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# Non-divisible point on a two-parameter family of elliptic curves

Valentin Petit

## Abstract

Let  $n$  be a positive integer and  $t$  a non-zero integer. We consider the elliptic curve over  $\mathbb{Q}$  given by

$$E: y^2 = x^3 + tx^2 - n^2(t + 3n^2)x + n^6.$$

It is a special case of an elliptic surface studied recently by Bettin, David and Delaunay [2] and it generalizes Washington's family. The point  $(0, n^3)$  belongs to  $E(\mathbb{Q})$  and we obtain some results about its non-divisibility in  $E(\mathbb{Q})$ . Our work extends to this two-parameter family of elliptic curves a previous study of Duquesne (mainly stated for  $n = 1$  and  $t > 0$ ).

MSC 2020: 11G05, 11G50.

Keywords: Elliptic curves, Integral points, Heights.

## 1 Introduction

We are concerned with proving the non-divisibility of a point on a family of elliptic curves defined over  $\mathbb{Q}$  with two integer parameters. This family generalizes Washington's family [11] which is connected to simplest cubic fields. Let  $n$  be a positive integer. We consider the elliptic surface over  $\mathbb{Q}(T)$  given by

$$\mathcal{E}: y^2 = x^3 + Tx^2 - n^2(T + 3n^2)x + n^6. \quad (1)$$

It is a special case of an elliptic surface studied recently by Bettin, David and Delaunay [2]. The case  $n = 1$  is precisely Washington's family studied by Washington [11] and Duquesne [5]. In [2] the authors obtain a formula for the average root numbers of the elliptic curves obtained by specializing  $\mathcal{E}$  at  $T = t \in \mathbb{Q}$ . Another elliptic surface over  $\mathbb{Q}(T)$  derived from their family is

$$\mathcal{F}: y^2 = x^3 + Tx^2 + n^2(T - 3n^2)x - n^6. \quad (2)$$

Both elliptic surfaces  $\mathcal{E}$  and  $\mathcal{F}$  are related. Indeed, for all positive integers  $n$  and all integers  $t$ , the curves  $\mathcal{E}(t - 3n^2)$  and  $\mathcal{F}(t)$  are isomorphic over  $\mathbb{Q}$ . Throughout the article, we denote by  $E$  the elliptic curve  $\mathcal{E}(t)$  over  $\mathbb{Q}$ , where  $t$  is a non-zero integer as in [2].

Theorem 5.7 of Duquesne in [5] asserts that if  $n = 1$ ,  $t$  is positive and  $t^2 + 3t + 9$  is squarefree, then the point  $(0, 1)$  is not divisible. In this article, we generalize this result to all  $n > 0$  whenever  $|t|$  is large enough compared to  $n$  (Theorem 21). In particular, we also take care of the case  $t < 0$ .

The strategy is similar to Duquesne's with an additional specific and careful treatment for the two parameters (we also notice that a small issue seems to occur in Section 5D of [5] during the computation of the local contribution of the height, which we can correct here). Suppose that there exist an integer  $k \geq 2$  and a point  $P \in E(\mathbb{Q})$  such that  $kP = (0, n^3)$ . The main idea is to minimize such an integer  $k$ . The strategy is to find a lower bound for the canonical height  $\hat{h}(P)$  of  $P$ , an upper bound for  $\hat{h}(0, n^3)$  (Section 4), and to obtain a contradiction. To do this, we will split the canonical height into local contributions (Section 4) and approximate the periods of  $E$  (Section 3).

## 2 Generalities about the family

Let  $n$  be a positive integer and  $t$  a non-zero integer. We consider the elliptic curve

$$E : y^2 = x^3 + tx^2 - n^2(t + 3n^2)x + n^6. \quad (3)$$

We set  $\delta = t^2 + 3n^2t + 9n^4$ . The discriminant and  $j$ -invariant of the elliptic curve  $E$  are given by

$$\begin{aligned} \Delta &= 16n^4\delta^2, \\ j &= \frac{256}{n^4}\delta. \end{aligned}$$

Let  $f(x)$  be the polynomial

$$f(x) = x^3 + tx^2 - n^2(t + 3n^2)x + n^6.$$

Its discriminant is  $n^4\delta^2$ , which is positive, so  $f$  has three real roots denoted by  $\alpha_1 < \alpha_2 < \alpha_3$ . They satisfy  $\alpha_1 < 0 < \alpha_2 < n^2 < \alpha_3$  and the following relations:

$$\alpha_2 = \frac{n^4}{n^2 - \alpha_1}, \quad \alpha_3 = n^2 - \frac{n^4}{\alpha_1}. \quad (4)$$

The polynomial  $f$  is irreducible over  $\mathbb{Q}$  if  $\delta$  is squarefree. Indeed, it suffices to see that the polynomial  $h(x) = 27f\left(\frac{x-t}{3}\right) = x^3 + 3\delta x + \delta(2t + 3n^2)$  is irreducible over  $\mathbb{Q}$  by Eisenstein's criterion.

We denote by  $E_0(\mathbb{R})$  the connected component of the identity element and by  $E(\mathbb{R}) - E_0(\mathbb{R})$  the bounded connected component of  $E(\mathbb{R})$ . Recall that  $E_0(\mathbb{R})$  is a subgroup of  $E(\mathbb{R})$  and that the sum of two points in  $E(\mathbb{R}) - E_0(\mathbb{R})$  lies in  $E_0(\mathbb{R})$ . The integral point  $(0, n^3)$  belongs to  $E(\mathbb{Q})$  for all  $t$  and  $n$ . More precisely the fact that  $\alpha_2 > 0$  implies that the point  $(0, n^3)$  belongs to  $E(\mathbb{Q}) - E_0(\mathbb{Q})$  for all  $t \in \mathbb{Z}_{\neq 0}$  and  $n \in \mathbb{N}$ .

We note that  $(0, n^3)$  is not a torsion point of  $E$ . Indeed, since  $(0, n^3) \in E(\mathbb{Q}) - E_0(\mathbb{Q})$ , the order of  $(0, n^3)$  cannot be odd if it is finite. Moreover, since the polynomial  $f(x)$  is irreducible over  $\mathbb{Q}$ , there is no torsion point of even order. Hence the point  $(0, n^3)$  has infinite order.

Throughout the article we assume that  $\delta$  is squarefree, which implies that  $n$  and  $t$  are coprime. This condition will play a key role later. Moreover we need (3) to be a minimal Weierstrass equation for  $E$  which, by Tate's algorithm [9, IV.9], occurs if we assume also that  $t$  is not congruent to 1 modulo 4 when  $4 \mid n$ . The minimal Weierstrass equation will be necessary to compute the non-archimedean local contributions.

