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Jean N Renault

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THE C*-ALGEBRA OF A TWISTED GROUPOID EXTENSION

JEAN N. RENAULT

ABSTRACT. This written version of a talk given in July 2020 at the Western Sydney Abend seminar and based on the joint work [6] gives a decomposition of the C*-algebra of a locally compact groupoid with Haar system, possibly endowed with a twist, in presence of a normal subgroupoid. The natural expression of this result uses Fell bundles over groupoids. When the normal subgroupoid and the twist over it are abelian, one obtains another twisted groupoid C*-algebra.

1. INTRODUCTION.

The Mackey normal subgroup analysis (also called the Mackey machine) describes the representations of a group G in terms of a normal subgroup S and the quotient $H = G/S$. A semidirect product of groups $G = S \rtimes H$ such as the group of rigid motions or the Poincaré group is the simplest example. From the C*-algebraic perspective, it gives a description of $C^*(G)$ as a crossed product. In the simple case of a semidirect product, we easily have

$$C^*(S \rtimes H) = C^*(S) \rtimes H = C^*(H, C^*(S))$$

A semidirect product is a trivial extension. When the extension is not trivial, a twist appears:

Theorem 1.1 (Green [5]).

$$C^*(G) = C^*(G, C^*(S), \tau_S)$$

where the right handside is a twisted crossed product.

When S is abelian, one can go one step further, namely use the Gelfand transform

$$C^*(S \rtimes H) = C^*(H, C^*(S)) = C^*(H, C_0(\hat{S})) = C^*(\hat{S} \rtimes H)$$

The last term is a groupoid C*-algebra, where the groupoid $\hat{S} \rtimes H$ has less isotropy than the initial group $S \rtimes H$. One may want to iterate the process. It is then necessary to extend the Mackey machine to a groupoid G rather than a group. The original motivation was the analysis of nilpotent group C*-algebras. It is limited here to the example of the Heisenberg group, which is presented in the last section.

This article is a written version of a talk I gave at the Western Sydney Abend seminar in July 2020. It is based on a joint work [6] with M. Ionescu, A. Kumjian, A. Sims and D. Williams, whom I thank for a stimulating and enjoyable collaboration. The situation considered in the present version is more general than that of [6], since the initial groupoid may be twisted. While most proofs are the same as in [6], it seems preferable to give a separate presentation of the general result because it requires some changes all along.

Key words and phrases. groupoid extension, twist, Mackey machine, Fell bundle.

Note however the case of a twisted groupoid extension considered here can be deduced directly from the untwisted case considered in [6]. This is done in [7, Proposition 3.5]. There is an attempt to make this article self-contained but the reader is directed to [6] (and also [7]) for some proofs. We also refer to [6] for unexplained notation. Here is a notation which is frequently used: given two maps $p : X \rightarrow T$ and $q : Y \rightarrow T$ with the same range, their fibre product over T is denoted by $X * Y$ when there is no ambiguity about the maps.

2. FELL BUNDLES AND GROUPOID C*-DYNAMICAL SYSTEMS

Definition 2.1. A groupoid extension is a short exact sequence of groupoids

$$S \twoheadrightarrow G \twoheadrightarrow H$$

with common unit space $G^{(0)}$. Equivalently, an extension of the groupoid H is a surjective homomorphism $\pi : G \twoheadrightarrow H$ such that $\pi^{(0)} : G^{(0)} \rightarrow H^{(0)}$ is a bijection (we shall assume that $G^{(0)} = H^{(0)}$ and that $\pi^{(0)}$ is the identity map). We shall write $\dot{\gamma} = \pi(\gamma)$ when there is no ambiguity about the projection map.

Then $S = \text{Ker}(\pi)$ is a subgroup bundle of the isotropy group bundle G' . Moreover, it is normal in the sense that for all compatible pair $(\gamma, s) \in G * S$, $\gamma s \gamma^{-1}$ belongs to S . Note that subgroupoids which are normal in this sense are necessarily subgroup bundles of the isotropy group bundle. Note also that H is naturally isomorphic to the quotient groupoid G/S . Since the normal subgroupoid S of G determines the extension, we shall often use the terminology of normal subgroupoid rather than extension. Then H is the quotient groupoid. We assume that H and G are locally compact Hausdorff groupoids and that π is continuous and open. In particular S is a closed normal subgroupoid of G . We also assume that H has a Haar system α and that S has a Haar system β . There is a homomorphism $\delta : G \rightarrow \mathbf{R}_+^*$ such that for all $\gamma \in G$, $\gamma \beta^{s(\gamma)} \gamma^{-1} = \delta(\gamma) \beta^{r(\gamma)}$. This homomorphism is called the modular cocycle of the extension. Its cohomology class does not depend on the choice of β . For all $x \in G^{(0)}$, its restriction to the group S_x is the modular function of S_x . It is continuous (see [6, Lemma 2.4]). Given α and β , we define the Haar system λ for G by the formula:

$$\int f(\gamma) d\lambda^x(\gamma) = \int_H \int_S f(\gamma t) d\beta^{s(\gamma)}(t) d\alpha^x(\dot{\gamma})$$

Definition 2.2. Under these assumptions, we say that $S \twoheadrightarrow G \twoheadrightarrow H$ is a locally compact groupoid extension with Haar systems.

It will be convenient in the sequel to define an extension with Haar system as a pair (G, S) where S is a closed normal subgroupoid of the locally compact groupoid G which admits a Haar system and such that the quotient groupoid $H = G/S$ admits a Haar system.

Definition 2.3. [16] A groupoid C*-dynamical system (or dynamical system for short) is a triple (G, S, \mathcal{A}) where G is a locally compact groupoid, S is a closed normal subgroupoid, and \mathcal{A} is an upper semi-continuous bundle of C*-algebras over $G^{(0)}$ endowed with a continuous action $G * \mathcal{A} \rightarrow \mathcal{A}$ such that S is unitarily implemented in the multiplier

algebra bundle $M(\mathcal{A})$, meaning the existence of a bundle homomorphism χ from S to the unitary bundle of $M(\mathcal{A})$, such that

- (i) the map $S * \mathcal{A} \rightarrow \mathcal{A}$ sending (s, a) to $\chi(s)a$ is continuous;
- (ii) $s.a = \chi(s)a\chi(s)^{-1}$ for all $(s, a) \in S * \mathcal{A}$;
- (iii) $\chi(\gamma s \gamma^{-1}) = \gamma.\chi(s)$ for all $(\gamma, s) \in G * S$.

In [16, Section 3], it was assumed that the kernel S was abelian, in the sense that it was a bundle of abelian groups. However, as shown in [6], this assumption is not necessary and most results of [16] remain valid.

To distinguish bundles of algebras (or of linear spaces) from algebras, algebra bundles will usually be denoted by calligraphic letters such as \mathcal{A} while algebras will be denoted by Roman letters such as A . For example, if $p : G \rightarrow X$ is a bundle of groupoids with Haar systems, $\mathcal{C}_c(G)$ and $\mathcal{C}^*(G)$ denote the bundles with respective fibers $C_c(G(x))$ and $C^*(G(x))$. On the other hand, $C_c(G)$ and $C^*(G)$ denote the usual $*$ -algebras of the groupoid G , which are the sectional algebras of the above bundle. Groupoid dynamical systems fit into the more general framework of Fell bundles over groupoids.

Definition 2.4. [22, 10] A Fell bundle over a groupoid is a bundle $\mathcal{B} \rightarrow H$ where H is a locally compact groupoid and \mathcal{B} is an upper semi-continuous bundle of Banach spaces over H endowed with a continuous multiplication $\mathcal{B} * \mathcal{B} \rightarrow \mathcal{B}$ and a continuous involution $\mathcal{B} \rightarrow \mathcal{B}$ satisfying the C^* -algebra axioms whenever they make sense.

This definition implies that the fibers B_x over $x \in H^{(0)}$ become C^* -algebras, the fibres B_h over $h \in H$ become $(B_{r(h)}, B_{s(h)})$ - C^* -bimodules and $h \mapsto B_h$ is functorial. One says that the Fell bundle is saturated if the B_h 's are equivalence C^* -bimodules.

A groupoid C^* -dynamical system (G, S, \mathcal{A}) as above gives rise to a Fell bundle \mathcal{B} over $H = G/S$. This is [12, Example 7.3] which we recall now. We form

$$\mathcal{A} * G = \{(a, \gamma) \in \mathcal{A} \times G : a \in A_{r(\gamma)}\}.$$

We let S act on it by $s(a, \gamma) = (a\chi(s^{-1}), s\gamma)$ and consider the quotient $\mathcal{B} = (\mathcal{A} * G)/S$. The image of (a, γ) in \mathcal{B} is denoted by $[a, \gamma]$. The bundle map $p : \mathcal{B} \rightarrow H$ sends $[a, \gamma]$ to $\pi(\gamma)$ where $\pi : G \rightarrow G/S$ is the quotient map. A choice of γ in $\pi^{-1}(h)$ gives a Banach space isomorphism $[a, \gamma] \mapsto a$ from B_h to $A_{r(h)}$. The multiplication in \mathcal{B} is given by

$$[a, \gamma][b, \gamma'] = [a(\gamma.b), \gamma\gamma']$$

and the involution by

$$[a, \gamma]^* = [\gamma^{-1}.a^*, \gamma^{-1}]$$

Lemma 2.1. [6, Lemma 1.5] *The above bundle $\mathcal{B} \rightarrow H$ is a saturated Fell bundle.*

Definition 2.5. This bundle $\mathcal{B} \rightarrow G/S$ is called the Fell bundle of the groupoid C^* -dynamical system (G, S, \mathcal{A}) .

