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Non-divisible point on a two-parameter family of elliptic curves

Valentin Petit

Abstract

Let n be a positive integer and t a non-zero integer. We consider the elliptic curve over \mathbb{Q} given by

$$E: y^2 = x^3 + tx^2 - n^2(t + 3n^2)x + n^6.$$

It is a special case of an elliptic surface studied recently by Bettin, David and Delaunay [2] and it generalizes Washington's family. The point $(0, n^3)$ belongs to $E(\mathbb{Q})$ and we obtain some results about its non-divisibility in $E(\mathbb{Q})$. Our work extends to this two-parameter family of elliptic curves a previous study of Duquesne (mainly stated for $n = 1$ and $t > 0$).

MSC 2020: 11G05, 11G50.

Keywords: Elliptic curves, Integral points, Heights.

1 Introduction

We are concerned with proving the non-divisibility of a point on a family of elliptic curves defined over \mathbb{Q} with two integer parameters. This family generalizes Washington's family [11] which is connected to simplest cubic fields. Let n be a positive integer. We consider the elliptic surface over $\mathbb{Q}(T)$ given by

$$\mathcal{E}: y^2 = x^3 + Tx^2 - n^2(T + 3n^2)x + n^6. \quad (1)$$

It is a special case of an elliptic surface studied recently by Bettin, David and Delaunay [2]. The case $n = 1$ is precisely Washington's family studied by Washington [11] and Duquesne [5]. In [2] the authors obtain a formula for the average root numbers of the elliptic curves obtained by specializing \mathcal{E} at $T = t \in \mathbb{Q}$. Another elliptic surface over $\mathbb{Q}(T)$ derived from their family is

$$\mathcal{F}: y^2 = x^3 + Tx^2 + n^2(T - 3n^2)x - n^6. \quad (2)$$

Both elliptic surfaces \mathcal{E} and \mathcal{F} are related. Indeed, for all positive integers n and all integers t , the curves $\mathcal{E}(t - 3n^2)$ and $\mathcal{F}(t)$ are isomorphic over \mathbb{Q} . Throughout the article, we denote by E the elliptic curve $\mathcal{E}(t)$ over \mathbb{Q} , where t is a non-zero integer as in [2].

Theorem 5.7 of Duquesne in [5] asserts that if $n = 1$, t is positive and $t^2 + 3t + 9$ is squarefree, then the point $(0, 1)$ is not divisible. In this article, we generalize this result to all $n > 0$ whenever $|t|$ is large enough compared to n (Theorem 21). In particular, we also take care of the case $t < 0$.

The strategy is similar to Duquesne's with an additional specific and careful treatment for the two parameters (we also notice that a small issue seems to occur in Section 5D of [5] during the computation of the local contribution of the height, which we can correct here). Suppose that there exist an integer $k \geq 2$ and a point $P \in E(\mathbb{Q})$ such that $kP = (0, n^3)$. The main idea is to minimize such an integer k . The strategy is to find a lower bound for the canonical height $\hat{h}(P)$ of P , an upper bound for $\hat{h}(0, n^3)$ (Section 4), and to obtain a contradiction. To do this, we will split the canonical height into local contributions (Section 4) and approximate the periods of E (Section 3).

2 Generalities about the family

Let n be a positive integer and t a non-zero integer. We consider the elliptic curve

$$E : y^2 = x^3 + tx^2 - n^2(t + 3n^2)x + n^6. \quad (3)$$

We set $\delta = t^2 + 3n^2t + 9n^4$. The discriminant and j -invariant of the elliptic curve E are given by

$$\begin{aligned} \Delta &= 16n^4\delta^2, \\ j &= \frac{256}{n^4}\delta. \end{aligned}$$

Let $f(x)$ be the polynomial

$$f(x) = x^3 + tx^2 - n^2(t + 3n^2)x + n^6.$$

Its discriminant is $n^4\delta^2$, which is positive, so f has three real roots denoted by $\alpha_1 < \alpha_2 < \alpha_3$. They satisfy $\alpha_1 < 0 < \alpha_2 < n^2 < \alpha_3$ and the following relations:

$$\alpha_2 = \frac{n^4}{n^2 - \alpha_1}, \quad \alpha_3 = n^2 - \frac{n^4}{\alpha_1}. \quad (4)$$

The polynomial f is irreducible over \mathbb{Q} if δ is squarefree. Indeed, it suffices to see that the polynomial $h(x) = 27f\left(\frac{x-t}{3}\right) = x^3 + 3\delta x + \delta(2t + 3n^2)$ is irreducible over \mathbb{Q} by Eisenstein's criterion.

We denote by $E_0(\mathbb{R})$ the connected component of the identity element and by $E(\mathbb{R}) - E_0(\mathbb{R})$ the bounded connected component of $E(\mathbb{R})$. Recall that $E_0(\mathbb{R})$ is a subgroup of $E(\mathbb{R})$ and that the sum of two points in $E(\mathbb{R}) - E_0(\mathbb{R})$ lies in $E_0(\mathbb{R})$. The integral point $(0, n^3)$ belongs to $E(\mathbb{Q})$ for all t and n . More precisely the fact that $\alpha_2 > 0$ implies that the point $(0, n^3)$ belongs to $E(\mathbb{Q}) - E_0(\mathbb{Q})$ for all $t \in \mathbb{Z}_{\neq 0}$ and $n \in \mathbb{N}$.

We note that $(0, n^3)$ is not a torsion point of E . Indeed, since $(0, n^3) \in E(\mathbb{Q}) - E_0(\mathbb{Q})$, the order of $(0, n^3)$ cannot be odd if it is finite. Moreover, since the polynomial $f(x)$ is irreducible over \mathbb{Q} , there is no torsion point of even order. Hence the point $(0, n^3)$ has infinite order.

Throughout the article we assume that δ is squarefree, which implies that n and t are coprime. This condition will play a key role later. Moreover we need (3) to be a minimal Weierstrass equation for E which, by Tate's algorithm [9, IV.9], occurs if we assume also that t is not congruent to 1 modulo 4 when $4 \mid n$. The minimal Weierstrass equation will be necessary to compute the non-archimedean local contributions.