For the case  $4 \mid n$  and  $t \equiv 1 \pmod{4}$ , write  $n = 4m$  and  $t = 4k + 1$  with  $m \in \mathbb{Z}_{>0}$  and  $k \in \mathbb{Z}$ . The elliptic curve  $E$  is then isomorphic over  $\mathbb{Q}$  to the curve

$$E' : y^2 + xy = x^3 + kx^2 - m^2(4k + 1 - 48m^2)x + 64m^6. \quad (5)$$

Note that (5) is a minimal model for  $E'$ . A change of variables maps the point  $(0, n^3) \in E(\mathbb{Q})$  to the point  $(0, 8m^3) \in E'(\mathbb{Q})$ . Therefore, if we want to prove the non-divisibility of  $(0, n^3)$  on  $E(\mathbb{Q})$ , it suffices to prove the non-divisibility of  $(0, 8m^3)$  on  $E'(\mathbb{Q})$ .

### 3 Approximation of periods

The goal of this section is to approximate the real period  $\omega_1$  and the imaginary period  $\omega_2$  of  $E$ . In order to compute  $\omega_1$  and  $\omega_2$ , the Weierstrass equation defining  $E$  does not need to be minimal. The elliptic curve  $E : y^2 = x^3 + tx^2 - n^2(t + 3n^2)x + n^6$  is isomorphic over  $\mathbb{Q}$  to the curve

$$y^2 = 4g(x),$$

where  $g(x) = x^3 - \frac{1}{3}\delta x + \frac{1}{27}(2t + 3n^2)\delta$ . The real roots  $e_1, e_2, e_3$  of  $g$  are given by  $e_i = \alpha_i + \frac{t}{3}$  for all  $i \in \{1, 2, 3\}$  and the periods  $\omega_1$  and  $\omega_2$  of  $E$  are (see [3, 7.3.2])

$$\omega_1 = \int_{e_1}^{e_2} \frac{dx}{\sqrt{g(x)}} \in \mathbb{R}, \quad \omega_2 = - \int_{e_2}^{e_3} \frac{dx}{\sqrt{g(x)}} \in i\mathbb{R}.$$

### 3.1 The case $t > 0$

When  $t > 0$ , the real roots of  $g$  satisfy  $e_1 < 0 < e_2 < e_3$  because  $e_2 = \alpha_2 + \frac{t}{3}$  and  $g\left(-\frac{2t}{3} - n^2\right) > 0$ . A straightforward study of the function  $g$  gives for  $t \geq 3n^2$ ,

$$\begin{aligned} -\frac{2}{3}t - n^2 - 2\frac{n^4}{t} &\leq e_1 \leq -\frac{2}{3}t - n^2 - \frac{n^4}{t}, \\ \frac{t}{3} &\leq e_2 \leq \frac{t}{3} + \frac{n^4}{t}, \\ \frac{t}{3} + n^2 &\leq e_3 \leq \frac{t}{3} + n^2 + \frac{n^4}{t}. \end{aligned} \tag{6}$$

**Lemma 1.** *Let  $n$  be a fixed positive integer. As  $t \rightarrow +\infty$ , we have  $\frac{\omega_2}{i} \sim \frac{\pi}{\sqrt{t}}$ .*

Moreover for  $t \geq 100n^2$ , we have

$$\frac{3.11}{\sqrt{t}} \leq \frac{\omega_2}{i} \leq \frac{3.15}{\sqrt{t}}.$$

*Proof.* We note that  $\frac{\omega_2}{i} = \int_{e_2}^{e_3} \frac{dx}{\sqrt{(x-e_1)(x-e_2)(e_3-x)}}$ . If  $x \in [e_2, e_3]$ , we have  $t + n^2 + \frac{n^4}{t} \leq x - e_1 \leq t + 2n^2 + \frac{3n^4}{t}$  by (6). So

$$\frac{1}{\sqrt{t + 2n^2 + \frac{3n^4}{t}}} \leq \frac{1}{\sqrt{x - e_1}} \leq \frac{1}{\sqrt{t + n^2 + \frac{n^4}{t}}},$$

and then

$$\frac{\int_{e_2}^{e_3} \frac{dx}{\sqrt{(x-e_2)(e_3-x)}}}{\sqrt{t + 2n^2 + \frac{3n^4}{t}}} \leq \frac{\omega_2}{i} \leq \frac{\int_{e_2}^{e_3} \frac{dx}{\sqrt{(x-e_2)(e_3-x)}}}{\sqrt{t + n^2 + \frac{n^4}{t}}}, \tag{7}$$

where  $\int_{e_2}^{e_3} \frac{dx}{\sqrt{(x-e_2)(e_3-x)}} = \pi$ . Moreover, as  $t \rightarrow +\infty$  the left-hand side and the right-hand side of (7) are both equivalent to  $\frac{\pi}{\sqrt{t}}$ . Finally, when  $t \geq 100n^2$ , we derive from (7)

$$\frac{3.11}{\sqrt{t}} \leq \frac{\omega_2}{i} \leq \frac{3.15}{\sqrt{t}}.$$

□

**Lemma 2.** For  $t \geq 100n^2$ , we have

$$\frac{1.88 + 0.99 \log\left(\frac{t}{n^2}\right)}{\sqrt{t}} \leq \omega_1 \leq \frac{5.35 + 1.23 \log\left(\frac{t}{n^2}\right)}{\sqrt{t}}.$$

*Proof.* We split the integral into two parts

$$\omega_1^- = \int_{e_1}^0 \frac{dx}{\sqrt{(x-e_1)(e_2-x)(e_3-x)}}, \omega_1^+ = \int_0^{e_2} \frac{dx}{\sqrt{(x-e_1)(e_2-x)(e_3-x)}}.$$

First we consider  $\omega_1^-$ . If  $x \in [e_1, 0]$ , we have  $\frac{t}{3} \leq e_2 - x \leq t + n^2 + \frac{3n^4}{t}$  and  $\frac{t}{3} + n^2 \leq e_3 - x \leq t + 2n^2 + \frac{3n^4}{t}$  by (6). So we get the lower bound

$$\begin{aligned} \omega_1^- \sqrt{\left(t + n^2 + \frac{3n^4}{t}\right) \left(t + 2n^2 + \frac{3n^4}{t}\right)} &\geq \int_{e_1}^0 \frac{dx}{\sqrt{x-e_1}} \\ &\geq 2\sqrt{-e_1} \\ &\geq 2\sqrt{\frac{2}{3}t + n^2 + \frac{n^4}{t}}. \end{aligned}$$

Furthermore using (6), we get the upper bound

$$\omega_1^- \leq \frac{\int_{e_1}^0 \frac{dx}{\sqrt{x-e_1}}}{\sqrt{\frac{t}{3} \left(\frac{t}{3} + n^2\right)}} \leq \frac{2\sqrt{-e_1}}{\sqrt{\frac{t}{3} \left(\frac{t}{3} + n^2\right)}} \leq \frac{2}{\sqrt{t}} \frac{\sqrt{\frac{2}{3} + \frac{n^2}{t} + \frac{2n^4}{t^2}}}{\sqrt{\frac{1}{9}}}.$$

