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Letter

The role of constant vorticity on weakly nonlinear surface gravity waves



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ABSTRACT

This manuscript describes the derivation of systems of equations for weakly nonlinear gravity waves in shallow water in the presence of constant vorticity. The derivation is based on a multi-layer generalization of the traditional columnar Ansatz. A perturbative development in a nonlinear parameter and a dispersive parameter allow us to obtain sets of equations, for the horizontal fluid velocity and the free surface, able to describe propagation of weakly nonlinear and dispersive surface waves moving in water with some prescribed initial constant vorticity. We have shown that vorticity plays a central role on the dispersive properties of the system. When it is weak, it acts as a correction in linear and nonlinear dispersive terms. When stronger, it can also influence the nondispersive behavior of the system. Explicit steady solutions of the system corresponding to zero, weak, normal or strong vorticity are obtained. They correspond to solitary waves. Evolution of the soliton celerity, amplitude and width for these four cases are discussed.

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1. Introduction

When propagating in coastal waters, surface waves often encounter currents. These currents, with intensities ranging from weak to very strong are generated through various mechanisms, such as oceanic circulation, tides, wind action or wave breaking. Given these generation processes, the currents are often observed to vary with depth, and result in an underlying vorticity. Such background vorticity, included within strong tidal currents [1] or in wind driven currents [2] could be important and should be taken into account in modeling the propagation of water waves [3]. This underlying vorticity is especially observed in shallow water environments. For instance, strong currents, linearly sheared, were observed in the surf zone, in strong rip currents, in situ [4] or in laboratory experiments [5]. More recently, a similar vertical structure of the current was observed over coral reefs [6].

From a theoretical point of view the role played by constant or variable vorticity constitutes a vast and classical subject in fluid mechanics. Da Silva and Peregrine have studied [7] steady surface waves in water of finite depth with constant vorticity. In Refs. [8–12], Constantin et al. have developed a complete study of exact steady periodic water waves with vorticity. Castro and Lannes [13] investigated fully nonlinear long-waves under the action of vorticity with a Green-Naghdi equation. Numerical studies of this problem can be found in references Ko and Strauss [14,15] and Milewski,

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Vanden-Broeck and Z. Wang [16] However very few works have concentrated to modeling nonlinear dynamics of long surface waves in finite depth under the actions of a sheared current. The pioneering work in this domain was that of Benjamin [17] who generalized the classical solitary wave theory to surface waves under the action of vorticity. Freeman and Johnson [18] derived a Korteweg-de Vries equation in a flow of arbitrary vorticity. Choi has derived the Green–Naghdi equation and the associated Boussinesq equation for small amplitude wave. In Ref. [2] Johnson studied the problem of two-way propagation as described by the Boussinesq equation and derived a new Boussinesq-type equation valid for constant vorticity. The solitary-wave solution for this new equation is exhibited. The Camassa–Holm equation for water waves with constant vorticity and its solitary-wave solution were described. Kharif, Abid and Touboul have studied the action of vorticity on the rogue waves dynamics [19]. More recently Kharif and Abid [20] have studied the role played by constant vorticity on surface waves using a generalized Whitham equation [21].

The main objective of this work is to derive higher order evolution equations for weakly nonlinear long surface gravity waves in shallow water in the presence of constant vorticity beyond all orders in nonlinearity, dispersion and nonlinear dispersion. The employed method is based on the Johnson's work [22]. A generalized columnar pattern Ansatz is introduced, together with the two perturbative parameters : the nonlinear parameter $\alpha = a/h < 1$ and the dispersive parameter $\delta = kh < 1$, where h refers to the constant depth of the fluid while a and k are a typical amplitude and wavenumber of the wave being studied. This generalization is based on two main hypothesis. Firstly, the horizontal velocity of the fluid is represented by means of an infinite series in powers of $z^2\delta^2$. Secondly, the conservation of vorticity is enforced, which provides a consistent closure of the problem. This results in a generalized Serre–Green–Naghdi model with vorticity equivalent to a *multi-layer model* in place of the single-layer model (see Kim et al. [23,24]). An interesting advantage of this approach is that it provides an inter-comparison of the terms describing frequency dispersion, and nonlinear dispersion, even when they involve vorticity. The main result in relation to previous studies is that Ω plays a fundamental role on the dispersive properties of the system. Furthermore, when it is of stronger order of magnitude, vorticity also acts as a corrective term in the non dispersive terms of the system. The role of vorticity can thus be discussed, considering nil, weak, normal or strong vorticity. Furthermore, solitary wave solutions are derived in these four cases, and the role of vorticity on the solitonic properties are fully investigated.

2. The combined effect of vorticity, dispersion, and nonlinearity on long water waves

2.1. General equations of the problem

We consider the water particles to be located by a two-dimensional Cartesian coordinate system with origin 0 and axes x, z , z being orientated upward in the vertical direction. The governing equations are the Euler equations and the equation of mass conservation in x, z and t (time) completed with appropriate surface and bottom boundary conditions. Before perturbations the fluid lies between the impermeable bottom at $z = 0$ and the still water free surface at $z = h$ with h constant. The undisturbed initial state is incompressible, inviscid, with zero surface tension and with a superimposed current profile in the x direction and depending only linearly on z i.e. $\vec{U} = \Omega z \vec{e}_x$ with \vec{e}_x the unitary vector in the x direction. The perturbations to the free surface, horizontal and vertical velocities and pressure (relative to the hydrostatic pressure in the undisturbed initial state) are $\eta(x, t)$, $u(x, z, t)$, $w(x, z, t)$ and $p(x, z, t)$ respectively. The governing equations are made nondimensional as follow (primes mean dimensionless variables): $z' = z/h$, $x' = kx$, $t' = kt\sqrt{gh}$, $\eta' = \eta/h$, with k an average or a typical wavenumber, a the amplitude and g the gravity. This brings to

$$u' = \frac{u}{\sqrt{gh}}, \quad w' = \frac{w}{kh\sqrt{gh}}, \quad p' = \frac{p}{\rho gh}, \quad \Omega' = \frac{\Omega}{k\sqrt{gh}},$$

We now introduce the two small parameters $\alpha = a/h \ll 1$ and $\delta = kh \ll 1$, which are used as perturbative parameters, as it is classically done in small perturbative methods for studying surface water waves problems. The final form of the governing equations require to scale u' , w' , p' and η' with α i.e.: $(u', w', p', \eta') \rightarrow \alpha(u', w', p', \eta')$. As a result we obtain for $0 \leq z \leq 1 + \alpha\eta$ and dropping the primes

$$u_x + w_z = 0, \quad u_t + (\delta\Omega z + \alpha u)u_x + w(\delta\Omega + \alpha u_z) + p_x = 0, \quad w_t + (\delta\Omega z + \alpha u)w_x + \alpha w w_z + \frac{\alpha p_z + 1}{\alpha\delta^2} = 0 \quad (1)$$

$$p = 0 \quad \text{and} \quad \eta_t + [\delta\Omega(1 + \alpha\eta) + \alpha u]\eta_x - w = 0, \quad \text{on } z = 1 + \alpha\eta; \quad \text{and } w = 0, \quad \text{on } z = 0. \quad (2)$$

A classical treatment of the system (1)–(2) relies on its linearization, which admits solutions for the horizontal and vertical velocities u_L and w_L as:

$$u_L(x, z, t) = 2ai\delta \exp\{i\theta(x, t)\} \cosh \delta z, \quad w_L(x, z, t) = 2a \exp\{i\theta(x, t)\} \sinh \delta z$$

with $\exp\{i\theta(x, t)\} = \exp\{i(x - \sigma t)\}$ and σ the linear frequency of expression

$$\sigma = \delta\Omega - \frac{\Omega}{2} \tanh \delta \pm \left\{ \frac{\Omega^2 \tanh^2 \delta}{4} + \frac{\tanh \delta}{2} \right\}^{\frac{1}{2}}.$$