For the case $4 \mid n$ and $t \equiv 1 \pmod{4}$, write $n = 4m$ and $t = 4k + 1$ with $m \in \mathbb{Z}_{>0}$ and $k \in \mathbb{Z}$. The elliptic curve E is then isomorphic over \mathbb{Q} to the curve

$$E' : y^2 + xy = x^3 + kx^2 - m^2(4k + 1 - 48m^2)x + 64m^6. \quad (5)$$

Note that (5) is a minimal model for E' . A change of variables maps the point $(0, n^3) \in E(\mathbb{Q})$ to the point $(0, 8m^3) \in E'(\mathbb{Q})$. Therefore, if we want to prove the non-divisibility of $(0, n^3)$ on $E(\mathbb{Q})$, it suffices to prove the non-divisibility of $(0, 8m^3)$ on $E'(\mathbb{Q})$.

3 Approximation of periods

The goal of this section is to approximate the real period ω_1 and the imaginary period ω_2 of E . In order to compute ω_1 and ω_2 , the Weierstrass equation defining E does not need to be minimal. The elliptic curve $E : y^2 = x^3 + tx^2 - n^2(t + 3n^2)x + n^6$ is isomorphic over \mathbb{Q} to the curve

$$y^2 = 4g(x),$$

where $g(x) = x^3 - \frac{1}{3}\delta x + \frac{1}{27}(2t + 3n^2)\delta$. The real roots e_1, e_2, e_3 of g are given by $e_i = \alpha_i + \frac{t}{3}$ for all $i \in \{1, 2, 3\}$ and the periods ω_1 and ω_2 of E are (see [3, 7.3.2])

$$\omega_1 = \int_{e_1}^{e_2} \frac{dx}{\sqrt{g(x)}} \in \mathbb{R}, \quad \omega_2 = - \int_{e_2}^{e_3} \frac{dx}{\sqrt{g(x)}} \in i\mathbb{R}.$$

3.1 The case $t > 0$

When $t > 0$, the real roots of g satisfy $e_1 < 0 < e_2 < e_3$ because $e_2 = \alpha_2 + \frac{t}{3}$ and $g\left(-\frac{2t}{3} - n^2\right) > 0$. A straightforward study of the function g gives for $t \geq 3n^2$,

$$\begin{aligned} -\frac{2}{3}t - n^2 - 2\frac{n^4}{t} &\leq e_1 \leq -\frac{2}{3}t - n^2 - \frac{n^4}{t}, \\ \frac{t}{3} &\leq e_2 \leq \frac{t}{3} + \frac{n^4}{t}, \\ \frac{t}{3} + n^2 &\leq e_3 \leq \frac{t}{3} + n^2 + \frac{n^4}{t}. \end{aligned} \tag{6}$$

Lemma 1. *Let n be a fixed positive integer. As $t \rightarrow +\infty$, we have $\frac{\omega_2}{i} \sim \frac{\pi}{\sqrt{t}}$.*

Moreover for $t \geq 100n^2$, we have

$$\frac{3.11}{\sqrt{t}} \leq \frac{\omega_2}{i} \leq \frac{3.15}{\sqrt{t}}.$$

Proof. We note that $\frac{\omega_2}{i} = \int_{e_2}^{e_3} \frac{dx}{\sqrt{(x-e_1)(x-e_2)(e_3-x)}}$. If $x \in [e_2, e_3]$, we have $t + n^2 + \frac{n^4}{t} \leq x - e_1 \leq t + 2n^2 + \frac{3n^4}{t}$ by (6). So

$$\frac{1}{\sqrt{t + 2n^2 + \frac{3n^4}{t}}} \leq \frac{1}{\sqrt{x - e_1}} \leq \frac{1}{\sqrt{t + n^2 + \frac{n^4}{t}}},$$

and then

$$\frac{\int_{e_2}^{e_3} \frac{dx}{\sqrt{(x-e_2)(e_3-x)}}}{\sqrt{t + 2n^2 + \frac{3n^4}{t}}} \leq \frac{\omega_2}{i} \leq \frac{\int_{e_2}^{e_3} \frac{dx}{\sqrt{(x-e_2)(e_3-x)}}}{\sqrt{t + n^2 + \frac{n^4}{t}}}, \tag{7}$$

where $\int_{e_2}^{e_3} \frac{dx}{\sqrt{(x-e_2)(e_3-x)}} = \pi$. Moreover, as $t \rightarrow +\infty$ the left-hand side and the right-hand side of (7) are both equivalent to $\frac{\pi}{\sqrt{t}}$. Finally, when $t \geq 100n^2$, we derive from (7)

$$\frac{3.11}{\sqrt{t}} \leq \frac{\omega_2}{i} \leq \frac{3.15}{\sqrt{t}}.$$

□

Lemma 2. For $t \geq 100n^2$, we have

$$\frac{1.88 + 0.99 \log\left(\frac{t}{n^2}\right)}{\sqrt{t}} \leq \omega_1 \leq \frac{5.35 + 1.23 \log\left(\frac{t}{n^2}\right)}{\sqrt{t}}.$$

Proof. We split the integral into two parts

$$\omega_1^- = \int_{e_1}^0 \frac{dx}{\sqrt{(x-e_1)(e_2-x)(e_3-x)}}, \omega_1^+ = \int_0^{e_2} \frac{dx}{\sqrt{(x-e_1)(e_2-x)(e_3-x)}}.$$

First we consider ω_1^- . If $x \in [e_1, 0]$, we have $\frac{t}{3} \leq e_2 - x \leq t + n^2 + \frac{3n^4}{t}$ and $\frac{t}{3} + n^2 \leq e_3 - x \leq t + 2n^2 + \frac{3n^4}{t}$ by (6). So we get the lower bound

$$\begin{aligned} \omega_1^- \sqrt{\left(t + n^2 + \frac{3n^4}{t}\right) \left(t + 2n^2 + \frac{3n^4}{t}\right)} &\geq \int_{e_1}^0 \frac{dx}{\sqrt{x-e_1}} \\ &\geq 2\sqrt{-e_1} \\ &\geq 2\sqrt{\frac{2}{3}t + n^2 + \frac{n^4}{t}}. \end{aligned}$$

Furthermore using (6), we get the upper bound

$$\omega_1^- \leq \frac{\int_{e_1}^0 \frac{dx}{\sqrt{x-e_1}}}{\sqrt{\frac{t}{3} \left(\frac{t}{3} + n^2\right)}} \leq \frac{2\sqrt{-e_1}}{\sqrt{\frac{t}{3} \left(\frac{t}{3} + n^2\right)}} \leq \frac{2}{\sqrt{t}} \frac{\sqrt{\frac{2}{3} + \frac{n^2}{t} + \frac{2n^4}{t^2}}}{\sqrt{\frac{1}{9}}}.$$