Thus when  $t \geq 100n^2$ , we obtain

$$\frac{1.60}{\sqrt{t}} \leq \omega_1^- \leq \frac{4.94}{\sqrt{t}}.$$

Now we consider  $\omega_1^+$ . If  $x \in [0, e_2]$ , we have  $\frac{2}{3}t + n^2 + \frac{n^4}{t} \leq x - e_1 \leq t + n^2 + \frac{n^4}{t}$  by (6). So we have

$$\frac{J}{\sqrt{t + n^2 + \frac{3n^4}{t}}} \leq \omega_1^+ \leq \frac{J}{\sqrt{\frac{2}{3}t + n^2 + \frac{n^4}{t}}},$$

with

$$J = \int_0^{e_2} \frac{dx}{\sqrt{(e_2-x)(e_3-x)}} = \log\left(\frac{\sqrt{e_3} + \sqrt{e_2}}{\sqrt{e_3} - \sqrt{e_2}}\right).$$

Moreover by (6), we have

$$\frac{\frac{4}{3}t + n^2}{n^2 + \frac{n^4}{t}} \leq \frac{\sqrt{e_3} + \sqrt{e_2}}{\sqrt{e_3} - \sqrt{e_2}} \leq \frac{\frac{2}{3}t + n^2 + \frac{2n^4}{t} + 2\sqrt{(\frac{t}{3} + n^2 + \frac{n^4}{t})(\frac{t}{3} + \frac{n^4}{t})}}{n^2 - \frac{n^4}{t}},$$

which implies

$$\frac{\frac{4}{3}t + n^2}{n^2 + \frac{n^4}{t}} \leq \frac{\sqrt{e_3} + \sqrt{e_2}}{\sqrt{e_3} - \sqrt{e_2}} \leq \frac{\frac{4}{3}t + 4n^2 + \frac{4n^4}{t}}{n^2 - \frac{n^4}{t}}.$$

So we get

$$\frac{1}{\sqrt{t + n^2 + \frac{3n^4}{t}}} \log \left( \frac{\frac{4}{3}t + n^2}{n^2 + \frac{n^4}{t}} \right) \leq \omega_1^+ \leq \frac{1}{\sqrt{\frac{2}{3}t + n^2 + \frac{n^4}{t}}} \log \left( \frac{\frac{4}{3}t + 4n^2 + \frac{4n^4}{t}}{n^2 - \frac{n^4}{t}} \right).$$

Assume now that  $t \geq 100n^2$ . We derive

$$\begin{aligned} \omega_1^+ &\leq \frac{1}{\sqrt{\frac{2}{3}t + n^2 + \frac{n^4}{t}}} \left( \log(t) + \log\left(\frac{4}{3}\right) + \log\left(1 + \frac{3n^2}{t} + \frac{3n^4}{t^2}\right) - \log\left(n^2 - \frac{n^4}{t}\right) \right) \\ &\leq \frac{0.41 + 1.23 \log\left(\frac{t}{n^2}\right)}{\sqrt{t}} \end{aligned}$$

and

$$\omega_1^+ \geq \frac{1}{\sqrt{t + n^2 + \frac{3n^4}{t}}} \log \left( \frac{4t}{3(n^2 + \frac{n^4}{t})} \right) \geq \frac{0.28 + 0.99 \log\left(\frac{t}{n^2}\right)}{\sqrt{t}}.$$

Finally we obtain

$$\frac{1.88 + 0.99 \log\left(\frac{t}{n^2}\right)}{\sqrt{t}} \leq \omega_1 \leq \frac{5.35 + 1.23 \log\left(\frac{t}{n^2}\right)}{\sqrt{t}}.$$

□

**Remark.** A numerical analysis suggests that  $\omega_1$  should be equivalent to  $\frac{\log\left(\frac{t}{n^2}\right)}{\sqrt{t}}$  as  $t \rightarrow +\infty$ .

### 3.2 The case $t < 0$

Once again we use the fact that  $E$  is isomorphic to the curve  $y^2 = 4g(x)$ . A straightforward study of the function  $g$  gives the following estimates when  $t \leq -3n^2$ :

$$\begin{aligned} \frac{t}{3} + \frac{2n^4}{t} &\leq e_1 \leq \frac{t}{3} + \frac{n^4}{t}, \\ \frac{t}{3} + n^2 + \frac{2n^4}{t} &\leq e_2 \leq \frac{t}{3} + n^2 + \frac{n^4}{t}, \\ -\frac{2t}{3} - n^2 &\leq e_3 \leq -\frac{2t}{3}. \end{aligned} \quad (8)$$

**Lemma 3.** *Let  $n$  be a fixed positive integer. As  $t \rightarrow -\infty$ , we have  $\omega_1 \sim \frac{\pi}{\sqrt{|t|}}$ .*

Moreover, for  $t \leq -100n^2$  we have

$$\frac{3.14}{\sqrt{|t|}} \leq \omega_1 \leq \frac{3.15}{\sqrt{|t|}}.$$

*Proof.* If  $x \in [e_1, e_2]$ , then  $|t| - 2n^2 + \frac{n^4}{|t|} \leq e_3 - x \leq |t| + \frac{2n^4}{|t|}$  by (8). So

$$\frac{1}{\sqrt{|t| + \frac{2n^4}{|t|}}} \leq \frac{1}{\sqrt{e_3 - x}} \leq \frac{1}{\sqrt{|t| - 2n^2 + \frac{n^4}{|t|}}}.$$

We obtain

$$\frac{J}{\sqrt{|t| + \frac{2n^4}{|t|}}} \leq \omega_1 \leq \frac{J}{\sqrt{|t| - 2n^2 + \frac{n^4}{|t|}}},$$

with

$$J = \int_{e_1}^{e_2} \frac{dx}{\sqrt{(x - e_1)(e_2 - x)}} = \pi.$$

Since  $\frac{\pi}{\sqrt{|t| + \frac{2n^4}{|t|}}}$  and  $\frac{\pi}{\sqrt{|t| - 2n^2 + \frac{n^4}{|t|}}}$  are both equivalent to  $\frac{\pi}{\sqrt{|t|}}$  as  $t \rightarrow -\infty$ , we deduce  $\omega_1 \underset{t \rightarrow -\infty}{\sim} \frac{\pi}{\sqrt{|t|}}$ . Furthermore if  $t \leq -100n^2$ , we get

$$\frac{3.14}{\sqrt{|t|}} \leq \omega_1 \leq \frac{3.15}{\sqrt{|t|}}.$$

□

**Lemma 4.** *If  $t \leq -100n^2$ , we have*

$$\frac{\omega_2}{i} \geq \frac{0.39 + \log\left(\frac{|t|}{n^2}\right)}{\sqrt{|t|}}.$$