Recall the construction of the crossed products (see [16, Section 3] and [22]). Let (G, S, \mathcal{A}) be a groupoid C^* -dynamical system. We assume that $H = G/S$ has a Haar system $(\alpha^x)_{x \in G^{(0)}}$ and that S has a Haar system $(\beta^x)_{x \in G^{(0)}}$. One first forms the $*$ -algebra $C_c(G, S, \mathcal{A})$. Its elements are continuous functions $f : G \rightarrow \mathcal{A}$ such that

- (i) $f(\gamma)$ belongs to $A_{r(\gamma)}$ for all $\gamma \in G$;
- (ii) $f(s\gamma) = f(\gamma)\chi(s^{-1})$ for all $(s, \gamma) \in S * G$;
- (iii) f has compact support modulo S .

The product and the involution are respectively given by

$$f * g(\gamma) = \int f(\tau)[\tau.g(\tau^{-1}\gamma)]d\alpha^{r(\gamma)}(\dot{\tau})$$

and

$$f^*(\gamma) = \gamma.(f(\gamma^{-1}))^*$$

The crossed product C*-algebra $C^*(G, S, \mathcal{A})$ is the completion of $C_c(G, S, \mathcal{A})$ for the full C*-norm.

On the other hand, the sectional C*-algebra of a Fell bundle \mathcal{B} over a locally groupoid H endowed with a Haar system α is constructed from the *-algebra $C_c(H, \mathcal{B})$ whose elements are continuous compactly supported sections $F : H \rightarrow \mathcal{B}$. The product and the involution are respectively given by

$$F * G(h) = \int F(\eta)G(\eta^{-1}h)d\alpha^{r(h)}(\eta)$$

and

$$F^*(h) = F(h^{-1})^*$$

Again, the C*-algebra $C^*(H, \mathcal{B})$ is obtained as the C*-completion for the full C*-norm.

Proposition 2.2. [6, Section 1.4] *Let (G, S, \mathcal{A}) be a groupoid C*-dynamical system where S and $H = G/S$ have a Haar system and let $\mathcal{B} \twoheadrightarrow H$ be its Fell bundle. Then the C*-algebras $C^*(G, S, \mathcal{A})$ and $C^*(H, \mathcal{B})$ are canonically isomorphic.*

Proof. This is just a sketch of the proof. We refer the reader to [6] and the references given there. Let $f \in C_c(G, S, \mathcal{A})$. Note that $[f(\gamma), \gamma] \in B_{\pi(\gamma)}$ depends on $h = \pi(\gamma)$ only. Call this element $F(h)$. Then check that

- F belongs to $C_c(H, \mathcal{B})$;
- $f \mapsto F$ is a *-homomorphism;
- this *-homomorphism extends to an isomorphism $C^*(G, S, \mathcal{A}) \rightarrow C^*(H, \mathcal{B})$.

□

Remark 2.1. The notion of groupoid Fell bundle (H, \mathcal{B}) generalizes that of groupoid C*-dynamical system (G, S, \mathcal{A}) , where $H = G/S$ as above. A Fell bundle over a groupoid H is sometimes called an action by C*-correspondences. The sectional C*-algebra $C^*(H, \mathcal{B})$ is then called its crossed-product C*-algebra (see for example [1]).

3. MACKEY ANALYSIS OF A TWISTED GROUPOID C*-ALGEBRA.

3.1. Twists. We have given earlier the general notion of an extension. The following special case has been introduced by Kumjian in [9] in the framework of groupoids.

Definition 3.1. A central groupoid extension

$$G^{(0)} \times \mathbf{T} \twoheadrightarrow \Sigma \twoheadrightarrow G$$

where \mathbf{T} is the group of complex numbers of module 1, is called a twist. Then, we say that (G, Σ) is a twisted groupoid.

We need to distinguish arbitrary extensions as above and twists, because they do not play the same role in this study. While a twisted groupoid is denoted by (G, Σ) where Σ is the middle term, an arbitrary extension will be determined by a closed normal subgroupoid and denoted for example by (G, S) where S is the kernel of the extension.

3.2. Twisted extensions. Let (G, Σ) be a twisted groupoid. Then $(\Sigma, G^{(0)} \times \mathbf{T}, G^{(0)} \times \mathbf{C})$ is a groupoid dynamical system with the action $\sigma(s(\sigma), a) = (r(\sigma), a)$ for $(\sigma, a) \in \Sigma \times \mathbf{C}$ and $\chi(x, \theta) = \theta$ for $(x, \theta) \in G^{(0)} \times \mathbf{T}$. If G is a locally compact groupoid with Haar system, we can construct the crossed product C^* -algebra, which we denote by $C^*(G, \Sigma)$ rather than $C^*(\Sigma, G^{(0)} \times \mathbf{T}, G^{(0)} \times \mathbf{C})$ and which we call the twisted groupoid C^* -algebra. The principle of our version of Mackey analysis is to decompose this C^* -algebra when G possesses a closed normal subgroupoid S endowed with a Haar system. This is a strong condition, which is often not satisfied by the isotropy bundle itself.

Definition 3.2. We call (G, Σ, S) a twisted extension with Haar systems when (G, Σ) is a twisted groupoid, S is a closed normal subgroupoid of G with Haar system and G/S has a Haar system.

In the sequel, we shall denote $H = G/S$ the quotient groupoid and $\pi : G \rightarrow H$ the quotient map. We shall denote by $(\alpha^x)_{x \in G^{(0)}}$ [resp. $(\beta^x)_{x \in G^{(0)}}$] the Haar system of H [resp. S]. We denote by $(\lambda^x)_{x \in G^{(0)}}$ the Haar system of G described earlier. The following diagram summarizes the situation:

$$\begin{array}{ccccc}
 G^{(0)} \times \mathbf{T} & \xlongequal{\quad} & G^{(0)} \times \mathbf{T} & & \\
 \downarrow & & \downarrow & & \\
 \Sigma|_S & \xrightarrow{\quad} & \Sigma & \xrightarrow{\pi_\Sigma} & H \\
 \downarrow & & \downarrow p & & \parallel \\
 S & \xrightarrow{\quad} & G & \xrightarrow{\pi} & H
 \end{array}$$

3.3. The tautological Fell bundle. There is a Fell bundle $\mathcal{C} \twoheadrightarrow H$ naturally associated to a twisted extension (G, Σ, S) . The construction below is a particular case of the construction of Section 6 of [1], where the authors consider the more general framework of fibrations. The idea is very clear: the twisted groupoid (G, Σ) defines a C^* -category over the groupoid H . We first note that Σ defines a Fell line bundle $\mathcal{L} = \Sigma \otimes_{\mathbf{T}} \mathbf{C}$ over G , with multiplication $(\sigma_1 \otimes \lambda_1)(\sigma_2 \otimes \lambda_2) = \sigma_1 \sigma_2 \otimes \lambda_1 \lambda_2$ and involution $(\sigma \otimes \lambda)^* = \sigma^{-1} \otimes \bar{\lambda}$. In the next section, we shall view a section of $\mathcal{L} \twoheadrightarrow G$ as a function $f : \Sigma \rightarrow \mathbf{C}$ such that $f(\theta\tau) = f(\tau)\bar{\theta}$ for $(\theta, \tau) \in \mathbf{T} \times \Sigma$, but here we use the line bundle framework. We associate to the unit $x \in H^{(0)}$ the C^* -algebra $C_x = C^*(S_x, \Sigma|_{S_x})$. We want to associate to the arrow $h \in H$ a suitable completion C_h of $C_c(G(h), \mathcal{L}(h))$ where $G(h) = \pi^{-1}(h)$ and $\mathcal{L}(h)$ is the restriction to $G(h)$ of the line bundle \mathcal{L} . Let us write $\Sigma(h) = \pi_\Sigma^{-1}(h)$. Since Σ is a right

principal $\Sigma|_S$ -space, a choice of $\tau \in \Sigma(h)$ defines a homeomorphism $L(\tau) : \Sigma(s(h)) \rightarrow \Sigma(h)$ sending σ to $\tau\sigma$. In fact, this defines a line bundle isomorphism, still denoted by $L(\tau)$, from the line bundle $\mathcal{L}(s(h)) \rightarrow S_{s(h)}$ to the line bundle $\mathcal{L}(h) \rightarrow G(h)$, hence an isomorphism $L(\tau)^* : C_c(G(h), \mathcal{L}(h)) \rightarrow C_c(S_{s(h)}, \mathcal{L}(s(h))) \subset C^*(S_{s(h)}, \Sigma|_{S_{s(h)}})$. The norm $\|L(\tau)^*(f)\|$ of $f \in C_c(G(h), \mathcal{L}(h))$ depends on h only and we write it $\|f\|_h$. We let C_h be the completion of $C_c(G(h), \mathcal{L}(h))$ with respect to this norm. We define \mathcal{C} as the disjoint union $\mathcal{C} = \bigsqcup_{h \in H} C_h$. We take $\Gamma = C_c(G, \mathcal{L})$ as a fundamental family of continuous sections to define its topology. One checks that the following conditions are satisfied.

