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# The stochastic Auxiliary Problem Principle in Banach spaces: measurability and convergence

Thomas Bittar, Pierre Carpentier, Jean-Philippe Chancelier, Jérôme Lonchamp

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## Abstract

The stochastic Auxiliary Problem Principle (APP) algorithm is a general Stochastic Approximation (SA) scheme that turns the resolution of an original optimization problem into the iterative resolution of a sequence of auxiliary problems. This framework has been introduced to design decomposition-coordination schemes but also encompasses many well-known SA algorithms such as stochastic gradient descent or stochastic mirror descent. We study the stochastic APP in the case where the iterates lie in a Banach space and we consider an additive error on the computation of the subgradient of the objective. In order to derive convergence results or efficiency estimates for a SA scheme, the iterates must be random variables. This is why we prove the measurability of the iterates of the stochastic APP algorithm. Then, we extend convergence results from the Hilbert space case to the Banach space case. Finally, we derive efficiency estimates for the function values taken at the averaged sequence of iterates or at the last iterate, the latter being obtained by adapting the concept of modified Fejér monotonicity to our framework.

## 1 Introduction

Let  $\mathbb{U}$  be a Banach space with a norm  $\|\cdot\|$ ,  $(\Omega, \mathcal{A}, \mathbb{P})$  be a probability space and  $(\mathbb{W}, \mathcal{B}(\mathbb{W}))$  be a measurable topological vector space with  $\mathcal{B}(\mathbb{W})$  being the Borel  $\sigma$ -field on  $\mathbb{W}$ . We refer to [3, 6] for the definitions of basic concepts in analysis and probability theory. We consider a stochastic optimization problem of the form:

$$\min_{u \in U^{\text{ad}}} \{J(u) := J^C(u) + J^\Sigma(u)\} \text{ where } \begin{cases} J^C(u) = \mathbb{E}(j^C(u, \mathbf{W})) , \\ J^\Sigma(u) = \mathbb{E}(j^\Sigma(u, \mathbf{W})) . \end{cases} \quad (1)$$

where  $U^{\text{ad}} \subset \mathbb{U}$  is a non-empty closed convex set,  $\mathbf{W} : \Omega \rightarrow \mathbb{W}$  is a random variable,  $j^\Sigma(\cdot, w)$  and  $j^C(\cdot, w)$  are proper, convex, lower-semicontinuous (l.s.c.) real-valued functions for all  $w \in \mathbb{W}$ .

Stochastic Approximation (SA) algorithms are the workhorse for solving Problem (1). The SA technique has been originally introduced in [20, 28] as an iterative method to find

the root of a monotone function which is known only through noisy estimates. SA algorithms have been the subject of many theoretical studies [2, 19, 24, 27] and have applications in various disciplines such as machine learning, signal processing or stochastic optimal control [4, 21]. Back in 1990, with decomposition applications in mind, Culioli and Cohen [13] proposed a general SA scheme in an infinite dimensional Hilbert space based on the so-called Auxiliary Problem Principle (APP), called the stochastic APP algorithm. This algorithm also encompasses several well-known algorithms such as stochastic gradient descent, the stochastic proximal gradient algorithm or stochastic mirror descent. Recently, [15, 23] apply SA methods to solve PDE-constrained optimization problems. In this paper, we extend the stochastic APP algorithm to the Banach case.

A SA algorithm is defined by a recursive stochastic update rule. For  $k \in \mathbb{N}$ , the  $k$ -th iterate of a SA algorithm is a mapping  $\mathbf{U}_k : \Omega \rightarrow \mathbb{U}$ , where the range of  $\mathbf{U}_k$  is included in  $U^{\text{ad}}$ . We denote by  $\langle \cdot, \cdot \rangle$  the duality pairing between  $\mathbb{U}$  and its topological dual space  $\mathbb{U}^*$ . In the case where  $j^C$  is differentiable, the  $k$ -th iteration of the stochastic APP algorithm computes a minimizer  $u_{k+1}$  such that:

$$u_{k+1} \in \arg \min_{u \in U^{\text{ad}}} K(u) + \langle \varepsilon_k \nabla_u j^C(u_k, w_{k+1}) - \nabla K(u_k), u \rangle + \varepsilon_k j^\Sigma(u, w_{k+1}), \quad (2)$$

where  $\varepsilon_k > 0$  is a positive real,  $w_{k+1}$  is a realization of the random variable  $\mathbf{W}$  and  $K$  is a user-defined Gateaux-differentiable convex function. The role of the function  $K$  is made clear in Section 2. In the context of the APP, Problem (2) is called the *auxiliary problem* and the function  $K$  is called the *auxiliary function*. Let us now briefly expose how this scheme reduces to well-known algorithms for particular values of  $K$  and  $j^\Sigma$ .

The most basic SA scheme is stochastic gradient descent. Assume that  $\mathbb{U}$  is a Hilbert space,  $U^{\text{ad}} = \mathbb{U}$  and  $j^\Sigma = 0$ . The  $k$ -th iteration is given by:

$$u_{k+1} = u_k - \varepsilon_k \nabla_u j^C(u_k, w_{k+1}). \quad (3)$$

This is exactly the stochastic APP algorithm (2) with  $j^\Sigma = 0$  and  $K = \frac{1}{2} \|\cdot\|^2$  where  $\|\cdot\|$  is the norm induced by the inner product in  $\mathbb{U}$ .

In the case where  $j^C$  is differentiable and  $j^\Sigma$  is non-smooth but with a proximal operator that is easy to compute, proximal methods [1, 26] are particularly efficient, even in a high-dimensional Hilbert space  $\mathbb{U}$ . An iteration of the stochastic proximal gradient algorithm is:

$$u_{k+1} \in \arg \min_{u \in \mathbb{U}} \frac{1}{2\varepsilon_k} \|u_k - u\|^2 + \langle \nabla_u j^C(u_k, w_{k+1}), u - u_k \rangle + j^\Sigma(u, w_{k+1}). \quad (4)$$

This is again the stochastic APP algorithm with  $K = \frac{1}{2} \|\cdot\|^2$  but with a non zero function  $j^\Sigma$ . The proximal term  $\frac{1}{2\varepsilon_k} \|u_k - u\|^2$  forces the next iterate  $u_{k+1}$  to be close to  $u_k$  with respect to the norm  $\|\cdot\|$ . When  $j^\Sigma$  is the indicator of a convex set, the stochastic proximal gradient method reduces to stochastic projected gradient descent and when  $j^\Sigma = 0$ , this is just the regular stochastic gradient descent (3). Proximal methods are well-suited for regularized regression problems in machine learning for example.

When  $\mathbb{U}$  is only a Banach space and not a Hilbert space, Equation (3) does not make sense as  $u_k \in \mathbb{U}$  while  $\nabla_u j^C(u_k, w_{k+1}) \in \mathbb{U}^*$  the topological dual of  $\mathbb{U}$ , thus the minus operation is not defined. This difficulty is addressed with the mirror descent algorithm [25]. The original insight of the method is to map the iterate  $u_k$  to  $\nabla K(u_k) \in \mathbb{U}^*$ , where  $K$  is a Gateaux-differentiable user-defined function. Then, we do a gradient step in  $\mathbb{U}^*$  and we map back the resulting point to the primal space  $\mathbb{U}$ . The function  $K$  is called the *mirror map* in this setting [9]. There is also a proximal interpretation of mirror descent: instead of defining proximity with the norm  $\|\cdot\|$ , the mirror descent algorithm and its stochastic counterpart [24] use a Bregman divergence [7] that captures the geometric properties of the problem:

$$u_{k+1} \in \arg \min_{u \in U^{\text{ad}}} \frac{1}{\varepsilon_k} D_K(u, u_k) + \langle \nabla_u j^C(u_k, w_{k+1}), u - u_k \rangle, \quad (5)$$

where  $D_K$  is the Bregman divergence associated with  $K$ :

$$D_K(u, u') = K(u) - K(u') - \langle \nabla K(u'), u - u' \rangle, \quad u, u' \in \mathbb{U},$$

The function  $K$  is sometimes called the *distance-generating function* as it defines the proximity between  $u$  and  $u'$ . With  $K = \frac{1}{2} \|\cdot\|^2$ , we get back to the setting of stochastic gradient descent. The mirror descent algorithm is particularly suited to the case where  $\nabla_u j^C$  has a Lipschitz constant which is large with respect to the norm  $\|\cdot\|$  but small with respect to some other norm that is better suited to the geometry of the problem [24]. For example, in the finite-dimensional case, the performance of stochastic gradient descent depends on the Lipschitz constant of  $\nabla_u j^C$  in the Euclidean geometry. Hence, if the problem exhibits a non-Euclidean geometric structure, stochastic mirror descent may be more efficient. Note that stochastic mirror descent corresponds to the stochastic APP with a general function  $K$  and  $j^\Sigma = 0$ .

In fact, the stochastic APP algorithm combines the ideas of mirror descent and of the proximal gradient method. The iteration defined by (2) can be equivalently written as:

$$u_{k+1} \in \arg \min_{u \in U^{\text{ad}}} \frac{1}{\varepsilon_k} D_K(u, u_k) + \langle \nabla_u j^C(u_k, w_{k+1}), u - u_k \rangle + j^\Sigma(u, w_{k+1}),$$

In the sequel, we stick to the formulation (2) and we consider a more general version as  $j^C$  is only assumed to be subdifferentiable and we allow for an additive error on the subgradient  $\partial_u j^C(u_k, w_{k+1})$ . Figure 1 summarizes the relationship between the four stochastic approximation algorithms that we have introduced.

The paper is organized as follows:

- In Section 2, we describe the setting of the stochastic APP algorithm considered in this paper along with some examples of application.
- In Section 3, we prove the measurability of the iterates of the stochastic APP algorithm in a Banach space. The issue of measurability is not often addressed in the literature,

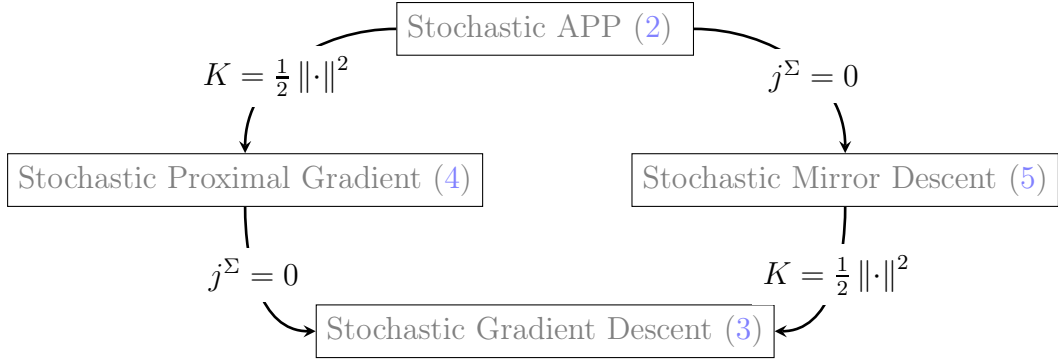


Figure 1: Links between the different stochastic approximation algorithms

yet it is essential from a theoretical point of view. When convergence results or efficiency estimates are derived for SA algorithms, the iterates must be random variables so that the probabilities or the expectations that appear in the computation are well-defined. For that purpose, we carry out a precise study based on [10, 17] and we adapt some results of [29] to the Banach case.

- Section 4 deals with convergence results and efficiency estimates. In §4.1, convergence results for the iterates and for the function values of the stochastic APP algorithm are extended to the Banach case. These results already appear in [13] for the Hilbert case. They are also given, again in the Hilbert case, for stochastic projected gradient in [15] or stochastic mirror descent in [24]. In §4.2, we derive efficiency estimates for the expected function value taken either for the averaged sequence of iterates or for the last iterate. These efficiency estimates take into account the additive error on the subgradient, using the technique from [16]. To obtain convergence rates for the expected function value of the last iterate, we adapt the concept of modified Fejér monotonicity [22] to the framework of the stochastic APP algorithm.
- The paper ends by some concluding remarks in Section 5.

## 2 Description of the algorithm and examples

In this section, we describe the version of the stochastic APP algorithm that is studied in this paper and we give some examples of problems that fit in the general framework of Problem (1).

### 2.1 Setting of the stochastic APP algorithm

We aim at solving Problem (1) that we call the *master problem*. The original idea of the APP, first introduced in [11] and extended to the stochastic case in [13], is to solve a sequence of auxiliary problems whose solutions converge to the optimal solution of the master problem.

Assume that  $j^C$  is subdifferentiable. At iteration  $k$  of the algorithm, a realization  $w_{k+1}$  of a random variable  $\mathbf{W}_{k+1}$  is drawn. The random variables  $\mathbf{W}_1, \dots, \mathbf{W}_{k+1}$  are independent and identically distributed as  $\mathbf{W}$ . Then, the following auxiliary problem is solved:

$$\min_{u \in U^{\text{ad}}} K(u) + \langle \varepsilon_k(g_k + r_k) - \nabla K(u_k), u \rangle + \varepsilon_k j^{\Sigma}(u, w_{k+1}), \quad (6)$$

where  $g_k \in \partial_u j^C(u_k, w_{k+1})$  and we allow for an additive error  $r_k$  on the gradient. The term  $r_k$  represents a numerical error or a bias due to an approximation of the gradient e.g. with a finite difference scheme. The auxiliary problem is characterized by the choice of the auxiliary function  $K$ . In the introduction, we have given particular choices for  $K$  that lead to well-known algorithms. Depending on the context, the function  $K$  allows for an adaptation of the algorithm to the geometric structure of the data or it can provide decomposition properties to the algorithm, see Example 2.2. The stochastic APP algorithm is given in Algorithm 1.

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**Algorithm 1** Stochastic APP algorithm

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- 1: Choose an initial point  $u_0 \in U^{\text{ad}}$ , and a positive sequence  $\{\varepsilon_k\}_{k \in \mathbb{N}}$ .
  - 2: At iteration  $k$ , draw a realization  $w_{k+1}$  of the random variable  $\mathbf{W}_{k+1}$ .
  - 3: Solve Problem (6), denote by  $u_{k+1}$  the solution.
  - 4:  $k \leftarrow k + 1$  and go back to 2.
- 

Note that no explicit stopping rule is provided in Algorithm 1. It is indeed difficult to know when to stop a stochastic algorithm as its properties are of statistical nature. Nevertheless, stopping rules have been developed in [32, 33] for the Robbins-Monro algorithm. In practice, the stopping criterion may be a maximal number of evaluations imposed by a budget limitation.

## 2.2 Some cases of interest for the stochastic APP

The structure of Problem (1) is very general and covers a wide class of problems that arise in machine learning or stochastic optimal control. We give some cases of interest that can be cast in this framework.