The Johnson derivation of the *single-layer model* equations (in Ref. [22]) means, from a physical mathematical point of view, the lower order in δ of the linear horizontal velocity solution u_L proportional to $\exp\{i\theta(x, t)\}$ is replaced by

an unknown function $u(x, t)$ which is supposed to bring an *approximate* solution of the entire nonlinear problem. Our approach can be viewed as a generalization of this columnar hypothesis where each one of the orders

$$\delta^0 z^0 \exp\{i\theta(x, t)\}, \quad \delta^2 z^2 \exp\{i\theta(x, t)\}, \delta^4 z^4 \exp\{i\theta(x, t)\}, \dots$$

is replaced by

$$u_0(x, t), \quad \delta^2 z^2 u_2(x, t), \quad \delta^4 z^4 u_4(x, t), \dots \text{ so, } u(x, z, t) = \sum_{n=0}^{\infty} u_{2n}(x, t) \delta^{2n} z^{2n}, \forall n \tag{3}$$

In these two developments, only even terms are retained in the decomposition of u , while uneven terms are kept for the decomposition in w . This assumption, classical in Boussinesq type equations, is justified by the Taylor's expansions of $\cosh(\delta z)$ and $\sinh(\delta z)$ respectively. From the continuity equation and the boundary condition $w = 0$ on $z = 0$ it follows that

$$w(x, z, t) = - \sum_{n=0}^{\infty} u_{2n,x} \frac{\delta^{2n} z^{2n+1}}{2n+1}. \tag{4}$$

Substituting $u(x, z, t)$ and $w(x, z, t)$ in the equation for w_t integrating for $z \in [z, 1 + \alpha\eta]$ and using $p = 0$ at $z = 1 + \alpha\eta$ we obtain $p(x, z, t)$

$$p = 1 + \alpha\eta - z - \frac{\delta^2}{2} [(1 + \alpha\eta)^2 - z^2] [u_{0,xt} + \alpha u_0 u_{0,xx} - \alpha u_{0,x}^2] - \frac{\delta^3}{3} [(1 + \alpha\eta)^3 - z^3] \Omega u_{0,xxx} + \mathcal{O}(\delta^4). \tag{5}$$

Using the expression of p in u_t and integrating for $z \in [0, 1 + \alpha\eta]$ lead to

$$u_{0,t} + \alpha u_0 u_{0,x} + \eta_x = \frac{\delta^2}{3(1 + \alpha\eta)} \left\{ [(1 + \alpha\eta)^3 (u_{0,xt} + \alpha u_0 u_{0,xx} - \alpha u_{0,x}^2)]_x - (1 + \alpha\eta)^3 (u_{2,t} + \alpha(u_0 u_{2,x} - u_2 u_{0,x})) \right\} + \delta^3 \Omega \left\{ (1 + \alpha\eta)^3 \left(\frac{u_{2,x}}{6} - \frac{u_{0,xxx}}{4} - \alpha u_{0,xx} \eta_x (1 + \alpha\eta)^{-1} \right) \right\} + \mathcal{O}(\Omega \delta^5, \delta^4). \tag{6}$$

Finally equation for $\eta(x, t)$ using $u(x, z, t)$ and $w(x, z, t)$ at $z = 1 + \alpha\eta$ gives

$$\eta_t + (1 + \alpha\eta) \delta \Omega \eta_x + [u_0 (1 + \alpha\eta)]_x + \frac{\delta^2}{3} [u_2 (1 + \alpha\eta)^3]_x + \mathcal{O}(\delta^4) = 0. \tag{7}$$

Expressions (6) and (7) are a system of two coupled equations in three fields (and their derivatives): u_0 , u_2 and η . In order to reduce this system to only two fields u_0 (and derivatives) and η we use the conservation of vorticity equation

$$\frac{D\vec{\omega}}{Dt} = 0$$

We obtain

$$\omega_t + (\delta \Omega z + \alpha u) \omega_x + \alpha w \omega_z = 0, \quad \text{with } \omega = |\vec{\omega}|, \quad \alpha \vec{\omega} = (\delta \Omega + \alpha u_z - \alpha \delta^2 w_x) \vec{e}_y, \tag{8}$$

with \vec{e}_y the unitary vector in the y direction. This yields the following expression

$$u_{2n+2} = - \frac{u_{2n,xx}}{(2n+1)(2n+2)}, \quad n = 0, 1, 2, \dots \tag{9}$$

This recurrence allows us to reduce (6) and (7), at any degree of approximation, in a system in u_0 , η and their derivatives. Coming back to Eqs. (6) and (7), we might substitute $u_2(x, t)$ with $u_0(x, t)$ by using the recurrence (9). Then, if dropping terms of greater order than δ^3 and rearranging, we obtain

$$u_{0,t} + \alpha u_0 u_{0,x} + \eta_x - \frac{\delta^2}{2} \left\{ (1 + \alpha\eta)^2 (u_{0,xt} + \alpha u_0 u_{0,xx} - \alpha u_{0,x}^2) \right\}_x - \delta^3 \Omega \left\{ \frac{1}{3} (1 + \alpha\eta)^3 u_{0,xxx} + \alpha (1 + \alpha\eta)^2 u_{0,xx} \eta_x \right\} + \mathcal{O}(\delta^4, \Omega \delta^4) = 0, \tag{10}$$

$$\eta_t + (1 + \alpha\eta) \delta \Omega \eta_x + [u_0 (1 + \alpha\eta)]_x - \frac{\delta^2}{6} [u_{0,xx} (1 + \alpha\eta)^3]_x + \mathcal{O}(\delta^4) = 0. \tag{11}$$

The system of Eqs. (10)–(11) is obtained for small values of the parameter α and δ . However, it is emphasized that no assumption was performed on the order of magnitude of Ω . The purpose of this work is to analyze the role played by the vorticity on this system. Following this purpose in the next section, a further assumption is to be introduced, the order of magnitude of the vorticity being compared to the dispersive parameter of the system δ . Indeed, three cases will be distinguished, which are weak, normal or strong vorticity, assuming respectively that $\Omega = \Omega_0 \delta = \mathcal{O}(\delta)$, $\Omega = \Omega_0 = \mathcal{O}(1)$, and $\Omega = \Omega_0 / \delta = \mathcal{O}(\delta^{-1})$. In this further decomposition, Ω_0 is, of course, of order 0 in δ , setting the value of the constant vorticity.

2.2. Long wave evolution equations

2.2.1. The irrotational case $\Omega = 0$

We shall start by considering the zero vorticity case, with $\Omega = \Omega_0 = 0$. Starting from Eqs. (10) and (11), and introducing this further assumption, we obtain

$$u_{0,t} + \alpha u_0 u_{0,x} + \eta_x - \frac{\delta^2}{2} \left\{ u_{0,xt} + \alpha u_0 u_{0,xx} - \alpha u_{0,x}^2 + 2\alpha \eta u_{0,xt} \right\}_x = 0, \tag{12}$$

$$\eta_t + u_{0,x} + \alpha(u_0 \eta)_x - \frac{\delta^2}{6} u_{0,xxx} - \frac{\alpha \delta^2}{2} (u_{0,xx} \eta)_x = 0. \tag{13}$$

These equations are different from the classical Green–Naghdi system because we have used the conservation of the vorticity theorem. *En passant*, we note that if $D\bar{\omega}/Dt = 0$ had been used independently from the order in δ , the Green–Naghdi system would be reduced (as it can easily be shown) to the Shallow Water Equation.