- (i) Γ is a linear subspace of the complex linear space $\prod_{h \in H} C_h$;
- (ii) for all $h \in H$, the evaluation map $ev_h : \Gamma \rightarrow C_h$ has a dense image;
- (iii) for all $\xi \in \Gamma$, the norm map $h \mapsto \|\xi(h)\|_h$ is upper semicontinuous;
- (iv) if $\eta \in \prod_{h \in H} C_h$ satisfies: for all $h \in H$ and all $\epsilon > 0$, there exists $\xi \in \Gamma$ and a neighborhood U of h in H such that $\|\eta(h') - \xi(h')\|_{h'} \leq \epsilon$ for all $h' \in U$, then η belongs to Γ .

Before describing the product and the involution of \mathcal{C} , it is useful to define a system of measures for the map $\pi : G \rightarrow H$. Such a system of measures already appears in [21] in the transitive case. Recall that, by assumption, the group bundle S is equipped with a Haar system $\beta = (\beta^x)_{x \in G^{(0)}}$. We extend it to a π -system by left invariance: for $h \in H$, we set $\beta^h = \gamma\beta^{s(h)}$, where γ is an arbitrary element of $G(h)$; it depends on h only. It is a continuous π -system of measures. It is left-invariant: for $(h, h') \in H^{(2)}$ and $\gamma \in G(h)$, $\gamma\beta^{h'} = \beta^{hh'}$. Note also the relation $(\delta(\gamma)\beta^h)^{-1} = \beta^{h^{-1}}$. Given $(h_1, h_2) \in H^{(2)}$, $f_1 \in C_c(G(h_1), \mathcal{L}(h_1))$, $f_2 \in C_c(G(h_2), \mathcal{L}(h_2))$ and $\gamma \in G(h_1 h_2)$, we define

$$f_1 *_\beta f_2(\gamma) = \int f_1(\gamma') f_2(\gamma'^{-1} \gamma) d\beta^{h_1}(\gamma')$$

and for $\gamma \in G(h^{-1})$,

$$f^*(\gamma) = f(\gamma^{-1})^*$$

One checks that these operations extend to \mathcal{C} and turn it into a Fell bundle over H . We can also use the fact proved below that \mathcal{C} is isomorphic to the Fell bundle of a groupoid dynamical system to complete the proof.

Definition 3.3. The Fell bundle \mathcal{C} constructed above is called the tautological Fell bundle of the twisted extension (G, Σ, S) .

The following result is a particular case of [1, Theorem 6.2].

Theorem 3.1 (Buss-Meyer [1]). *Let (G, Σ, S) be a locally compact twisted groupoid extension with Haar systems. Let $H = G/S$ be the quotient groupoid. Then the twisted groupoid C^* -algebra $C^*(G, \Sigma)$ is canonically isomorphic to the sectional C^* -algebra $C^*(H, \mathcal{C})$ of the tautological Fell bundle.*

Proof. Recall that the topology of \mathcal{C} has been defined by the fundamental family of continuous sections Γ . The bijective map which associates to $f \in C_c(G, \mathcal{L})$ the corresponding

section $j(f) \in \Gamma$ is an isomorphism of $*$ -algebras: let $f, g \in C_c(G, \mathcal{L})$. On one hand,

$$\begin{aligned} f *_{\lambda} g(\gamma) &= \int_{G^{r(\gamma)}} f(\gamma') g(\gamma'^{-1}\gamma) d\lambda^{r(\gamma)}(\gamma') \\ &= \int_{H^{r(\gamma)}} \int_{G(\dot{\gamma}')} f(\gamma') g(\gamma'^{-1}\gamma) d\beta^{\dot{\gamma}'}(\gamma') d\alpha^{r(\dot{\gamma})}(\dot{\gamma}') \end{aligned}$$

On the other hand, for $h \in H$,

$$(j(f) *_{\alpha} j(g))_h = \int_{H^{r(h)}} j(f)(h') *_{\beta} j(g)(h'^{-1}h) d\alpha^{r(h)}(h')$$

Hence, for $\gamma \in G(h)$

$$(j(f) *_{\alpha} j(g))_h(\gamma) = \int_{H^{r(\gamma)}} \int_{G(\dot{\gamma}')} f(\gamma') g(\gamma'^{-1}\gamma) d\beta^{\dot{\gamma}'}(\gamma') d\alpha^{r(\dot{\gamma})}(\dot{\gamma}')$$

The involution in $C_c(G, \mathcal{L})$ and in $C_c(H, \mathcal{C})$ are given by the same formula. The $*$ -homomorphism $j : C_c(G, \Sigma) \rightarrow C^*(H, \mathcal{C})$ extends to $C^*(G, \Sigma)$ by continuity. It is surjective since its range contains the dense subalgebra Γ . To show its injectivity, one shows by using the disintegration theorem [16, Théorème 4.1.] that for every representation L of $C^*(G, \Sigma)$, there exists a representation L' of $C^*(H, \mathcal{C})$ such that $L = L' \circ j$. \square

3.4. The tautological groupoid dynamical system. The tautological Fell bundle of the twisted extension (G, Σ, S) is in fact the Fell bundle of the following groupoid dynamical system $(\Sigma, \Sigma|_S, \mathcal{C}^*(S, \Sigma|_S))$. It can be shown just as in the untwisted case (see for example [11, Theorem 5.5]) that the twisted group bundle $(S, \Sigma|_S)$ with Haar system β defines an upper semi-continuous bundle of C^* -algebras over $G^{(0)}$, with fibres $C^*(S_x, \Sigma|_{S_x})$ which we denote by $\mathcal{C}^*(S, \Sigma|_S)$. Moreover, when the groups are amenable, it is a continuous bundle. It is endowed with an action $\Sigma * \mathcal{C}^*(S, \Sigma|_S) \rightarrow \mathcal{C}^*(S, \Sigma|_S)$, where

$$(\tau.f)(\sigma) = \delta(p(\tau))f(\tau^{-1}\sigma\tau), \quad \text{for } \tau \in \Sigma, f \in C_c(S_{s(\tau)}, \Sigma|_{S_{s(\tau)}}), \sigma \in p^{-1}(S_{r(\tau)})$$

The introduction of δ is necessary in order to preserve the convolution product of the twisted group C^* -algebras $C^*(S_x, \Sigma|_{S_x})$.

Proposition 3.2. *The triple $(\Sigma, \Sigma|_S, \mathcal{C}^*(S, \Sigma|_S))$ is a groupoid C^* -dynamical system, which we call the tautological groupoid dynamical system of the twisted extension.*

Proof. The continuity of the action map $\Sigma * \mathcal{C}^*(S, \Sigma|_S) \rightarrow \mathcal{C}^*(S, \Sigma|_S)$ is proved just as [6, Proposition 2.7]. The action of $\Sigma|_S$ on $\mathcal{C}^*(S, \Sigma|_S)$ is implemented by the bundle homomorphism $\chi : \Sigma|_S \rightarrow UM(\mathcal{C}^*(S, \Sigma|_S))$ which associates to $\sigma \in p^{-1}(S_x)$ the canonical unitary $\chi(\sigma)$ in the multiplier algebra of $C^*(S_x, p^{-1}(S_x))$. Explicitly, for $f \in C_c(S_x, \Sigma|_{S_x})$ and $\sigma, \tau \in p^{-1}(S_x)$,

$$(\chi(\sigma)f)(\tau) = \delta^{1/2}(p(\sigma))f(\sigma^{-1}\tau)$$

One checks just as in [6, Section 2] that the conditions of Definition 2.3 are satisfied. \square

Theorem 3.3. *Let (G, Σ, S) be a locally compact twisted groupoid extension with Haar systems. Its tautological Fell bundle \mathcal{C} is isomorphic to the Fell bundle \mathcal{B} of its tautological groupoid dynamical system.*

Proof. As before, we denote by $\mathcal{L} = \Sigma \otimes_{\mathbf{T}} \mathbf{C}$ the Fell line bundle over G associated with Σ . Here, the sections of this bundle are viewed as functions $f : \Sigma \rightarrow \mathbf{C}$ such that $f(\theta\tau) = f(\tau)\bar{\theta}$ for all $(\theta, \tau) \in \mathbf{T} \times \Sigma$. Recall that $\mathcal{B} = (\mathcal{C}^*(S, \Sigma|_S) * \Sigma) / \Sigma|_S$. Let $\tau \in \Sigma$ and $h = \pi_{\Sigma}(\tau)$. Given $f \in C_c(G(h), \mathcal{L}(h))$, we define $\rho_{\tau}(f) \in C_c(G(r(h)), \mathcal{L}(r(h)))$ by

$$\rho_{\tau}(f)(\sigma) = \delta^{1/2}(p(\tau))f(\sigma\tau), \quad \forall \sigma \in p^{-1}(G(r(h)))$$