Thus when $t \geq 100n^2$, we obtain

$$\frac{1.60}{\sqrt{t}} \leq \omega_1^- \leq \frac{4.94}{\sqrt{t}}.$$

Now we consider ω_1^+ . If $x \in [0, e_2]$, we have $\frac{2}{3}t + n^2 + \frac{n^4}{t} \leq x - e_1 \leq t + n^2 + \frac{n^4}{t}$ by (6). So we have

$$\frac{J}{\sqrt{t + n^2 + \frac{3n^4}{t}}} \leq \omega_1^+ \leq \frac{J}{\sqrt{\frac{2}{3}t + n^2 + \frac{n^4}{t}}},$$

with

$$J = \int_0^{e_2} \frac{dx}{\sqrt{(e_2-x)(e_3-x)}} = \log\left(\frac{\sqrt{e_3} + \sqrt{e_2}}{\sqrt{e_3} - \sqrt{e_2}}\right).$$

Moreover by (6), we have

$$\frac{\frac{4}{3}t + n^2}{n^2 + \frac{n^4}{t}} \leq \frac{\sqrt{e_3} + \sqrt{e_2}}{\sqrt{e_3} - \sqrt{e_2}} \leq \frac{\frac{2}{3}t + n^2 + \frac{2n^4}{t} + 2\sqrt{(\frac{t}{3} + n^2 + \frac{n^4}{t})(\frac{t}{3} + \frac{n^4}{t})}}{n^2 - \frac{n^4}{t}},$$

which implies

$$\frac{\frac{4}{3}t + n^2}{n^2 + \frac{n^4}{t}} \leq \frac{\sqrt{e_3} + \sqrt{e_2}}{\sqrt{e_3} - \sqrt{e_2}} \leq \frac{\frac{4}{3}t + 4n^2 + \frac{4n^4}{t}}{n^2 - \frac{n^4}{t}}.$$

So we get

$$\frac{1}{\sqrt{t + n^2 + \frac{3n^4}{t}}} \log \left(\frac{\frac{4}{3}t + n^2}{n^2 + \frac{n^4}{t}} \right) \leq \omega_1^+ \leq \frac{1}{\sqrt{\frac{2}{3}t + n^2 + \frac{n^4}{t}}} \log \left(\frac{\frac{4}{3}t + 4n^2 + \frac{4n^4}{t}}{n^2 - \frac{n^4}{t}} \right).$$

Assume now that $t \geq 100n^2$. We derive

$$\begin{aligned} \omega_1^+ &\leq \frac{1}{\sqrt{\frac{2}{3}t + n^2 + \frac{n^4}{t}}} \left(\log(t) + \log\left(\frac{4}{3}\right) + \log\left(1 + \frac{3n^2}{t} + \frac{3n^4}{t^2}\right) - \log\left(n^2 - \frac{n^4}{t}\right) \right) \\ &\leq \frac{0.41 + 1.23 \log\left(\frac{t}{n^2}\right)}{\sqrt{t}} \end{aligned}$$

and

$$\omega_1^+ \geq \frac{1}{\sqrt{t + n^2 + \frac{3n^4}{t}}} \log \left(\frac{4t}{3(n^2 + \frac{n^4}{t})} \right) \geq \frac{0.28 + 0.99 \log\left(\frac{t}{n^2}\right)}{\sqrt{t}}.$$

Finally we obtain

$$\frac{1.88 + 0.99 \log\left(\frac{t}{n^2}\right)}{\sqrt{t}} \leq \omega_1 \leq \frac{5.35 + 1.23 \log\left(\frac{t}{n^2}\right)}{\sqrt{t}}.$$

□

Remark. A numerical analysis suggests that ω_1 should be equivalent to $\frac{\log\left(\frac{t}{n^2}\right)}{\sqrt{t}}$ as $t \rightarrow +\infty$.

3.2 The case $t < 0$

Once again we use the fact that E is isomorphic to the curve $y^2 = 4g(x)$. A straightforward study of the function g gives the following estimates when $t \leq -3n^2$:

$$\begin{aligned} \frac{t}{3} + \frac{2n^4}{t} &\leq e_1 \leq \frac{t}{3} + \frac{n^4}{t}, \\ \frac{t}{3} + n^2 + \frac{2n^4}{t} &\leq e_2 \leq \frac{t}{3} + n^2 + \frac{n^4}{t}, \\ -\frac{2t}{3} - n^2 &\leq e_3 \leq -\frac{2t}{3}. \end{aligned} \quad (8)$$

Lemma 3. *Let n be a fixed positive integer. As $t \rightarrow -\infty$, we have $\omega_1 \sim \frac{\pi}{\sqrt{|t|}}$.*

Moreover, for $t \leq -100n^2$ we have

$$\frac{3.14}{\sqrt{|t|}} \leq \omega_1 \leq \frac{3.15}{\sqrt{|t|}}.$$

Proof. If $x \in [e_1, e_2]$, then $|t| - 2n^2 + \frac{n^4}{|t|} \leq e_3 - x \leq |t| + \frac{2n^4}{|t|}$ by (8). So

$$\frac{1}{\sqrt{|t| + \frac{2n^4}{|t|}}} \leq \frac{1}{\sqrt{e_3 - x}} \leq \frac{1}{\sqrt{|t| - 2n^2 + \frac{n^4}{|t|}}}.$$

We obtain

$$\frac{J}{\sqrt{|t| + \frac{2n^4}{|t|}}} \leq \omega_1 \leq \frac{J}{\sqrt{|t| - 2n^2 + \frac{n^4}{|t|}}},$$

with

$$J = \int_{e_1}^{e_2} \frac{dx}{\sqrt{(x - e_1)(e_2 - x)}} = \pi.$$

Since $\frac{\pi}{\sqrt{|t| + \frac{2n^4}{|t|}}}$ and $\frac{\pi}{\sqrt{|t| - 2n^2 + \frac{n^4}{|t|}}}$ are both equivalent to $\frac{\pi}{\sqrt{|t|}}$ as $t \rightarrow -\infty$, we deduce $\omega_1 \underset{t \rightarrow -\infty}{\sim} \frac{\pi}{\sqrt{|t|}}$. Furthermore if $t \leq -100n^2$, we get

$$\frac{3.14}{\sqrt{|t|}} \leq \omega_1 \leq \frac{3.15}{\sqrt{|t|}}.$$

□

Lemma 4. *If $t \leq -100n^2$, we have*

$$\frac{\omega_2}{i} \geq \frac{0.39 + \log\left(\frac{|t|}{n^2}\right)}{\sqrt{|t|}}.$$