*Proof.* To approximate  $\omega_2$ , we need to split the integral into two parts:

$$\frac{\omega_2}{i} = \underbrace{\int_{e_2}^0 \frac{dx}{\sqrt{(x-e_1)(x-e_2)(e_3-x)}}}_{W^-} + \underbrace{\int_0^{e_3} \frac{dx}{\sqrt{(x-e_1)(x-e_2)(e_3-x)}}}_{W^+}.$$

We begin with estimating  $W^-$ . If  $x \in [e_2, 0]$ , we get by (8)

$$-\frac{2}{3}t - n^2 \leq e_3 - x \leq -t - n^2 - \frac{2n^4}{t}.$$

So

$$\frac{L}{\sqrt{-t - n^2 - \frac{2n^4}{t}}} \leq W^- \leq \frac{L}{\sqrt{-\frac{2}{3}t - n^2}},$$

with

$$L = -\int_0^{e_2} \frac{dx}{\sqrt{(x-e_1)(x-e_2)}} = -\log(e_2 - e_1) + \log(-e_1 - e_2 + \sqrt{e_1 e_2}).$$

Moreover by (8), we have

$$-\log\left(n^2 - \frac{n^4}{t}\right) + \log\left(-\frac{2}{3}t - n^2 - \frac{2n^4}{t}\right) \leq L.$$

For  $t \leq -100n^2$ , we get

$$\begin{aligned} W^- \sqrt{-t - n^2 - \frac{2n^4}{t}} &\geq -\log\left(n^2 - \frac{n^4}{t}\right) + \log\left(-\frac{2}{3}t - n^2 - \frac{2n^4}{t}\right) \\ &\geq -2\log(n) - \log\left(1 - \frac{1}{100}\right) + \log(|t|) + \log\left(\frac{2}{3} - \frac{1}{100}\right), \end{aligned}$$

from which we derive

$$W^- \geq \frac{\log\left(\frac{|t|}{n^2}\right) - 0.42}{\sqrt{|t|}}. \quad (9)$$

If  $x \in [0, e_3]$ , we obtain by (8)

$$\begin{aligned} -\frac{t}{3} - \frac{n^4}{t} &\leq x - e_1 \leq -t - \frac{2n^4}{t}, \\ -\frac{t}{3} - n^2 - \frac{n^4}{t} &\leq x - e_2 \leq -t - n^2 - \frac{n^4}{t}. \end{aligned}$$

Thus

$$W^+ \sqrt{\left(-t - \frac{2n^4}{t}\right) \left(-t - n^2 - \frac{n^4}{t}\right)} \geq \sqrt{e_3},$$

from which we derive

$$W^+ \geq \frac{\sqrt{-\frac{2}{3}t - n^2}}{\sqrt{\left(-t - \frac{2n^4}{t}\right) \left(-t - n^2 - \frac{n^4}{t}\right)}} \geq \frac{\sqrt{|t|} \sqrt{\frac{2}{3} - \frac{n^2}{|t|}}}{|t| \sqrt{\left(1 + \frac{2n^4}{|t|^2}\right)^2}}.$$

So we obtain, for  $t \leq -100n^2$

$$W^+ \geq \frac{0.81}{\sqrt{|t|}}. \tag{10}$$

Finally by adding both inequalities (9) and (10), we have

$$\frac{\omega_2}{i} \geq \frac{0.39 + \log\left(\frac{|t|}{n^2}\right)}{\sqrt{|t|}}.$$

□

**Remark.** When  $t < 0$ , we just give a lower bound for  $\frac{\omega_2}{i}$  since no upper bound is required in order estimate the height.

## 4 Estimates on the heights

We need to estimate the heights of some points in  $E(\mathbb{Q})$ . For this purpose, we will decompose the heights as a sum of local contributions. Depending on the conventions, there are different ways to split the canonical height into a sum of local heights. However in this section, a special care was taken during the computations of local heights to make sure that their sum agrees with the definition of the canonical height.

### 4.1 Lower bound for the height

We want to prove that there does not exist a point  $P = (\alpha, \beta) \in E(\mathbb{Q})$  and an integer  $\ell \geq 2$  such that  $\ell P = (0, n^3)$ . The goal of this part is to find, if such a point exists, a lower bound for the canonical height of  $P$ . Since  $(0, n^3) \in E(\mathbb{Q}) - E_0(\mathbb{Q})$ , the point  $P$  has to belong to  $E(\mathbb{Q}) - E_0(\mathbb{Q})$  and  $\ell$  must be odd (see Section 2).

**Lemma 5.** *Let  $F : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$  be an elliptic curve over  $\mathbb{Q}$  with  $a_i \in \mathbb{Z}$  for all  $i \in \{1, 2, 3, 4, 6\}$ . Let  $P \in F(\mathbb{Q})$  be a point such that  $nP$  is an integral point for some integer  $n \geq 1$ . Then  $P$  is an integral point on  $F$ .*

*Proof.* The proof of [1, 4.4, p. 18] for models of the form  $y^2 = x^3 + Ax + B$  using division polynomials can be generalized for longer models of the form  $y^2 = x^3 + a_2x^2 + a_4x + a_6$  without any difficulty.  $\square$

Lemma 5 implies that if  $(0, n^3)$  is the multiple of a rational point  $P$  then  $P \in E(\mathbb{Z})$ . Moreover the point  $(0, n^3)$  is singular modulo  $p$  for all primes  $p \mid n$ . This implies that  $P$  is singular modulo  $p$  for all  $p \mid n$  since the multiple of a non-singular point is always non-singular. If  $p \mid n$ , the only singular point is  $(0, 0)$  so we have  $p \mid \alpha$ . This remark will be used in Lemmas 6, 7 and Proposition 10.

**Lemma 6.** *Let  $P = (\alpha, \beta) \in E(\mathbb{Z}) - E_0(\mathbb{Z})$  be such that there exists  $\ell \geq 2$  with  $\ell P = (0, n^3)$ . If  $t \geq n^4$  then  $|\beta| \geq n\sqrt{2t}$ .*

*Proof.* Assume that  $t \geq n^4$  and  $n \geq 2$ . Recall that  $\alpha_1, \alpha_2, \alpha_3$  are the real roots of the polynomial  $f(x) = x^3 + tx^2 - n^2(t + 3n^2)x + n^6$  with  $\alpha_1 < 0 < \alpha_2 < \alpha_3$ . By studying the function  $f$ , we have for  $t \geq n^4$

$$-n^2 - t - 1 < \alpha_1 < -n^2 - t \quad \text{and} \quad 0 < \alpha_2 < 1.$$

Since  $P \in E(\mathbb{Q}) - E_0(\mathbb{Q})$ , we have  $\alpha_1 < \alpha < \alpha_2$ , and then  $\alpha \in [-n^2 - t, -1]$ . Moreover since  $p \mid \alpha$  for all prime  $p \mid n$ , we get  $\alpha \leq -2$  and then

$$\begin{aligned} f(\alpha) &\geq \min(f(-n^2 - t), f(-2)) \\ &\geq \min(2n^4t + 3n^6, (2n^2 + 4)t + n^6 + 6n^4 - 8) \\ &\geq 2n^2t. \end{aligned}$$