Then $(\rho_{\tau}(f), \tau) \in \mathcal{C}^*(S, \Sigma|_S) * \Sigma$ and for all $\sigma \in \Sigma|_S$, we have:

$$\begin{aligned} (\rho_{\sigma\tau}(f), \sigma\tau) &= (\rho_{\tau}(f)\chi(\sigma^{-1}), \sigma\tau) \\ &= \sigma(\rho_{\tau}(f), \tau) \end{aligned}$$

Therefore, for all $h \in H$, there is a map $j_h : C_c(G(h), \mathcal{L}(h)) \rightarrow B_h$ such that

$$j_h(f) = [\rho_{\tau}(f), \tau]$$

where $\pi_{\Sigma}(\tau) = h$. By definition of the norm on $C_c(G(h), \mathcal{L}(h))$, j_h extends to a Banach space isomorphism from C_h onto B_h and that this defines a bundle isomorphism j from \mathcal{C} to \mathcal{B} . Then one deduces from [4, Propositions 13.6 and 13.7] that j is a Banach bundle isomorphism. Let us check that j preserves the product and the involution. Let $(h_1, h_2) \in H^{(2)}$. Choose $\tau_i \in \Sigma$ such that $\pi_{\Sigma}(\tau_i) = h_i$ for $i = 1, 2$. Let $f_i \in C_c(G(h_i), \mathcal{L}(h_i))$ for $i = 1, 2$. The equality

$$j_{h_1 h_2}(f_1 *_{\beta} f_2) = j_{h_1}(f_1)j_{h_2}(f_2)$$

amounts to the equality

$$\rho_{\tau_1 \tau_2}(f_1 *_{\beta} f_2) = \rho_{\tau_1}(f_1)[\tau_1 \cdot \rho_{\tau_2}(f_2)]$$

We have for $\sigma \in p^{-1}(G(r(h_1)))$:

$$\begin{aligned} \rho_{\tau_1 \tau_2}(f_1 *_{\beta} f_2)(\sigma) &= \delta^{1/2}(p(\tau_1 \tau_2))(f_1 *_{\beta} f_2)(\sigma\tau_1 \tau_2) \\ &= \delta^{1/2}(p(\tau_1 \tau_2)) \int f_1(\tau) f_2(\tau^{-1} \sigma\tau_1 \tau_2) d\beta^{h_1}(\dot{\tau}) \end{aligned}$$

where $\dot{\tau} = p(\tau)$. On the other hand

$$\begin{aligned} \rho_{\tau_1}(f_1)[\tau_1 \cdot \rho_{\tau_2}(f_2)](\sigma) &= \int \rho_{\tau_1}(f_1)(\sigma') [\tau_1 \cdot \rho_{\tau_2}(f_2)](\sigma'^{-1} \sigma) d\beta^{r(h_1)}(\dot{\sigma}') \\ &= \int \delta^{1/2}(\dot{\tau}_1) f_1(\sigma' \tau_1) \delta(\dot{\tau}_1) \delta^{1/2}(\dot{\tau}_2) f_2(\tau_1^{-1} \sigma'^{-1} \sigma\tau_1 \tau_2) d\beta^{r(h_1)}(\dot{\sigma}') \\ &= \delta^{1/2}(\dot{\tau}_1) \delta^{1/2}(\dot{\tau}_2) \int f_1(\sigma' \tau_1) f_2(\tau_1^{-1} \sigma'^{-1} \sigma\tau_1 \tau_2) \delta(\dot{\tau}_1) d\beta^{r(h_1)}(\dot{\sigma}') \end{aligned}$$

This last integral is of the form $\int g(s\dot{\tau}_1)\delta(\dot{\tau}_1)d\beta^{r(h_1)}(s)$ where $g(\dot{\tau}) = f_1(\tau)f_2(\tau^{-1}\sigma\tau_1\tau_2)$. The change of variable $s = \dot{\tau}_1 t \dot{\tau}_1^{-1}$ gives

$$\begin{aligned} \int g(s\dot{\tau}_1)\delta(\dot{\tau}_1)d\beta^{r(h_1)}(u) &= \int g(\dot{\tau}_1 t)d\beta^{s(h_1)}(t) \\ &= \int g(t)d\beta^{h_1}(t) \end{aligned}$$

which is the desired equality.

Let $f \in C_c(G(h), \mathcal{L}(h))$. The equality

$$(j_h(f))^* = j_{h^{-1}}(f^*)$$

amounts to the equality

$$\tau^{-1} \cdot (\rho_\tau(f))^* = \rho_{\tau^{-1}}(f^*)$$

where $\pi_\Sigma(\tau) = h$. For $\sigma \in p^{-1}(G(s(h)))$, we have

$$\begin{aligned} \tau^{-1} \cdot (\rho_\tau(f))^*(\sigma) &= \delta(\dot{\tau}^{-1})(\rho_\tau(f))^*(\tau\sigma\tau^{-1}) \\ &= \delta(\dot{\tau}^{-1})(\rho_\tau(f)(\tau\sigma^{-1}\tau^{-1}))^* \\ &= \delta(\dot{\tau}^{-1})\delta^{1/2}(\dot{\tau})f(\tau\sigma^{-1})^* \\ &= \delta^{1/2}(\dot{\tau}^{-1})f(\tau\sigma^{-1})^* \end{aligned}$$

On the other hand,

$$\begin{aligned} \rho_{\tau^{-1}}(f^*)(\sigma) &= \delta^{1/2}(\dot{\tau}^{-1})f^*(\sigma\tau^{-1}) \\ &= \delta^{1/2}(\dot{\tau}^{-1})f(\tau\sigma^{-1})^* \end{aligned}$$

□

Corollary 3.4. *Let (G, Σ, S) be a locally compact twisted groupoid extension with Haar systems. Then the twisted groupoid C^* -algebra $C^*(G, \Sigma)$ is isomorphic to the crossed product C^* -algebra $C^*(\Sigma, \Sigma|_S, \mathcal{C}^*(S, \Sigma|_S))$ of the dynamical system $(\Sigma, \Sigma|_S, \mathcal{C}^*(S, \Sigma|_S))$.*

Remark 3.1. This result can be obtained directly (without introducing the tautological Fell bundle) by using the disintegration theorem of [16] (or rather its generalization to a non abelian extension). This is the road followed in [6], where this result (in the untwisted case) appears as Theorem 2.11. The authors also consider the reduced C^* -algebras. The main advantage of the Fell bundle is that it gives simple and natural formulas.

4. ABELIAN FELL BUNDLES

When the normal subgroupoid S in the locally compact twisted groupoid extension with Haar systems (G, Σ, S) is abelian and the restriction $\Sigma|_S$ of the twist is also abelian, one can go one step further by using the Gelfand transform for the bundle of abelian C^* -algebras $\mathcal{C}^*(S, \Sigma|_S)$. Again, instead of doing this step directly, we make a detour via abelian Fell bundles.

4.1. Abelian Fell bundles. The structure of saturated abelian Fell bundles over a groupoid has been established by V. Deaconu, A. Kumjian and B. Ramazan [2, Theorem 5.6]. It is a direct consequence of the well-known following facts about Morita equivalence of commutative C^* -algebras.

Lemma 4.1. [14, Appendix A]

- (i) *Let X, Y be locally compact spaces, $\varphi : Y \rightarrow X$ a homeomorphism and \mathcal{L} a hermitian line bundle over X (with scalar product linear in the first variable). Then $(A = C_0(X), E = C_0(X, \mathcal{L}), B = C_0(Y))$ is an imprimitivity bimodule where*

(a) *for $(a, \xi, b) \in A \times E \times B$,*

$$(b\xi)(x) = b(x)\xi(x) \quad (\xi b)(x) = b(\varphi^{-1}(x))\xi(x)$$

(b) *for $(\xi, \eta) \in E \times E$,*

$${}_A\langle \xi, \eta \rangle(x) = (\xi(x) | \eta(x))_x \quad \langle \xi, \eta \rangle_B(y) = (\eta(\varphi(y)) | \xi(\varphi(y)))_{\varphi(y)}$$

- (ii) *Conversely, every imprimitivity bimodule (A, E, B) , where the C^* -algebras A and B are abelian, is isomorphic to an imprimitivity bimodule*

$$(C_0(X), C_0(X, \mathcal{L}), C_0(Y))$$

with (X, Y, h, \mathcal{L}) as in (i). The homeomorphism $\varphi : Y \rightarrow X$ is uniquely determined by the relation $\xi b = \alpha(b)\xi$ for all $\xi \in E$ and $b \in B$ and $\alpha(b) = b \circ \varphi^{-1}$. The quadruple $(X, Y, \varphi, \mathcal{L})$, which is unique up to isomorphism, is called the spatial realization of the Morita equivalence.