**Example 2.1** Regularized risk minimization in machine learning

Let  $(\mathbb{X}, \mathcal{X})$  and  $(\mathbb{Y}, \mathcal{Y})$  be two measurable spaces, where  $\mathcal{X}$  and  $\mathcal{Y}$  denote respectively the  $\sigma$ -fields on  $\mathbb{X}$  and  $\mathbb{Y}$ . Let  $X \subset \mathbb{X}$  and  $Y \subset \mathbb{Y}$  and assume there is a probability distribution  $\nu$  on  $X \times Y$ . Suppose that we have a training set  $\{(x_i, y_i)\}_{1 \leq i \leq N} \in (X \times Y)^N$  which consists in independent and identically distributed samples of a random vector  $(\mathbf{X}, \mathbf{Y})$  following the distribution  $\nu$ . Consider a convex loss function  $\ell : Y \times Y \rightarrow \mathbb{R}_+$  and let  $\mathbb{U}$  be a space of functions from  $X$  to  $Y$ . The goal of regularized expected loss minimization is to find a regression function  $u^\sharp \in U^{\text{ad}}$ , where  $U^{\text{ad}} \subset \mathbb{U}$ , such that:

$$u^\sharp \in \arg \min_{u \in U^{\text{ad}}} \int_{X \times Y} \ell(y, u(x)) \nu(dx, dy) + R(u), \quad (7)$$

where  $R$  is a regularization term. In practice, as the distribution  $\nu$  is unknown, we solve an approximate problem, called the regularized empirical risk minimization problem:

$$u^\# \in \arg \min_{u \in U^{\text{ad}}} \frac{1}{N} \sum_{i=1}^N \ell(y_i, u(x_i)) + R(u), \quad (8)$$

Note that Problem (8) is in fact exactly of the form of Problem (7) if the distribution  $\nu$  is taken to be the empirical measure  $\nu = 1/N \sum_{i=1}^N \delta_{(x_i, y_i)}$ , where  $\delta_{(x_i, y_i)}$  denotes the measure of mass one at  $(x_i, y_i)$  and zero elsewhere.

The regularized expected loss minimization Problem (7) is of the form of Problem (1) with the smooth term  $J^C(u) = \int_{X \times Y} \ell(y, u(x)) \nu(dx, dy)$  and the possibly non-smooth term  $J^\Sigma(u) = R(u)$ .

**Example 2.2** Decomposition aspects of the stochastic APP algorithm

Let  $n > 0$  be a given positive integer. Suppose that  $\mathbb{U} = \mathbb{U}_1 \times \dots \times \mathbb{U}_n$  and  $U^{\text{ad}} = U_1^{\text{ad}} \times \dots \times U_n^{\text{ad}}$  with  $U_i^{\text{ad}} \subset \mathbb{U}_i$  for all  $i \in \{1, \dots, n\}$ . Moreover, assume that  $j^\Sigma$  is an additive function, that is,  $j^\Sigma(u, \mathbf{W}) = \sum_{i=1}^n j_i^\Sigma(u^i, \mathbf{W})$  with  $u^i \in \mathbb{U}_i$ , whereas  $j^C$  induces a non-additive coupling. In this case, Problem (1) is:

$$\min_{u \in U^{\text{ad}}} J^C(u) + \sum_{i=1}^n J_i^\Sigma(u^i).$$

where  $J_i^\Sigma(u^i) = \mathbb{E}(j_i^\Sigma(u^i, \mathbf{W}))$ . We apply the stochastic APP algorithm with an additive auxiliary function  $K(u) = \sum_{i=1}^n K_i(u^i)$ . Let  $\bar{u} \in \mathbb{U}$  be given, a canonical choice for  $K_i$  is:

$$K_i(u^i) = J^C(\bar{u}^{1:i-1}, u^i, \bar{u}^{i+1:n}), \quad i \in \{1, \dots, n\},$$

where  $\bar{u}^{i:j} = (\bar{u}^i, \dots, \bar{u}^j)$  for  $1 \leq i \leq j \leq n$  and  $\bar{u}^{1:0}$  denotes the empty vector by convention. Another classical choice is  $K = \frac{1}{2} \|\cdot\|^2$ . With an additive function  $K$ , the auxiliary problem (6) can be split in  $n$  independent subproblems that can be solved in parallel. At iteration  $k$  of the stochastic APP algorithm, the  $i$ -th subproblem is:

$$\min_{u^i \in U_i^{\text{ad}}} K_i(u^i) + \langle \varepsilon_k (g_k^i + r_k^i) - \nabla K_i(u_k^i), u^i \rangle + \varepsilon_k j_i^\Sigma(u^i, w_{k+1}), \quad (9)$$

where  $g_k^i \in \partial_{u^i} j^C(u_k, w_{k+1})$  and  $r_k^i$  is an additive error on  $\partial_{u^i} j^C(u_k, w_{k+1})$ . This example shows that the stochastic APP framework encompasses decomposition techniques.

### 3 Measurability of the iterates of the stochastic APP algorithm

Convergence results for SA algorithms often consist in proving the almost sure convergence of the sequence of iterates  $\{\mathbf{U}_k\}_{k \in \mathbb{N}}$  to the optimal value  $u^\#$ . Other results provide non-asymptotic bounds for the expectation of function values  $\mathbb{E}(J(\mathbf{U}_k) - J(u^\#))$ , the quadratic

mean  $\mathbb{E} \left( \|\mathbf{U}_k - u^\sharp\|^2 \right)$  or the probability of large deviation  $\mathbb{P}(J(\mathbf{U}_k) - J(u^\sharp) > \eta)$  for some  $\eta > 0$ . In order for these expectations and probabilities to be well-defined,  $\mathbf{U}_k$  must be a measurable mapping from  $\Omega$  to  $\mathbb{U}$ . Hence, the measurability of the iterates is a key theoretical issue. However, it is hardly addressed in the literature. In this section, we prove the measurability of the iterates of the stochastic APP algorithm.

### 3.1 A general measurability result

The aim of this section is to prove that we can obtain the measurability of the iterates of the stochastic APP algorithm. For that purpose, we prove a general measurability result in Theorem 3.22 and obtain the measurability of the iterates of the stochastic APP algorithm as a consequence in Theorem 3.26.

Recall that  $(\Omega, \mathcal{A}, \mathbb{P})$  is a probability space and that  $(\mathbb{W}, \mathcal{B}(\mathbb{W}))$  is a measurable topological vector space. The Banach space  $\mathbb{U}$  is equipped with the Borel  $\sigma$ -field  $\mathcal{B}(\mathbb{U})$ . The topological dual of  $\mathbb{U}$  is denoted by  $\mathbb{U}^*$ , and its Borel  $\sigma$ -field is  $\mathcal{B}(\mathbb{U}^*)$ . We consider the following problem:

$$\min_{u \in U^{\text{ad}}} \left\{ \Phi(\omega, u) := K(u) + \langle \varphi(\omega), u \rangle + \varepsilon j^\Sigma(u, \mathbf{W}(\omega)) \right\}, \quad (10)$$

where  $\varepsilon > 0$  is a given positive real and  $\varphi : \Omega \rightarrow \mathbb{U}^*$  is a given measurable function. The goal is to show the existence of a measurable mapping  $\tilde{\mathbf{U}}$  such that for all  $\omega \in \Omega$ ,  $\tilde{\mathbf{U}}(\omega) \in \arg \min_{u \in U^{\text{ad}}} \Phi(\omega, u)$ . The mapping  $\omega \mapsto \arg \min_{u \in U^{\text{ad}}} \Phi(\omega, u)$  is a set-valued mapping. We recall some useful results on set-valued mappings in the next section.

#### 3.1.1 Some tools from the theory of set-valued mappings

We introduce some tools from the theory of set-valued mappings that are used to state and prove the measurability result of Theorem 3.22. The definitions and propositions are mostly taken from [10, 17]. For two sets  $X, Y$ , we denote by  $\Gamma : X \rightrightarrows Y$  a set-valued mapping  $\Gamma$  from  $X$  to  $Y$ . This means that for  $x \in X$ ,  $\Gamma(x) \subset Y$  or in other words that  $\Gamma(x) \in \mathcal{P}(Y)$  where  $\mathcal{P}(Y)$  is the power set of  $Y$ .

**Definition 3.1 (Measure completion)** *Let  $(\Omega, \mathcal{A})$  be a measurable space.*

- *Let  $\mu$  be a measure on  $(\Omega, \mathcal{A})$ . The  $\mu$ -completion of  $\mathcal{A}$  is the  $\sigma$ -field  $\mathcal{A}_\mu$  generated by  $\mathcal{A} \cup \{A' \in \mathcal{P}(\Omega) \mid A' \subset A, A \in \mathcal{A} \text{ and } \mu(A) = 0\}$ , that is, the union of  $\mathcal{A}$  and the  $\mu$ -negligible sets. The  $\sigma$ -field  $\mathcal{A}$  is said to be complete for the measure  $\mu$  if  $\mathcal{A} = \mathcal{A}_\mu$ .*
- *The  $\sigma$ -field  $\hat{\mathcal{A}}$  of universally measurable sets is defined by  $\hat{\mathcal{A}} = \bigcap_{\mu} \mathcal{A}_\mu$  where  $\mu$  ranges over the set of positive  $\sigma$ -finite measures on the measurable space  $(\Omega, \mathcal{A})$ .*

**Definition 3.2 (Measurable selection)** *Let  $(\Omega, \mathcal{A})$  be a measurable space and  $\mathbb{U}$  be a separable metric space. Let  $\Gamma : \Omega \rightrightarrows \mathbb{U}$  be a set-valued mapping. A function  $\gamma : \Omega \rightarrow \mathbb{U}$  is a measurable selection of  $\Gamma$  if  $\gamma(\omega) \in \Gamma(\omega)$  for all  $\omega \in \Omega$  and  $\gamma$  is measurable.*



**Definition 3.3 (Measurable mapping)** Let  $(\Omega, \mathcal{A})$  be a measurable space and  $\mathbb{U}$  be a separable metric space. A set-valued mapping  $\Gamma : \Omega \rightrightarrows \mathbb{U}$  is Effros-measurable if, for every open set  $O \subset \mathbb{U}$ , we have:

$$\Gamma^-(O) = \{\omega \in \Omega, \Gamma(\omega) \cap O \neq \emptyset\} \in \mathcal{A} .$$

**Remark 3.4** The Effros-measurability of a set-valued mapping  $\Gamma : \Omega \rightrightarrows \mathbb{U}$  is equivalent to the measurability of  $\Gamma$  viewed as a function from  $\Omega$  to  $\mathcal{P}(\mathbb{U})$ .

**Proposition 3.5** [10, Theorem III.9] Let  $(\Omega, \mathcal{A})$  be a measurable space and  $\mathbb{U}$  be a separable Banach space. Let  $\Gamma : \Omega \rightrightarrows \mathbb{U}$  be a non-empty-valued and closed-valued mapping. Then the following statements are equivalent:

- (i)  $\Gamma$  is Effros-measurable.
- (ii)  $\Gamma$  admits a Castaing representation: there exists a sequence of measurable functions  $\{\gamma_n\}_{n \in \mathbb{N}}$  such that for all  $\omega \in \Omega$ ,  $\Gamma(\omega) = \text{cl} \{\gamma_n(\omega), n \in \mathbb{N}\}$  where  $\text{cl}$  denotes the closure of a set.

**Proposition 3.6** [10, Proposition III.23: Sainte-Beuve's projection theorem] Let  $(\Omega, \mathcal{A})$  be a measurable space and  $(\mathbb{U}, \mathcal{B}(\mathbb{U}))$  be a separable Banach space equipped with its Borel  $\sigma$ -field. Let  $G \in \mathcal{A} \otimes \mathcal{B}(\mathbb{U})$ . Denote by  $\text{proj}_\Omega(G)$  the projection of  $G$  on  $\Omega$ . Then,  $\text{proj}_\Omega(G) \in \hat{\mathcal{A}}$ , where we recall that  $\hat{\mathcal{A}}$  is the  $\sigma$ -field of universally measurable sets.

**Proposition 3.7** [10, Proposition III.30] Let  $(\Omega, \mathcal{A}, \mathbb{P})$  be a measure space where  $\mathcal{A}$  is a complete  $\sigma$ -field, that is,  $\mathcal{A} = \mathcal{A}_\mathbb{P}$ . Let  $\Gamma : \Omega \rightrightarrows \mathbb{U}$  be a non-empty valued and closed-valued mapping. The following statements are equivalent:

- (i)  $\Gamma$  is Effros-measurable.
- (ii) For every closed set  $C \subset \mathbb{U}$ , we have:

$$\Gamma^-(C) = \{\omega \in \Omega, \Gamma(\omega) \cap C \neq \emptyset\} \in \mathcal{A} .$$

**Remark 3.8** When  $\mathbb{U}$  is finite-dimensional, Proposition 3.7 is true in any measurable space  $(\Omega, \mathcal{A})$ , that is, the completeness assumption of the  $\sigma$ -field  $\mathcal{A}$  is not needed [29, Theorem 14.3]. In the infinite-dimensional setting, (ii) implies (i) remains true in any measurable space  $(\Omega, \mathcal{A})$  [10, Proposition III.11]. The completeness assumption is only required to prove (i) implies (ii) when  $\mathbb{U}$  is infinite-dimensional. Essentially, in the finite-dimensional case, the proof of (i) implies (ii) relies on the fact that  $\mathbb{U}$  is locally compact. In the infinite-dimensional case,  $\mathbb{U}$  is not locally compact and the proof uses the Sainte Beuve's projection theorem.

**Definition 3.9 (Graph and epigraph)** Let  $(\mathbb{X}, \mathcal{X})$  be a measurable space and  $\mathbb{U}$  be a Banach space. Let  $h : \mathbb{X} \rightarrow \mathbb{R} \cup \{+\infty\}$  be a function and  $\Gamma : \mathbb{X} \rightrightarrows \mathbb{U}$  be a set-valued mapping.

- The graph and the epigraph of  $h$  are respectively defined by:

$$\begin{aligned} \text{gph } h &= \{(x, \alpha) \in \mathbb{X} \times \mathbb{R}, h(x) = \alpha\}, \\ \text{epi } h &= \{(x, \alpha) \in \mathbb{X} \times \mathbb{R}, h(x) \leq \alpha\}. \end{aligned}$$

- The graph of  $\Gamma$  is defined by:

$$\text{gph } \Gamma = \{(x, u) \in \mathbb{X} \times \mathbb{U}, u \in \Gamma(x)\}.$$

**Definition 3.10 (Normal integrand)** Let  $(\Omega, \mathcal{A})$  be a measurable space and  $\mathbb{U}$  be a Banach space. A function  $f : \Omega \times \mathbb{U} \rightarrow \mathbb{R} \cup \{+\infty\}$  is a normal integrand if it satisfies the following conditions:

- (i) For all  $\omega \in \Omega$ ,  $f(\omega, \cdot)$  is l.s.c.
- (ii) The epigraphical mapping  $S_f : \Omega \rightrightarrows \mathbb{U} \times \mathbb{R}$  defined by  $S_f(\omega) = \text{epi } f(\omega, \cdot)$  is Effros-measurable.