Thus  $|\beta| \geq n\sqrt{2t}$ . If  $n = 1$ , we have

$$f(\alpha) \geq \min(f(-t - 1), f(-1)) = 2t + 3,$$

hence  $|\beta| \geq \sqrt{2t + 3} \geq \sqrt{2t}$ .  $\square$

**Lemma 7.** *Suppose that  $t \leq -2n^4$ . Then*

$$|\beta| \geq \begin{cases} \sqrt{2|t|} & \text{if } n = 2, \\ n\sqrt{|t|} & \text{if } n \geq 3. \end{cases}$$

*Proof.* We use an argument similar to the case  $t \geq n^4$  (Lemma 6).  $\square$

**Remark.** Assume  $n = 1$ . If  $t \leq -2$ , the point  $(0, 1)$  is the only integral point in  $E(\mathbb{Q}) - E_0(\mathbb{Q})$  because  $-1 < \alpha_1 < \alpha_2$ . Hence  $(0, 1)$  is not divisible. If  $t = -1$ , it is easy to prove that this point is not divisible as well. Therefore when  $t$  is negative, we can assume that  $n \geq 2$ .

In order to find a lower bound for the canonical height of an integral point of  $E$ , we use the decomposition of the canonical height into local contributions (see [7, Theorem 5.2], [4, 7.5.7]). For  $P \in E(\mathbb{Q})$  we have

$$\widehat{h}(P) = \sum_{p \leq \infty} \lambda_p(P),$$

where the sum runs through the places of  $\mathbb{Q}$ . We recall the definitions of  $\lambda_p$  for finite places  $p$  in Definition/Proposition 8 and  $\lambda_\infty$  in Definition/Proposition 11. For this, we set

$$\begin{aligned} A &= 3\alpha^2 + 2t\alpha - n^2(t + 3n^2), \\ B &= 2\beta, \\ C &= 3\alpha^4 + 4t\alpha^3 - 6n^2(t + 3n^2)\alpha^2 + 12n^6\alpha - n^4(t^2 + 2n^2t + 9n^4), \\ D &= \gcd(A, B), \\ c_4 &= 16\delta. \end{aligned}$$

**Definition/Proposition 8.** ([7, Theorem 5.2],[4, 7.5.6]) *Let  $p$  be a prime number. The local non-archimedean contribution  $\lambda_p(P)$  is non-zero only if  $p \mid D$ . If  $p \mid D$ , we set  $m_p = \min\left(\frac{v_p(\Delta)}{2}, v_p(B)\right)$  and the local contribution at  $p$  is then given by*

$$\lambda_p(P) = \begin{cases} -\frac{m_p(v_p(\Delta) - 2m_p)}{2v_p(\Delta)} \log(p) & \text{if } p \nmid c_4, \\ -\frac{v_p(B)}{3} \log(p) & \text{if } p \mid c_4 \text{ and } v_p(C) \geq 3v_p(B), \\ -\frac{v_p(C)}{8} \log(p) & \text{otherwise,} \end{cases}$$

where  $v_p$  is the  $p$ -adic valuation.

**Lemma 9.** *Assume  $4 \nmid n$  or  $t \not\equiv 1[4]$ . Let  $P \in E(\mathbb{Q})$  be an integral point. If  $p \nmid 2n$ , then  $\lambda_p(P) = 0$ .*

*Proof.* Let  $p \mid D$ . We assume that  $p \nmid n$  and  $p \neq 2$ . First we have  $4A^2 = B^2(9\alpha + 3t) + 4\delta(\alpha^2 - n^2\alpha + n^4)$ . Since  $\delta$  is squarefree, we obtain that  $p \mid (\alpha^2 - n^2\alpha + n^4)$ . We also have

$$B^2 = 4(\alpha^2 - n^2\alpha + n^4) - 4n^2(3\alpha + t)$$

so  $p \mid (3\alpha + t)$ . Furthermore  $p$  divides the resultant of  $A$  and  $B^2$ , viewed as polynomials in  $\mathbb{Z}[\alpha]$ , which is equal to  $\Delta = 16n^4\delta^2$ . Therefore  $p \mid \delta$ . Moreover,

$$27B^2 = 4(3\alpha + t)^3 - 4\delta(9\alpha + t - 3n^2),$$

which implies that  $p \mid (9\alpha + t - 3n^2)$ . Since  $(9\alpha + t - 3n^2) = 3(3\alpha + t) - (2t + 3n^2)$ , we obtain that  $p \mid (2t + 3n^2)$ . But  $4\delta = (2t + 3n^2)^2 + 27n^4$ . We get  $p = 3$  hence  $\delta$  is not squarefree, which is a contradiction.  $\square$

**Proposition 10.** *Let  $P = (\alpha, \beta)$  be an integral point on  $E$  such that  $\ell P = (0, n^3)$  for some integer  $\ell \geq 1$ . Then the following inequality holds*

$$\sum_{p < \infty} \lambda_p(P) \geq -\frac{1}{2} \log(n) - \frac{1}{3} \log(2).$$

*Proof.* By Lemma 9, it suffices to compute the local contributions at primes  $p \mid 2n$ . For  $p \mid n$ , we first assume  $p \neq 2$ . Let  $m = v_p(n)$ . Since  $\delta$  is squarefree, we have  $p \nmid c_4$  and  $v_p(\Delta) = 4m$ . Put  $m_p = \min(2m, v_p(B))$ . By Definition/Proposition 8, the local contribution of  $P$  is then given by

$$\lambda_p(P) = -\frac{m_p(4m - m_p)}{8m} \log(p).$$

We easily conclude that

$$\lambda_p(P) \geq -\frac{m}{2} \log(p) \geq -\frac{1}{2} \log(p^m).$$

Now we want to compute the 2-local contribution  $\lambda_2(P)$ . Since  $2 \mid c_4$ , it is given by

$$\lambda_2(P) = \begin{cases} -\frac{v_2(B)}{3} \log(2) & \text{if } v_2(C) \geq 3v_2(B), \\ -\frac{v_2(C)}{8} \log(2) & \text{otherwise.} \end{cases}$$

If  $2 \nmid n$  then  $v_2(B) = 1$ , so  $\lambda_2(P) \geq -\frac{1}{3} \log(2)$ .

Suppose now that  $2 \mid n$  and  $m = v_2(n)$ .

If  $m \in \{1, 2\}$  it is easy to check that  $\lambda_2(P) \geq -\frac{1}{2} \log(2^m) - \frac{1}{3} \log(2)$ . So we may assume that  $m \geq 3$ .

If  $v_2(\alpha) < m$  then  $v_2(B) = 1 + v_2(\alpha)$ ; in that case we have  $\lambda_2(P) \geq \frac{-m}{3} \log(2)$  if  $v_2(C) \geq 3v_2(B)$ , else  $\lambda_2(P) \geq -\frac{m}{2} \log(2)$ .