We now make a further assumption, by searching for traveling waves. Though, we might introduce the change in variables $r = x - ct$, c being the speed of propagation. We thus exclude waves traveling to the $(-x)$ direction. Inserting this new variable in the system above, and integrating equation (12), we can eliminate the unknown δ from the system. We thus obtain the equation

$$(1 - c^2) u_0 + \frac{3\alpha}{2} c u_0^2 + \delta^2 \left(\frac{c^2}{2} - \frac{1}{6} \right) u_{0,rr} + \frac{\alpha \delta^2}{2} [(2c^3 - 3c) u_0 u_{0,rr} + c u_{0,r}^2] = 0. \tag{14}$$

2.2.2. The weak vorticity case, $\Omega = \Omega_0 \delta$

Now, we investigate the weak vorticity case, by introducing the relation $\Omega = \Omega_0 \delta = \mathcal{O}(\delta)$ in Eqs. (10) and (11). To introduce this assumption in the system, we proceed as done in Section 2.2.1. By replacing the unknown u_2 with its expression in u_0 , introducing the traveling wave hypothesis, integrating the system, and eliminating the unknown η , we obtain the equation on u_0 , which reads

$$(1 - c^2) u_0 + \frac{3\alpha}{2} c u_0^2 + \delta^2 \left[\left(\frac{c^2}{2} - \frac{1}{6} \right) u_{0,rr} + \Omega_0 c u_0 \right] + \frac{\alpha \delta^2}{2} [(2c^3 - 3c) u_0 u_{0,rr} + c u_{0,r}^2 + (c^2 - 1) \Omega_0 u_0^2] = 0. \tag{15}$$

2.2.3. The rotational case, $\Omega = \Omega_0 \neq 0$

Again, we shall start from Eqs. (10) and (11), and introduce the assumption $\Omega = \Omega_0 = \mathcal{O}(1)$. We shall now follow the procedure described in Section 2.2.1, and obtain the equation

$$(1 - c^2) u_0 + \frac{3\alpha}{2} c u_0^2 + \delta \Omega_0 c u_0 - \frac{\alpha \delta}{2} (1 - c^2) \Omega_0 u_0^2 + \delta^2 \left(\frac{c^2}{2} - \frac{1}{6} \right) u_{0,rr} + \frac{\alpha \delta^2}{2} [(2c^3 - 3c) u_0 u_{0,rr} + c u_{0,r}^2] = 0. \tag{16}$$

2.2.4. The strong vorticity case, $\Omega = \Omega_0 \delta^{-1}$

Finally, in this section, we examine the hypothesis where the vorticity is of great order of magnitude with respect to the dispersive properties. This assumption writes $\Omega = \Omega_0 / \delta = \mathcal{O}(\delta^{-1})$. Again, the procedure applied in the previous subsections, with the further assumption of traveling waves, leads to the following equation

$$(1 - c^2 + \Omega_0 c) u_0 + \frac{\alpha}{2} (3c + \Omega_0 c^2 - \Omega_0) u_0^2 + \delta^2 \left[\frac{c^2}{2} - \frac{1}{6} - \frac{\Omega_0}{6} (2\Omega_0 - c) \right] u_{0,rr} + \frac{\alpha \delta^2}{2} \left[(2c^3 - 3c + \frac{\Omega_0 + \Omega_0 c (4\Omega_0 - 15c)}{3}) u_0 u_{0,rr} + (c - \Omega_0) u_{0,r}^2 \right] = 0. \tag{17}$$

Yet, it is emphasized that the terms of order $\mathcal{O}(\Omega \delta^4)$, which were dropped in the system (10)–(11) for sake of brevity were kept in this derivation, to insure the consistency of Eq. (17).

2.3. The combined effect of vorticity, dispersion, and nonlinearity on long water waves

The four Eqs. (14)–(17) correspond to four long wave evolution equations, involving the influence of the vorticity, for various order of magnitude. The differences between the four equations are presented in Table 1, which collects each and every term, as a function of their order of magnitude in α , δ and their combinations.

In this table, the line denoted $\Omega = 0$ corresponds to our reference case, in the absence of vorticity. Collecting the terms of Eq. (14), as it is classically done in Boussinesq like models, we identify the contribution of each and every term on the dispersive properties of the system. The term in $\mathcal{O}(1)$ corresponds to the non-dispersive contribution, the term of

Table 1
Modifications of the long wave equations in the presence of weak, normal or strong vorticity.

	$\mathcal{O}(1)$	$\mathcal{O}(\alpha)$	$\mathcal{O}(\delta)$	$\mathcal{O}(\alpha\delta)$	$\mathcal{O}(\delta^2)$	$\mathcal{O}(\alpha\delta^2)$
$\Omega = 0$	$(1 - c^2)u_0$	$\frac{3}{2}cu_0^2$			$\left(\frac{c^2}{2} - \frac{1}{6}\right)u_{0,rr}$	$(2c^3 - 3c)u_0u_{0,rr} + cu_{0,r}^2$
$\Omega = \mathcal{O}(\delta)$	$(1 - c^2)u_0$	$\frac{3}{2}cu_0^2$			$\left(\frac{c^2}{2} - \frac{1}{6}\right)u_{0,rr} + \Omega_0cu_0$	$(2c^3 - 3c)u_0u_{0,rr} + cu_{0,r}^2 + (c^2 - 1)\Omega_0u_0^2$
$\Omega = \mathcal{O}(1)$	$(1 - c^2)u_0$	$\frac{3}{2}cu_0^2$	Ω_0cu_0	$-\frac{1}{2}(1 - c^2)\Omega_0u_0^2$	$\left(\frac{c^2}{2} - \frac{1}{6}\right)u_{0,rr}$	$(2c^3 - 3c)u_0u_{0,rr} + cu_{0,r}^2$
$\Omega = \mathcal{O}(\delta^{-1})$	$(1 - c^2)u_0 + \Omega_0cu_0$	$\frac{3c}{2}u_0^2 - \Omega_0(c^2 - 1)u_0^2$			$\left(\frac{c^2}{2} - \frac{1}{6}\right)u_{0,rr} - \frac{\Omega_0}{6}(2\Omega_0 - c)u_{0,rr}$	$(2c^3 - 3c)u_0u_{0,rr} + \frac{\Omega_0 + \Omega_0c(4\Omega_0 - 15c)}{3}u_0u_{0,rr} + (c - \Omega_0)u_{0,r}^2$

order $\mathcal{O}(\alpha)$ is the nonlinear correction, also known as shoaling effect, and the terms in $\mathcal{O}(\delta^2)$ and $\mathcal{O}(\alpha\delta^2)$ are respectively known as the linear dispersion (or frequency dispersion) and nonlinear dispersion. More details about this classification can be found in [25]. Table 1 provides a direct comparison of these terms, when vorticity is involved at various order, namely $\Omega = \mathcal{O}(\delta)$, $\mathcal{O}(\delta^0)$ and $\mathcal{O}(\delta^{-1})$.

When weak vorticity is considered ($\Omega = \mathcal{O}(\delta)$), the behaviour of the system is almost unmodified. The vorticity enters the equation as a direct correction in terms $\mathcal{O}(\delta^2)$ and $\mathcal{O}(\alpha\delta^2)$, but the structure of the equation remains unchanged. This is understood to be a modification of the dispersive properties of long waves, both linear and nonlinear.