**Remark 3.11** The point (i) of Definition 3.10 is equivalent to  $S_f$  being closed-valued. In this paper, we consider the definition of the normal integrand used by Hess [17]. It differs from the definition of Castaing [10] where the point (ii) is replaced by the  $\mathcal{A} \otimes \mathcal{B}(\mathbb{U})$ -measurability of  $f$ . We shall see in Proposition 3.16 that the Effros-measurability of the epigraphical mapping  $S_f$  implies the  $\mathcal{A} \otimes \mathcal{B}(\mathbb{U})$ -measurability of  $f$ . Note also that if  $\mathcal{A}$  is complete for a positive  $\sigma$ -finite measure  $\mathbb{P}$ , these two definitions are equivalent, see [10, Proposition III.30].

**Definition 3.12 (Carathéodory integrand)** Let  $(\Omega, \mathcal{A})$  be a measurable space and  $\mathbb{U}$  be a separable Banach space. A function  $f : \Omega \times \mathbb{U} \rightarrow \mathbb{R}$  (finite-valued) is a Carathéodory integrand if it satisfies the following conditions:

- (i) For all  $u \in \mathbb{U}$ ,  $f(\cdot, u)$  is measurable.
- (ii) For all  $\omega \in \Omega$ ,  $f(\omega, \cdot)$  is continuous.

**Proposition 3.13** [17, Proposition 2.5] If  $f$  is a Carathéodory integrand, then it is a normal integrand.

**Proposition 3.14** [10, Proposition III.13] Let  $(\Omega, \mathcal{A})$  be a measurable space and  $(\mathbb{U}, \mathcal{B}(\mathbb{U}))$  be a separable Banach space equipped with its Borel  $\sigma$ -field. If  $\Gamma : \Omega \rightrightarrows \mathbb{U}$  is an Effros-measurable, closed-valued mapping, then  $\text{gph } \Gamma \in \mathcal{A} \otimes \mathcal{B}(\mathbb{U})$ .

We now recall a technical result on the Borel  $\sigma$ -field of a product space that is used in the proof of subsequent propositions.

**Proposition 3.15** [5, Proposition 7.13] *Let  $\{(\mathbb{X}_i, \mathcal{B}(\mathbb{X}_i))\}_{i \in \mathbb{N}}$  be a sequence of measurable separable topological spaces equipped with their Borel  $\sigma$ -fields. For  $n \in \mathbb{N}$ , let  $\mathbb{Y}_n = \prod_{i=1}^n \mathbb{X}_i$  and let  $\mathbb{Y} = \prod_{i \in \mathbb{N}} \mathbb{X}_i$ . Then, the Borel  $\sigma$ -field of the product space  $\mathbb{Y}_n$  (resp.  $\mathbb{Y}$ ) coincides with the product of the Borel  $\sigma$ -fields of  $\{\mathbb{X}_i\}_{i=1}^n$  (resp.  $\{\mathbb{X}_i\}_{i \in \mathbb{N}}$ ), that is:*

$$\mathcal{B}(\mathbb{Y}_n) = \bigotimes_{i=1}^n \mathcal{B}(\mathbb{X}_i) \quad \text{and} \quad \mathcal{B}(\mathbb{Y}) = \bigotimes_{i \in \mathbb{N}} \mathcal{B}(\mathbb{X}_i).$$

The following proposition shows that a normal integrand  $f : \Omega \times \mathbb{U} \rightarrow \mathbb{R} \cup \{+\infty\}$ , as defined in [17], is jointly  $\mathcal{A} \otimes \mathcal{B}(\mathbb{U})$ -measurable. This result is given in [29, Corollary 14.34] when  $\mathbb{U} = \mathbb{R}^n$  but is extended here in the Banach case.

**Proposition 3.16** *Let  $(\Omega, \mathcal{A})$  be a measurable space and  $(\mathbb{U}, \mathcal{B}(\mathbb{U}))$  be a separable Banach space equipped with its Borel  $\sigma$ -field. If  $f : \Omega \times \mathbb{U} \rightarrow \mathbb{R} \cup \{+\infty\}$  is a normal integrand, then  $f$  is  $\mathcal{A} \otimes \mathcal{B}(\mathbb{U})$ -measurable.*

**Proof.** The function  $f$  is a normal integrand so its epigraphical mapping  $S_f$  is Effros-measurable and closed-valued. Moreover  $\mathbb{U}$  is separable, so by Proposition 3.14, we get that:

$$\text{gph } S_f = \{(\omega, u, \alpha) \in \Omega \times \mathbb{U} \times \mathbb{R}, f(\omega, u) \leq \alpha\} \in \mathcal{A} \otimes \mathcal{B}(\mathbb{U} \times \mathbb{R}).$$

Using that  $\mathbb{U}$  and  $\mathbb{R}$  are separable, we have  $\mathcal{B}(\mathbb{U} \times \mathbb{R}) = \mathcal{B}(\mathbb{U}) \otimes \mathcal{B}(\mathbb{R})$  by Proposition 3.15. Then, for each  $\alpha \in \mathbb{R}$ , we get:

$$f^{-1}([\!-\infty, \alpha]) = \{(\omega, u) \in \Omega \times \mathbb{U}, f(\omega, u) \leq \alpha\} \in \mathcal{A} \otimes \mathcal{B}(\mathbb{U}).$$

This shows that  $f$  is  $\mathcal{A} \otimes \mathcal{B}(\mathbb{U})$ -measurable. □

The following proposition is an adaptation of [29, Proposition 14.45(c)] on the composition operations on normal integrands to the Banach case. Note that the separability of  $\mathbb{U}$  is a crucial assumption that is used explicitly in the proof of Proposition 3.17 and that appears in most of the results of this part. Essentially, as only a countable union of measurable sets is measurable, countable dense subsets of a separable space are often used in proofs of measurability. Moreover, in the infinite-dimensional setting, we must assume the completeness of the  $\sigma$ -field  $\mathcal{A}$  because we appeal to Proposition 3.7 in the proof.

**Proposition 3.17** *Let  $(\Omega, \mathcal{A}, \mathbb{P})$  be a measure space where  $\mathcal{A}$  is a complete  $\sigma$ -field, that is,  $\mathcal{A} = \mathcal{A}_{\mathbb{P}}$ . Let  $(\mathbb{W}, \mathcal{B}(\mathbb{W}))$  be a topological measurable space and  $(\mathbb{U}, \mathcal{B}(\mathbb{U}))$  be a separable Banach space equipped with its Borel  $\sigma$ -field. Let  $h : \mathbb{U} \times \mathbb{W} \rightarrow \mathbb{R} \cup \{+\infty\}$  be l.s.c. and  $\mathbf{W} : \Omega \rightarrow \mathbb{W}$  be a measurable mapping. Then:*

$$f : (\omega, u) \in \Omega \times \mathbb{U} \mapsto h(u, \mathbf{W}(\omega)) \in \mathbb{R} \cup \{+\infty\}$$

*is a normal integrand.*

**Proof.** We have that  $h$  is l.s.c. so  $f(\omega, \cdot) = h(\cdot, \mathbf{W}(\omega))$  is l.s.c. for all  $\omega \in \Omega$ . It remains to prove that the epigraphical mapping  $S_f$  is Effros-measurable. As  $h$  is l.s.c., the set  $\text{epi } h$  is closed. Define:

$$G : (\omega, u, \alpha) \in \Omega \times \mathbb{U} \times \mathbb{R} \mapsto (u, \mathbf{W}(\omega), \alpha) \in \mathbb{U} \times \mathbb{W} \times \mathbb{R} ,$$

Then, let:

$$\begin{aligned} Q(\omega) &= [(\mathbb{U} \times \mathbb{R}) \times \text{epi } h] \cap \text{gph } G(\omega, \cdot, \cdot) , \\ &= \{ ((u, \alpha), (u, \mathbf{W}(\omega), \alpha)) \text{ such that } h(u, \mathbf{W}(\omega)) \leq \alpha, (u, \alpha) \in \mathbb{U} \times \mathbb{R} \} , \\ &= \{ ((u, \alpha), (u, \mathbf{W}(\omega), \alpha)) \text{ such that } f(\omega, u) \leq \alpha, (u, \alpha) \in \mathbb{U} \times \mathbb{R} \} . \end{aligned}$$

Now, define the projection operator  $P$  as:

$$\begin{aligned} P : (\mathbb{U} \times \mathbb{R}) \times (\mathbb{U} \times \mathbb{W} \times \mathbb{R}) &\rightarrow (\mathbb{U} \times \mathbb{R}) , \\ ((u, \alpha), (v, w, \beta)) &\mapsto (u, \alpha) \end{aligned}$$

so that we have:

$$S_f(\omega) = \{(u, \alpha) \in \mathbb{U} \times \mathbb{R}, f(\omega, u) \leq \alpha\} = P(Q(\omega)) .$$

- Let  $\Gamma$  be the set valued mapping defined by  $\Gamma : \omega \in \Omega \mapsto \text{gph } G(\omega, \cdot, \cdot) \in (\mathbb{U} \times \mathbb{R}) \times (\mathbb{U} \times \mathbb{W} \times \mathbb{R})$ . We show that  $\Gamma$  is Effros-measurable. As  $\mathbb{U}$  is separable, there exists a countable dense subset  $\{(b_n, r_n), n \in \mathbb{N}\}$  of  $\mathbb{U} \times \mathbb{R}$ . For  $n \in \mathbb{N}$ , let  $\gamma_n(\omega) = ((b_n, r_n), G(\omega, b_n, r_n))$ . As  $G(\omega, b_n, r_n) = (b_n, \mathbf{W}(\omega), r_n)$  and  $\mathbf{W}$  is measurable, we get that  $\gamma_n$  is measurable. Then, we have  $\Gamma(\omega) = \text{cl}\{\gamma_n(\omega), n \in \mathbb{N}\}$ . Hence,  $\{\gamma_n\}_{n \in \mathbb{N}}$  is a Castaing representation of  $\Gamma$ . Moreover,  $\Gamma$  is closed-valued and non-empty valued so by Proposition 3.5, we deduce that  $\Gamma$  is Effros-measurable.
- Let  $C \subset (\mathbb{U} \times \mathbb{R}) \times (\mathbb{U} \times \mathbb{W} \times \mathbb{R})$  be a closed set. We have:

$$\begin{aligned} Q^-(C) &= \{\omega \in \Omega, [(\mathbb{U} \times \mathbb{R}) \times \text{epi } h] \cap \Gamma(\omega) \cap C \neq \emptyset\} , \\ &= \Gamma^-(C \cap [(\mathbb{U} \times \mathbb{R}) \times \text{epi } h]) . \end{aligned}$$

As  $\text{epi } h$  is closed, the set  $C \cap [(\mathbb{U} \times \mathbb{R}) \times \text{epi } h]$  is closed. By assumption, the  $\sigma$ -field  $\mathcal{A}$  is complete and we have shown that  $\Gamma$  is Effros-measurable, therefore by Proposition 3.7, we get that  $\Gamma^-(C \cap [(\mathbb{U} \times \mathbb{R}) \times \text{epi } h]) = Q^-(C) \in \mathcal{A}$ . Hence,  $Q$  is Effros-measurable.

- Finally, for every open set  $V \subset \mathbb{U} \times \mathbb{R}$ , as  $S_f(\omega) = P(Q(\omega))$ , we have:

$$S_f^-(V) = \{\omega \in \Omega, Q(\omega) \cap P^{-1}(V) \neq \emptyset\} .$$

The projection  $P$  is continuous so  $P^{-1}(V)$  is open. As  $Q$  is Effros-measurable, we get that  $S_f^-(V) \in \mathcal{A}$ , that is,  $S_f$  is Effros-measurable.

This completes the proof. □

We now give the main results that are used to prove the measurability of the iterates of the stochastic APP. The following proposition is a slight extension of [18, Proposition 4.2(c)].

**Proposition 3.18** *Let  $(\Omega, \mathcal{A})$  be a measurable space and  $(\mathbb{U}, \mathcal{B}(\mathbb{U}))$  be a separable Banach space equipped with its Borel  $\sigma$ -field. Let  $U^{\text{ad}}$  be a closed subset of  $\mathbb{U}$ . Let  $f : \Omega \times \mathbb{U} \rightarrow \mathbb{R} \cup \{+\infty\}$  be an  $\mathcal{A} \otimes \mathcal{B}(\mathbb{U})$ -measurable function. Let  $M : \Omega \rightrightarrows \mathbb{U}$  be the argmin set-valued mapping:*

$$M(\omega) = \arg \min_{u \in U^{\text{ad}}} f(\omega, u) .$$

*Assume that the argmin mapping  $M$  is non-empty valued, then  $M$  admits an  $\hat{\mathcal{A}}$ -measurable selection.*

**Proof.** Let  $\alpha \in \mathbb{R}$  and  $m(\omega) = \min_{u \in U^{\text{ad}}} f(\omega, u)$ . The function  $m$  is well-defined as  $M$  is non-empty valued. Let:

$$H = (\Omega \times U^{\text{ad}}) \cap \{(\omega, u) \in \Omega \times \mathbb{U}, f(\omega, u) < \alpha\} .$$

We have:

$$\{\omega \in \Omega, m(\omega) < \alpha\} = \text{proj}_{\Omega}(H) ,$$

where  $\text{proj}_{\Omega}(H)$  is the projection of  $H$  on  $\Omega$ . As  $f$  is  $\mathcal{A} \otimes \mathcal{B}(\mathbb{U})$ -measurable and  $U^{\text{ad}}$  is closed hence measurable, we get that  $H \in \mathcal{A} \otimes \mathcal{B}(\mathbb{U})$ . From Proposition 3.6, we deduce that  $m^{-1}(] - \infty, \alpha])$  is  $\hat{\mathcal{A}}$ -measurable so that  $m$  is  $\hat{\mathcal{A}}$ -measurable. As  $\mathcal{A} \subset \hat{\mathcal{A}}$ , the function  $f$  is  $\hat{\mathcal{A}} \otimes \mathcal{B}(\mathbb{U})$ -measurable. We can write:

$$M(\omega) = \left\{ u \in U^{\text{ad}}, f(\omega, u) = m(\omega) \right\} ,$$

so,  $\text{gph } M = \{(\omega, u) \in \Omega \times U^{\text{ad}}, f(\omega, u) = m(\omega)\}$ . Therefore,  $\text{gph } M$  is  $\hat{\mathcal{A}} \otimes \mathcal{B}(\mathbb{U})$ -measurable as the inverse image of  $\{0\}$  under the  $\hat{\mathcal{A}} \otimes \mathcal{B}(\mathbb{U})$ -measurable mapping  $(\omega, u) \mapsto f(\omega, u) - m(\omega)$ . Let  $O$  be an open subset of  $\mathbb{U}$ . We have:

$$M^{-}(O) = \text{proj}_{\Omega}((\Omega \times O) \cap \text{gph } M) .$$

As  $(\Omega \times O) \cap \text{gph } M \in \hat{\mathcal{A}} \otimes \mathcal{B}(\mathbb{U})$ , by Proposition 3.6, we get that  $M^{-}(O) \in \hat{\mathcal{A}} = \hat{\mathcal{A}}$ . Hence,  $M$  is Effros-measurable for the  $\sigma$ -field  $\hat{\mathcal{A}}$  and is non-empty-valued by assumption, so by Proposition 3.5,  $M$  admits an  $\hat{\mathcal{A}}$ -measurable selection.  $\square$