If  $m \leq v_2(\alpha) < \frac{4m-2}{3}$  then  $v_2(B) = 1 + v_2(\alpha)$  and  $v_2(C) = 3v_2(\alpha) + 2$ ; in that case we have  $\lambda_2(P) = -\frac{3v_2(\alpha) + 2}{8} \log(2) \geq -\frac{1}{2} \log(2^m)$ .

If  $v_2(\alpha) = \frac{4m-2}{3}$  then  $v_2(C) > 4m$  and  $v_2(B) = \frac{4m+1}{3}$ ; in that case we have  $\lambda_2(P) = -\frac{4m+1}{9} \log(2) \geq -\frac{m}{2} \log(2)$ .

If  $v_2(\alpha) > \frac{4m-2}{3}$  then  $v_2(C) = 4m$  and  $3v_2(B) > 4m$ ; in that case we have  $\lambda_2(P) \geq -\frac{1}{2} \log(2^m)$ .

Finally we conclude by summing all local non-archimedean contributions.  $\square$

Now we study the local archimedean contribution.

**Definition/Proposition 11.** ([4, 7.5.7]) Let  $P = (\alpha, \beta) \in E(\mathbb{Q})$ . Let  $z$  be the elliptic logarithm of  $P$ . Let  $\mu = \frac{2\pi}{\omega_1}$ ,  $s = \mu \operatorname{Re}(z)$ ,  $q = \exp\left(\frac{2i\pi\omega_2}{\omega_1}\right)$ , and

$$\theta = \sum_{k=0}^{\infty} \sin((2k+1)s)(-1)^k q^{\frac{k(k+1)}{2}}.$$

Then the local archimedean contribution is given by

$$\lambda_{\infty}(P) = \frac{1}{32} \log \left| \frac{\Delta}{q} \right| - \frac{1}{4} \log |\theta| + \frac{1}{8} \log \left| \frac{\alpha^3 + \frac{b_2}{4}\alpha^2 + \frac{b_4}{2}\alpha + \frac{b_6}{4}}{\mu} \right|$$

where  $b_2 = a_1^2 + 4a_2$ ,  $b_4 = a_1a_3 + 2a_4$ ,  $b_6 = a_1a_2a_3 + 4a_6$  and  $a_1, a_2, a_3, a_4$  and  $a_6$  are defined as in Lemma 5.

**Proposition 12.** Let  $P$  be an integral point on  $E$  such that  $\ell P = (0, n^3)$  for some integer  $\ell \geq 2$ . If  $t \geq \max(100n^2, n^4)$ , we have

$$\lambda_{\infty}(P) \geq \frac{13}{80} \log(t) + \frac{3}{8} \log(n) + 0.30.$$

If  $t \leq \min(-100n^2, -2n^4)$ , we have

$$\lambda_{\infty}(P) \geq \frac{3}{16} \log(|t|) + \frac{3}{8} \log(n) + 0.27.$$

*Proof.* We first note that

$$|\theta| \leq \sum_{k=0}^{\infty} q^{\frac{k(k+1)}{2}} \leq \frac{1}{1-q}. \quad (11)$$

To find a lower bound for  $\lambda_\infty(P)$ , we need to find an upper bound for  $q$ . If  $t \geq 100n^2$ , by Lemmas 1 and 2, we have

$$\frac{2i\pi\omega_2}{\omega_1} \leq \frac{-3.11 \times 2\pi}{5.35 + 1.23 \log\left(\frac{t}{n^2}\right)} \leq \frac{-15.88}{\log\left(\frac{t}{n^2}\right)}$$

thus

$$q \leq \exp\left(\frac{-15.88}{\log\left(\frac{t}{n^2}\right)}\right) \leq 1 - \frac{4.3}{\log\left(\frac{t}{n^2}\right)}. \quad (12)$$

On the one hand, by definition of  $\Delta$  we have

$$\frac{1}{32} \log(\Delta) = \frac{1}{32} \log(16n^4(t^2 + 3n^2t + 9n^4)^2) \geq \frac{1}{8} \log(t) + \frac{1}{8} \log(n) + \frac{1}{8} \log(2).$$

On the other hand, with (12) and (11) we get

$$\frac{1}{32} \log\left|\frac{1}{q}\right| \geq \frac{1}{32} \log\left(\exp\left(\frac{15.88}{\log\left(\frac{t}{n^2}\right)}\right)\right) \geq \frac{15.88}{32 \log\left(\frac{t}{n^2}\right)},$$

$$-\frac{1}{4} \log|\theta| \geq -\frac{1}{4} \log\left|\frac{1}{1-q}\right| \geq \frac{1}{4} \log(4.3) - \frac{1}{4} \log \log\left(\frac{t}{n^2}\right),$$

and by Lemma 2,

$$\begin{aligned} -\frac{1}{8} \log(\mu) = \frac{1}{8} \log\left(\frac{\omega_1}{2\pi}\right) &\geq -\frac{1}{8} \log(2\pi) + \frac{1}{8} \log\left(\frac{1.88 + 0.99 \log\left(\frac{t}{n^2}\right)}{\sqrt{t}}\right), \\ &\geq -\frac{1}{16} \log(t) - \frac{1}{8} \log(2\pi) + \frac{1}{8} \log \log\left(\frac{t}{n^2}\right). \end{aligned}$$

Moreover, we have

$$\frac{1}{8} \log(2) + \frac{1}{4} \log(4.3) - \frac{1}{8} \log(2\pi) + \frac{1}{8} \log(0.99) \geq 0.22.$$

Finally using Definition/Proposition 11 and the fact that  $\alpha^3 + \frac{b_2}{4}\alpha^2 + \frac{b_4}{2}\alpha + \frac{b_6}{4} = \beta^2$ , we obtain the lower bound for  $\lambda_\infty(P)$ :

$$\begin{aligned} \lambda_\infty(P) &\geq \frac{1}{16} \log(t) - \frac{1}{8} \log \log\left(\frac{t}{n^2}\right) + \frac{15.88}{32 \log\left(\frac{t}{n^2}\right)} \\ &\quad + \frac{1}{4} \log|\beta| + \frac{1}{8} \log(n) + 0.22. \end{aligned}$$

Moreover, by Lemma 6, for  $t \geq n^4$  we have  $|\beta| \geq n\sqrt{2t}$ . If  $t \geq \max(n^4, 100n^2)$ , we obtain

$$\lambda_\infty(P) \geq \frac{13}{80} \log(t) + \frac{3}{8} \log(n) + 0.30.$$

When  $t$  is negative, we proceed similarly to find a lower bound to  $\lambda_\infty(P)$ . If  $t \leq -100n^2$ , we have by Lemmas 3 and 4

$$\frac{2i\pi\omega_2}{\omega_1} \leq \frac{-2\pi \left( 0.39 + \log\left(\frac{|t|}{n^2}\right) \right)}{3.15} \leq -\frac{2\pi}{3.15} \log\left(\frac{t}{n^2}\right),$$

and

$$q \leq \exp\left(-\frac{2\pi}{3.15} \log\left(\frac{t}{n^2}\right)\right) \leq 0.0002. \quad (13)$$