On the other hand, the cases of normal and strong vorticity are more complicated. For the first case, if $\Omega = \mathcal{O}(\delta^0)$, two new terms appear in the evolution equation, in the sense these terms are expressed at an order of magnitude which was not involved precedently. These two terms are respectively of order $\mathcal{O}(\delta)$ and $\mathcal{O}(\alpha\delta)$, and are respectively proportional to u_0 and u_0^2 . This is very similar to the first terms observed in Eqs. (14) and (15), expressed at orders $\mathcal{O}(1)$ and $\mathcal{O}(\alpha)$. It is thus understood that vorticity, when involved to this order of magnitude, acts as a non dispersive correction, at lower order, though ($\mathcal{O}(\delta)$). It also enters as a correction at lower in δ in the nonlinear term. In the meantime, it is emphasized that the dispersive behaviour of the equation (the terms in $\mathcal{O}(\delta^2)$ and $\mathcal{O}(\alpha\delta^2)$) are unaffected by the vorticity.

Finally, the strong vorticity case, involving $\Omega = \mathcal{O}(\delta^{-1})$, presents the largest complexity. In this case, the vorticity is expressed in each and every term. First, it plays a role on non dispersive effects and nonlinear effects, at the leading order. In the meantime, vorticity is also involved in the dispersive terms of order $\mathcal{O}(\delta^2)$ and $\mathcal{O}(\alpha\delta^2)$, also at the leading order. It is thus understood that when vorticity is strong, it changes completely the nature of the evolution equation, playing a significant role on non-dispersive and nonlinear terms, but also on the dispersive properties of the system, both in frequency and nonlinear dispersion.

3. Effect of the vorticity on traveling waves solutions

3.1. Solitonic solutions of the evolution equations

The four long wave evolution equations obtained in Section 2.2 exhibit an imbalance between linear dispersion and nonlinear dispersion terms. Thus, we expect these four equations to admit solutions describing solitary waves. Finding these solutions is the purpose of this subsection.

3.1.1. The irrotational case $\Omega = 0$

Starting from Eq. (14), we now seek for solitary waves solutions. This might be done by assuming the existence of solutions in the form

$$u_0(r) = A\text{sech}^2(pr) + \alpha B\text{sech}^4(pr), \tag{18}$$

with c a supercritical velocity defined by

$$c = c_0 + \alpha V = 1 + \alpha V,$$

and where V , p and B are constants in r , yet functions of A . Substituting $u_0(r)$ in (14), we obtain the unknown V , p , B , which read

$$V = \frac{A}{2},$$

$$p = \frac{\sqrt{3\alpha A}}{2\delta} \left[1 - \frac{3}{4}\alpha A \right],$$

$$B = \frac{3A^2}{20}.$$

Thus, the soliton solution for $u_0(x, t)$ reads

$$u_0(x, t) = A \operatorname{sech}^2 [p(x - ct)] + \alpha \frac{3A^2}{20} \operatorname{sech}^4 [p(x - ct)], \quad c = c_0 + \alpha \frac{A}{2}.$$

Finally, the soliton solution in terms of η is obtained by introducing the expression of $u_0(r)$ (and its derivatives) in Eq. (13). We obtain

$$\eta(x, t) = A(1 - \alpha A) \operatorname{sech}^2 [p(x - ct)] + \alpha \frac{19}{10} A^2 \operatorname{sech}^4 [p(x - ct)]. \tag{19}$$

Furthermore, the parameter $\alpha = a/h$ is, by definition, the dimensionless amplitude of η . The maximum of η being observed at $x - ct = 0$, the relation $\alpha = \eta(0)$ has to be fulfilled. Thus, the dependence of A with respect to α is necessarily the positive solution of the quadratic polynomial

$$A^2 \left(\frac{19}{20} - 1 \right) \alpha + A - \alpha = 0. \tag{20}$$

3.1.2. The weak vorticity case, $\Omega = \Omega_0 \delta$

We now seek for solitary solutions for the cases involving weak vorticity, as described by Eq. (15). This is achieved by reproducing the previous, and seeking for solitary solutions in the form of (18). Doing so, we obtain

$$c_0 = 1 + \frac{1}{2} \delta^2 \Omega_0,$$

$$V = A \left[\frac{1}{2} + \frac{1}{4} \delta^2 \Omega_0 \right],$$

$$p = \frac{\sqrt{3\alpha A}}{2\delta} \left[1 + \frac{1}{4} \delta^2 \Omega_0 \right] \left[1 - \frac{3}{4} \alpha A \left(1 + \frac{1}{2} \delta^2 \Omega_0 \right) \right],$$

$$B = \frac{3A^2}{20},$$

the solitary wave, in terms of velocity, being given by

$$u_0(x, t) = A \operatorname{sech}^2 [p(x - ct)] + \alpha \frac{3A^2}{20} \operatorname{sech}^4 [p(x - ct)], \quad c = c_0 + \alpha \frac{A}{2} \left[1 + \frac{1}{2} \delta^2 \Omega_0 \right],$$

and in terms of elevation,

$$\eta(x, t) = A \left(c_0 - \alpha A \left[1 + \frac{5}{4} \delta^2 \Omega_0 \right] \right) \operatorname{sech}^2 [p(x - ct)]$$

$$+ \alpha A^2 \left[\frac{19}{10} + \frac{93}{40} \delta^2 \Omega_0 \right] \operatorname{sech}^4 [p(x - ct)]. \tag{21}$$

Finally, the wave elevation at the crest has to be α , which compels the value of A . Indeed, the latter is obtained as the positive root of the quadratic polynomial

$$A^2 \left(-\frac{1}{20} + \frac{179}{120} \delta^2 \Omega_0^2 \right) \alpha + c_0 A - \alpha = 0. \tag{22}$$

3.1.3. The rotational case, $\Omega = \Omega_0 \neq 0$

Again, we reproduce the procedure described in Section 3.1.1. We seek for solutions of Eq. (16) in the form (18). In this equation, the critical velocity satisfies: $1 + \delta \Omega_0 c_0 - c_0^2 = 0$, which admits the positive solution

$$c_0 = 1 + \frac{1}{2} \delta \Omega_0 + \frac{1}{8} \delta^2 \Omega_0^2.$$

Thus, if $u_0(r)$ is a solitary wave of the form (18) with a supercritical velocity $c = c_0 + \alpha V$, we find

$$V = A \left[\frac{1}{2} + \frac{1}{4} \delta \Omega_0 + \frac{1}{6} \delta^2 \Omega_0^2 \right],$$

$$p = \frac{\sqrt{3\alpha A}}{2\delta} \left[1 + \frac{1}{4} \delta \Omega_0 + \frac{19}{96} \delta^2 \Omega_0^2 \right] \left[1 - \frac{3}{4} \alpha A \left(1 + \frac{1}{2} \delta \Omega_0 + \frac{1}{3} \delta^2 \Omega_0^2 \right) \right],$$

$$B = \frac{3A^2}{20}.$$

Hence, examining the case of constant vorticity Ω of order $\mathcal{O}(\delta^0)$, we obtain the expression

$$u_0(x, t) = A \operatorname{sech}^2[p(x - ct)] + \alpha \frac{3A^2}{20} \operatorname{sech}^4[p(x - ct)], \quad c = c_0 + \alpha \frac{A}{2} \left[1 + \frac{1}{2} \delta \Omega_0 + \frac{1}{3} \delta^2 \Omega_0^2 \right],$$

for $u_0(x, t)$, and for $\eta(x, t)$,

$$\begin{aligned} \eta(x, t) = A \operatorname{sech}^2[p(x - ct)] & \left(c_0 - \alpha A \left[1 + \frac{5}{6} \delta \Omega_0 + \frac{13}{12} \delta^2 \Omega_0^2 \right] \right) \\ & + \alpha A^2 \left[\frac{19}{10} + \frac{93}{40} \delta \Omega_0 + \frac{303}{160} \delta^2 \Omega_0^2 \right] \operatorname{sech}^4[p(x - ct)], \end{aligned} \tag{23}$$