**Corollary 3.19** *Let  $(\Omega, \mathcal{A}, \mathbb{P})$  be a complete probability space, i.e.  $\mathcal{A} = \mathcal{A}_{\mathbb{P}}$ . Let  $(\mathbb{U}, \mathcal{B}(\mathbb{U}))$  be a separable Banach space equipped with its Borel  $\sigma$ -field. Let  $f : \Omega \times \mathbb{U} \rightarrow \mathbb{R} \cup \{+\infty\}$  be an  $\mathcal{A} \otimes \mathcal{B}(\mathbb{U})$ -measurable function. Suppose that the argmin mapping  $M$  is non-empty valued. Then,  $M$  admits an  $\mathcal{A}$ -measurable selection.*

**Proof.** As  $\mathbb{P}$  is a positive  $\sigma$ -finite measure, we have  $\hat{\mathcal{A}} = \bigcap_{\mu} \mathcal{A}_{\mu} \subset \mathcal{A}_{\mathbb{P}} = \mathcal{A}$ . By Proposition 3.18,  $M$  admits an  $\hat{\mathcal{A}}$ -measurable selection, which is also an  $\mathcal{A}$ -measurable selection.  $\square$

**Proposition 3.20** [17, Theorem 4.6] *Let  $(\Omega, \mathcal{A})$  be a measurable space,  $\mathbb{U}$  be a separable Banach space with separable topological dual  $\mathbb{U}^*$ . Let  $f : \Omega \times \mathbb{U} \rightarrow \mathbb{R} \cup \{+\infty\}$  be a normal integrand and assume that  $f(\omega, \cdot)$  is proper, convex and subdifferentiable for all  $\omega \in \Omega$ . Let  $\mathbf{U} : \Omega \rightarrow \mathbb{U}$  be a measurable mapping. Then, the set-valued mapping  $D_{\mathbf{U}} : \Omega \rightrightarrows \mathbb{U}^*$  such that:*

$$\begin{aligned} D_{\mathbf{U}}(\omega) &= \partial_u f(\omega, \mathbf{U}(\omega)) \\ &= \{v \in \mathbb{U}^*, f(\omega, u) \geq f(\omega, \mathbf{U}(\omega)) + \langle v, u - \mathbf{U}(\omega) \rangle, \forall u \in \mathbb{U}\}, \end{aligned}$$

is Effros-measurable.

### 3.1.2 Existence of a measurable selection for the argmin mapping of $\Phi$

In this section, we make use of the tools introduced in §3.1.1 to prove our main measurability result. We introduce the argmin set-valued mapping  $M : \Omega \rightrightarrows \mathbb{U}$  for Problem (10):

$$M(\omega) = \arg \min_{u \in U^{\text{ad}}} \{ \Phi(\omega, u) := K(u) + \langle \varphi(\omega), u \rangle + \varepsilon j^\Sigma(u, \mathbf{W}(\omega)) \}. \quad (11)$$

We consider the following assumptions:

- (A1) The space  $\mathbb{U}$  is a reflexive, separable Banach space.
- (A2)  $U^{\text{ad}}$  is a non-empty closed convex subset of  $\mathbb{U}$ .
- (A3)  $j^\Sigma : \mathbb{U} \times \mathbb{W} \rightarrow \mathbb{R}$  is jointly l.s.c. and for all  $w \in \mathbb{W}$ ,  $j^\Sigma(\cdot, w)$  is proper and convex.
- (A4) The function  $K : \mathbb{U} \rightarrow \mathbb{R}$  is proper, convex, l.s.c. and Gateaux-differentiable on an open set containing  $U^{\text{ad}}$ .
- (A5) For all  $\omega \in \Omega$ , the function  $u \mapsto \Phi(\omega, u)$  is coercive on  $U^{\text{ad}}$  meaning that when  $\|u\| \rightarrow +\infty$  with  $u \in U^{\text{ad}}$ , we have  $\Phi(\omega, u) \rightarrow +\infty$ . This assumption is automatically satisfied if  $U^{\text{ad}}$  is bounded.
- (A6) The  $\sigma$ -field  $\mathcal{A}$  is complete for the measure  $\mathbb{P}$ , that is,  $\mathcal{A} = \mathcal{A}_{\mathbb{P}}$ .
- (A7) The function  $\mathbf{W} : \Omega \rightarrow \mathbb{W}$  is measurable.
- (A8) The function  $\varphi : \Omega \rightarrow \mathbb{U}^*$  is measurable.

The objective of this part is to prove that  $M$  defined in Equation (11) admits a measurable selection. We start by a classical theorem from optimization theory giving conditions for the existence and unicity of a minimizer  $\Phi(\omega, \cdot)$ .

**Theorem 3.21** *Let  $\omega \in \Omega$ . Under Assumptions (A1)-(A5),  $M(\omega)$  is non-empty, closed and convex. Moreover, if  $K$  is strongly convex, then  $M(\omega)$  is a singleton, meaning that  $\Phi(\omega, \cdot)$ , defined in (11), has a unique minimizer.*

**Proof.** The objective function  $\Phi(\omega, \cdot)$  is the sum of three convex, l.s.c. functions, it is then convex and l.s.c. By (A5), the objective function is also coercive. As  $\mathbb{U}$  is a reflexive Banach space (A1) and  $U^{\text{ad}}$  is non-empty, closed and convex (A2), the set of minimizers  $M(\omega)$  is non-empty [8, Corollary III.20]. The convexity of  $\Phi(\omega, \cdot)$  ensures that  $M(\omega)$  is convex and the lower-semicontinuity of  $\Phi(\omega, \cdot)$  ensures that  $M(\omega)$  is closed.

If  $K$  is strongly convex, then  $\Phi(\omega, \cdot)$  is strongly convex, hence the minimizer of  $\Phi(\omega, \cdot)$  is unique so  $M(\omega)$  is a singleton.<sup>1</sup>  $\square$

**Theorem 3.22** *Under Assumptions (A1)-(A8), the mapping  $M$  defined in Equation (11) admits a measurable selection.*

**Proof.** We start by proving that  $\Phi(\omega, u) = K(u) + \langle \varphi(\omega), u \rangle + \varepsilon j^\Sigma(u, \mathbf{W}(\omega))$  is a normal integrand:

- As the function  $K$  is l.s.c. (A4),  $(\omega, u) \mapsto K(u)$  is a normal integrand. Indeed, its epigraphical mapping  $\omega \mapsto \{(u, \alpha) \in \mathbb{U} \times \mathbb{R}, K(u) \leq \alpha\}$  is a constant function of  $\omega$  and is then measurable.
- We have that the Banach space  $\mathbb{U}$  is separable (A1) and that  $\mathcal{A}$  is complete (A6). The space  $\mathbb{U}^*$  equipped with its Borel  $\sigma$ -field  $\mathcal{B}(\mathbb{U}^*)$  is a measurable space. The function  $\varphi$  is measurable (A8) and the function  $(u, v) \in \mathbb{U} \times \mathbb{U}^* \mapsto \langle v, u \rangle \in \mathbb{R}$  is continuous hence l.s.c. in particular. Then, Proposition 3.17 applies, showing that the function  $(\omega, u) \mapsto \langle \varphi(\omega), u \rangle$  is a normal integrand.
- With the same reasoning, using that  $\mathbb{U}$  is separable (A1),  $\mathbf{W}$  is measurable (A7),  $\mathcal{A}$  is complete (A6) and  $j^\Sigma$  is l.s.c. (A3), we appeal to Proposition 3.17 with  $h = j^\Sigma$  to deduce that  $(\omega, u) \mapsto j^\Sigma(u, \mathbf{W}(\omega))$  is a normal integrand.

The function  $\Phi$  is then a normal integrand as the sum of three normal integrands. By Proposition 3.16,  $\Phi$  is then  $\mathcal{A} \otimes \mathcal{B}(\mathbb{U})$ -measurable. Moreover, the  $\sigma$ -field  $\mathcal{A}$  is complete for  $\mathbb{P}$  (A6) and  $\mathbb{U}$  is separable (A1). In addition, using (A2)-(A5) to apply Theorem 3.21 ensures that  $M$  is non-empty valued. Hence, by Corollary 3.19, we conclude that  $M : \omega \mapsto \arg \min_{u \in U^{\text{ad}}} \Phi(\omega, u)$  admits a measurable selection.  $\square$

**Corollary 3.23** *Under Assumptions (A1)-(A8) and if we additionally assume that  $K$  is strongly convex, then for all  $\omega \in \Omega$ ,  $\Phi(\omega, \cdot)$ , defined in (11), has a unique minimizer and the mapping:*

$$\tilde{U}(\omega) = \arg \min_{u \in U^{\text{ad}}} \Phi(\omega, u) \in \mathbb{U}$$

*is measurable, that is,  $\tilde{U}$  is a random variable.*

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<sup>1</sup>In the case where  $K$  is strongly convex, the coercivity assumption is not needed as it is implied by the strong convexity of  $\Phi(\omega, \cdot)$ .



### 3.2 Application to the stochastic APP algorithm

We aim at studying the iterations of the stochastic APP in terms of random variables so we consider the argmin set-valued mapping  $M : \Omega \rightrightarrows \mathbb{U}$  defined by:

$$M(\omega) = \arg \min_{u \in U^{\text{ad}}} K(u) + \langle \varepsilon(\mathbf{G}(\omega) + \mathbf{R}(\omega)) - \nabla K(\mathbf{U}(\omega)), u \rangle + \varepsilon j^\Sigma(u, \mathbf{W}(\omega)), \quad (12)$$

with  $\varepsilon > 0$ ,  $\mathbf{U}(\omega) \in U^{\text{ad}}$ ,  $\mathbf{W}(\omega) \in \mathbb{W}$ ,  $\mathbf{G}(\omega) \in \partial_u j^C(\mathbf{U}(\omega), \mathbf{W}(\omega))$  and  $\mathbf{R}(\omega) \in \mathbb{U}^*$ . An iteration of the stochastic APP algorithm consists in solving Problem (12), which is exactly of the form of Problem (11) with:

$$\varphi(\omega) = \varepsilon(\mathbf{G}(\omega) + \mathbf{R}(\omega)) - \nabla K(\mathbf{U}(\omega)). \quad (13)$$

In addition to (A1)-(A7), we assume now:

- (A9) The dual space  $\mathbb{U}^*$  is separable.
- (A10) The function  $j^C : \mathbb{U} \times \mathbb{W} \rightarrow \mathbb{R}$  that appears in Problem (1) is jointly l.s.c. and for all  $w \in \mathbb{W}$ ,  $j^C(\cdot, w)$  is proper, convex and subdifferentiable on an open set containing  $U^{\text{ad}}$ .
- (A11) The mappings  $\mathbf{U} : \Omega \rightarrow U^{\text{ad}}$  and  $\mathbf{R} : \Omega \rightarrow \mathbb{U}^*$  are measurable.

In (A11), we assume that the mappings  $\mathbf{U}$  and  $\mathbf{R}$  are random variables. We cannot do the same for the mapping  $\mathbf{G}$  as it must satisfy  $\mathbf{G}(\omega) \in \partial_u j^C(\mathbf{U}(\omega), \mathbf{W}(\omega))$  for all  $\omega \in \Omega$ . Hence, we need to ensure that there exists a measurable mapping satisfying this constraint. This is the object of the following proposition.

**Proposition 3.24** *Under Assumptions (A1), (A6), (A7), (A9)-(A11), the subgradient mapping  $\Gamma : \omega \mapsto \partial_u j^C(\mathbf{U}(\omega), \mathbf{W}(\omega)) \subset \mathbb{U}^*$  admits a measurable selection  $\mathbf{G} : \Omega \rightarrow \mathbb{U}^*$ .*

**Proof.** Let  $f(\omega, u) = j^C(u, \mathbf{W}(\omega))$  for  $\omega \in \Omega$ ,  $u \in \mathbb{U}$ .

- Using that  $\mathbb{U}$  is separable (A1),  $\mathbf{W}$  is measurable (A7),  $\mathcal{A}$  is complete (A6) and  $j^C$  is l.s.c. (A10), Proposition 3.17 with  $h = j^C$  shows that  $f$  is a normal integrand.
- We have that for all  $\omega \in \Omega$ ,  $\Gamma(\omega) = \partial_u f(\omega, \mathbf{U}(\omega))$ . With (A10), we get that  $f(\omega, \cdot)$  is proper, convex and subdifferentiable for all  $\omega \in \Omega$ . We have that  $\mathbb{U}$  and  $\mathbb{U}^*$  are separable (A1), (A9),  $\mathbf{U}$  is measurable (A11) and  $f$  is a normal integrand, so by Proposition 3.20,  $\Gamma$  is Effros-measurable.

Assumption (A10) ensures that  $\Gamma$  is non-empty valued. In addition,  $\Gamma$  is Effros-measurable and closed-valued in  $\mathbb{U}^*$  which is separable (A9). By Proposition 3.5,  $\Gamma$  admits a measurable selection. This means that there exists a measurable function  $\mathbf{G} : \Omega \rightarrow \mathbb{U}^*$  such that for all  $\omega \in \Omega$ ,  $\mathbf{G}(\omega) \in \Gamma(\omega) = \partial_u j^C(\mathbf{U}(\omega), \mathbf{W}(\omega))$ .  $\square$

In the sequel,  $\mathbf{G}$  denotes a measurable selection of  $\Gamma$ . In order to apply Theorem 3.22 to prove that the iterates of the stochastic APP algorithm are measurable, we must ensure that



Assumption (A8) is satisfied, that is, we must show that the mapping  $\varphi$  defined in (13) is measurable. We prove in Proposition 3.25 that Assumption (A8) can be deduced from the other assumptions.