On the one hand, by definition of  $\Delta$  we have

$$\frac{1}{32} \log(\Delta) \geq \frac{1}{8} \log(|t|) + \frac{1}{8} \log(n) + \frac{1}{8} \log(2) + \frac{1}{32} \log\left(1 - \frac{3}{100}\right)$$

On the other hand, with (13) and (11) we get

$$\begin{aligned} \frac{1}{32} \log\left|\frac{1}{q}\right| &\geq \frac{1}{32} \log \exp\left(\frac{2\pi}{3.15} \log\left(\frac{|t|}{n^2}\right)\right) \geq 0.28, \\ -\frac{1}{4} \log|\theta| &\geq -\frac{1}{4} \log\left|\frac{1}{1-q}\right| \geq -\frac{1}{4} \log\left(\frac{1}{1-0.0002}\right), \end{aligned}$$

and, by Lemma 3,

$$\begin{aligned} -\frac{1}{8} \log(\mu) &\geq -\frac{1}{8} \log(2\pi) + \frac{1}{8} \log\left(\frac{3.14}{\sqrt{|t|}}\right) \\ &\geq -\frac{1}{16} \log(|t|) - \frac{1}{8} \log(2\pi) + \frac{1}{8} \log(3.14). \end{aligned}$$

Moreover we have

$$\frac{1}{8} \log(2) + \frac{1}{8} \log(3.14) - \frac{1}{8} \log(2\pi) + \frac{1}{4} \log(1 - 0.0002) + 0.28 \geq 0.27.$$

Then we obtain

$$\lambda_\infty(P) \geq \frac{1}{16} \log(|t|) + \frac{1}{4} \log|\beta| + \frac{1}{8} \log(n) + 0.27.$$

Moreover by Lemma 7, for  $t \leq -2n^4$ , we have  $|\beta| \geq n\sqrt{|t|}$ . If  $t \leq \min(-2n^4, -100n^2)$ , we obtain

$$\lambda_\infty(P) \geq \frac{3}{16} \log(|t|) + \frac{3}{8} \log(n) + 0.27.$$

□

**Theorem 13.** *Let  $P$  be an integral point on  $E$  such that  $\ell P = (0, n^3)$  for some integer  $\ell \geq 2$ . If  $t \geq \max(100n^2, n^4)$ , we have*

$$\widehat{h}(P) \geq \frac{13}{80} \log(t) - \frac{1}{8} \log(n) + 0.06.$$

*If  $t \leq \min(-100n^2, -n^4)$ , we have*

$$\widehat{h}(P) \geq \frac{3}{16} \log(|t|) - \frac{1}{8} \log(n).$$

*Proof.* It suffices to sum the inequalities obtained in Proposition 10 and Proposition 12.  $\square$

By similar arguments, we obtain the following statements for the case  $t = 4k + 1$  and  $n = 4m$ .

**Proposition 14.** *If  $P = (\alpha, \beta)$  is an integral point of  $E'$  such that  $\ell P = (0, 8m^3)$  for some integer  $\ell \geq 1$ , then the following inequality holds*

$$\sum_{p < \infty} \lambda_p(P) \geq -\frac{1}{2} \log(m) \geq -\frac{1}{2} \log(n) + \log(2).$$

**Proposition 15.** *Let  $P \in E'(\mathbb{Q})$  be an integral point such that  $\ell P = (0, 8m^3)$  for some integer  $\ell \geq 2$ . If  $t \geq \max(n^4, 100n^2)$ , we have*

$$\lambda_\infty(P) \geq \frac{13}{80} \log(t) + \frac{3}{8} \log(n) - \frac{5}{8} \log(2) + 0.04.$$

*If  $t \leq \min(-100n^2, -2n^4)$ , we have*

$$\lambda_\infty(P) \geq \frac{3}{16} \log(|t|) + \frac{3}{8} \log(n) - \frac{3}{4} \log(2) + 0.10.$$

Again after summing the inequalities obtained in Propositions 14 and 15, we obtain the following statement.

**Theorem 16.** *Let  $P$  be an integral point on  $E'$  such that  $\ell P = (0, 8m^3)$ , for some integer  $\ell \geq 2$ . If  $t \geq \max(100n^2, n^4)$ , we have*

$$\widehat{h}(P) \geq \frac{13}{80} \log(t) + \frac{3}{8} \log(n) + 0.04.$$

*If  $t \leq \min(-100n^2, -2n^4)$ , we have*

$$\widehat{h}(P) \geq \frac{3}{16} \log(|t|) + \frac{1}{4} \log(n) + 0.10.$$

## 4.2 Upper bound for the height

**Proposition 17.** *If  $|t| \geq 100n^2$ , we have the following inequality*

$$\widehat{h}((0, n^3)) \leq \log(|t|) + 1.57.$$

*Proof.* We use the result of Silverman [8, Theorem 1.1]. Let  $P \in E(\mathbb{Q})$ . We have

$$\widehat{h}(P) - \frac{1}{2}h(P) \leq \frac{1}{12}h(\Delta) + \frac{1}{12}h_\infty(j) + \frac{1}{2}h_\infty\left(\frac{b_2}{12}\right) + \frac{1}{2}\log(2) + 1.07$$

where  $h(P) = h(x(P))$  is the logarithmic height over  $\mathbb{Q}$ ,  $h_\infty(x) = \max(\log|x|, 0)$  and  $b_2$  is defined as in Definition/Proposition 11. Since  $h((0, n^3)) = 0$ , in our case we have

$$\begin{aligned} \widehat{h}((0, n^3)) &\leq \frac{1}{12}\log(16n^4(t^2 + 3n^2t + 9n^4)^2) + \frac{1}{12}\log\left(\frac{256}{n^4}(t^2 + 3n^2t + 9n^4)\right) \\ &\quad + \frac{1}{2}\log\left(\frac{|t|}{3}\right) + \frac{1}{2}\log(2) + 1.07 \\ &\leq \frac{1}{4}\log(t^2 + 3n^2t + 9n^4) + \frac{1}{2}\log(|t|) + 1.561 \\ &\leq \log(|t|) + \frac{1}{4}\log\left(1 + \frac{3n^2}{|t|} + \frac{9n^4}{t^2}\right) + 1.561. \end{aligned}$$

Finally, if  $|t| \geq 100n^2$ , we obtain

$$\widehat{h}((0, n^3)) \leq \log(|t|) + 1.57.$$

□

**Remark.** We note that the upper bound of Proposition 17 is independent of  $n$ . In fact, this bound is not optimal. A numerical analysis suggests that  $\widehat{h}((0, n^3))$  should be equivalent to  $\frac{1}{2}\log\left(\frac{|t|}{n^2}\right)$  as  $|t| \rightarrow +\infty$ .