Proof. To approximate ω_2 , we need to split the integral into two parts:

$$\frac{\omega_2}{i} = \underbrace{\int_{e_2}^0 \frac{dx}{\sqrt{(x-e_1)(x-e_2)(e_3-x)}}}_{W^-} + \underbrace{\int_0^{e_3} \frac{dx}{\sqrt{(x-e_1)(x-e_2)(e_3-x)}}}_{W^+}.$$

We begin with estimating W^- . If $x \in [e_2, 0]$, we get by (8)

$$-\frac{2}{3}t - n^2 \leq e_3 - x \leq -t - n^2 - \frac{2n^4}{t}.$$

So

$$\frac{L}{\sqrt{-t - n^2 - \frac{2n^4}{t}}} \leq W^- \leq \frac{L}{\sqrt{-\frac{2}{3}t - n^2}},$$

with

$$L = -\int_0^{e_2} \frac{dx}{\sqrt{(x-e_1)(x-e_2)}} = -\log(e_2 - e_1) + \log(-e_1 - e_2 + \sqrt{e_1 e_2}).$$

Moreover by (8), we have

$$-\log\left(n^2 - \frac{n^4}{t}\right) + \log\left(-\frac{2}{3}t - n^2 - \frac{2n^4}{t}\right) \leq L.$$

For $t \leq -100n^2$, we get

$$\begin{aligned} W^- \sqrt{-t - n^2 - \frac{2n^4}{t}} &\geq -\log\left(n^2 - \frac{n^4}{t}\right) + \log\left(-\frac{2}{3}t - n^2 - \frac{2n^4}{t}\right) \\ &\geq -2\log(n) - \log\left(1 - \frac{1}{100}\right) + \log(|t|) + \log\left(\frac{2}{3} - \frac{1}{100}\right), \end{aligned}$$

from which we derive

$$W^- \geq \frac{\log\left(\frac{|t|}{n^2}\right) - 0.42}{\sqrt{|t|}}. \quad (9)$$

If $x \in [0, e_3]$, we obtain by (8)

$$\begin{aligned} -\frac{t}{3} - \frac{n^4}{t} &\leq x - e_1 \leq -t - \frac{2n^4}{t}, \\ -\frac{t}{3} - n^2 - \frac{n^4}{t} &\leq x - e_2 \leq -t - n^2 - \frac{n^4}{t}. \end{aligned}$$

Thus

$$W^+ \sqrt{\left(-t - \frac{2n^4}{t}\right) \left(-t - n^2 - \frac{n^4}{t}\right)} \geq \sqrt{e_3},$$

from which we derive

$$W^+ \geq \frac{\sqrt{-\frac{2}{3}t - n^2}}{\sqrt{\left(-t - \frac{2n^4}{t}\right) \left(-t - n^2 - \frac{n^4}{t}\right)}} \geq \frac{\sqrt{|t|} \sqrt{\frac{2}{3} - \frac{n^2}{|t|}}}{|t| \sqrt{\left(1 + \frac{2n^4}{|t|^2}\right)^2}}.$$

So we obtain, for $t \leq -100n^2$

$$W^+ \geq \frac{0.81}{\sqrt{|t|}}. \tag{10}$$

Finally by adding both inequalities (9) and (10), we have

$$\frac{\omega_2}{i} \geq \frac{0.39 + \log\left(\frac{|t|}{n^2}\right)}{\sqrt{|t|}}.$$

□

Remark. When $t < 0$, we just give a lower bound for $\frac{\omega_2}{i}$ since no upper bound is required in order estimate the height.

4 Estimates on the heights

We need to estimate the heights of some points in $E(\mathbb{Q})$. For this purpose, we will decompose the heights as a sum of local contributions. Depending on the conventions, there are different ways to split the canonical height into a sum of local heights. However in this section, a special care was taken during the computations of local heights to make sure that their sum agrees with the definition of the canonical height.

4.1 Lower bound for the height

We want to prove that there does not exist a point $P = (\alpha, \beta) \in E(\mathbb{Q})$ and an integer $\ell \geq 2$ such that $\ell P = (0, n^3)$. The goal of this part is to find, if such a point exists, a lower bound for the canonical height of P . Since $(0, n^3) \in E(\mathbb{Q}) - E_0(\mathbb{Q})$, the point P has to belong to $E(\mathbb{Q}) - E_0(\mathbb{Q})$ and ℓ must be odd (see Section 2).

Lemma 5. Let $F : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$ be an elliptic curve over \mathbb{Q} with $a_i \in \mathbb{Z}$ for all $i \in \{1, 2, 3, 4, 6\}$. Let $P \in F(\mathbb{Q})$ be a point such that nP is an integral point for some integer $n \geq 1$. Then P is an integral point on F .

Proof. The proof of [1, 4.4, p. 18] for models of the form $y^2 = x^3 + Ax + B$ using division polynomials can be generalized for longer models of the form $y^2 = x^3 + a_2x^2 + a_4x + a_6$ without any difficulty. \square

Lemma 5 implies that if $(0, n^3)$ is the multiple of a rational point P then $P \in E(\mathbb{Z})$. Moreover the point $(0, n^3)$ is singular modulo p for all primes $p \mid n$. This implies that P is singular modulo p for all $p \mid n$ since the multiple of a non-singular point is always non-singular. If $p \mid n$, the only singular point is $(0, 0)$ so we have $p \mid \alpha$. This remark will be used in Lemmas 6, 7 and Proposition 10.

Lemma 6. Let $P = (\alpha, \beta) \in E(\mathbb{Z}) - E_0(\mathbb{Z})$ be such that there exists $\ell \geq 2$ with $\ell P = (0, n^3)$. If $t \geq n^4$ then $|\beta| \geq n\sqrt{2t}$.

