $x \in X$, let $Y_x = \pi^{-1}(x)$ be the fibre over x . Endowed with the fundamental family

of continuous sections $C_c(Y)$, $x \mapsto C_0(Y_x)$ is a continuous field of C^* -algebras. Its C^* -algebra of continuous sections is identified to $C_0(Y)$. Thus $C_0(Y)$ is a $C_0(X)$ - C^* -algebra. The following proposition gives a description of the densely defined lower semi-continuous $C_0(X)$ -weights of $C_0(Y)$.

Proposition 3.8. *In the above situation, we have natural one-to-one correspondence between*

- (i) *the continuous π -systems of measures, by which we mean families of Radon measures $\alpha = (\alpha_x)_{x \in X}$ on Y , where α_x is supported on $\pi^{-1}(x)$, and such that for all $f \in C_c(Y)$, the function $x \mapsto \int f d\alpha_x$ is continuous;*
- (ii) *the densely defined and lower semi-continuous $C_0(X)$ -weights of $C_0(Y)$.*

Proof. When X is reduced to a point, this is a well-known result given for example in the introduction of [1]. A crucial point which we shall use again below is that a densely defined weight on $C_0(Y)$ is necessarily finite on $C_c(Y)_+$ because the linear span of its domain of definition is a dense ideal, hence it contains the minimal dense ideal, called the Pedersen ideal, which in our case is $C_c(Y)$ ([13, 5.6.3.]). The proof of the general case is essentially the same. Suppose that $\alpha = (\alpha_x)_{x \in X}$ is a continuous π -system. For $f \in C_0(Y)_+$, we define $\Phi(f)(x) = \int f d\alpha_x$. There exists an increasing sequence (f_n) in $C_c(Y)_+$ converging uniformly to f . Therefore, $\Phi(f) = \sup_n \Phi(f_n)$ is lower semi-continuous. It is clear that the other properties of a $C_0(X)$ -weight are satisfied. Since its domain contains $C_c(Y)$, Φ is densely defined. The lower semi-continuity of the Radon measures gives the lower semi-continuity of Φ . Conversely, let Φ be a densely defined and lower semi-continuous $C_0(X)$ weight on $C_0(Y)$. For all $x \in X$, we can define $\varphi_x : C_0(Y_x)_+ \rightarrow [0, \infty]$ such that $\Phi(f)(x) = \varphi_x(f|_{Y_x})$ for all f . Then φ_x is a densely defined and lower semi-continuous weight on $C_0(Y_x)$. As we recalled at the beginning, there exists a unique Radon measure α_x such that $\varphi_x(f) = \int f d\alpha_x$ for all $f \in C_0(Y_x)$. By assumption, the linear span of the domain of definition P of Φ is dense. Since it is also an ideal, it contains the Pedersen ideal $C_c(Y)$ of $C_0(Y)$. This shows that $\alpha = (\alpha_x)_{x \in X}$ is a continuous π -system of measures. \square

Example 3.9. *The Haar $C_0(X)$ -weight of a groupoid.* This example is a particular case of the previous example. Let G be a locally compact groupoid. We use the range map $r : G \rightarrow G^{(0)}$ to turn the commutative C^* -algebra $C_0(G)$ into a $C_0(X)$ -algebra, where $X = G^{(0)}$. Then a continuous Haar system $(\lambda^x)_{x \in X}$ defines a densely defined and lower semi-continuous $C_0(X)$ -weight of $C_0(G)$, which in accordance with the group case, can be called the Haar $C_0(X)$ -weight of G . On the other hand, we cannot define a Plancherel $C_0(X)$ -weight for an arbitrary groupoid G since $C^*(G)$ is usually not a $C_0(X)$ -algebra.

We now have all the elements to prove the Proposition 3.6 in [12].

Corollary 3.10. *Let $p : G \rightarrow X$ a locally compact bundle of abelian groups, equipped with a continuous Haar system $\lambda = (\lambda^x)_{x \in X}$. Then the family of dual Haar measures $\hat{\lambda} = (\hat{\lambda}^x)_{x \in X}$ is a continuous Haar system for $\hat{p} : \hat{G} \rightarrow X$.*

Proof. From Theorem 3.6, we know that the Plancherel $C_0(X)$ -weight \mathcal{T} on $C^*(G)$ is lower semi-continuous and densely defined. Therefore $\Phi = \mathcal{T} \circ \mathcal{G}^{-1}$, where

$\mathcal{G} : C^*(G) \rightarrow C_0(\hat{G})$ is the Gelfand transform, is a lower semi-continuous and densely defined $C_0(X)$ -weight on $C_0(\hat{G})$. Moreover, for all $x \in X$, the Plancherel weight τ_x of G_x correspond to the Haar weight of \hat{G}_x , which is given by the Radon measure $\hat{\lambda}^x$. From Proposition 3.8, the \hat{p} -system $\hat{\lambda} = (\hat{\lambda}^x)_{x \in X}$ is continuous. \square

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