**Proposition 3.25** *Under Assumptions (A1), (A4), (A7), (A9)-(A11), the function  $\varphi$  is measurable.*

**Proof.** We have already seen in the proof of Theorem 3.22 that  $\Lambda : (\omega, u) \mapsto K(u)$  is a normal integrand. Assumption (A4) ensures that  $\Lambda(\omega, \cdot)$  is proper, convex and subdifferentiable for all  $\omega \in \Omega$ . We have that  $\mathbb{U}$  and  $\mathbb{U}^*$  are separable (A1), (A9),  $\mathbf{U}$  is measurable (A11), so  $\omega \mapsto \nabla_u \Lambda(\omega, \mathbf{U}(\omega)) = \nabla K(\mathbf{U}(\omega))$  is measurable by Proposition 3.20. Finally,  $\mathbf{R}$  is also measurable (A11), so  $\varphi$  is measurable as a sum of measurable functions.  $\square$

Putting Theorem 3.22 and Proposition 3.25 together, we have obtained that under Assumptions (A1)-(A7), (A9)-(A11), the mapping  $M : \Omega \rightrightarrows \mathbb{U}$  defined in (12) admits a measurable selection. We can now give the measurability result for the iterates of the stochastic APP algorithm, which is defined by the following recursion for  $\omega \in \Omega$  and  $k \in \mathbb{N}$ :

$$\begin{aligned} M_0(\omega) &= \{u_0\} \subset U^{\text{ad}}, \\ M_{k+1}(\omega) &= \arg \min_{u \in U^{\text{ad}}} K(u) + \langle \varepsilon_k (\mathbf{G}_k(\omega) + \mathbf{R}_k(\omega)) - \nabla K(\mathbf{U}_k(\omega)), u \rangle \\ &\quad + \varepsilon_k j^\Sigma(u, \mathbf{W}_{k+1}(\omega)), \end{aligned} \tag{14}$$

**Theorem 3.26** *Under Assumptions (A1)-(A7), (A9)-(A11), for all  $k \in \mathbb{N}$ , the mapping  $M_k$  that defines the  $k$ -th iteration of the stochastic APP algorithm (14) admits a measurable selection.*

**Proof.** The mapping  $M_0$  admits a measurable selection defined by  $\mathbf{U}_0(\omega) = u_0$ . Then, by iteratively using the fact that (12) admits a measurable selection, we deduce that for all  $k \in \mathbb{N}$ ,  $M_k$  admits a measurable selection.  $\square$

**Corollary 3.27** *Assume that (A1)-(A7), (A9)-(A11) are satisfied and that the auxiliary mapping  $K$  is strongly convex. Then, for all  $k \in \mathbb{N}$ , the unique mapping  $\mathbf{U}_k$  that defines the  $k$ -th iterate of the stochastic APP algorithm is measurable.*

**Proof.** If  $K$  is strongly convex, from Corollary 3.23, we get that  $M_k$  is single-valued, so the iterate  $\mathbf{U}_k$  is uniquely defined. The measurability of  $\mathbf{U}_k$  follows from Theorem 3.26. This concludes the proof of the measurability of the iterates of the stochastic APP algorithm.  $\square$

**Remark 3.28** *In [29, Chapter 14], Rockafellar exposes a whole set of measurability results in the case where  $\mathbb{U}$  is finite-dimensional. The finite-dimensional framework allows to avoid some technicalities of the infinite-dimensional case. In particular, the completeness assumption (A6) is not needed as shown by [29, Proposition 14.37] which is the analogues of Proposition 3.18 in the finite-dimensional case.*

**Remark 3.29** In Problem (1), when  $\mathbb{U}$  is a Hilbert space (and hence  $\mathbb{U}^* = \mathbb{U}$ ),  $U^{\text{ad}} = \mathbb{U}$  and  $j^\Sigma = 0$ , we can use stochastic gradient descent. Then, we have the explicit formula:

$$\mathbf{U}_{k+1} = \mathbf{U}_k - \varepsilon_k \nabla j^C(\mathbf{U}_k, \mathbf{W}_{k+1}) . \quad (15)$$

Under Assumptions (A1), (A7), (A10), the measurability of the iterates is directly obtained by induction using the explicit formula (15).

## 4 Convergence results and efficiency estimates

In this section, we prove the convergence of the stochastic APP algorithm for solving Problem (1) that we recall here:

$$\min_{u \in U^{\text{ad}}} \{J(u) := J^C(u) + J^\Sigma(u)\} \text{ where } \begin{cases} J^C(u) = \mathbb{E}(j^C(u, \mathbf{W})) , \\ J^\Sigma(u) = \mathbb{E}(j^\Sigma(u, \mathbf{W})) . \end{cases}$$

In addition, we give efficiency estimates for the convergence of function values. Some technical results for the proofs of this section are given in the appendix.

### 4.1 Convergence of the stochastic APP algorithm

We introduce a filtration  $\{\mathcal{F}_k\}_{k \in \mathbb{N}}$ , where for  $k \in \mathbb{N}$ , the  $\sigma$ -field  $\mathcal{F}_k$  is defined by:

$$\mathcal{F}_k = \sigma(\mathbf{W}_1, \dots, \mathbf{W}_k) ,$$

where  $(\mathbf{W}_1, \dots, \mathbf{W}_k)$  are the random variables that appear in the successive iterations of the stochastic APP algorithm (14). Recall that, in (14),  $\mathbf{G}_k \in \partial_u j^C(\mathbf{U}_k, \mathbf{W}_{k+1})$ <sup>2</sup> is an unbiased stochastic gradient, whereas the term  $\mathbf{R}_k$  represents a bias on the gradient.

The convergence results for the iterates and the function values of the stochastic APP algorithm are already proved in [12, 13] in the case where  $\mathbb{U}$  is a Hilbert space (possibly infinite-dimensional) and when there is no bias  $\mathbf{R}_k$ . In [15], convergence of the projected stochastic gradient descent is proved in a Hilbert space and with a bias  $\mathbf{R}_k$ . For stochastic mirror descent, convergence results and efficiency estimates can be found in [24], but no bias is considered. Here, we present convergence results in the Banach case for the stochastic APP algorithm and we allow for a bias  $\mathbf{R}_k$ , hence generalizing the previous results.

In the sequel, in addition to (A1)-(A7), (A9)-(A11), we make the following assumptions:

(A12) The functions  $j^C(\cdot, w) : \mathbb{U} \rightarrow \mathbb{R}$  and  $j^\Sigma(\cdot, w) : \mathbb{U} \rightarrow \mathbb{R}$  have linearly bounded subgradient in  $u$ , uniformly in  $w \in \mathbb{W}$ :

$$\begin{cases} \exists c_1, c_2 > 0 , \quad \forall (u, w) \in U^{\text{ad}} \times \mathbb{W} , \quad \forall r \in \partial_u j^C(u, w) , \quad \|r\| \leq c_1 \|u\| + c_2 . \\ \exists d_1, d_2 > 0 , \quad \forall (u, w) \in U^{\text{ad}} \times \mathbb{W} , \quad \forall s \in \partial_u j^\Sigma(u, w) , \quad \|s\| \leq d_1 \|u\| + d_2 . \end{cases}$$

---

<sup>2</sup>In this expression, the  $\in$  relationship is to be understood  $\omega$  by  $\omega$ .

(A13) The objective function  $J$  is coercive on  $U^{\text{ad}}$ .

(A14) The function  $K$  is  $b$ -strongly convex for  $b > 0$ , meaning that for all  $u, v \in \mathbb{U}$ , we have:

$$K(v) \geq K(u) + \langle \nabla K(u), v - u \rangle + \frac{b}{2} \|u - v\|^2 ,$$

and  $\nabla K$  is  $L_K$ -Lipschitz continuous with  $L_K > 0$ , that is, for all  $u, v \in \mathbb{U}$ :

$$\|\nabla K(v) - \nabla K(u)\|_* \leq L_K \|v - u\| ,$$

where  $\|\cdot\|_*$  is the dual norm on  $\mathbb{U}^*$ .

(A15) The sequence of step sizes  $\{\varepsilon_k\}_{k \in \mathbb{N}}$  is such that:

$$\sum_{k \in \mathbb{N}} \varepsilon_k = +\infty , \quad \sum_{k \in \mathbb{N}} \varepsilon_k^2 < +\infty . \quad (16)$$

(A16) Each  $\mathbf{R}_k$  is measurable with respect to  $\mathcal{F}_{k+1}$ , the sequence of random variables  $\{\mathbf{R}_k\}_{k \in \mathbb{N}}$  is  $\mathbb{P}$ -almost surely ( $\mathbb{P}$ -a.s.) bounded,<sup>3</sup> and we have:

$$\sum_{k \in \mathbb{N}} \varepsilon_k \mathbb{E}(\|\mathbf{R}_k\| \mid \mathcal{F}_k) < +\infty \quad \mathbb{P}\text{-a.s.}$$

Assumptions (A1)-(A3), (A10) and (A13) ensure that  $J$  is well-defined, convex, l.s.c., coercive and attains its minimum on  $U^{\text{ad}}$ . Hence, Problem (1) has a non-empty set of solutions  $U^\sharp$ . We denote by  $J^\sharp$  the value of  $J$  on  $U^\sharp$ . Note that from now on,  $K$  is supposed to be  $b$ -strongly convex, so by Corollary 3.27, the problem solved at each iteration  $k$  of the stochastic APP algorithm admits a unique solution  $\mathbf{U}_{k+1}$ , which is measurable.

We start by a technical lemma where we give an inequality that is satisfied by a Lyapunov function for the stochastic APP algorithm. This inequality will be used for the proof of convergence of the stochastic APP algorithm in Theorem 4.2 but also to derive efficiency estimates in Theorems 4.5 and 4.6.

**Lemma 4.1** *Let  $v \in U^{\text{ad}}$  and consider the Lyapunov function:*

$$\ell_v(u) = K(v) - K(u) - \langle \nabla K(u), v - u \rangle, \quad u \in U^{\text{ad}} .$$

*Let  $\{u_k\}_{k \in \mathbb{N}}$  be the sequence of iterates generated by Algorithm 1 corresponding to the realization  $\{w_k\}_{k \in \mathbb{N}}$  of the stochastic process  $\{\mathbf{W}_k\}_{k \in \mathbb{N}}$ . Then, under Assumptions (A10), (A12) and (A14), there exists constants  $\alpha, \beta, \gamma, \delta > 0$  such that, for all  $k \in \mathbb{N}$ :*

$$\begin{aligned} \ell_v(u_{k+1}) \leq & \left(1 + \alpha \varepsilon_k^2 + \frac{2}{b} \varepsilon_k \|r_k\|\right) \ell_v(u_k) + \beta \varepsilon_k^2 \ell_v(u_{k+1}) \\ & + \left(\gamma \varepsilon_k^2 + \varepsilon_k \|r_k\| + \delta (\varepsilon_k \|r_k\|)^2\right) \\ & + \varepsilon_k \left( (j^C + j^\Sigma)(v, w_{k+1}) - (j^C + j^\Sigma)(u_k, w_{k+1}) \right) , \quad (17) \end{aligned}$$

---

<sup>3</sup>The set  $\{\omega \in \Omega, \{\mathbf{R}_k(\omega)\}_{k \in \mathbb{N}} \text{ is unbounded}\}$  is negligible.

where we recall that  $b > 0$  is the strong convexity constant of  $K$ ,  $\varepsilon_k$  is the step size and  $r_k$  is the additive error on the stochastic gradient at iteration  $k$  of the stochastic APP algorithm.

**Proof.** By (A14),  $K$  is  $b$ -strongly convex implying that:

$$\frac{b}{2}\|u - v\|^2 \leq \ell_v(u). \quad (18)$$

This shows that  $\ell_v$  is lower bounded and coercive.

Let  $k \in \mathbb{N}$ , as  $u_{k+1}$  is solution of (6), it solves the following variational inequality: for all  $u \in U^{\text{ad}}$ ,

$$\begin{aligned} \langle \nabla K(u_{k+1}) - \nabla K(u_k) + \varepsilon_k(g_k + r_k), u - u_{k+1} \rangle \\ + \varepsilon_k(j^\Sigma(u, w_{k+1}) - j^\Sigma(u_{k+1}, w_{k+1})) \geq 0. \end{aligned} \quad (19)$$

Then, we have:

$$\begin{aligned} \ell_v(u_{k+1}) - \ell_v(u_k) = \underbrace{K(u_k) - K(u_{k+1}) - \langle \nabla K(u_k), u_k - u_{k+1} \rangle}_{T_1} \\ + \underbrace{\langle \nabla K(u_k) - \nabla K(u_{k+1}), v - u_{k+1} \rangle}_{T_2}. \end{aligned} \quad (20)$$

- By the convexity of  $K$  (A14), we get:

$$T_1 \leq 0.$$

- The optimality condition (19) at  $u = v$  implies:

$$\begin{aligned} T_2 &\leq \varepsilon_k \langle g_k + r_k, v - u_{k+1} \rangle + \varepsilon_k (j^\Sigma(v, w_{k+1}) - j^\Sigma(u_{k+1}, w_{k+1})) \\ &\leq \varepsilon_k \left( \underbrace{\langle g_k, v - u_k \rangle + j^\Sigma(v, w_{k+1}) - j^\Sigma(u_k, w_{k+1})}_{T_3} + \underbrace{\langle r_k, v - u_k \rangle}_{T_4} \right. \\ &\quad \left. + \underbrace{\langle g_k + r_k, u_k - u_{k+1} \rangle + j^\Sigma(u_k, w_{k+1}) - j^\Sigma(u_{k+1}, w_{k+1})}_{T_5} \right). \end{aligned}$$

- As  $j^C(\cdot, w_{k+1})$  is convex (A10), we get:

$$T_3 \leq (j^C + j^\Sigma)(v, w_{k+1}) - (j^C + j^\Sigma)(u_k, w_{k+1}).$$

- By Schwarz inequality, using  $a \leq a^2 + 1$  for all  $a \geq 0$  and the upper bound (18), we get:

$$\begin{aligned} T_4 &\leq \|r_k\| \|v - u_k\| \\ &\leq \|r_k\| (\|v - u_k\|^2 + 1) \\ &\leq \|r_k\| + \frac{2}{b} \ell_v(u_k) \|r_k\|. \end{aligned}$$

- The optimality condition (19) at  $u = u_k$  and the strong monotonicity of  $\nabla K$ , that arises from (A14), imply:

$$b\|u_{k+1} - u_k\|^2 \leq \varepsilon_k (\langle g_k + r_k, u_k - u_{k+1} \rangle + j^\Sigma(u_k, w_{k+1}) - j^\Sigma(u_{k+1}, w_{k+1})) , \quad (21)$$

where we recognize  $T_5$  as the right-hand side. Using the linearly bounded subgradient property of  $j^\Sigma$  (A12) with the technical result of Proposition A.4, we deduce that:

$$\begin{aligned} & |j^\Sigma(u_k, w_{k+1}) - j^\Sigma(u_{k+1}, w_{k+1})| \\ & \leq \left( d_1 \max \{ \|u_k\|, \|u_{k+1}\| \} + d_2 \right) \|u_k - u_{k+1}\| , \\ & \leq \left( d_1 (\|u_k\| + \|u_{k+1}\|) + d_2 \right) \|u_k - u_{k+1}\| . \end{aligned}$$