Proof. Assume that $t \geq n^4$ and $n \geq 2$. Recall that $\alpha_1, \alpha_2, \alpha_3$ are the real roots of the polynomial $f(x) = x^3 + tx^2 - n^2(t + 3n^2)x + n^6$ with $\alpha_1 < 0 < \alpha_2 < \alpha_3$. By studying the function f , we have for $t \geq n^4$

$$-n^2 - t - 1 < \alpha_1 < -n^2 - t \quad \text{and} \quad 0 < \alpha_2 < 1.$$

Since $P \in E(\mathbb{Q}) - E_0(\mathbb{Q})$, we have $\alpha_1 < \alpha < \alpha_2$, and then $\alpha \in [-n^2 - t, -1]$. Moreover since $p \mid \alpha$ for all prime $p \mid n$, we get $\alpha \leq -2$ and then

$$\begin{aligned} f(\alpha) &\geq \min(f(-n^2 - t), f(-2)) \\ &\geq \min(2n^4t + 3n^6, (2n^2 + 4)t + n^6 + 6n^4 - 8) \\ &\geq 2n^2t. \end{aligned}$$

Thus $|\beta| \geq n\sqrt{2t}$. If $n = 1$, we have

$$f(\alpha) \geq \min(f(-t - 1), f(-1)) = 2t + 3,$$

hence $|\beta| \geq \sqrt{2t + 3} \geq \sqrt{2t}$. \square

Lemma 7. Suppose that $t \leq -2n^4$. Then

$$|\beta| \geq \begin{cases} \sqrt{2|t|} & \text{if } n = 2, \\ n\sqrt{|t|} & \text{if } n \geq 3. \end{cases}$$

Proof. We use an argument similar to the case $t \geq n^4$ (Lemma 6). \square

Remark. Assume $n = 1$. If $t \leq -2$, the point $(0, 1)$ is the only integral point in $E(\mathbb{Q}) - E_0(\mathbb{Q})$ because $-1 < \alpha_1 < \alpha_2$. Hence $(0, 1)$ is not divisible. If $t = -1$, it is easy to prove that this point is not divisible as well. Therefore when t is negative, we can assume that $n \geq 2$.

In order to find a lower bound for the canonical height of an integral point of E , we use the decomposition of the canonical height into local contributions (see [7, Theorem 5.2], [4, 7.5.7]). For $P \in E(\mathbb{Q})$ we have

$$\widehat{h}(P) = \sum_{p \leq \infty} \lambda_p(P),$$

where the sum runs through the places of \mathbb{Q} . We recall the definitions of λ_p for finite places p in Definition/Proposition 8 and λ_∞ in Definition/Proposition 11. For this, we set

$$\begin{aligned} A &= 3\alpha^2 + 2t\alpha - n^2(t + 3n^2), \\ B &= 2\beta, \\ C &= 3\alpha^4 + 4t\alpha^3 - 6n^2(t + 3n^2)\alpha^2 + 12n^6\alpha - n^4(t^2 + 2n^2t + 9n^4), \\ D &= \gcd(A, B), \\ c_4 &= 16\delta. \end{aligned}$$

Definition/Proposition 8. ([7, Theorem 5.2], [4, 7.5.6]) *Let p be a prime number. The local non-archimedean contribution $\lambda_p(P)$ is non-zero only if $p \mid D$. If $p \mid D$, we set $m_p = \min\left(\frac{v_p(\Delta)}{2}, v_p(B)\right)$ and the local contribution at p is then given by*

$$\lambda_p(P) = \begin{cases} -\frac{m_p(v_p(\Delta) - 2m_p)}{2v_p(\Delta)} \log(p) & \text{if } p \nmid c_4, \\ -\frac{v_p(B)}{3} \log(p) & \text{if } p \mid c_4 \text{ and } v_p(C) \geq 3v_p(B), \\ -\frac{v_p(C)}{8} \log(p) & \text{otherwise,} \end{cases}$$

where v_p is the p -adic valuation.

Lemma 9. *Assume $4 \nmid n$ or $t \not\equiv 1[4]$. Let $P \in E(\mathbb{Q})$ be an integral point. If $p \nmid 2n$, then $\lambda_p(P) = 0$.*

Proof. Let $p \mid D$. We assume that $p \nmid n$ and $p \neq 2$. First we have $4A^2 = B^2(9\alpha + 3t) + 4\delta(\alpha^2 - n^2\alpha + n^4)$. Since δ is squarefree, we obtain that $p \mid (\alpha^2 - n^2\alpha + n^4)$. We also have

$$B^2 = 4(\alpha^2 - n^2\alpha + n^4) - 4n^2(3\alpha + t)$$

so $p \mid (3\alpha + t)$. Furthermore p divides the resultant of A and B^2 , viewed as polynomials in $\mathbb{Z}[\alpha]$, which is equal to $\Delta = 16n^4\delta^2$. Therefore $p \mid \delta$. Moreover,

$$27B^2 = 4(3\alpha + t)^3 - 4\delta(9\alpha + t - 3n^2),$$

which implies that $p \mid (9\alpha + t - 3n^2)$. Since $(9\alpha + t - 3n^2) = 3(3\alpha + t) - (2t + 3n^2)$, we obtain that $p \mid (2t + 3n^2)$. But $4\delta = (2t + 3n^2)^2 + 27n^4$. We get $p = 3$ hence δ is not squarefree, which is a contradiction. \square

Proposition 10. *Let $P = (\alpha, \beta)$ be an integral point on E such that $\ell P = (0, n^3)$ for some integer $\ell \geq 1$. Then the following inequality holds*

$$\sum_{p < \infty} \lambda_p(P) \geq -\frac{1}{2} \log(n) - \frac{1}{3} \log(2).$$

Proof. By Lemma 9, it suffices to compute the local contributions at primes $p \mid 2n$. For $p \mid n$, we first assume $p \neq 2$. Let $m = v_p(n)$. Since δ is squarefree, we have $p \nmid c_4$ and $v_p(\Delta) = 4m$. Put $m_p = \min(2m, v_p(B))$. By Definition/Proposition 8, the local contribution of P is then given by

$$\lambda_p(P) = -\frac{m_p(4m - m_p)}{8m} \log(p).$$

We easily conclude that

$$\lambda_p(P) \geq -\frac{m}{2} \log(p) \geq -\frac{1}{2} \log(p^m).$$

Now we want to compute the 2-local contribution $\lambda_2(P)$. Since $2 \mid c_4$, it is given by

$$\lambda_2(P) = \begin{cases} -\frac{v_2(B)}{3} \log(2) & \text{if } v_2(C) \geq 3v_2(B), \\ -\frac{v_2(C)}{8} \log(2) & \text{otherwise.} \end{cases}$$

If $2 \nmid n$ then $v_2(B) = 1$, so $\lambda_2(P) \geq -\frac{1}{3} \log(2)$.

Suppose now that $2 \mid n$ and $m = v_2(n)$.

If $m \in \{1, 2\}$ it is easy to check that $\lambda_2(P) \geq -\frac{1}{2} \log(2^m) - \frac{1}{3} \log(2)$. So we may assume that $m \geq 3$.