- (iii) *The spatial realization of the composition $(A, E \otimes_B F, C)$ of two Morita equivalences (A, E, B) and (B, F, C) having $(X, Y, \varphi, \mathcal{L})$ and $(Y, Z, \psi, \mathcal{M})$ as spatial realizations is the quadruple $(X, Z, \varphi \circ \psi, \mathcal{L} \otimes_X \varphi^* \mathcal{M})$.*
- (iv) *If the imprimitivity bimodule (A, E, B) admits the spatial realization $(X, Y, \varphi, \mathcal{L})$, then its inverse (B, E^*, A) admits the spatial realization $(Y, X, \varphi^{-1}, \varphi^* \overline{\mathcal{L}})$, where $\overline{\mathcal{L}}$ is the conjugate line bundle.*

Proof. The assertions (i), (iii) and (iv) are straightforward. A direct proof of the assertion (ii) is given in [2]. It can also be obtained by introducing the imprimitivity algebra and by using [17], to which we refer the reader for unexplained notations. There is a C^* -algebra C and complementary full projections $p, q \in M(C)$ such that (A, E, B) is isomorphic to (pCp, pCq, qCq) . It is easily checked that $D = pCp + qCq$ is a Cartan subalgebra of C having the map $c \mapsto pcp + qcq$ as an expectation onto it. The spectrum Z of D is the disjoint union of the open subsets X and Y , which are respectively the spectra of A and B . Thus, according to [17, Theorem 5.9], there is an isomorphism Φ from C onto $C_r^*(G(D), \Sigma(D))$, the reduced groupoid C^* -algebra of the Weyl twist $(G(D), \Sigma(D))$, which carries D onto $C_0(Z)$. The subsets pCp, pCq, qCp and qCq are contained in the normalizer $N(D)$ of D . Since X and Y are open, every germ of a partial homeomorphism induced by an element of the normalizer can be obtained by an element of one of these subsets. The partial homeomorphisms obtained from elements of pCp and qCq are partial identity maps. Let $n \in pCq$ and $y \in Y$ such that $n^*n(y) > 0$. Then the germ of α_n at y does not depend on n . Indeed, let $m \in pCq$ such that $m^*m(y) > 0$. Without changing

the germ of α_m at y , we may assume that the closed support of m^*m is contained in an open neighborhood of y on which $n^*n \geq \epsilon > 0$. Then, there exists $b \in B$ with $b(y) \neq 0$ and $n^*m = (n^*n)b$. Then, the equality $(nn^*)m = (nn^*)nb$ implies that α_m and α_n have the same germ at y . Moreover, since the projection q is full, every $y \in Y$ is the domain of some normalizer $n \in pCq$. Therefore there exists a unique homeomorphism $\varphi : Y \rightarrow X$ such that $\alpha_n(y) = \varphi(y)$ for all $n \in pCq$ and $y \in Y$ such that $n^*n(y) > 0$. This shows that $G(D)$ is the graph of the equivalence relation on $Z = X \sqcup Y$ whose classes are $\{y, \varphi(y)\}$ for $y \in Y$ and $\{x, \varphi^{-1}(x)\}$ for $x \in X$. It is a closed subset of $Z \times Z$. Let \mathcal{F} be the Fell line bundle associated to the $\Sigma(D)$. The restriction \mathcal{L} of \mathcal{F} to the subset $\{(x, \varphi^{-1}(x)), x \in X\}$, which we identify to X , is a hermitian line bundle with scalar product $(\xi|\eta) = \xi\eta^*$. Then Φ maps isomorphically (pCp, pCq, qCq) onto $(C_0(X), C_0(X, \mathcal{L}), C_0(Y))$. \square

We give now a construction of a saturated abelian Fell bundle over a groupoid. It will turn out that every saturated abelian Fell bundle over a groupoid can be constructed in this fashion. Let (H, α) be a locally compact groupoid with Haar system and let Z be a right locally compact H -space. As usual, we write $s : Z \rightarrow H^{(0)}$ the projection and $Z_x = s^{-1}(x)$ for $x \in H^{(0)}$. Let Σ be a twist over the semi-direct product $Z \rtimes H$. We denote by \mathcal{L} the associated Fell line bundle. Then, for each $h \in H$, we obtain by restriction to $Z_{r(h)} \times \{h\}$ a line bundle \mathcal{L}_h over $Z_{r(h)}$ and consider the space of continuous sections vanishing at infinity $B_h := C_0(Z_{r(h)}, \mathcal{L}_h)$ endowed with the sup-norm. We turn $\mathcal{B} := \bigsqcup_{h \in H} B_h$ into a Banach bundle with $C_c(Z \rtimes H, \mathcal{L})$ as fundamental space of continuous sections, where we identify $f \in C_c(Z \rtimes H, \mathcal{L})$ with the section $h \rightarrow f_h$, where $f_h \in C_c(Z_{r(h)}, \mathcal{L}_h)$ is defined by $f_h(z) = f(z, h)$. Given $(h, h') \in H^{(2)}$, we define the product

$$B_h \otimes B_{h'} \rightarrow B_{hh'} : b \otimes b' \mapsto b(hb')$$

where, for $z \in Z_{r(h)}$, $(b(hb'))(z) = b(z)b'(zh)$. For $h \in H$, we define the involution

$$B_h \rightarrow B_{h^{-1}} : b \rightarrow b^*$$

where, for $z \in Z_{s(h)}$, $b^*(z) = (b(zh^{-1}))^*$.

Proposition 4.2. *Let Σ be a twist over the semi-direct product $Z \rtimes H$, where (H, α) is a locally compact groupoid with Haar system and Z a right locally compact H -space Z . Construct \mathcal{B} as above. Then*

- (i) \mathcal{B} is a saturated abelian Fell bundle over H ;
- (ii) $C^*(H, \mathcal{B})$ is isomorphic to $C^*(Z \rtimes H, \Sigma)$.

Proof. The first assertion is a straightforward verification. For the second assertion, one checks that the map which sends $f \in C_c(Z \rtimes H, \mathcal{L})$ into the section $h \mapsto f_h$ is a $*$ -homomorphism from $C_c(Z \rtimes H, \Sigma)$ to $C_c(H, \mathcal{B})$. It is continuous in the inductive limit topology and has a dense image. Hence it extends to a $*$ -isomorphism from $C^*(Z \rtimes H, \Sigma)$ to $C^*(H, \mathcal{B})$. \square

Definition 4.1. We say that \mathcal{B} is the Fell bundle of the twisted semi-direct product $(Z \rtimes H, \Sigma)$.

Note that this terminology does not agree with Definition 2.5, because here \mathcal{B} is a Fell bundle over H and not over $Z \rtimes H$. A coherent terminology would require the notion of

fibration as in [1]. Let us show now that every saturated abelian Fell bundle is the Fell bundle of a twisted semi-direct product.

Theorem 4.3. [2, Theorem 5.6] *Let \mathcal{B} be a saturated abelian Fell bundle over a locally compact groupoid H . Then there exist a right locally compact H -space Z and a twist Σ over the semi-direct product $Z \rtimes H$ such that \mathcal{B} is isomorphic to the Fell bundle of $(Z \rtimes H, \Sigma)$. The pair (Z, Σ) is unique up to isomorphism. We call it the spatial realization of \mathcal{B} .*

Proof. Let \mathcal{B} be a saturated abelian Fell bundle over a locally compact groupoid H . Its restriction to $H^{(0)}$ is an abelian C^* -bundle $\mathcal{B}^{(0)}$ over $H^{(0)}$. The sectional C^* -algebra $C_0(H^{(0)}, \mathcal{B}^{(0)})$ is abelian, hence isomorphic to $C_0(Z)$, where Z is its spectrum. The space Z is fibred above $H^{(0)}$. The bundle map, which is continuous, open and onto, is denoted by $s : Z \rightarrow H^{(0)}$ and the fibre above $x \in H^{(0)}$ is written Z_x . We write $B_x = C_0(Z_x)$. For each $h \in H$, $(B_{r(h)}, B_h, B_{s(h)})$ is a Morita equivalence. We let $(Z_{r(h)}, Z_{s(h)}, \varphi_h, \mathcal{L}_h)$ be its spatial realization. For $z \in Z_{s(h)}$, we define $zh = \varphi_h(z)$. This defines a map $Z * H \rightarrow Z$. Given $(z, h) \in Z * H$, we write $\mathcal{L}_{(z,h)} := (\mathcal{L}_h)_z$ and define the algebraic line bundle $\mathcal{L} = \bigsqcup_{(z,h) \in Z * H} \mathcal{L}_{(z,h)}$ over $Z * H$. A section $\xi \in C_c(H, \mathcal{B})$ defines a section $\tilde{\xi}$ of \mathcal{L} according to $\tilde{\xi}(z, h) = \xi(h)(z)$. This defines a Banach bundle structure on \mathcal{L} . Let us show that $(z, h) \mapsto zh$ is an action map. The relation $(zh)h' = z(hh')$ results from the lemma. The continuity of the action can be obtained by applying the relation

$$b(zh)\tilde{\xi}(z, h) = (\tilde{\xi}b)(z, h)$$

where $b \in C_c(Z) \subset C_c(H^{(0)}, \mathcal{B}^{(0)})$ and $\xi \in C_c(H, \mathcal{B})$. The isomorphism $B_h \otimes_{B_{s(h)}} B_{h'} \rightarrow B_{hh'}$ defined by the product in the Fell bundle \mathcal{B} gives an isomorphism $\mathcal{L}_h \otimes_{Z_{r(h)}} \mathcal{L}_{h'} \rightarrow \mathcal{L}_{hh'}$ which defines a product on \mathcal{L} . Similarly, the involution $B_h \rightarrow B_{h^{-1}}$ gives an involution $\mathcal{L}_h \rightarrow \mathcal{L}_{h^{-1}}$ on \mathcal{L} . This turns \mathcal{L} into a Fell line bundle over $Z \rtimes H$. We let Σ be the unitary bundle of \mathcal{L} . Then, by construction, \mathcal{B} is isomorphic to the Fell bundle of $(Z \rtimes H, \Sigma)$. \square