With Schwarz inequality on the first term of  $T_5$ , we have:

$$T_5 \leq \|g_k + r_k\| \|u_k - u_{k+1}\| + (d_1 \|u_k\| + d_1 \|u_{k+1}\| + d_2) \|u_k - u_{k+1}\| .$$

By the triangular inequality and Assumption (A12) for  $j^C$ , we deduce that there exist positive constants  $e_1, e_2$  and  $e_3$  such that:

$$T_5 \leq (e_1 \|u_k\| + e_2 \|u_{k+1}\| + e_3 + \|r_k\|) \|u_{k+1} - u_k\| .$$

By the inequality (21), we then get:

$$\|u_{k+1} - u_k\| \leq \frac{\varepsilon_k}{b} (e_1 \|u_k\| + e_2 \|u_{k+1}\| + e_3 + \|r_k\|) , \quad (22)$$

and therefore by a repeated use of  $(a + b)^2 \leq 2(a^2 + b^2)$ ,

$$\begin{aligned} T_5 & \leq \frac{\varepsilon_k}{b} (e_1 \|u_k\| + e_2 \|u_{k+1}\| + e_3 + \|r_k\|)^2 , \\ & \leq \frac{4\varepsilon_k}{b} (e_1^2 \|u_k\|^2 + e_2^2 \|u_{k+1}\|^2 + e_3^2 + \|r_k\|^2) . \end{aligned}$$

Finally we bound  $\|u_k\|$  (resp.  $\|u_{k+1}\|$ ) by  $\|u_k - v\| + \|v\|$  (resp.  $\|u_{k+1} - v\| + \|v\|$ ) and we use (18) again to deduce that there exist four positive constants  $\alpha, \beta, \gamma$  and  $\delta$  such that:

$$T_5 \leq \varepsilon_k (\alpha \ell_v(u_k) + \beta \ell_v(u_{k+1}) + \gamma + \delta \|r_k\|^2) .$$

We collect the bounds we have obtained for  $T_1, T_3, T_4$  and  $T_5$  to get:

$$\begin{aligned} \ell_v(u_{k+1}) & \leq \left( 1 + \alpha \varepsilon_k^2 + \frac{2}{b} \varepsilon_k \|r_k\| \right) \ell_v(u_k) + \beta \varepsilon_k^2 \ell_v(u_{k+1}) \\ & \quad + \left( \gamma \varepsilon_k^2 + \varepsilon_k \|r_k\| + \delta (\varepsilon_k \|r_k\|)^2 \right) \\ & \quad + \varepsilon_k ((j^C + j^\Sigma)(v, w_{k+1}) - (j^C + j^\Sigma)(u_k, w_{k+1})) . \quad (23) \end{aligned}$$

□

When no bias is present,  $r_k = 0$ , we retrieve the same inequality as in the PhD thesis of Culioli [12, §2.5.1]. In the proofs of the subsequent theorems, Inequality (17) will be fundamental to derive boundedness properties or convergence results for the Lyapunov function  $\ell_v$ , using variants of the Robbins-Siegmund theorem.

Now, we give convergence results for the stochastic APP algorithm, in terms of function values as well as for the iterates. The proof is similar to that in [12, 13] (case of a Hilbert space, no bias considered). The assumption that the Banach  $\mathbb{U}$  is reflexive (A1) allows for a similar treatment as in the Hilbert case. The additional contribution of the bias is already taken care of by Inequality (17).

**Theorem 4.2** *Under Assumptions (A1)-(A7), (A9)-(A16), we have the following statements:*

- *The sequence of random variables  $\{J(\mathbf{U}_k)\}_{k \in \mathbb{N}}$  converges to  $J^\sharp$  almost surely.*
- *The sequence of iterates  $\{\mathbf{U}_k\}_{k \in \mathbb{N}}$  of the stochastic APP algorithm is bounded almost surely and every weak cluster point of a bounded realization of this sequence belongs to the optimal set  $U^\sharp$ .*

**Proof.** Let  $u^\sharp \in U^\sharp$  be a solution of Problem (1) and let  $\{u_k\}_{k \in \mathbb{N}}$  be the sequence of iterates generated by Algorithm 1 for an outcome  $\omega \in \Omega$  corresponding to the realization  $\{w_k\}_{k \in \mathbb{N}}$  of the stochastic process  $\{\mathbf{W}_k\}_{k \in \mathbb{N}}$ .

### 1. Upper bound on the variation of the Lyapunov function.

Lemma 4.1 with  $v = u^\sharp$  yields:

$$\begin{aligned} \ell_{u^\sharp}(u_{k+1}) \leq & \left(1 + \alpha \varepsilon_k^2 + \frac{2}{b} \varepsilon_k \|r_k\|\right) \ell_{u^\sharp}(u_k) + \beta \varepsilon_k^2 \ell_{u^\sharp}(u_{k+1}) \\ & + \left(\gamma \varepsilon_k^2 + \varepsilon_k \|r_k\| + \delta(\varepsilon_k \|r_k\|)^2\right) \\ & + \varepsilon_k \left((j^C + j^\Sigma)(u^\sharp, w_{k+1}) - (j^C + j^\Sigma)(u_k, w_{k+1})\right). \end{aligned} \quad (24)$$

We write this inequality in terms of random variables and take the conditional expectation on both sides with respect to the  $\sigma$ -field  $\mathcal{F}_k$  generated by the random variables  $(\mathbf{W}_1, \dots, \mathbf{W}_k)$ . By construction  $\mathbf{U}_k$  is  $\mathcal{F}_k$ -measurable, so  $\mathbb{E}(\ell_{u^\sharp}(\mathbf{U}_k) \mid \mathcal{F}_k) = \ell_{u^\sharp}(\mathbf{U}_k)$ . The random variable  $\mathbf{W}_{k+1}$  is independent of the past random variables  $\{\mathbf{W}_l\}_{l \leq k}$  and therefore of  $\mathbf{U}_k$ , thus we have  $\mathbb{E}((j^C + j^\Sigma)(\mathbf{U}_k, \mathbf{W}_{k+1}) \mid \mathcal{F}_k) = (j^C + j^\Sigma)(\mathbf{U}_k) = J(\mathbf{U}_k)$ . We finally get:

$$\begin{aligned} \mathbb{E}(\ell_{u^\sharp}(\mathbf{U}_{k+1}) \mid \mathcal{F}_k) \leq & (1 + \alpha_k) \ell_{u^\sharp}(\mathbf{U}_k) + \beta_k \mathbb{E}(\ell_{u^\sharp}(\mathbf{U}_{k+1}) \mid \mathcal{F}_k) + \gamma_k \\ & - \varepsilon_k (J(\mathbf{U}_k) - J(u^\sharp)), \end{aligned} \quad (25)$$

where we have:

$$\begin{aligned} \alpha_k &= \alpha \varepsilon_k^2 + \frac{2}{b} \varepsilon_k \mathbb{E}(\|\mathbf{R}_k\| \mid \mathcal{F}_k), \\ \beta_k &= \beta \varepsilon_k^2, \\ \gamma_k &= \gamma \varepsilon_k^2 + \varepsilon_k \mathbb{E}(\|\mathbf{R}_k\| \mid \mathcal{F}_k) + \delta(\varepsilon_k \mathbb{E}(\|\mathbf{R}_k\| \mid \mathcal{F}_k))^2, \end{aligned}$$

By Assumptions (A15) and (A16),  $\alpha_k, \beta_k$  and  $\gamma_k$  are the terms of convergent series. Recall that  $J(\mathbf{U}_k) - J(u^\sharp)$  is almost surely nonnegative as  $u^\sharp$  is solution of (1).

2. **Convergence analysis.** Applying Corollary A.3 (Robbins-Siegmund theorem), we get that the sequence of random variables  $\{\ell_{u^\sharp}(\mathbf{U}_k)\}_{k \in \mathbb{N}}$  converges  $\mathbb{P}$ -a.s. to a random variable  $\ell_{u^\sharp}^\infty$  almost surely bounded, and we have:

$$\sum_{k=0}^{+\infty} \varepsilon_k (J(\mathbf{U}_k) - J(u^\sharp)) < +\infty \quad \mathbb{P}\text{-a.s.} . \quad (26)$$

3. **Limits of sequences.** The sequence  $\{\ell_{u^\sharp}(\mathbf{U}_k)\}_{k \in \mathbb{N}}$  is  $\mathbb{P}$ -a.s. bounded, so by (18), we get that the sequence  $\{\mathbf{U}_k\}_{k \in \mathbb{N}}$  is also  $\mathbb{P}$ -a.s. bounded. Assumption (A12) then implies that the sequence  $\{\mathbf{G}_k\}_{k \in \mathbb{N}}$  is also  $\mathbb{P}$ -a.s. bounded. Finally, as the sequence  $\{\mathbf{R}_k\}_{k \in \mathbb{N}}$  is assumed to be  $\mathbb{P}$ -a.s. bounded (A16), we deduce from (22) that the sequence  $\{\|\mathbf{U}_{k+1} - \mathbf{U}_k\|/\varepsilon_k\}_{k \in \mathbb{N}}$  is also  $\mathbb{P}$ -a.s. bounded. This last property ensures that Assumption (c) of Proposition A.6, is satisfied. Assumption (b) of Proposition A.6 is exactly (26) and Assumption (a) is satisfied as we have (A15). On a bounded set containing the sequence  $\{\mathbf{U}_k\}_{k \in \mathbb{N}}$ , for instance the convex hull of this sequence, the function  $J$  is Lipschitz continuous by Corollary A.5. This ensures the continuity assumption required to apply Proposition A.6. We conclude that  $\{J(\mathbf{U}_k)\}_{k \in \mathbb{N}}$  converges almost surely to  $J(u^\sharp) = J^\sharp$ , the optimal value of Problem (1).

Let  $\Omega_0$  be the negligible subset of  $\Omega$  on which the sequence  $\{\ell_{u^\sharp}(\mathbf{U}_k)\}_{k \in \mathbb{N}}$  is unbounded and  $\Omega_1$  be the negligible subset of  $\Omega$  on which the relation (26) is not satisfied. We have  $\mathbb{P}(\Omega_0 \cup \Omega_1) = 0$ . Let  $\omega \notin \Omega_0 \cup \Omega_1$ . The sequence  $\{u_k\}_{k \in \mathbb{N}}$  associated to this element  $\omega$  is bounded and each  $u_k$  is in  $U^{\text{ad}}$ , a closed subset of  $\mathbb{U}$ . As  $\mathbb{U}$  is reflexive (A1), there exists a weakly converging subsequence  $\{u_{\xi(k)}\}_{k \in \mathbb{N}}$ . Note that  $\{\xi(k)\}_{k \in \mathbb{N}}$  depends on  $\omega$ . Let  $\bar{u}$  be the weak limit of the sequence  $\{u_{\xi(k)}\}_{k \in \mathbb{N}}$ . The function  $J$  is l.s.c. and convex, it is then weakly l.s.c. by [14, Corollary 2.2]. Thus we have:

$$J(\bar{u}) \leq \liminf_{k \rightarrow +\infty} J(u_{\xi(k)}) = J(u^\sharp) .$$

We conclude that  $\bar{u} \in U^\sharp$ . □

When the differential of  $K$  is weakly continuous, we can prove stronger convergence results for the sequence of iterates of the stochastic APP algorithm. These results already appear in [12] and remain valid for our more general version of the algorithm.

**Theorem 4.3** *Consider again (A1)-(A7), (A9)-(A16) and suppose that the differential of  $K$  is weakly continuous. Then, the sequence of iterates  $\{\mathbf{U}_k\}$  converges weakly  $\mathbb{P}$ -a.s. to a single element of  $U^\sharp$ . If moreover, the function  $J^C$  is strongly convex, then, the sequence of iterates  $\{\mathbf{U}_k\}$  converges strongly  $\mathbb{P}$ -a.s. to the unique solution  $u^\sharp$  of Problem (1).*

**Proof.** Consider the case where the differential of  $K$  is weakly continuous. Let  $\{u_k\}_{k \in \mathbb{N}}$  be a sequence generated by the algorithm. Suppose that there exist two subsequences  $\{u_{\xi(k)}\}_{k \in \mathbb{N}}$

and  $\{u_{\psi(k)}\}_{k \in \mathbb{N}}$  converging weakly respectively to two solutions  $\bar{u}_\xi$  and  $\bar{u}_\psi$  of the problem, with  $\bar{u}_\xi \neq \bar{u}_\psi$ . Then we have:

$$\begin{aligned} K(\bar{u}_\psi) - K(u_{\xi(k)}) - \langle \nabla K(u_{\xi(k)}), \bar{u}_\psi - u_{\xi(k)} \rangle &= K(\bar{u}_\psi) - K(\bar{u}_\xi) - \langle \nabla K(u_{\xi(k)}), \bar{u}_\psi - \bar{u}_\xi \rangle \\ &\quad + (K(\bar{u}_\xi) - K(u_{\xi(k)}) - \langle \nabla K(u_{\xi(k)}), \bar{u}_\xi - u_{\xi(k)} \rangle). \end{aligned} \quad (27)$$

By the point 2 of the proof of Theorem 4.2,

$$\begin{aligned} \lim_{k \rightarrow +\infty} K(\bar{u}_\psi) - K(u_{\xi(k)}) - \langle \nabla K(u_{\xi(k)}), \bar{u}_\psi - u_{\xi(k)} \rangle &= \lim_{k \rightarrow +\infty} \ell_{\bar{u}_\psi}(u_k) = \ell_{\bar{u}_\psi}, \\ \lim_{k \rightarrow +\infty} K(\bar{u}_\xi) - K(u_{\xi(k)}) - \langle \nabla K(u_{\xi(k)}), \bar{u}_\xi - u_{\xi(k)} \rangle &= \lim_{k \rightarrow +\infty} \ell_{\bar{u}_\xi}(u_k) = \ell_{\bar{u}_\xi}, \end{aligned}$$

therefore by weak continuity of the differential of  $K$  and strong convexity of  $K$ , we get:

$$\begin{aligned} \ell_{\bar{u}_\psi} - \ell_{\bar{u}_\xi} &= \lim_{k \rightarrow +\infty} K(\bar{u}_\psi) - K(\bar{u}_\xi) - \langle \nabla K(u_{\xi(k)}), \bar{u}_\psi - \bar{u}_\xi \rangle, \\ &= K(\bar{u}_\psi) - K(\bar{u}_\xi) - \langle \nabla K(\bar{u}_\xi), \bar{u}_\psi - \bar{u}_\xi \rangle, \\ &\geq \frac{b}{2} \|\bar{u}_\xi - \bar{u}_\psi\|^2. \end{aligned}$$

Inverting the roles of  $\bar{u}_\psi$  and  $\bar{u}_\xi$ , by a similar calculation as previously we get:

$$\ell_{\bar{u}_\xi} - \ell_{\bar{u}_\psi} \geq \frac{b}{2} \|\bar{u}_\xi - \bar{u}_\psi\|^2,$$

We then deduce that  $\bar{u}_\xi = \bar{u}_\psi$ , which contradicts the initial assumption. We conclude that all weakly converging subsequences of the sequence  $\{u_k\}$  converge to the same limit, hence we have the weak convergence of the whole sequence  $\{u_k\}$  to a single element of  $U^\sharp$ .