If $v_2(\alpha) < m$ then $v_2(B) = 1 + v_2(\alpha)$; in that case we have $\lambda_2(P) \geq \frac{-m}{3} \log(2)$ if $v_2(C) \geq 3v_2(B)$, else $\lambda_2(P) \geq -\frac{m}{2} \log(2)$.

If $m \leq v_2(\alpha) < \frac{4m-2}{3}$ then $v_2(B) = 1 + v_2(\alpha)$ and $v_2(C) = 3v_2(\alpha) + 2$; in that case we have $\lambda_2(P) = -\frac{3v_2(\alpha) + 2}{8} \log(2) \geq -\frac{1}{2} \log(2^m)$.

If $v_2(\alpha) = \frac{4m-2}{3}$ then $v_2(C) > 4m$ and $v_2(B) = \frac{4m+1}{3}$; in that case we have $\lambda_2(P) = -\frac{4m+1}{9} \log(2) \geq -\frac{m}{2} \log(2)$.

If $v_2(\alpha) > \frac{4m-2}{3}$ then $v_2(C) = 4m$ and $3v_2(B) > 4m$; in that case we have $\lambda_2(P) \geq -\frac{1}{2} \log(2^m)$.

Finally we conclude by summing all local non-archimedean contributions. \square

Now we study the local archimedean contribution.

Definition/Proposition 11. ([4, 7.5.7]) Let $P = (\alpha, \beta) \in E(\mathbb{Q})$. Let z be the elliptic logarithm of P . Let $\mu = \frac{2\pi}{\omega_1}$, $s = \mu \operatorname{Re}(z)$, $q = \exp\left(\frac{2i\pi\omega_2}{\omega_1}\right)$, and

$$\theta = \sum_{k=0}^{\infty} \sin((2k+1)s)(-1)^k q^{\frac{k(k+1)}{2}}.$$

Then the local archimedean contribution is given by

$$\lambda_{\infty}(P) = \frac{1}{32} \log \left| \frac{\Delta}{q} \right| - \frac{1}{4} \log |\theta| + \frac{1}{8} \log \left| \frac{\alpha^3 + \frac{b_2}{4}\alpha^2 + \frac{b_4}{2}\alpha + \frac{b_6}{4}}{\mu} \right|$$

where $b_2 = a_1^2 + 4a_2$, $b_4 = a_1a_3 + 2a_4$, $b_6 = a_1a_2a_3 + 4a_6$ and a_1, a_2, a_3, a_4 and a_6 are defined as in Lemma 5.

Proposition 12. Let P be an integral point on E such that $\ell P = (0, n^3)$ for some integer $\ell \geq 2$. If $t \geq \max(100n^2, n^4)$, we have

$$\lambda_{\infty}(P) \geq \frac{13}{80} \log(t) + \frac{3}{8} \log(n) + 0.30.$$

If $t \leq \min(-100n^2, -2n^4)$, we have

$$\lambda_{\infty}(P) \geq \frac{3}{16} \log(|t|) + \frac{3}{8} \log(n) + 0.27.$$

Proof. We first note that

$$|\theta| \leq \sum_{k=0}^{\infty} q^{\frac{k(k+1)}{2}} \leq \frac{1}{1-q}. \quad (11)$$

To find a lower bound for $\lambda_\infty(P)$, we need to find an upper bound for q . If $t \geq 100n^2$, by Lemmas 1 and 2, we have

$$\frac{2i\pi\omega_2}{\omega_1} \leq \frac{-3.11 \times 2\pi}{5.35 + 1.23 \log\left(\frac{t}{n^2}\right)} \leq \frac{-15.88}{\log\left(\frac{t}{n^2}\right)}$$

thus

$$q \leq \exp\left(\frac{-15.88}{\log\left(\frac{t}{n^2}\right)}\right) \leq 1 - \frac{4.3}{\log\left(\frac{t}{n^2}\right)}. \quad (12)$$

On the one hand, by definition of Δ we have

$$\frac{1}{32} \log(\Delta) = \frac{1}{32} \log(16n^4(t^2 + 3n^2t + 9n^4)^2) \geq \frac{1}{8} \log(t) + \frac{1}{8} \log(n) + \frac{1}{8} \log(2).$$

On the other hand, with (12) and (11) we get

$$\frac{1}{32} \log\left|\frac{1}{q}\right| \geq \frac{1}{32} \log\left(\exp\left(\frac{15.88}{\log\left(\frac{t}{n^2}\right)}\right)\right) \geq \frac{15.88}{32 \log\left(\frac{t}{n^2}\right)},$$

$$-\frac{1}{4} \log|\theta| \geq -\frac{1}{4} \log\left|\frac{1}{1-q}\right| \geq \frac{1}{4} \log(4.3) - \frac{1}{4} \log \log\left(\frac{t}{n^2}\right),$$

and by Lemma 2,

$$\begin{aligned} -\frac{1}{8} \log(\mu) = \frac{1}{8} \log\left(\frac{\omega_1}{2\pi}\right) &\geq -\frac{1}{8} \log(2\pi) + \frac{1}{8} \log\left(\frac{1.88 + 0.99 \log\left(\frac{t}{n^2}\right)}{\sqrt{t}}\right), \\ &\geq -\frac{1}{16} \log(t) - \frac{1}{8} \log(2\pi) + \frac{1}{8} \log \log\left(\frac{t}{n^2}\right). \end{aligned}$$

Moreover, we have

$$\frac{1}{8} \log(2) + \frac{1}{4} \log(4.3) - \frac{1}{8} \log(2\pi) + \frac{1}{8} \log(0.99) \geq 0.22.$$

Finally using Definition/Proposition 11 and the fact that $\alpha^3 + \frac{b_2}{4}\alpha^2 + \frac{b_4}{2}\alpha + \frac{b_6}{4} = \beta^2$, we obtain the lower bound for $\lambda_\infty(P)$:

$$\begin{aligned} \lambda_\infty(P) &\geq \frac{1}{16} \log(t) - \frac{1}{8} \log \log\left(\frac{t}{n^2}\right) + \frac{15.88}{32 \log\left(\frac{t}{n^2}\right)} \\ &\quad + \frac{1}{4} \log|\beta| + \frac{1}{8} \log(n) + 0.22. \end{aligned}$$