Finally, the crest amplitude of the solitary wave having an elevation of α , the value of A is provided by the positive root of the polynomial

$$A^2 \left(-\frac{1}{20} + \frac{179}{120} \delta \Omega_0 + \frac{389}{480} \delta^2 \Omega_0^2 \right) \alpha + c_0 A - \alpha = 0. \tag{24}$$

3.1.4. The strong vorticity case, $\Omega = \Omega_0 \delta^{-1}$

Finally, we examine the case of strong vorticity, when $\Omega = \mathcal{O}(\delta^{-1})$. Starting from a solution in the form

$$u_0(r) = A \operatorname{sech}^2(pr) + \alpha B \operatorname{sech}^4(pr),$$

we obtain the expression of the constants

$$\begin{aligned} c_0 &= \frac{\Omega_0}{2} + \sqrt{1 + \frac{\Omega_0^2}{4}} \\ V &= A c_0 \frac{3 + \Omega_0^2}{3(2c_0 - \Omega_0)}, \\ p &= \frac{1}{2\delta} \sqrt{\frac{\alpha A c_0 (3 + \Omega_0^2)}{\Omega_0^2 - c_0^2 + 2}} \left[1 + \frac{\alpha A c_0 (3 + \Omega_0^2)}{2(2c_0 - \Omega_0)(\Omega_0^2 - c_0^2 + 2)} \left(\frac{5\Omega_0}{6} - c_0 \right) \right], \\ B &= \frac{3A^2}{20} \left[\frac{\Omega_0^2 c_0 + 6\Omega_0 + c_0}{\Omega_0^2 - c_0^2 + 2} \right]. \end{aligned}$$

Following the procedure described previously, we obtain the expression of the solitary wave, in terms of velocity $u_0(x, t)$,

$$u_0(x, t) = A \operatorname{sech}^2[p(x - ct)] + \alpha \frac{3A^2}{20} \left[\frac{\Omega_0^2 c_0 + 6\Omega_0 + c_0}{\Omega_0^2 - c_0^2 + 2} \right] \operatorname{sech}^4[p(x - ct)], \quad c = c_0 + \frac{\alpha A c_0 (3 + \Omega_0^2)}{3(2c_0 - \Omega_0)},$$

and in terms of elevation $\eta(x, t)$,

$$\begin{aligned} \eta(x, t) = c_0 A \operatorname{sech}^2[p(x - ct)] & + c_0 (3 + \Omega_0^2) \left(\frac{1}{3(2c_0 - \Omega_0)} + \frac{2\Omega_0 - 3c_0}{6(\Omega_0^2 - c_0^2 + 2)} \right) \alpha A^2 \operatorname{sech}^2[p(x - ct)] \\ & + \frac{3c_0}{\Omega_0^2 - c_0^2 + 2} \left(\frac{\Omega_0^2 c_0 + 6\Omega_0 + c_0}{20} + \frac{(3 + \Omega_0^2)(3c_0 - 2\Omega_0)}{12} \right) \alpha A^2 \operatorname{sech}^4[p(x - ct)], \end{aligned} \tag{25}$$

A being determined as the positive root of the polynomial

$$A^2 \left(\frac{c_0(3 + \Omega_0^2)}{3(2c_0 - \Omega_0)} - \frac{c_0(3 + \Omega_0^2)(2\Omega_0 - 3c_0)}{12(\Omega_0^2 - c_0^2 + 2)} + \frac{3c_0(\Omega_0^2 c_0 + 6\Omega_0 + c_0)}{20(\Omega_0^2 - c_0^2 + 2)} \right) \alpha + c_0 A - \alpha = 0. \tag{26}$$

3.2. Characteristics of the solitary waves

The quasi-steady solutions obtained in the previous section present various behavior, depending on the order of magnitude considered for the vorticity. The purpose of this section is to describe the influence of vorticity magnitude on the properties of solitary waves.

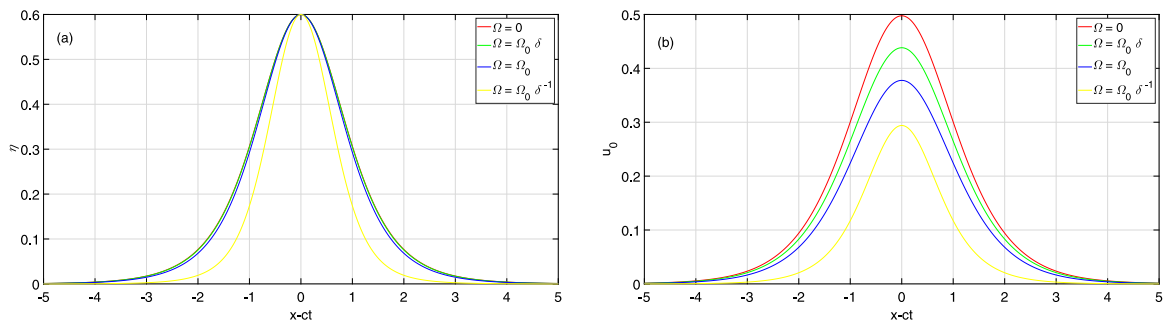


Fig. 1. Comparison of the free surface elevation η (a) and the mean velocity u_0 (b) obtained with the four model equations. The red lines correspond to the case without any vorticity ($\Omega = 0$), the green lines are the model with weak vorticity ($\Omega = \Omega_0\delta$), the blue lines describe the model with unit vorticity ($\Omega = \Omega_0$), and the yellow lines correspond to the strong vorticity case ($\Omega = \Omega_0/\delta$). The parameters used are $\alpha = 0.6$, and $\delta = 0.5$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

3.2.1. Effect of the vorticity on the soliton properties

The purpose of this section is to describe the influence of vorticity magnitude on the properties of solitary waves. Fig. 1 presents the main characteristic of the solitary waves obtained for all cases of vorticity. Fig. 1(a) corresponds to the water elevation η , while Fig. 1(b) illustrates the distribution of the mean horizontal velocity u_0 induced by the perturbation. In both subfigures, non dimensional parameters are chosen such as $\alpha = 0.6$, $\delta = 0.5$. In every case considered the reference vorticity Ω_0 is chosen to be unity. In these figures red lines describe the results obtained when no vorticity is considered. Green lines correspond to the weak vorticity case, the blue lines are the result obtained with the normal vorticity, while the yellow lines describe the results obtained when the vorticity is strong. Since the value of α is prescribed, the four curves describing water elevation (Fig. 1(a)) present the same maximum values. Differences appearing only concern the waves width, and thus the local slopes. The four curves presented in Fig. 1(a) show a similar tendency. Solitary waves are narrower in the presence of vorticity. Indeed, if the curves obtained with no vorticity, and with weak vorticity are almost perfectly superimposed, one can notice that the model involving a unit vorticity describes a slightly narrower wave. The effect of vorticity when considering these two vortical conditions is almost insignificant, in terms of wave width. On the other hand, when considering the model assuming strong vorticity, the wave obtained is clearly narrower than the previous ones. The trend observed previously emphasized, and is more visible here.