Now let us consider the case where  $J^C$  is strongly convex, with constant  $a$ . Then, Problem (1) admits a unique solution  $u^\sharp$  which is characterized by the following variational inequality:

$$\exists r^\sharp \in \partial J^C(u^\sharp), \quad \forall u \in U^{\text{ad}}, \quad \langle r^\sharp, u - u^\sharp \rangle + J^\Sigma(u) - J^\Sigma(u^\sharp) \geq 0.$$

The strong convexity assumption on  $J^C$  yields:

$$\begin{aligned} J(\mathbf{U}_k) - J(u^\sharp) &\geq \langle r^\sharp, \mathbf{U}_k - u^\sharp \rangle + \frac{a}{2} \|\mathbf{U}_k - u^\sharp\|^2 + J^\Sigma(\mathbf{U}_k) - J^\Sigma(u^\sharp) \\ &\geq \frac{a}{2} \|\mathbf{U}_k - u^\sharp\|^2. \end{aligned}$$

As  $\{J(\mathbf{U}_k)\}_{k \in \mathbb{N}}$  converges almost surely to  $J(u^\sharp)$ , we get that  $\|\mathbf{U}_k - u^\sharp\|$  converges to zero. Thus, we have the strong convergence of the sequence  $\{\mathbf{U}_k\}_{k \in \mathbb{N}}$  to the unique solution  $u^\sharp$  of the problem.

□

## 4.2 Efficiency estimates

In this section, we derive efficiency estimates for the convergence of the expectation of function values. In Theorem 4.5, we consider the expected function value taken for the averaged



iterates following the technique of Polyak-Ruppert [27, 31]. We take a step size  $\varepsilon_k$  of the order  $\mathcal{O}(k^{-\theta})$  with  $1/2 < \theta < 1$ , ensuring the convergence of the algorithm, and leading to a better convergence rate than with a small step size  $\varepsilon_k = \mathcal{O}(k^{-1})$ . The efficiency estimate is obtained using a similar technique as in [24] but without requiring the boundedness of  $U^{\text{ad}}$ . Moreover, we are able to take into account the bias on the gradient with the following assumption, inspired from [16]:

(A17) For  $k \in \mathbb{N}$ , let  $Q_k = \text{ess sup}_{\omega \in \Omega} \|\mathbf{R}_k(\omega)\|$  be the essential supremum of  $\|\mathbf{R}_k\|$  and assume that:

$$\sum_{k \in \mathbb{N}} Q_k \varepsilon_k < \infty .$$

We start by a lemma that proves the boundedness of the expectation of the Lyapunov function. This result will be used multiple times in this part.

**Lemma 4.4** *Under Assumptions (A10), (A12), (A14), (A15) and (A17), the sequence of expectations of the Lyapunov function  $\{\mathbb{E}(\ell_{u^\#}(\mathbf{U}_k))\}_{k \in \mathbb{N}}$  is bounded.*

**Proof.** We start from Lemma 4.1 with  $v = u^\#$  where we use  $\|r_k\| \leq Q_k$  and then take the full expectation. This yields:

$$\mathbb{E}(\ell_{u^\#}(\mathbf{U}_{k+1})) \leq (1 + \alpha_k)\mathbb{E}(\ell_{u^\#}(\mathbf{U}_k)) + \beta_k\mathbb{E}(\ell_{u^\#}(\mathbf{U}_{k+1})) + \gamma_k - \varepsilon_k\mathbb{E}\left(J(\mathbf{U}_k) - J(u^\#)\right) , \quad (28)$$

where

$$\alpha_k = \alpha\varepsilon_k^2 + \frac{2}{b}\varepsilon_k Q_k, \quad \beta_k = \beta\varepsilon_k^2, \quad \gamma_k = (\gamma + \delta Q_k^2)\varepsilon_k^2 + Q_k\varepsilon_k ,$$

From (A15) and (A17),  $\alpha_k$ ,  $\beta_k$  and  $\gamma_k$  are the terms of convergent series. Using a deterministic version of Corollary A.3, we get that the sequence  $\{\mathbb{E}(\ell_{u^\#}(\mathbf{U}_k))\}_{k \in \mathbb{N}}$  converges and is therefore bounded.  $\square$

**Theorem 4.5** *Suppose that Assumptions (A1)-(A7), (A9)-(A17) are satisfied. Let  $n \in \mathbb{N}$  and let  $\{\mathbf{U}_k\}_{k \in \mathbb{N}}$  be the sequence of iterates of the stochastic APP algorithm. Define the averaged iterate as:*

$$\tilde{\mathbf{U}}_i^n = \sum_{k=i}^n \eta_k^i \mathbf{U}_k \quad \text{with} \quad \eta_k^i = \frac{\varepsilon_k}{\sum_{l=i}^n \varepsilon_l} .$$

*Suppose that for all  $k \in \mathbb{N}$ ,  $\varepsilon_k = ck^{-\theta}$  with  $1/2 < \theta < 1$  and a constant  $c > 0$ . Then for any minimizer  $u^\#$  of  $J$ , we have:*

$$\mathbb{E}\left(J\left(\tilde{\mathbf{U}}_1^n\right) - J\left(u^\#\right)\right) = \mathcal{O}\left(n^{\theta-1}\right) .$$

*In particular, the rate of convergence can be arbitrarily close to the order  $n^{-1/2}$  if  $\theta$  is chosen to be arbitrarily close to  $1/2$ .*

**Proof.** From Lemma 4.4, we get that Inequality (28) is satisfied and the sequence  $\{\mathbb{E}(\ell_{u^\#}(\mathbf{U}_k))\}_{k \in \mathbb{N}}$  is bounded. Then, there exists a constant  $M \geq 0$  such that  $\mathbb{E}(\ell_{u^\#}(\mathbf{U}_k)) \leq M$  for all  $k \in \mathbb{N}$ . Summing (28) over  $i \leq k \leq n$  and using  $\mathbb{E}(\ell_{u^\#}(\mathbf{U}_k)) \leq M$ , we get:

$$\sum_{k=i}^n \varepsilon_k \mathbb{E} \left( J(\mathbf{U}_k) - J(u^\#) \right) \leq \sum_{k=i}^n (M(\alpha + \beta) + \gamma + \delta Q_k^2) \varepsilon_k^2 + \left( \frac{2}{b} M + 1 \right) Q_k \varepsilon_k .$$

In the sequel, let  $R = M(\alpha + \beta) + \gamma$  and  $S = \frac{2}{b} M + 1$ . By convexity of  $J$ , we get:

$$\mathbb{E} \left( J(\tilde{\mathbf{U}}_i^n) - J(u^\#) \right) \leq \frac{\sum_{k=i}^n (R + \delta Q_k^2) \varepsilon_k^2 + S Q_k \varepsilon_k}{\sum_{k=i}^n \varepsilon_k} .$$

We have  $\varepsilon_k = ck^{-\theta}$  with  $1/2 < \theta < 1$  and:

$$\sum_{k=1}^n k^{-\theta} \geq \frac{(n+1)^{1-\theta} - 1}{1-\theta} \geq \tilde{C}_\theta n^{1-\theta} ,$$

for some  $\tilde{C}_\theta > 0$ . Moreover, from (A15) and (A17),  $\varepsilon_k^2$ ,  $Q_k \varepsilon_k$  and  $Q_k^2 \varepsilon_k^2$  are the terms of convergent series. Thus, there exists a constant  $C_\theta > 0$  such that:

$$\mathbb{E} \left( J(\tilde{\mathbf{U}}_1^n) - J(u^\#) \right) \leq \frac{C_\theta}{n^{1-\theta}} ,$$

which gives the desired rate of convergence.  $\square$

Theorem 4.5 proves a convergence rate of order  $\mathcal{O}(n^{\theta-1})$  for the stochastic APP algorithm without assuming strong convexity of the objective. This rate appears for stochastic gradient descent in [2] where it is stated that the combination of large step sizes of order  $\mathcal{O}(n^{-\theta})$  with  $1/2 < \theta < 1$ , together with averaging lead to the best convergence behavior. A similar rate is also given for stochastic proximal gradient in [30].

In the following theorem, we show that this rate also holds when we consider the expected function value taken at the last iterate  $\mathbf{U}_n$  instead of the averaged iterate  $\tilde{\mathbf{U}}_1^n$ . Using the concept of modified Fejér monotone sequences, Lin and al. [22] have been able to give convergence rates of the expected function value of the last iterate for many algorithms, such as the projected subgradient method or the proximal gradient algorithm. The idea of modified Fejér sequence is adapted to the stochastic case in [30, Theorem 3.1]. We further adapt this concept for the stochastic APP algorithm which allows to derive a convergence rate for the expected function value of the last iterate.

**Theorem 4.6** *Suppose that Assumptions (A1)-(A7), (A9)-(A17) are satisfied. Let  $n \in \mathbb{N}$  and let  $\{\mathbf{U}_k\}_{k \in \mathbb{N}}$  be the sequence of the iterates of the stochastic APP algorithm. Suppose that for all  $k \in \mathbb{N}$ ,  $\varepsilon_k = ck^{-\theta}$  with  $1/2 < \theta < 1$  and a constant  $c > 0$ . Assume also that  $Q_k \leq qk^{-\nu}$  for  $\nu > 1 - \theta$  and a constant  $q > 0$ . Then, for any minimizer  $u^\#$  of  $J$  we have:*

$$\mathbb{E} \left( J(\mathbf{U}_n) - J(u^\#) \right) = \mathcal{O}(n^{\theta-1}) .$$

*In particular, the rate of convergence can be arbitrarily close to the order  $n^{-1/2}$  if  $\theta$  is chosen to be arbitrarily close to  $1/2$ .*

**Proof.** Let  $a_k = \mathbb{E} (J (\mathbf{U}_k) - J (u^\sharp))$ , from Lemma A.1 (see in the Appendix), we can write:

$$\varepsilon_n a_n = \frac{1}{n} \sum_{k=1}^n \varepsilon_k a_k + \sum_{i=1}^{n-1} \frac{1}{i(i+1)} \left( \sum_{k=n-i+1}^n \varepsilon_k a_k - i \varepsilon_{n-i} a_{n-i} \right).$$

We have:

$$\begin{aligned} \frac{1}{i(i+1)} \left( \sum_{k=n-i+1}^n \varepsilon_k a_k - i \varepsilon_{n-i} a_{n-i} \right) &= \frac{1}{i(i+1)} \sum_{k=n-i+1}^n \varepsilon_k \mathbb{E} (J (\mathbf{U}_k) - J (\mathbf{U}_{n-i})) \\ &\quad + \frac{1}{i+1} \left( \frac{1}{i} \sum_{k=n-i+1}^n \varepsilon_k - \varepsilon_{n-i} \right) a_{n-i}. \end{aligned} \quad (29)$$

By choice of  $\varepsilon_k$ , the sequence  $\{\varepsilon_k\}_{k \in \mathbb{N}}$  is decreasing so,

$$\frac{1}{i} \sum_{k=n-i+1}^n \varepsilon_k - \varepsilon_{n-i} \leq 0.$$

Moreover, by optimality of  $u^\sharp$ , we have  $a_{n-i} \geq 0$  so,

$$\varepsilon_n a_n \leq \frac{1}{n} \sum_{k=1}^n \varepsilon_k a_k + \sum_{i=1}^{n-1} \frac{1}{i(i+1)} \sum_{k=n-i+1}^n \varepsilon_k \mathbb{E} (J (\mathbf{U}_k) - J (\mathbf{U}_{n-i})).$$

Again by optimality of  $u^\sharp$ , we have  $\mathbb{E} (J (\mathbf{U}_k) - J (\mathbf{U}_{n-i})) \leq \mathbb{E} (J (\mathbf{U}_k) - J (u^\sharp)) = a_k$ . This yields:

$$\varepsilon_n a_n \leq \frac{1}{n} \sum_{k=1}^n \varepsilon_k a_k + \sum_{i=1}^{n-1} \frac{1}{i(i+1)} \sum_{k=n-i+1}^n \varepsilon_k a_k. \quad (30)$$

From Lemma 4.4, Inequality (28) is verified and there exists a constant  $M \geq 0$  such that  $\mathbb{E} (\ell_{u^\sharp} (\mathbf{U}_k)) \leq M$  for all  $k \in \mathbb{N}$ . Using this bound into (28) and summing over  $j \leq k \leq n$ , we get:

$$\sum_{k=j}^n \varepsilon_k a_k \leq \sum_{k=j}^n M (\alpha_k + \beta_k) + \gamma_k. \quad (31)$$

Taking  $j = 1$  or  $j = n - i + 1$  in (31) allows to bound both terms in (30). Define:

$$\bar{\alpha}_k = \alpha \varepsilon_k^2 + \frac{2}{b} \varepsilon_k q k^{-\nu}, \quad \bar{\gamma}_k = (\gamma + \delta q^2 k^{-2\nu}) \varepsilon_k^2 + \varepsilon_k q k^{-\nu},$$

As  $Q_k \leq q k^{-\nu}$ , we have  $\alpha_k \leq \bar{\alpha}_k$  and  $\gamma_k \leq \bar{\gamma}_k$ . We let  $\xi_k = M (\bar{\alpha}_k + \beta_k) + \bar{\gamma}_k$ , so that we have:

$$\sum_{k=1}^n \varepsilon_k a_k \leq \sum_{k=1}^n \xi_k \quad \text{and} \quad \sum_{i=1}^{n-1} \frac{1}{i(i+1)} \sum_{k=n-i+1}^n \varepsilon_k a_k \leq \sum_{i=1}^{n-1} \frac{1}{i(i+1)} \sum_{k=n-i+1}^n \xi_k. \quad (32)$$

Exchanging the order in the sum yields:

$$\sum_{i=1}^{n-1} \frac{1}{i(i+1)} \sum_{k=n-i+1}^n \xi_k = \sum_{k=2}^n \sum_{i=n-k+1}^{n-1} \left( \frac{1}{i} - \frac{1}{i+1} \right) \xi_k = \sum_{k=2}^n \frac{1}{n-k+1} \xi_k - \frac{1}{n} \sum_{k=2}^n \xi_k. \quad (33)$$

Plugging (32) and (33) into (30), we get:

$$\varepsilon_n a_n \leq \frac{1}{n} \sum_{k=1}^n \xi_k + \sum_{k=2}^n \frac{1}{n-k+1} \xi_k - \frac{1}{n} \sum_{k=2}^n \xi_k = \sum_{k=1}^n \frac{1}{n-k+1} \xi_k.$$