Moreover, by Lemma 6, for $t \geq n^4$ we have $|\beta| \geq n\sqrt{2t}$. If $t \geq \max(n^4, 100n^2)$, we obtain

$$\lambda_\infty(P) \geq \frac{13}{80} \log(t) + \frac{3}{8} \log(n) + 0.30.$$

When t is negative, we proceed similarly to find a lower bound to $\lambda_\infty(P)$. If $t \leq -100n^2$, we have by Lemmas 3 and 4

$$\frac{2i\pi\omega_2}{\omega_1} \leq \frac{-2\pi \left(0.39 + \log\left(\frac{|t|}{n^2}\right) \right)}{3.15} \leq -\frac{2\pi}{3.15} \log\left(\frac{t}{n^2}\right),$$

and

$$q \leq \exp\left(-\frac{2\pi}{3.15} \log\left(\frac{t}{n^2}\right)\right) \leq 0.0002. \quad (13)$$

On the one hand, by definition of Δ we have

$$\frac{1}{32} \log(\Delta) \geq \frac{1}{8} \log(|t|) + \frac{1}{8} \log(n) + \frac{1}{8} \log(2) + \frac{1}{32} \log\left(1 - \frac{3}{100}\right)$$

On the other hand, with (13) and (11) we get

$$\begin{aligned} \frac{1}{32} \log\left|\frac{1}{q}\right| &\geq \frac{1}{32} \log \exp\left(\frac{2\pi}{3.15} \log\left(\frac{|t|}{n^2}\right)\right) \geq 0.28, \\ -\frac{1}{4} \log|\theta| &\geq -\frac{1}{4} \log\left|\frac{1}{1-q}\right| \geq -\frac{1}{4} \log\left(\frac{1}{1-0.0002}\right), \end{aligned}$$

and, by Lemma 3,

$$\begin{aligned} -\frac{1}{8} \log(\mu) &\geq -\frac{1}{8} \log(2\pi) + \frac{1}{8} \log\left(\frac{3.14}{\sqrt{|t|}}\right) \\ &\geq -\frac{1}{16} \log(|t|) - \frac{1}{8} \log(2\pi) + \frac{1}{8} \log(3.14). \end{aligned}$$

Moreover we have

$$\frac{1}{8} \log(2) + \frac{1}{8} \log(3.14) - \frac{1}{8} \log(2\pi) + \frac{1}{4} \log(1 - 0.0002) + 0.28 \geq 0.27.$$

Then we obtain

$$\lambda_\infty(P) \geq \frac{1}{16} \log(|t|) + \frac{1}{4} \log|\beta| + \frac{1}{8} \log(n) + 0.27.$$

Moreover by Lemma 7, for $t \leq -2n^4$, we have $|\beta| \geq n\sqrt{|t|}$. If $t \leq \min(-2n^4, -100n^2)$, we obtain

$$\lambda_\infty(P) \geq \frac{3}{16} \log(|t|) + \frac{3}{8} \log(n) + 0.27.$$

□

Theorem 13. *Let P be an integral point on E such that $\ell P = (0, n^3)$ for some integer $\ell \geq 2$. If $t \geq \max(100n^2, n^4)$, we have*

$$\widehat{h}(P) \geq \frac{13}{80} \log(t) - \frac{1}{8} \log(n) + 0.06.$$

If $t \leq \min(-100n^2, -n^4)$, we have

$$\widehat{h}(P) \geq \frac{3}{16} \log(|t|) - \frac{1}{8} \log(n).$$

Proof. It suffices to sum the inequalities obtained in Proposition 10 and Proposition 12. \square

By similar arguments, we obtain the following statements for the case $t = 4k + 1$ and $n = 4m$.

Proposition 14. *If $P = (\alpha, \beta)$ is an integral point of E' such that $\ell P = (0, 8m^3)$ for some integer $\ell \geq 1$, then the following inequality holds*

$$\sum_{p < \infty} \lambda_p(P) \geq -\frac{1}{2} \log(m) \geq -\frac{1}{2} \log(n) + \log(2).$$

Proposition 15. *Let $P \in E'(\mathbb{Q})$ be an integral point such that $\ell P = (0, 8m^3)$ for some integer $\ell \geq 2$. If $t \geq \max(n^4, 100n^2)$, we have*

$$\lambda_\infty(P) \geq \frac{13}{80} \log(t) + \frac{3}{8} \log(n) - \frac{5}{8} \log(2) + 0.04.$$

If $t \leq \min(-100n^2, -2n^4)$, we have

$$\lambda_\infty(P) \geq \frac{3}{16} \log(|t|) + \frac{3}{8} \log(n) - \frac{3}{4} \log(2) + 0.10.$$

Again after summing the inequalities obtained in Propositions 14 and 15, we obtain the following statement.

Theorem 16. *Let P be an integral point on E' such that $\ell P = (0, 8m^3)$, for some integer $\ell \geq 2$. If $t \geq \max(100n^2, n^4)$, we have*

$$\widehat{h}(P) \geq \frac{13}{80} \log(t) + \frac{3}{8} \log(n) + 0.04.$$

If $t \leq \min(-100n^2, -2n^4)$, we have

$$\widehat{h}(P) \geq \frac{3}{16} \log(|t|) + \frac{1}{4} \log(n) + 0.10.$$

4.2 Upper bound for the height

Proposition 17. *If $|t| \geq 100n^2$, we have the following inequality*

$$\widehat{h}((0, n^3)) \leq \log(|t|) + 1.57.$$

Proof. We use the result of Silverman [8, Theorem 1.1]. Let $P \in E(\mathbb{Q})$. We have

$$\widehat{h}(P) - \frac{1}{2}h(P) \leq \frac{1}{12}h(\Delta) + \frac{1}{12}h_\infty(j) + \frac{1}{2}h_\infty\left(\frac{b_2}{12}\right) + \frac{1}{2}\log(2) + 1.07$$

where $h(P) = h(x(P))$ is the logarithmic height over \mathbb{Q} , $h_\infty(x) = \max(\log|x|, 0)$ and b_2 is defined as in Definition/Proposition 11. Since $h((0, n^3)) = 0$, in our case we have

$$\begin{aligned} \widehat{h}((0, n^3)) &\leq \frac{1}{12}\log(16n^4(t^2 + 3n^2t + 9n^4)^2) + \frac{1}{12}\log\left(\frac{256}{n^4}(t^2 + 3n^2t + 9n^4)\right) \\ &\quad + \frac{1}{2}\log\left(\frac{|t|}{3}\right) + \frac{1}{2}\log(2) + 1.07 \\ &\leq \frac{1}{4}\log(t^2 + 3n^2t + 9n^4) + \frac{1}{2}\log(|t|) + 1.561 \\ &\leq \log(|t|) + \frac{1}{4}\log\left(1 + \frac{3n^2}{|t|} + \frac{9n^4}{t^2}\right) + 1.561. \end{aligned}$$

Finally, if $|t| \geq 100n^2$, we obtain

$$\widehat{h}((0, n^3)) \leq \log(|t|) + 1.57.$$

□

Remark. We note that the upper bound of Proposition 17 is independent of n . In fact, this bound is not optimal. A numerical analysis suggests that $\widehat{h}((0, n^3))$ should be equivalent to $\frac{1}{2}\log\left(\frac{|t|}{n^2}\right)$ as $|t| \rightarrow +\infty$.