When considering the average velocity u_0 , differences are much more significant induced by the solitary wave. These velocities are presented in Fig. 1(b). From this figure, it clearly appears that the maximum velocity induced by the free surface disturbance is strongly affected by the magnitude of the vorticity. Namely, the stronger is the vorticity, the weaker is the fluid disturbance. It appears that the maximum of the velocity disturbance can be reduced up to 40% when the vorticity becomes strong. From Eq. (17), we observed that strong vorticity adds a dispersive term and a nonlinear dispersive term. It is responsible for the trend leading to the weakest solitonic profile downstream propagating for strong vorticity. The effects on $u_0(x, t)$ of normal and weak vorticities occur only through $\eta(x, t)$. For strong vorticity $u_0(x, t)$ directly depends on Ω through the new dispersive and nonlinear dispersive term.

The evolution of the fluid velocity, observed in previous figures has to be related with the kinematic properties of the solitary wave. The evolution of its velocity and width are depicted in Fig. 2. On Fig. 2(a), the evolution of the wave celerity c of the solitary wave is plotted as a function of the vorticity Ω , for the four models. Here again, the nonlinear and dispersive parameters (α, δ) are set equal to (0.6, 0.5), while Ω varies. Since the models do not cover the same range of values for Ω , four colors are used, in order to distinguish the area described by each model. Obviously, the model without vorticity corresponds to a single point in this diagram, while the green line correspond to the model with weak vorticity, the blue line is the model of vorticity of order unity, and the yellow line is the strong vorticity assumption. The first result observed is an opposite trend to the behaviour shown by the fluid disturbance velocity u_0 . Namely, the celerity of the wave increases with vorticity, while the components of the fluid disturbance decreases with vorticity. For the parameters considered, the celerity is increased by about 85%. Another interesting remark, here, is to notice the relative continuity exhibited by the four models. Indeed, they have been derived independently, under various asymptotic assumptions, but present very compatible trends, and values. In Fig. 2(b), the evolution of the width of the wave, λ , defined as $\lambda = 1/p$, is presented versus vorticity Ω for the four models. This figure confirms the trend forebode on Fig. 1(a). Here again, the four models are juxtaposed, where the model without vorticity corresponds to a single point, a red square, in this diagram, the green line correspond to the model with weak vorticity, the blue line is the model of vorticity of order unity, and the yellow line is the strong vorticity assumption. The three models involving vorticity present a similar trend, the width decreasing when vorticity increases.

3.2.2. Effect of the vorticity on the korteweg–de Vries soliton

The behavior of $\lambda(\Omega)$ is quite difficult to analyze from the soliton expressions of $u_0(x, t)$ and $\eta(x, t)$ of (10)–(11). However this becomes possible with soliton solutions of the Korteweg–de Vries equations associated to the system (10)–(11). Let us have a look at $\lambda(\Omega)$ of the soliton solutions of KdV equations for $u_0(x, t)$ in the two extreme cases $\Omega = 0$ and

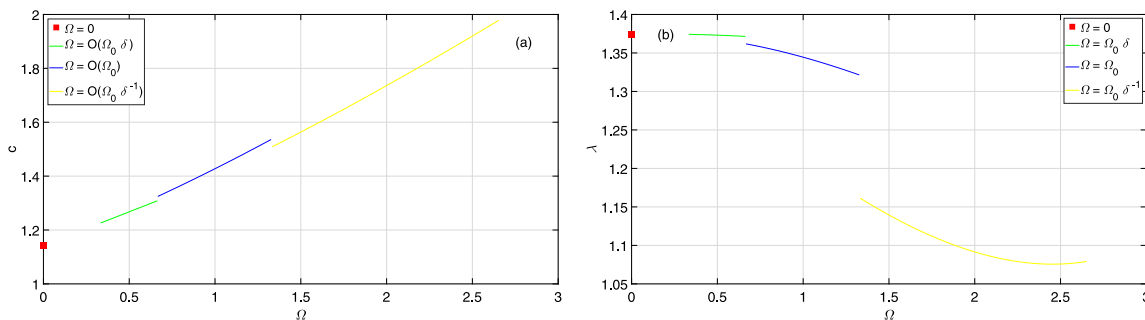


Fig. 2. Evolution of the wave velocity (a), and the wave width (b) as a function of the vorticity. The red squares correspond to the case without any vorticity ($\Omega = 0$), the green lines are the model with weak vorticity ($\Omega = \Omega_0\delta$), the blue lines describe the model with unit vorticity ($\Omega = \Omega_0$), and the yellow lines correspond to the strong vorticity case ($\Omega = \Omega_0\delta^{-1}$). The parameters used are $\alpha = 0.6$, and $\delta = 0.5$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

$\Omega = \Omega_0\delta^{-1}$. Using asymptotic methods (see [21]) in (14) and (17) we obtain $u_{0,t}$ for these two extreme cases. They read

$$u_{0,t} + \frac{3}{2}\alpha u_0 u_{0,x} + \frac{\delta^2}{6}u_{0,xxx} = 0, \quad \Omega = 0, \tag{27}$$

$$(\Omega_0^2 + 4)^{\frac{1}{2}}u_{0,t} + c_0(\Omega_0^2 + 3)\alpha u_0 u_{0,x} + \frac{\delta^2}{6}(c_0^2 - \Omega_0^2)u_{0,xxx} = 0, \quad \Omega = \Omega_0/\delta, \quad c_0 = \frac{1}{2}[\Omega_0 + (\Omega_0^2 + 4)^{1/2}]. \tag{28}$$

The soliton solutions of (27) and (28) read

$$u_0 = A_s \operatorname{sech}^2\left[\frac{1}{2\delta}(3\alpha A_s)^{\frac{1}{2}}\left(x - \frac{A_s\alpha}{2}t\right)\right], \quad u_0 = A_s \operatorname{sech}^2\left\{\frac{1}{2\delta}(3\alpha A_s)^{\frac{1}{2}}\left[\frac{c_0(3 + \Omega_0^2)}{3(c_0^2 - \Omega_0^2)}\right]^{\frac{1}{2}}\left[x - \left(\frac{\alpha A_s c_0(3 + \Omega_0^2)}{3(4 + \Omega_0^2)}\right)t\right]\right\}, \tag{29}$$

with A_s an arbitrary constant amplitude. The associated soliton widths read

$$\lambda(\Omega = 0) = \frac{2\delta}{(3\alpha A_s)^{\frac{1}{2}}}, \quad \lambda(\Omega = \frac{\Omega_0}{\delta}) = \frac{2\delta}{(3\alpha A_s)^{\frac{1}{2}}} \left[\frac{c_0^2 - \Omega_0^2}{c_0(1 + \frac{\Omega_0^2}{3})}\right]^{\frac{1}{2}}. \tag{30}$$

From expressions (30) (taken $\Omega_0 = 1$) we obtain $\frac{\lambda(\Omega = \frac{1}{\delta})}{\lambda(\Omega = 0)} \sim 0,5 < 1$, and (29) in the case $\Omega = \Omega_0\delta^{-1}$ is a downstream wave narrower than in the case $\Omega = 0$. So for downstream propagation the stronger the vorticity the narrower the width. This is a very well known result not only for KdV but also for the Boussinesq equation and the Camassa–Holm equation (see [22]).