Corollary 4.4. *Let \mathcal{B} be a saturated abelian Fell bundle over a locally compact groupoid H . Then the spectrum Z of the sectional algebra $C_0(H^{(0)}, \mathcal{B}^{(0)})$ is a right locally compact H -space and there exists a twist Σ over the semi-direct product $Z \rtimes H$ such that the sectional algebra $C^*(H, \mathcal{B})$ is isomorphic to the twisted groupoid C^* -algebra $C^*(Z \rtimes H, \Sigma)$.*

4.2. Abelian groupoid dynamical system. We say that a groupoid dynamical system (G, S, \mathcal{A}) is abelian when \mathcal{A} is a bundle of commutative C^* -algebras over $G^{(0)}$. Note that we do not assume here that the groups S_x are abelian. Then the associated Fell bundle \mathcal{B} over $H = G/S$ is abelian and admits the above spatial realization. The twisted semi-direct product $(Z \rtimes H, \Sigma)$ of the previous section admits a convenient description. The sectional C^* -algebra $A = C_0(G^{(0)}, \mathcal{A})$ is abelian, hence isomorphic to $C_0(Z)$, where Z is the spectrum of A . As said earlier, the space Z is fibred above $G^{(0)}$. The bundle map is denoted by $s : Z \rightarrow G^{(0)}$ and the fibre above $x \in G^{(0)}$ is written Z_x . We have $A = C_0(Z)$ and $A_x = C_0(Z_x)$. The action of G on \mathcal{A} induces an action on Z which we write as a right action so that $(\sigma.f)(z) = f(z\sigma)$ where $f \in C_0(Z_{s(\sigma)}) = A_{s(\sigma)}$ and $z \in Z_{r(\sigma)}$. Because the action of S is unitarily implemented and \mathcal{A} is abelian, S acts trivially on \mathcal{A} . Therefore, the action of G is in fact an action of $H = G/S$. This gives the semi-direct product $Z \rtimes H$. Let us describe now the twist Σ over this semi-direct product. It is given by a

pushout construction. We first observe that the homomorphism $\chi : S \rightarrow UM(\mathcal{A})$ which implements the restriction of the action to S gives a map

$$\underline{\chi} : Z * S \rightarrow \mathbf{T} \quad (z, t) \mapsto (\chi(t))(z)$$

It is a continuous groupoid homomorphism which satisfies $\underline{\chi}(z, \gamma t \gamma^{-1}) = \underline{\chi}(z \gamma, t)$ for all $(\gamma, t) \in G * S$. Here is a general definition.

Definition 4.2. Given a groupoid extension $S \twoheadrightarrow G \twoheadrightarrow H$ and an H -bundle of abelian groups T , we say that a group bundle morphism $\varphi : S \rightarrow T$ is equivariant if $\varphi(\gamma s \gamma^{-1}) = \dot{\gamma} \varphi(s)$ for all $(\gamma, s) \in G * S$.

We give now the general pushout construction (the reader is directed to [7] for a full exposition). It is summarized by the following diagram.

$$\begin{array}{ccccc} S & \longrightarrow & G & \longrightarrow & H \\ \varphi \downarrow & & \varphi_* \downarrow & & \parallel \\ T & \longrightarrow & \underline{G} & \longrightarrow & H \end{array}$$

Here are the details.

Proposition 4.5. *Let $S \twoheadrightarrow G \twoheadrightarrow H$ be a groupoid extension, let T be locally compact abelian group bundle endowed with an H -action and let $\varphi : S \rightarrow T$ be an equivariant group bundle morphism. Then there is an extension $T \twoheadrightarrow \underline{G} \twoheadrightarrow H$ and a morphism $\varphi_* : G \rightarrow \underline{G}$ that is compatible with φ . They are unique up to isomorphism.*

Proof. We define

$$T * G = \{ (t, \gamma) \in T \times G, | p_T(t) = r(\gamma) \}$$

It is a groupoid over $G^{(0)}$ with multiplication

$$(t, \gamma)(t', \gamma') = (t(\dot{\gamma}t'), \gamma\gamma')$$

and inverse

$$(t, \gamma)^{-1} = (\dot{\gamma}^{-1}(t^{-1}), \gamma^{-1})$$

Endowed with the relative topology, it is a locally compact topological groupoid. Then S embeds into it as a closed normal subgroupoid via $i : S \rightarrow T * G$ given by $i(s) = (f(s^{-1}), s)$. We define $\underline{G} := T * G / i(S)$. Equivalently, \underline{G} is the quotient of $T * G$ for the left action of S given by $s(t, \gamma) = (t\varphi(s^{-1}), s\gamma)$. Its elements are of the form $[t, \gamma]$ where $(t, \gamma) \in T * G$ and satisfy $[t, \gamma] = [t\varphi(s^{-1}), s\gamma]$ for $s \in S_{r(\gamma)}$. Let us spell out its groupoid structure. Its unit space is $G^{(0)}$ with obvious range and source maps. The multiplication is given by

$$[t, \gamma][t', \gamma'] = [t(\dot{\gamma}t'), \gamma\gamma']$$

and its inverse map is given

$$[t, \gamma]^{-1} = [\dot{\gamma}^{-1}(t^{-1}), \gamma^{-1}]$$

The map $\underline{\pi} : \underline{G} \rightarrow H$ given by $\underline{\pi}[t, \gamma] = \pi(\gamma)$ is a surjective homomorphism and $\underline{\pi}^{(0)}$ is the identity map. Its kernel is identified to T via the map $\underline{j} : T \rightarrow \underline{G}$ defined by $\underline{j}(t) = [t, p_T(t)]$. The map $\varphi_* : G \rightarrow \underline{G}$ is given by $\varphi_*(\gamma) = [r(\gamma), \gamma]$ for $\gamma \in G$.

□

Definition 4.3. The above extension $T \twoheadrightarrow \underline{G} \twoheadrightarrow H$ is called the pushout of the extension $S \twoheadrightarrow G \twoheadrightarrow H$ by the morphism $\varphi : S \rightarrow T$.

To apply this construction to our abelian groupoid C^* -dynamical system (G, S, \mathcal{A}) , we first consider the extension

$$S \twoheadrightarrow G \twoheadrightarrow H$$

Taking the semi-direct product, we obtain a new extension

$$Z * S \twoheadrightarrow Z \rtimes G \twoheadrightarrow Z \rtimes H$$

We view $Z \times \mathbf{T}$ as a group bundle over Z with the trivial action of $Z \rtimes H$. The map

$$\varphi : Z * S \rightarrow Z \times \mathbf{T} \quad \text{given by} \quad \varphi(z, t) = (z, \underline{\chi}(z, t))$$

is a group bundle morphism which is equivariant in the above sense. We define the extension

$$Z \times \mathbf{T} \twoheadrightarrow \Sigma \twoheadrightarrow Z \rtimes H$$

as the pushout by this morphism. Explicitly,

$$\Sigma = \{[\theta, z, \gamma] : \theta \in \mathbf{T}, (z, \gamma) \in Z \rtimes G\}$$

where

$$[\theta, z, t\gamma] = [\theta \underline{\chi}(z, t), z, \gamma], \quad \forall (t, \gamma) \in S * G$$

Note that Σ is a twist over the semi-direct product $Z \rtimes H$

Theorem 4.6. *Let (G, S, \mathcal{A}) a groupoid C^* -dynamical system where \mathcal{A} is abelian. Let Z be the spectrum of the abelian C^* -algebra $C_0(G^{(0)}, \mathcal{A})$. Then the twisted crossed product $C^*(G, S, \mathcal{A})$ is isomorphic to $C^*(Z \rtimes H, \Sigma)$, where Σ is the above twist.*

Proof. This is a particular case of Corollary 4.4 but we give here an independent proof. We shall identify both $C_c(G, S, \mathcal{A})$ and $C_c(Z \rtimes H, \Sigma)$ as $*$ -algebras of complex-valued functions on $Z \rtimes G$ and observe that these $*$ -algebras essentially coincide.