By an argument similar to Proposition 17, we obtain the following statement.

**Proposition 18.** *For the case  $t = 4k + 1$ ,  $n = 4m$ . If  $|t| \geq 100n^2$ , we have*

$$\widehat{h}((0, 8m^3)) \leq \log(|t|) + 0.19.$$

## 5 Main results

**Proposition 19.** *Suppose that  $t \geq \max(100n^2, n^4)$  or  $t \leq \min(-100n^2, -2n^4)$ . Suppose also that the equation  $y^2 = f(x)$  is a minimal Weierstrass model for  $E$  when  $\delta$  is squarefree (which occurs if  $t \not\equiv 1 \pmod{4}$  when  $4 \mid n$ ). Then the point  $(0, n^3)$  is not divisible.*

*Proof.* Assume that  $t$  is positive. Let  $P$  be a point in  $E(\mathbb{Q}) - E_0(\mathbb{Q})$  and  $\ell \geq 2$  such that  $\ell P = (0, n^3)$ . By Lemma 5,  $P$  must be an integral point. By Theorem 13 we know that  $\widehat{h}(P) \geq \frac{13}{80} \log(t) - \frac{1}{8} \log(n) + 0.06$  and by Proposition 17, we have  $\widehat{h}((0, n^3)) \leq \log(t) + 1.57$ . We note that  $P$  has infinite order and then  $\widehat{h}(P) \neq 0$ . Since  $\widehat{h}$  is quadratic, we have

$$\begin{aligned} \ell^2 = \frac{\widehat{h}((0, n^3))}{\widehat{h}(P)} &\leq \frac{\log(t) + 1.57}{\frac{13}{80} \log(t) - \frac{1}{8} \log(n) + 0.06} \\ &\leq \frac{80}{13} + \frac{\frac{2}{3} \log(n) + 1.21}{\frac{13}{80} \log(t) - \frac{1}{8} \log(n) + 0.06} \\ &\leq 8.6 \end{aligned}$$

thus  $\ell \leq 2$ . However we have seen at the beginning of Section 4.1 that  $\ell$  is odd, hence the point  $(0, n^3)$  is not divisible.

Assume now that  $t$  is negative. We have by a similar argument (see Theorem 13 and Proposition 17),

$$\begin{aligned} \ell^2 &\leq \frac{\log(|t|) + 1.57}{\frac{3}{16} \log(|t|) - \frac{1}{8} \log(n)} \\ &\leq 5.34 + \frac{\frac{2}{3} \log(n) + 1.57}{\frac{3}{16} \log(|t|) - \frac{1}{8} \log(n)} \\ &\leq 8.85. \end{aligned}$$

Again we get  $\ell \leq 2$  which is impossible. Thus the point  $(0, n^3)$  is not divisible.  $\square$

Using Theorem 16 and Proposition 18, we obtain the following statement.

**Proposition 20.** *We assume  $n = 4m$  and  $t = 4k + 1$ , for some  $m \in \mathbb{Z}_{>0}$  and  $k \in \mathbb{Z}$ . If  $\delta$  is squarefree, and  $t \geq \max(100n^2, n^4)$  or  $t \leq \min(-100n^2, -2n^4)$ , then the point  $(0, n^3)$  is not divisible on  $E$ .*

By combining Propositions 19 and 20, we obtain the main result.

**Theorem 21.** *Suppose that  $t \geq \max(100n^2, n^4)$  or  $t \leq \min(-100n^2, -2n^4)$ , and  $\delta$  is squarefree. Then the point  $(0, n^3)$  is not divisible on  $E$ .*

Now, to extend Theorem 21 to the case  $|t| < \max(100n^2, n^4)$ , we can use the lower bound for the height in [6, Proposition 1.7]: for all  $P \in E(\mathbb{Q})$  we have

$$\widehat{h}(P) \geq \frac{1}{768C_E^2} \log |\Delta_{\min}|,$$

where  $C_E$  is the lowest common multiple of the Tamagawa numbers of  $E$  and  $\Delta_{\min}$  is the minimal discriminant of  $E$ . Let us denote by  $B_E$  this lower bound. In our case, if  $n$  is squarefree we have

$$B_E = \frac{\log |\Delta|}{110592}.$$

So for a given  $t$ , it suffices to check that  $(0, n^3)$  is not equal to  $\ell P$  for all  $P \in E(\mathbb{Q})$  and for all primes  $\ell \leq \sqrt{\frac{\widehat{h}(0, n^3)}{B_E}}$ .

We use the following method: if there exists a prime  $\ell$  and  $P \in E(\mathbb{Q})$  such that  $\ell P = (0, n^3)$  then for all primes  $p \nmid \Delta$ , the reduction of  $E$  modulo  $p$  is an elliptic curve over  $\mathbb{F}_p$  and  $\ell \overline{P} = \overline{(0, n^3)}$ , where  $\overline{P}$  denote the reduction of the point  $P$ . If  $p$  is a prime number such that  $\ell$  divides the exponent of group  $E(\mathbb{F}_p)$  denoted by  $r_p$ , we shall have  $\frac{r_p}{\ell} \overline{(0, n^3)} = O$ . So, if we find a prime  $p$  such that  $\ell \mid r_p$  and  $\frac{r_p}{\ell} \overline{(0, n^3)} \neq O$ , there does not exist  $P \in E(\mathbb{Q})$  such that  $\ell P = (0, n^3)$ .

We obtain the following statement.

**Theorem 22.** *If  $n \leq 10$  and  $\delta$  is squarefree, then the point  $(0, n^3)$  is not divisible.*

*Proof.* Assume  $n = 1$ . Theorem 21 implies that if  $|t| \geq 100$ , the point is not divisible. We also note that the result is true for negative  $t$  because  $(0, 1)$  is the only integer point in  $E(\mathbb{Q}) - E_0(\mathbb{Q})$ . For  $1 \leq t < 100$ , it suffices to use the method described above with the help of PARI/GP [10]. For  $2 \leq n \leq 10$ , we use a similar method.  $\square$

**Remark.** Let  $n \geq 1$ . If  $t = 5n^2$ , the point  $(0, n^3)$  on  $E$  is divisible and satisfies  $(0, n^3) = 3(-4n^2, 7n^3)$ . The elliptic curve  $E$  is isomorphic over  $\mathbb{Q}$  to the elliptic curve  $y^2 = x^3 + 5x^2 - 8x + 1$ . Therefore in this case, the rank of  $E$  over  $\mathbb{Q}$  is 1 and the point  $(-4n^2, 7n^3)$  is a generator of  $E(\mathbb{Q})$ . Note that we have not found another example where the point  $(0, n^3)$  is the multiple of some point.

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