3.2.3. Effect of the vorticity on the inner flow

Another interesting point with this approach is to take into account the evolution of the velocity field within the fluid layer. Indeed, accordingly to Eqs. (3) and (4), the velocity can be reconstructed as a power series of $\delta^2 z^2$. If closed with the recurrence relation (9), the velocity in the total fluid layer might be reconstructed, as

$$u(x, z, t) = u_0(x, t) - \frac{1}{2}u_{0,xx}(x, t)\delta^2 z^2 + O(\delta^4 z^4), \tag{31}$$

$$w(x, z, t) = -u_{0,x}(x, t)z + \frac{1}{6}u_{0,xxx}(x, t)\delta^2 z^3 + O(\delta^4 z^4). \tag{32}$$

These are the fields shown in Fig. 3, for a solitary wave of nonlinear parameter $\alpha = 0.6$ and dispersive parameter $\delta = 0.5$, where Ω_0 is taken equal to unity. In this figure, the horizontal components of the velocities within the fluid layer are presented on the left column, while the vertical components of the velocity are shown on the right column. Each line corresponds to one of the models. Thus, sub-figures (a-e) present the two components of the velocity obtained in the absence of vorticity, sub-figures (b-f) are the fields obtained under the assumption of weak vorticity, sub-figures (c-g) describe the velocity fields obtained with the assumption of unit vorticity, while sub-figures (d-h) correspond to the fields obtained under the assumption of a strong vorticity. From these figures, two main results are observed. First, the magnitude of the velocity field decreases when vorticity increases. This result was obtained when considering the mean velocity u_0 . It is confirmed here within the entire fluid layer, concerning both components (u, w) of the velocity. Secondly, these figures allow to compare the evolution of the velocity fields with depth. It appears that the velocity disturbance vanishes faster with depth when vorticity increases. This comes from the fact that in the presence of vorticity, for waves of a given height α , the velocity disturbance diminishes when vorticity increases. Thus, the second term in the decomposition (31),(32) is relatively smaller with respect to the first term, than it is when considering no vorticity.

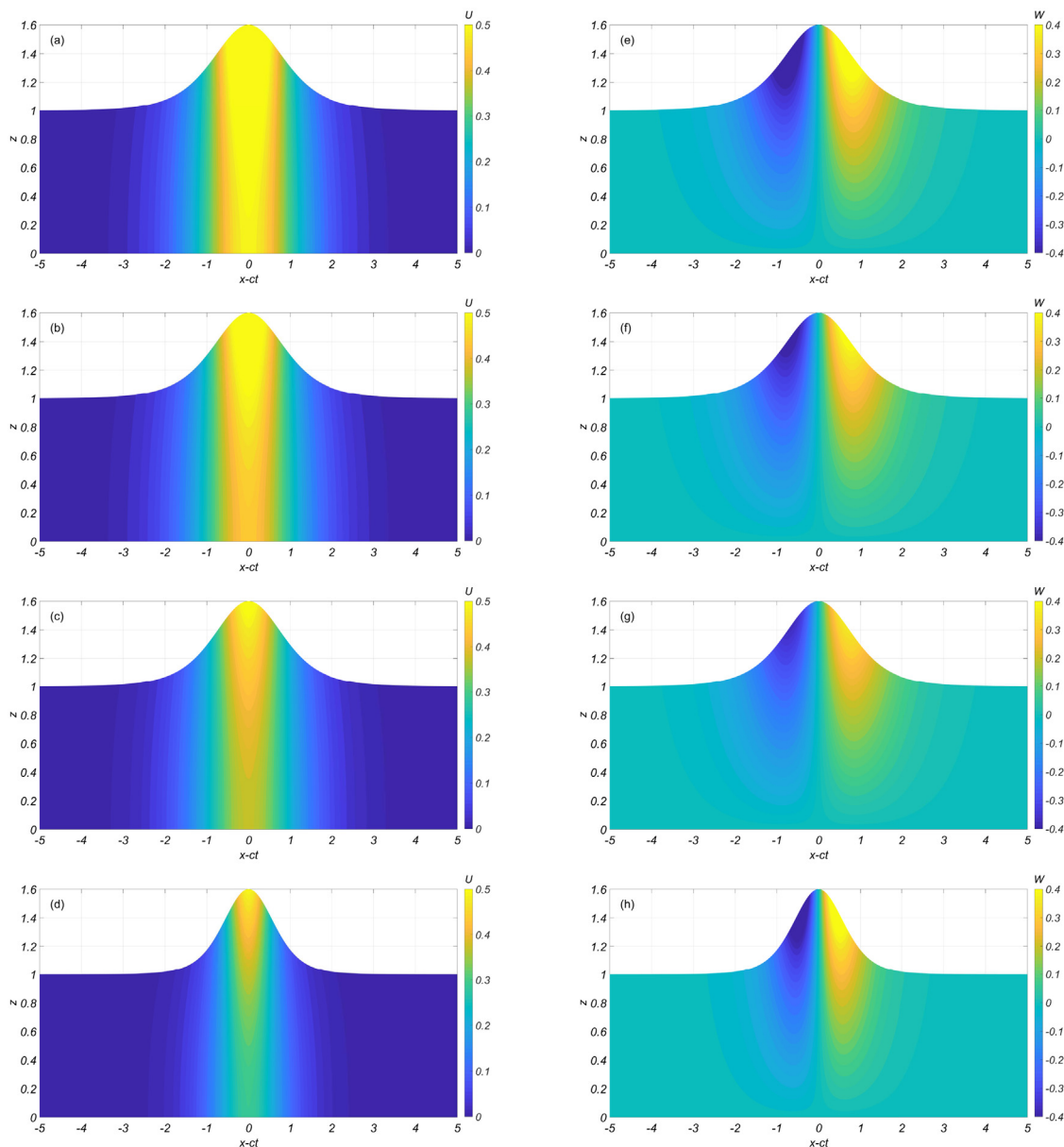


Fig. 3. Distribution of the horizontal (left column) and vertical (right column) components of the velocity within the fluid layer. (a–e) $\Omega = 0$, (b–f) $\Omega = \Omega_0\delta$, (c–g) $\Omega = \Omega_0$, (d–h) $\Omega = \Omega_0\delta^{-1}$. The parameters used are $\alpha = 0.6$, and $\delta = 0.5$.

3.2.4. Effect of α and δ on the soliton properties

Another interesting aspect of this approach is to allow comparisons of the influence of the parameters α and δ on the wave properties. Following that purpose, Fig. 4 describes the evolution of the wave width λ in the parameter map (α, δ) . Sub-figure (a) corresponds to the behavior of a solitary wave propagating in the absence of vorticity, sub-figure (b) is the weak vorticity case, sub-figure (c) is the unit vorticity model, while sub-figure (d) is the strong vorticity configuration. First, it has to be emphasized that these figures present very different color scales, confirming the trends observed in Fig. 2. In the meantime, it is striking to notice that the qualitative dependence on both the nonlinear and the dispersive parameters are very similar. Oppositely, Fig. 5 illustrates the evolution of the wave celerity in the parameter map (α, δ) . Sub-figure (a) corresponds to the behavior of a solitary wave propagating in the absence of vorticity, sub-figure (b) is the weak vorticity case, sub-figure (c) is the unit vorticity model, while sub-figure (d) is the strong vorticity configuration. It is interesting to notice that, in the absence of vorticity, the traveling celerity does not depend on the dispersive parameter δ , which confirms classical behavior of traveling solitary waves in the absence of vorticity. Noteworthy, this result does not hold when vorticity is involved. Indeed, it is interesting to notice that, when weak or unit vorticity are considered, the