By definition, an element $f \in C_c(G, S, \mathcal{A})$ is a map $f : G \rightarrow \mathcal{A}$, continuous with compact support modulo S and satisfying $f(\gamma) \in A_{r(\gamma)}$ and $f(s\gamma) = f(\gamma)\chi(s^{-1})$. Writing $A_{r(\gamma)}$ as $C_0(Z_{r(\gamma)})$, we define

$$f(z, \gamma) = f(\gamma)(z), \quad (z, \gamma) \in Z \rtimes G$$

One can check that this complex-valued function defined on $Z \rtimes G$ is continuous, satisfies

$$f(z, s\gamma) = f(s, \gamma)\underline{\chi}(z, s^{-1}), \quad (s, \gamma) \in S * G$$

there is a compact subset K of H such that $f(z, \gamma) = 0$ for all $(z, \gamma) \in Z \rtimes G$ such that $\dot{\gamma} \notin K$ and for all $\epsilon > 0$, there exists a compact subset L of Z such that $|f(z, \gamma)| \leq \epsilon$ for all $(z, \gamma) \in Z \rtimes G$ such that $z \notin L$. Conversely, every complex-valued function defined on $Z \rtimes G$ satisfying these conditions defines an element of $C_c(G, S, \mathcal{A})$. With this identification of $C_c(G, S, \mathcal{A})$, the $*$ -algebra structure is given by

$$f * g(z, \gamma) = \int f(z, \tau)g(z\tau, \tau^{-1}\gamma) d\alpha^{r(\gamma)}(\dot{\tau}), \quad f^*(z, \gamma) = \overline{f(z\gamma, \gamma^{-1})}$$

On the other hand, an element of $C_c(Z \rtimes H, \Sigma)$ is a continuous function $f : \Sigma \rightarrow \mathbf{C}$ which is compactly supported modulo \mathbf{T} (since \mathbf{T} is compact, this equivalent to be compactly supported) and which satisfies $f[\theta'\theta, z, \gamma] = f[\theta, z, \gamma]\theta'^{-1}$ for $\theta, \theta' \in \mathbf{T}$ and $(z, \gamma) \in Z \rtimes G$.

It is completely determined by its restriction to $\theta = 1$. Thus, with a slight abuse of notation, we write $f(z, \gamma) = f[1, z, \gamma]$. We obtain a complex-valued function f defined on $Z \rtimes G$ which is continuous with compact support and satisfies

$$f(z, s\gamma) = f(z, \gamma)\underline{\chi}(z, s)^{-1} \quad \forall (s, z, \gamma) \in S * Z \rtimes G$$

Conversely, given such a function f , we retrieve the original element of $C_c(Z \rtimes H, \Sigma)$ by defining $f(\theta, s, z) = f(s, z)\theta^{-1}$. When we express the product and the involution of $C_c(Z \rtimes H, \Sigma)$ in terms of these functions, we obtain the same expressions as above. Thus the elements of $C_c(G, S, \mathcal{A})$ and $C_c(Z \rtimes H, \Sigma)$ are both continuous functions on $Z \rtimes G$ satisfying

$$f(z, s\gamma) = f(z, \gamma)\gamma(z, s)^{-1} \quad \forall (s, z, \gamma) \in S * Z \rtimes G$$

The only difference between the elements of $C_c(G, S, \mathcal{A})$ and those of $C_c(Z \rtimes H, \Sigma)$ is their supports. Note that, for these functions f , the absolute value $|f|$ is defined on $Z \rtimes H$. For $f \in C_c(Z \rtimes H, \Sigma)$, $|f|$ has compact support. Therefore, f belongs to $C_c(G, S, \mathcal{A})$. Thus, we have realized $C_c(Z \rtimes H, \Sigma)$ as a $*$ -subalgebra of $C_c(H, \Sigma, \mathcal{A})$. Let us return to the original description of $C_c(G, S, \mathcal{A})$ as the space of compactly supported continuous sections of a Banach bundle \mathcal{B} over G . $C_c(Z \rtimes G, \Sigma)$ is a linear subspace of $C_c(G, S, \mathcal{A})$. It satisfies conditions (I) and (II) of Proposition 14.6 of [Fell-Doran, vol I, page 139]. Indeed, for h continuous function on G and $f \in C_c(Z \rtimes G, \Sigma)$, the function hf defined by

$$(hf)(z, \gamma) = h(\dot{\gamma})f(z, \gamma) \quad \forall (z, \gamma) \in Z \rtimes G$$

belongs to $C_c(Z \rtimes G, \Sigma)$. The fibre B_h of the bundle \mathcal{B} can be identified to the Banach space $C_0(Z_{r(h)})$. In this identification, the evaluation at h of the elements of $C_c(Z \rtimes G, \Sigma)$ gives the whole subspace $C_c(Z_{r(h)})$, which is dense in $C_0(Z_{r(h)})$. Therefore, $C_c(Z \rtimes G, \Sigma)$ is dense in $C_c(G, S, \mathcal{A})$ in the inductive limit topology. Since $C_c(G, S, \mathcal{A})$ is complete in the inductive limit topology, representations of $C_c(Z \rtimes G, \Sigma)$ which are continuous in the inductive limit topology extend by continuity. Therefore, the inclusion of $C_c(Z \rtimes G, \Sigma)$ into $C_c(G, S, \mathcal{A})$ gives an isomorphism $C^*(Z \rtimes G, \Sigma) \simeq C^*(G, S, \mathcal{A})$. \square

4.3. Abelian twisted extensions. After this digression about abelian Fell bundles and abelian groupoid dynamical systems, we return to our initial problem, which is the analysis of a twisted groupoid C^* -algebra $C^*(G, \Sigma)$ in presence of a closed normal subgroupoid S having a Haar system. As said earlier, we make a further assumption, whose present form I owe to Alex Kumjian.

Definition 4.4. We say that a twisted extension (G, Σ, S) is abelian if

- (i) the group bundle S is abelian and
- (ii) the group bundle $\Sigma|_S$ is abelian.

We stated condition (i) for convenience only since it is implied by condition (ii). When the twisted extension is abelian, the C^* -algebra $C^*(S, \Sigma|_S)$ is abelian and Corollary 3.4 can be completed. This gives our main result.

Theorem 4.7. *Let (G, Σ, S) be a locally compact abelian twisted groupoid extension. Then the twisted groupoid C^* -algebra $C^*(G, \Sigma)$ is isomorphic to the twisted groupoid C^* -algebra $C^*(Z \rtimes H, \underline{\Sigma})$ where Z is the spectrum of $C^*(S, \Sigma|_S)$, $H = G/S$ and the twist $\underline{\Sigma}$ is obtained by a pushout construction.*

Proof. We have seen that $C^*(G, \Sigma)$ is isomorphic to the crossed product C^* -algebra $C^*(\Sigma, \Sigma|_S, C^*(S, \Sigma|_S))$ of the tautological dynamical system $(\Sigma, \Sigma|_S, C^*(S, \Sigma|_S))$. Since the bundle of C^* -algebras $C^*(S, \Sigma|_S)$ is abelian, we can apply Theorem 4.6. \square

For applications, it is necessary to be more explicit about the space Z , the action of H on it and the twist $\underline{\Sigma}$. Recall that we assume that the abelian group bundle S has a Haar system. Therefore, we endow its dual group bundle \hat{S} with the topology of the spectrum of $C^*(S)$, as in [13, Section 3]. Since the abelian group bundle $\Sigma|_S$, as an extension of S by $G^{(0)} \times \mathbf{T}$ has also a Haar system, its dual group bundle $\hat{\Sigma}|_S$ has also a natural locally compact topology. An element of \hat{S} [resp. $\hat{\Sigma}|_S$] will be denoted by (x, χ) , where x is a base point and $\chi \in \hat{S}_x$ [resp. $(\hat{\Sigma}|_S)_x$]. We first consider the general case of an abelian twist (S, Σ) .

Definition 4.5. Let $X \times \mathbf{T} \rightarrow \Sigma \rightarrow S$ be an abelian twist over an abelian group bundle S . Its twisted spectrum is defined as

$$\hat{S}^\Sigma = \{(x, \chi) \in \hat{\Sigma} \text{ such that } \chi(\theta) = \theta \quad \forall \theta \in \mathbf{T}\}$$

The twisted spectrum \hat{S}^Σ is an affine space over the dual group bundle \hat{S} : the action of \hat{S} on \hat{S}^Σ is the usual multiplication: given $\chi \in \hat{S}_x^\Sigma$ and $\rho \in \hat{S}_x$, $\chi\rho \in \hat{S}_x^\Sigma$ is defined by $(\chi\rho)(\sigma) = \chi(\sigma)\rho(\dot{\sigma})$ for $\sigma \in \Sigma_x$ and where $\dot{\sigma}$ is the image of σ in S_x .

Remark 4.1. In [3], the authors give a similar description of the twisted spectrum when the twist Σ is given by a symmetric 2-cocycle. Then, the twist is obviously abelian. It may be useful to recall here that, according to [8, Lemma 7.2], a Borel 2-cocycle on a locally compact abelian group is trivial if and only if it is symmetric. Moreover, if the topology of the group is second countable, every twist is given by a Borel 2-cocycle. Thus, a twist over a bundle of second countable locally compact abelian groups is abelian if and only if it is pointwise trivial.