From the assumptions on  $\varepsilon_k$ ,  $\{\xi_k\}_{k \in \mathbb{N}}$  is non-increasing. Thus,

$$\begin{aligned} \sum_{k=1}^n \frac{1}{n-k+1} \xi_k &\leq \xi_{\lfloor \frac{n}{2} + 1 \rfloor} \sum_{n/2+1 \leq k \leq n} \frac{1}{n-k+1} + \frac{2}{n} \sum_{1 \leq k < n/2+1} \xi_k, \\ &\leq \xi_{\lfloor \frac{n}{2} + 1 \rfloor} \left( \log \left( \frac{n}{2} \right) + 1 \right) + \frac{2}{n} \sum_{k=1}^n \xi_k. \end{aligned}$$

Hence,

$$a_n \leq \frac{\xi_{\lfloor \frac{n}{2} + 1 \rfloor}}{\varepsilon_n} \left( \log \left( \frac{n}{2} \right) + 1 \right) + \frac{2}{n \varepsilon_n} \sum_{k=1}^n \xi_k.$$

Recall that,

$$\begin{aligned} \xi_k &= \bar{M}(\bar{\alpha}_k + \beta_k) + \bar{\gamma}_k \\ &= (\bar{M}(\alpha + \beta) + \gamma + \delta q^2 k^{-2\nu}) c^2 k^{-2\theta} + \left( \frac{2}{b} \bar{M} + 1 \right) c q k^{-(\nu+\theta)} \leq \xi k^{-\mu}, \end{aligned}$$

for  $\mu = \min \{2\theta, \nu + \theta\}$  and some constant  $\xi > 0$  so that,

$$a_n \leq 2^\mu \frac{\xi}{c} n^{\theta-\mu} \left( \log \left( \frac{n}{2} \right) + 1 \right) + 2 \frac{\xi}{c} n^{\theta-1} \sum_{k=1}^n k^{-\mu}.$$

As  $\theta > 1/2$  and  $\nu > 1 - \theta$ , we have  $\mu > 1$  so,

$$\sum_{k=1}^n k^{-\mu} \leq \frac{\mu}{\mu-1}.$$

Thus, noting that  $\theta - \mu < \theta - 1$ , we have:

$$a_n \leq 2^\mu \frac{\xi}{c} n^{\theta-\mu} \left( \log \left( \frac{n}{2} \right) + 1 \right) + \frac{2\mu\xi}{c(\mu-1)} n^{\theta-1} = \mathcal{O} \left( n^{\theta-1} \right).$$

This concludes the proof. □

**Remark 4.7** Inequality (28) (which holds in fact for any  $u \in U^{\text{ad}}$  in place of  $u^\sharp$ ) is the counterpart of modified Fejér monotonicity [22]. The main differences are that (28) involves a Bregman divergence instead of the Euclidean distance. Moreover, there are coefficients  $\alpha_k, \beta_k > 0$  that slightly degrade the inequality compared to what we obtain with Fejér monotone sequences where  $\alpha_k = \beta_k = 0$ . The summability of  $\alpha_k$  and  $\beta_k$  in addition with the boundedness of the expectation of the Bregman divergence  $\{\mathbb{E}(\ell_{u^\sharp}(\mathbf{U}_k))\}_{k \in \mathbb{N}}$  allow us to proceed in the same way as in [22, 30] to get the convergence rate of Theorem 4.6.

## 5 Conclusion

We have studied the stochastic APP algorithm in a Banach case. This framework generalizes many stochastic optimization algorithms. We have proved the measurability of the iterates of the algorithm, hence filling a theoretical gap to ensure that the quantities we manipulate when deriving efficiency estimates are well-defined. We have shown the convergence of the stochastic APP algorithm in the case where a bias on the gradient is considered. Finally, efficiency estimates are derived while taking the bias into account. Assuming a sufficiently fast decay of this bias, we get a convergence rate for the expectation of the function values that is similar to that of well-known stochastic optimization algorithms when no bias is present, such as stochastic gradient descent [2], stochastic mirror descent [24] or the stochastic proximal gradient algorithm [30]. Future work will consist in an application the stochastic APP algorithm to an optimization problem in a Banach space with decomposition aspects in mind.

## A Technical results used in the proofs

**Lemma A.1** *Let  $\{a_i\}_{i \in \mathbb{N}}$  be a sequence in  $\mathbb{R}$ . Let  $n \in \mathbb{N}$  and for  $i \in \{1, \dots, n-1\}$ , let  $s_i = \sum_{k=n-i}^n a_k$ . Then,*

$$a_n = \frac{s_{n-1}}{n} + \sum_{i=1}^{n-1} \frac{1}{i(i+1)} (s_{i-1} - ia_{n-i}) .$$

**Proof.** We have  $s_i = s_{i-1} + a_{n-i}$ , so:

$$\begin{aligned} \frac{1}{i} s_{i-1} - \frac{1}{i+1} s_i &= \frac{1}{i(i+1)} ((i+1)s_{i-1} - is_i) \\ &= \frac{1}{i(i+1)} (s_{i-1} - ia_{n-i}) . \end{aligned}$$

Summing over  $1 \leq i \leq n-1$ , we get:

$$a_n - \frac{s_{n-1}}{n} = \sum_{i=1}^{n-1} \frac{1}{i(i+1)} (s_{i-1} - ia_{n-i}) ,$$

giving the desired result. □

**Theorem A.2** *(Robbins-Siegmund) Consider four sequences of nonnegative random variables  $\{\Lambda_k\}_{k \in \mathbb{N}}$ ,  $\{\alpha_k\}_{k \in \mathbb{N}}$ ,  $\{\beta_k\}_{k \in \mathbb{N}}$  and  $\{\eta_k\}_{k \in \mathbb{N}}$ , that are all adapted to a given filtration  $\{\mathcal{F}_k\}_{k \in \mathbb{N}}$ . Moreover, suppose that:*

$$\mathbb{E}(\Lambda_{k+1} \mid \mathcal{F}_k) \leq (1 + \alpha_k)\Lambda_k + \beta_k - \eta_k, \quad \forall k \in \mathbb{N} ,$$

and that we have:

$$\sum_{k \in \mathbb{N}} \alpha_k < +\infty, \quad \sum_{k \in \mathbb{N}} \beta_k < +\infty, \quad \mathbb{P}\text{-a.s. .}$$

Then, the sequence of random variables  $\{\Lambda_k\}_{k \in \mathbb{N}}$  converges almost surely to  $\Lambda^\infty$ , an almost surely bounded random variable<sup>4</sup>, and we have in addition that:

$$\sum_{k \in \mathbb{N}} \eta_k < +\infty, \quad \mathbb{P}\text{-a.s. .}$$

An extension of Robbins-Siegmund theorem is given by the following corollary.

**Corollary A.3** Consider the following sequences of nonnegative random variables  $\{\Lambda_k\}_{k \in \mathbb{N}}$ ,  $\{\alpha_k\}_{k \in \mathbb{N}}$ ,  $\{\beta_k\}_{k \in \mathbb{N}}$ ,  $\{\gamma_k\}_{k \in \mathbb{N}}$ , and  $\{\eta_k\}_{k \in \mathbb{N}}$ , that are all adapted to a given filtration  $\{\mathcal{F}_k\}_{k \in \mathbb{N}}$ . Moreover suppose that:

$$\mathbb{E}(\Lambda_{k+1} \mid \mathcal{F}_k) \leq (1 + \alpha_k)\Lambda_k + \beta_k \mathbb{E}(\Lambda_{k+1} \mid \mathcal{F}_k) + \gamma_k - \eta_k,$$

and that we have:

$$\sum_{k \in \mathbb{N}} \alpha_k < +\infty, \quad \sum_{k \in \mathbb{N}} \beta_k < +\infty, \quad \sum_{k \in \mathbb{N}} \gamma_k < +\infty, \quad \mathbb{P}\text{-p.s. .}$$

Then, the sequence of random variables  $\{\Lambda_k\}_{k \in \mathbb{N}}$  converges almost surely to  $\Lambda^\infty$ , an almost surely bounded random variable and we have in addition that:

$$\sum_{k \in \mathbb{N}} \eta_k < +\infty, \quad \mathbb{P}\text{-p.s. .}$$

**Proof.** Consider a realization of the different sequences satisfying the assumptions of the corollary, and define three sequences  $\{\tilde{\alpha}_k\}_{k \in \mathbb{N}}$ ,  $\{\tilde{\gamma}_k\}_{k \in \mathbb{N}}$  and  $\{\tilde{\eta}_k\}_{k \in \mathbb{N}}$  such that:

$$1 + \tilde{\alpha}_k = \frac{1 + \alpha_k}{1 - \beta_k}, \quad \tilde{\gamma}_k = \frac{\gamma_k}{1 - \beta_k}, \quad \tilde{\eta}_k = \frac{\eta_k}{1 - \beta_k}.$$

As the sequence  $\{\beta_k\}$  converges to zero, we have that  $\beta_k \leq 1/2$  for  $k$  large enough. For such  $k$ , we get:

$$\frac{1}{1 - \beta_k} \leq 1 + 2\beta_k \quad \text{and} \quad 1 \leq \frac{1}{1 - \beta_k} \leq 2.$$

Then, we deduce that  $\tilde{\alpha}_k \leq 2(\alpha_k + \beta_k)$ ,  $\tilde{\gamma}_k \leq 2\gamma_k$  and  $\tilde{\eta}_k \geq \eta_k$ . The conclusions of the corollary are then obtained by applying Theorem A.2 directly.  $\square$

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<sup>4</sup>A random variable  $\mathbf{X}$  is bounded  $\mathbb{P}$ -a.s. if it is such that:  $\mathbb{P}(\{\omega \in \Omega \mid \mathbf{X}(\omega) = +\infty\}) = 0$ .

**Proposition A.4** Consider a function  $J : \mathbb{U} \rightarrow \mathbb{R}$  that is subdifferentiable on a non-empty, closed, convex subset  $U^{\text{ad}}$  of  $\mathbb{U}$ , with linearly bounded subgradient. Then, we have:

$$\forall (u, v) \in U^{\text{ad}} \times U^{\text{ad}}, \quad |J(u) - J(v)| \leq \left( c_1 \max \{ \|u\|, \|v\| \} + c_2 \right) \|u - v\|. \quad (34)$$

**Proof.** Let  $(u, v) \in U^{\text{ad}} \times U^{\text{ad}}$ . By the definition of subdifferentiability,

$$\begin{aligned} \forall r \in \partial J(u), \quad J(v) &\geq J(u) + \langle r, v - u \rangle, \\ \forall s \in \partial J(v), \quad J(u) &\geq J(v) + \langle s, u - v \rangle, \end{aligned}$$

from which we get:

$$\langle s, u - v \rangle \leq J(u) - J(v) \leq \langle r, u - v \rangle,$$

and therefore:

$$|J(u) - J(v)| \leq \max \{ \langle r, u - v \rangle, \langle s, v - u \rangle \}.$$

Using Schwarz inequality and the linearly bounded subgradient assumption we have:

$$\begin{aligned} |J(u) - J(v)| &\leq \max \{ \|r\| \|u - v\|, \|s\| \|v - u\| \}, \\ &\leq \left( c_1 \max \{ \|u\|, \|v\| \} + c_2 \right) \|u - v\|. \end{aligned}$$

giving the desired result. □

**Corollary A.5** A function  $J : \mathbb{U} \rightarrow \mathbb{R}$  that satisfies the assumptions of Proposition A.4 is Lipschitz continuous on every bounded subset that is contained in  $U^{\text{ad}}$ .

**Proposition A.6** Let  $J : \mathbb{U} \rightarrow \mathbb{R}$  be a Lipschitz continuous function with constant  $L > 0$ . Let  $\{u_k\}_{k \in \mathbb{N}}$  be a sequence of elements in  $\mathbb{U}$  and let  $\{\varepsilon_k\}_{k \in \mathbb{N}}$  a real positive sequence such that:

- (a)  $\sum_{k \in \mathbb{N}} \varepsilon_k = +\infty$ ,
- (b)  $\exists \mu \in \mathbb{R}, \sum_{k \in \mathbb{N}} \varepsilon_k |J(u_k) - \mu| < +\infty$ ,
- (c)  $\exists \delta > 0, \forall k \in \mathbb{N}, \|u_{k+1} - u_k\| \leq \delta \varepsilon_k$ .

Then, the sequence  $\{J(u_k)\}_{k \in \mathbb{N}}$  converges to  $\mu$ .

**Proof.** For  $\alpha > 0$ , define:

$$N_\alpha = \{k \in \mathbb{N}, |J(u_k) - \mu| \leq \alpha\}, \quad N_\alpha^c = \mathbb{N} \setminus N_\alpha.$$

(i) From Assumption (b), we have:

$$+\infty > \sum_{k \in \mathbb{N}} \varepsilon_k |J(u_k) - \mu| \geq \sum_{k \in N_\alpha^c} \varepsilon_k |J(u_k) - \mu| \geq \alpha \sum_{k \in N_\alpha^c} \varepsilon_k ,$$

from which we get that:

$$\forall \beta > 0 , \exists n_\beta \in \mathbb{N} \text{ such that } \sum_{k \geq n_\beta, k \in N_\alpha^c} \varepsilon_k \leq \beta .$$

(ii) From Assumption (a), we have:

$$+\infty = \sum_{k \in \mathbb{N}} \varepsilon_k = \sum_{k \in N_\alpha} \varepsilon_k + \sum_{k \in N_\alpha^c} \varepsilon_k ,$$

but we have just proved that the last sum in the above equality is finite, hence the first sum of the right hand side is infinite, which implies that  $N_\alpha$  is infinite.

Let  $\epsilon > 0$ , choose  $\alpha = \epsilon/2$  and  $\beta = \epsilon/(2L\delta)$  (where  $L$  is the Lipschitz constant of  $J$ ). Let  $n_\beta$  be the integer defined in (i). For  $k \geq n_\beta$ , there are two possible cases:

- $k \in N_\alpha$ : then, by definition of  $N_\alpha$ :

$$|J(u_k) - \mu| \leq \alpha < \epsilon ,$$

- $k \notin N_\alpha$ : let  $m$  be the smallest element of  $N_\alpha$  such that  $m \geq k$ , this element exists by (ii). Using the fact that  $J$  is Lipschitz continuous jointly with Assumption (c) and condition (i), it comes:

$$\begin{aligned} |J(u_k) - \mu| &\leq |J(u_k) - J(u_m)| + |J(u_m) - \mu| \\ &\leq L\|u_k - u_m\| + \alpha \\ &\leq L\delta \left( \sum_{l=k}^{m-1} \varepsilon_l \right) + \alpha \\ &\leq L\delta \left( \sum_{l \geq n_\beta, l \in N_\alpha^c} \varepsilon_l \right) + \alpha \\ &\leq \epsilon , \end{aligned}$$

so, we get  $|J(u_k) - \mu| \leq \epsilon$  for all  $k \geq n_\beta$ , giving the desired result.  $\square$

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