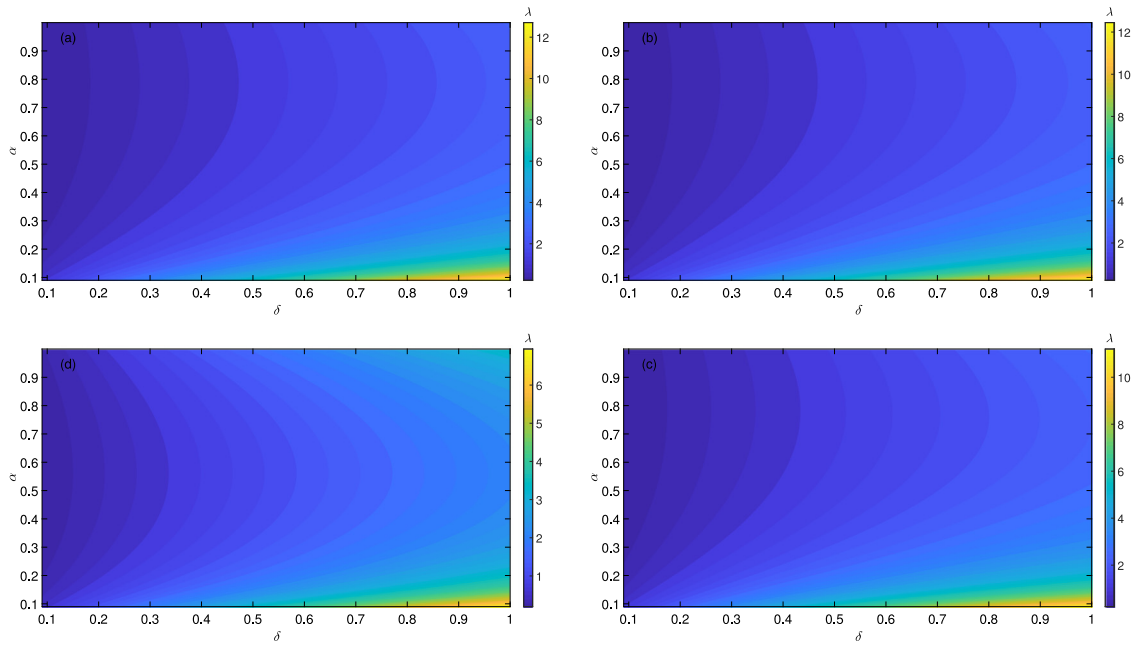


Fig. 4. Evolution of the width of the wave as a function of the dispersive parameter δ and the nonlinear parameter α . (a) $\Omega = 0$, (b) $\Omega = \Omega_0\delta$, (c) $\Omega = \Omega_0$, (d) $\Omega = \Omega_0\delta^{-1}$.

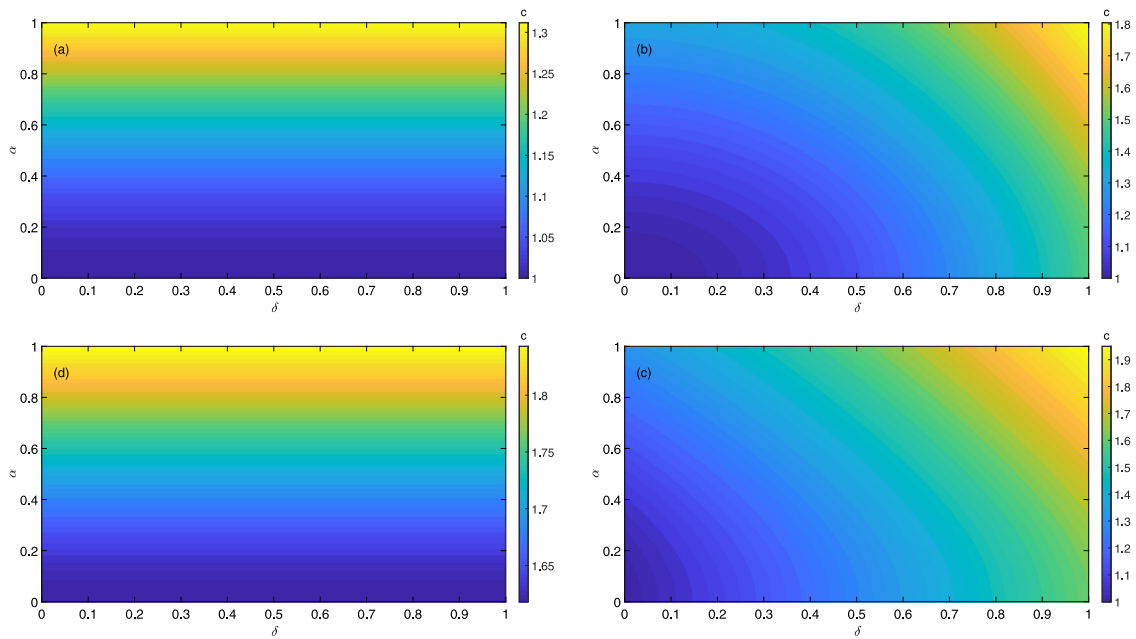


Fig. 5. Evolution of the wave celerity as a function of the dispersive parameter δ and the nonlinear parameter α . (a) $\Omega = 0$, (b) $\Omega = \Omega_0\delta$, (c) $\Omega = \Omega_0$, (d) $\Omega = \Omega_0\delta^{-1}$.

wave celerity exhibits a strong dependence on α and δ . However, the respective influence of α and δ on this quantity is not completely similar. In the weak vorticity model, the dispersive parameter δ seems to play a less significant role than in the unit vorticity approach. Finally, when considering the strong vorticity case, the celerity becomes constant again with respect to δ . This asymptotic case exhibits characteristics very similar to those observed in the absence of vorticity.

4. Conclusions and perspectives

We have introduced a new hierarchy of weakly nonlinear equations for surface gravity in finite depth in the presence of constant vorticity Ω . The members of the hierarchy are systems of equations for the z -integrated horizontal velocity $u(x, z, t)$ and the wave amplitude $\eta(x, t)$. They involve weak nonlinearities, dispersion and nonlinear dispersion and are characterized by the powers of two perturbative parameters α and δ (nonlinearity and dispersion) and their products (nonlinear dispersion). The vorticity Ω appears in the equations of motion in factors $\Omega\delta^s$ ($s = 1, 3, \dots$), so we have scaled Ω as $\Omega = \Omega_0\delta^q$ with $\Omega_0 \sim \mathcal{O}(\delta^0)$ and q integer. The negative values of q must be chosen such that the neglected term $\mathcal{O}(\Omega\delta^5, \delta^4)$ in Eq. (6) remains negligible. A remarkable finding is that the physical origin of dispersion and nonlinear dispersion for $u_0(x, t)$ can be associated or not with the vorticity Ω . In the $\eta(x, t)$ dynamics, vorticity is never directly associated with nonlinear dispersion. In order to analyze some physical-mathematical consequences while remaining relatively simple in formalism, we have only discussed equations at orders α , δ^n and $\alpha\delta^n$ with $n = 1, 2$. Non zero cases clearly shown that vorticity changes linear and nonlinear dispersive behavior of the system with respect to the zero vorticity case. Normal vorticity adds a weak correction to the non-dispersive and nonlinear terms. For strong vorticity the same modification is observed, but at the leading order. Furthermore, a modification in the linear and nonlinear dispersion terms is present simultaneously.

Furthermore, the solitonic solutions of these long wave evolution equations have been obtained analytically when involving nil, weak, normal and strong vorticity. For each case we have numerically analyzed these solitonic solutions, and their properties (celerity, amplitude and wave width).

CRedit authorship contribution statement

M.A. Manna: Conception and design of study, Writing - original draft, Writing - review & editing. **S. Noubissié:** Conception and design of study, Acquisition of data, Writing - original draft. **J. Touboul:** Conception and design of study, Acquisition of data, Analysis and/or interpretation of data, Writing - original draft, Writing - review & editing. **B. Simon:** Acquisition of data, Analysis and/or interpretation of data, Writing - original draft. **R.A. Kraenkel:** Conception and design of study, Analysis and/or interpretation of data, Writing - original draft.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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