By an argument similar to Proposition 17, we obtain the following statement.

Proposition 18. *For the case $t = 4k + 1$, $n = 4m$. If $|t| \geq 100n^2$, we have*

$$\widehat{h}((0, 8m^3)) \leq \log(|t|) + 0.19.$$

5 Main results

Proposition 19. *Suppose that $t \geq \max(100n^2, n^4)$ or $t \leq \min(-100n^2, -2n^4)$. Suppose also that the equation $y^2 = f(x)$ is a minimal Weierstrass model for E when δ is squarefree (which occurs if $t \not\equiv 1 \pmod{4}$ when $4 \mid n$). Then the point $(0, n^3)$ is not divisible.*

Proof. Assume that t is positive. Let P be a point in $E(\mathbb{Q}) - E_0(\mathbb{Q})$ and $\ell \geq 2$ such that $\ell P = (0, n^3)$. By Lemma 5, P must be an integral point. By Theorem 13 we know that $\widehat{h}(P) \geq \frac{13}{80} \log(t) - \frac{1}{8} \log(n) + 0.06$ and by Proposition 17, we have $\widehat{h}((0, n^3)) \leq \log(t) + 1.57$. We note that P has infinite order and then $\widehat{h}(P) \neq 0$. Since \widehat{h} is quadratic, we have

$$\begin{aligned} \ell^2 = \frac{\widehat{h}((0, n^3))}{\widehat{h}(P)} &\leq \frac{\log(t) + 1.57}{\frac{13}{80} \log(t) - \frac{1}{8} \log(n) + 0.06} \\ &\leq \frac{80}{13} + \frac{\frac{2}{3} \log(n) + 1.21}{\frac{13}{80} \log(t) - \frac{1}{8} \log(n) + 0.06} \\ &\leq 8.6 \end{aligned}$$

thus $\ell \leq 2$. However we have seen at the beginning of Section 4.1 that ℓ is odd, hence the point $(0, n^3)$ is not divisible.

Assume now that t is negative. We have by a similar argument (see Theorem 13 and Proposition 17),

$$\begin{aligned} \ell^2 &\leq \frac{\log(|t|) + 1.57}{\frac{3}{16} \log(|t|) - \frac{1}{8} \log(n)} \\ &\leq 5.34 + \frac{\frac{2}{3} \log(n) + 1.57}{\frac{3}{16} \log(|t|) - \frac{1}{8} \log(n)} \\ &\leq 8.85. \end{aligned}$$

Again we get $\ell \leq 2$ which is impossible. Thus the point $(0, n^3)$ is not divisible. \square

Using Theorem 16 and Proposition 18, we obtain the following statement.

Proposition 20. *We assume $n = 4m$ and $t = 4k + 1$, for some $m \in \mathbb{Z}_{>0}$ and $k \in \mathbb{Z}$. If δ is squarefree, and $t \geq \max(100n^2, n^4)$ or $t \leq \min(-100n^2, -2n^4)$, then the point $(0, n^3)$ is not divisible on E .*

By combining Propositions 19 and 20, we obtain the main result.

Theorem 21. *Suppose that $t \geq \max(100n^2, n^4)$ or $t \leq \min(-100n^2, -2n^4)$, and δ is squarefree. Then the point $(0, n^3)$ is not divisible on E .*

Now, to extend Theorem 21 to the case $|t| < \max(100n^2, n^4)$, we can use the lower bound for the height in [6, Proposition 1.7]: for all $P \in E(\mathbb{Q})$ we have

$$\widehat{h}(P) \geq \frac{1}{768C_E^2} \log |\Delta_{\min}|,$$

where C_E is the lowest common multiple of the Tamagawa numbers of E and Δ_{\min} is the minimal discriminant of E . Let us denote by B_E this lower bound. In our case, if n is squarefree we have

$$B_E = \frac{\log |\Delta|}{110592}.$$

So for a given t , it suffices to check that $(0, n^3)$ is not equal to ℓP for all $P \in E(\mathbb{Q})$ and for all primes $\ell \leq \sqrt{\frac{\widehat{h}(0, n^3)}{B_E}}$.

We use the following method: if there exists a prime ℓ and $P \in E(\mathbb{Q})$ such that $\ell P = (0, n^3)$ then for all primes $p \nmid \Delta$, the reduction of E modulo p is an elliptic curve over \mathbb{F}_p and $\ell \overline{P} = \overline{(0, n^3)}$, where \overline{P} denote the reduction of the point P . If p is a prime number such that ℓ divides the exponent of group $E(\mathbb{F}_p)$ denoted by r_p , we shall have $\frac{r_p}{\ell} \overline{(0, n^3)} = O$. So, if we find a prime p such that $\ell \mid r_p$ and $\frac{r_p}{\ell} \overline{(0, n^3)} \neq O$, there does not exist $P \in E(\mathbb{Q})$ such that $\ell P = (0, n^3)$.

We obtain the following statement.

Theorem 22. *If $n \leq 10$ and δ is squarefree, then the point $(0, n^3)$ is not divisible.*

Proof. Assume $n = 1$. Theorem 21 implies that if $|t| \geq 100$, the point is not divisible. We also note that the result is true for negative t because $(0, 1)$ is the only integer point in $E(\mathbb{Q}) - E_0(\mathbb{Q})$. For $1 \leq t < 100$, it suffices to use the method described above with the help of PARI/GP [10]. For $2 \leq n \leq 10$, we use a similar method. \square

Remark. Let $n \geq 1$. If $t = 5n^2$, the point $(0, n^3)$ on E is divisible and satisfies $(0, n^3) = 3(-4n^2, 7n^3)$. The elliptic curve E is isomorphic over \mathbb{Q} to the elliptic curve $y^2 = x^3 + 5x^2 - 8x + 1$. Therefore in this case, the rank of E over \mathbb{Q} is 1 and the point $(-4n^2, 7n^3)$ is a generator of $E(\mathbb{Q})$. Note that we have not found another example where the point $(0, n^3)$ is the multiple of some point.

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