Lemma 4.8. *Let (S, Σ) be an abelian twist over an abelian group bundle S . Assume that S has a Haar system β . Then the twisted spectrum \hat{S}^Σ is the spectrum of the abelian C^* -algebra $C^*(S, \Sigma)$.*

Proof. When S is an abelian group and the twist Σ is trivial, this is, by the choice of a trivialization, the well-known result that \hat{S} is the spectrum of $C^*(S)$. The explicit correspondence between \hat{S}^Σ and the spectrum of $C^*(S, \Sigma)$ is given by

$$\chi(f) = \int f(\sigma)\chi(\sigma)d\beta(\dot{\sigma})$$

for $\chi \in \hat{S}^\Sigma$ and $f \in C_c(S, \Sigma)$.

When S is an abelian group bundle, the C^* -algebra $C^*(S, \Sigma)$ is the C^* -algebra defined by the continuous field of C^* -algebras $x \mapsto C^*(S_x, \Sigma|_{S_x})$. As a set, its spectrum is the disjoint union over X of the above spectra, which is \hat{S}^Σ . \square

This turns \hat{S}^Σ into a locally compact space (in fact, a locally compact affine bundle). We now return to our situation. We denote by \hat{S}^Σ rather than $\hat{S}^{\Sigma|_S}$ the twisted spectrum

of $(S, \Sigma|_S)$. Let us describe the action of $H = G/S$ on \hat{S}^Σ . The groupoid H acts on the group bundle $\Sigma|_S$ by conjugation: $h \cdot \sigma = \tau \sigma \tau^{-1}$, where $\pi_\Sigma(\tau) = h$. The transposed action on the dual group bundle $\widehat{\Sigma|_S}$, defined by $(\chi h)(\sigma) = \chi(\tau \sigma \tau^{-1})$, preserves the twisted spectrum \hat{S}^Σ . It is easily checked that this is the action arising from the action of H on the bundle of C^* -algebras $\mathcal{C}^*(S, \Sigma|_S)$.

The above pushout diagram defining the twist $\underline{\Sigma}$ becomes:

$$\begin{array}{ccccc} \hat{S}^\Sigma * \Sigma|_S & \longrightarrow & \hat{S}^\Sigma \rtimes \Sigma & \longrightarrow & \hat{S}^\Sigma \rtimes H \\ \varphi \downarrow & & \downarrow & & \parallel \\ \hat{S}^\Sigma \times \mathbf{T} & \longrightarrow & \underline{\Sigma} & \longrightarrow & \hat{S}^\Sigma \rtimes H \end{array}$$

where $\varphi(\chi, \sigma) = (\chi, \chi(\sigma))$ for $(\chi, \sigma) \in \hat{S}^\Sigma * \Sigma|_S$.

Explicitly, $\underline{\Sigma}$ is the quotient of the groupoid $(\hat{S}^\Sigma \rtimes \Sigma) \times \mathbf{T}$ by the equivalence relation

$$(\chi, \sigma \tau, \theta) \sim (\chi, \tau, \overline{\chi(\sigma)\theta}), \quad \forall \sigma \in \Sigma|_S.$$

Remark 4.2. In [7, Proposition 3.5], the above Theorem 4.7 (the twisted case) is deduced from the similar result for the untwisted case, established in the previous work [6, Theorem 3.3].

5. AN APPLICATION: DEFORMATION QUANTIZATION

Rieffel has introduced a notion of C^* -algebraic deformation quantization and illustrated it by a number of examples in [18]. On the other hand, Ramazan, generalizing Connes' tangent groupoid, has produced deformation quantization of Lie-Poisson manifolds by using groupoid techniques (see [15, 11]). Our Theorem 4.7 shows that the two approaches agree on some important examples. We consider here the basic example of a symplectic finite-dimensional real vector space (V, ω) . Then, for every $\hbar \in \mathbf{R}$, $\sigma_\hbar = e^{i\hbar\omega/2}$ is a \mathbf{T} -valued 2-cocycle on $(V, +)$. It is shown in [19] that $\hbar \mapsto C^*(V, \sigma_\hbar)$ can be made into a continuous field of C^* -algebras and in [20] that it gives a C^* -algebraic deformation quantization of the Lie-Poisson manifold (V, ω) . Note that its sectional algebra can be viewed as a twisted groupoid C^* -algebra $C^*(G, \Sigma)$, where G is the trivial group bundle $\mathbf{R} \times V$ over \mathbf{R} and Σ is the twist defined by the 2-cocycle $\sigma(\hbar, \cdot) = \sigma_\hbar$. Let $V = L \oplus L'$ be a direct sum decomposition, where L and L' are complementary Lagrangian subspaces. This gives the extension

$$\mathbf{R} \times L \hookrightarrow G \twoheadrightarrow \mathbf{R} \times L'$$

The abelian group bundle $S = \mathbf{R} \times L$ satisfies the conditions of Definition 3.2 with respect to the twisted groupoid (G, Σ) . Therefore, by Theorem 4.7, $C^*(G, \Sigma)$ is isomorphic to $C^*(Z \rtimes (\mathbf{R} \times L'), \underline{\Sigma})$, where Z is the twisted spectrum and the twist $\underline{\Sigma}$ is obtained by the pushout construction. Let us determine them explicitly. The action of $H = \mathbf{R} \times L'$ on $\Sigma|_S = \mathbf{R} \times L \times \mathbf{T}$ is given by

$$(\hbar, y) \cdot (\hbar, x, \theta) = (\hbar, x, e^{-i\hbar\omega(x,y)}\theta), \quad \text{where } \hbar \in \mathbf{R}, \quad y \in L', \quad x \in L, \quad \theta \in \mathbf{T}$$

Since $\Sigma|_S = \mathbf{R} \times L \times \mathbf{T}$, the twisted spectrum Z is $\mathbf{R} \times \hat{L}$, where \hat{L} denotes the dual group of the abelian locally compact group L . The action of H on Z is given by

$$(\hbar, \chi)(\hbar, y) = (\hbar, \chi\varphi_\hbar(y)), \quad \text{where } \hbar \in \mathbf{R}, \chi \in \hat{L}, y \in L'$$

and for $\hbar \in \mathbf{R}$, φ_\hbar is the group homomorphism from L' to \hat{L} such that

$$\langle \varphi_\hbar(y), x \rangle = e^{-i\hbar\omega(x,y)} \quad \text{where } x \in L, y \in L'$$

The semi-direct product $Z \rtimes H$ is a bundle of semi-direct products $\hat{L} \rtimes_{\hbar} L'$. For $\hbar \neq 0$, φ_\hbar is an isomorphism and $\hat{L} \rtimes_{\hbar} L'$ is isomorphic to the trivial groupoid $\hat{L} \times \hat{L}$. For $\hbar = 0$, we get $\hat{L} \times L'$, where the first term is a space and the second is a group. We use again ω to identify L' and the dual L^* , which is the tangent space of \hat{L} . Thus $\hat{L} \times L'$ is isomorphic to the tangent bundle $T\hat{L}$ and the groupoid $Z \rtimes H$ is isomorphic to the tangent groupoid of the manifold \hat{L} . One can check that we have an isomorphism of topological groupoids.

It remains to determine the twist $\underline{\Sigma}$.

Proposition 5.1. *The above twist is trivial.*

Proof. By construction, $\underline{\Sigma}$ is the quotient of $(Z \rtimes \Sigma) \times \mathbf{T}$ by the equivalence relation

$$(\hbar, \chi, x + v, \sigma_\hbar(x, v)\varphi\psi, \chi(x)\varphi\theta) \sim (\hbar, \chi, v, \psi, \theta)$$

where $\hbar \in \mathbf{R}$, $\chi \in \hat{L}$, $x \in L, v \in V$ and $\varphi, \psi, \theta \in \mathbf{T}$. The map

$$(Z \rtimes \Sigma) \times \mathbf{T} \rightarrow (Z \rtimes H) \times \mathbf{T}$$

sending $(\hbar, \chi, x + y, \psi, \theta)$ to $(\hbar, \chi, y, \overline{\psi\chi(x)\sigma_\hbar(x, y)\theta})$ where $\hbar \in \mathbf{R}$, $\chi \in \hat{L}$, $x \in L, y \in L'$, and $\varphi, \theta \in \mathbf{T}$ identifies topologically this quotient. This is also a groupoid homomorphism. Therefore, $\underline{\Sigma}$ is isomorphic to $(Z \rtimes H) \times \mathbf{T}$. □

The above example can also be presented via the Heisenberg group $\mathcal{H} = V \times \mathbf{R}$ with multiplication $(v, s)(w, t) = (v + w, \omega(v, w) + s + t)$. Mackey's normal subgroup analysis (i.e. Theorem 4.7) applied to the center $\{0\} \times \mathbf{R}$ gives the first deformation. The second deformation can be obtained by applying this analysis to the subgroup $L \times \mathbf{R}$. In conclusion, we have three isomorphic C*-algebras: $C^*(\mathcal{H})$, $C^*(G, \Sigma)$ and $C^*(Z \rtimes H)$.

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INSTITUT DENIS POISSON (UMR 7013), UNIVERSITÉ D'ORLÉANS ET CNRS, 45067 ORLÉANS
CEDEX 2, FRANCE

Email address: `jean.renault@univ-orleans.fr`