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# Towards nonuniform distributions of unisolvent weights for Whitney finite element spaces on simplices: the edge element case

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## Abstract

We propose new sets of degrees of freedom, called weights, for the interpolation of a differential  $k$ -form in Whitney finite elements of arbitrary polynomial degree on simplices. They have a clear physical interpretation as integrals of the  $k$ -form on  $k$ -chains. This allows to consider quite general distributions of the supports, that are  $k$ -subsimplices not necessarily uniform, in a way here defined. We exploit this flexibility to investigate distributions that minimize the growth of the generalized Lebesgue constant when the polynomial degree increases. Preliminary numerical results for the edge element case support the nonuniform choice, in agreement with the well-known nodal case.

*MSC 2020* : 65N30 65D05

Key words : Polynomial differential forms Lebesgue constant weights interpolation uniform and nonuniform degrees of freedom edge finite elements

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## 1. Introduction

New degrees of freedom, called weights to distinguish them from the classical moments introduced in [17], have been firstly proposed in [20] for the interpolation, on simplicial meshes, of physical fields, intended as  $k$ -differential forms, in Whitney finite element spaces of high polynomial degree  $r \geq 1$  (see, [7], [3]). These weights are integrals of the field under consideration on a distribution of *small*  $k$ -simplices, that are particular subsimplices of dimension  $k$  in each element of the mesh. In [1] we have generalised this construction and now we develop that idea to establish how to select minimal and unisolvent sets of such small simplices as supports of the weights. This new methodology yields a flexibility, in the choice of the small simplices, that opens the way to nonuniform distributions, where the term nonuniform needs some care for  $k \neq 0$  and  $k \neq n$ , being  $n$  the ambient dimension. The quality of the interpolation on uniform and nonuniform distributions of small  $k$ -simplices can be analysed in terms of the generalised Lebesgue constant defined in [2].

Finite element spaces extending Whitney forms to higher degrees are widely used for discretizing physical balance laws in electromagnetism, fluid dynamics

or elasticity. The degrees of freedom (dofs) associated with Whitney differential forms have a direct physical relevance. When considering polynomial interpolation of higher degree, dofs can be chosen in different ways. In [17] and its extension [15] (see also [16]), higher moments are used. They are also considered in the general framework of the finite element exterior calculus [3]. In [20], the localization issue has been addressed, namely, the relationship between dofs and measurable quantities (such as circulations, fluxes, densities) for the field they are related with. In the framework of high order Whitney finite element spaces, integrals on suitable subsimplices of the mesh are a valid alternative as dofs to the classical moments. Their definition is based on the introduction of the small simplices, that are subsimplices resulting from homothetic contractions of the elements of the mesh<sup>1</sup>. New dofs are then the weights, integrals of a  $k$ -form on these small  $k$ -simplices. The weights make the connection between physics and geometry: the concept of small  $k$ -simplex was born from the necessity of extending to  $r > 1$  the geometrical construction proposed for  $r = 1$  by Weil-Whitney in a context other than finite elements but more related with algebraic cohomology and the proof of the de Rham's theorem (see [23, 24]). Understanding and generalizing this construction has been fundamental to provide explicit bases for high order finite element spaces involved in the discretization of problems from electromagnetism and other areas of physics.

The concept of weight on a small  $k$ -simplex has been an important theoretical achievement to see that it was indeed possible to define an interpolation theory for  $k$ -forms with  $k > 0$  in the same way as the one for the more familiar case  $k = 0$ . In the particular case of 0-forms the Whitney finite elements are in fact the Lagrange ones and the integrals on the small 0-simplices are the dofs used in the classical description of the Lagrange finite elements (see e.g. [9]), namely the values at the points of the principal lattices of the elements of the mesh. The weights of  $k$ -forms on small  $k$ -simplices for  $k > 0$ , can be intended as a generalization of the values at nodes (small 0-simplices) of continuous scalar functions (differential 0-forms). It is well-known that the Lagrange interpolation at uniformly distributed points in the mesh elements yields poor approximation as soon as the polynomial degree increases, even when the interpolated function is rather smooth. This is due to the Runge's phenomenon and mathematically stressed by a rapid increase of the Lebesgue constant. For this reason there have been several attempts in the literature (see [21, 22, 11, 18] and the references therein) to produce nodal sets in a triangle or tetrahedron using direct (with explicit formula) and indirect (undergoing an optimization procedure) methods, satisfying criteria of low computational complexity and minimization of the Lebesgue constant. It naturally arises the following question: "Is it possible to

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<sup>1</sup>The small simplices are *never* constructed in reality, in the sense that they do not constitute a refinement of the considered mesh. Even if in the text we propose a visualization for some particular distributions of such subsimplices, they have to be intended virtually. For  $k > 0$ , the number of subsimplices, and consequently of dofs, per element increases with the degree  $r$  of the approximation, in the same way as it occurs for  $k = 0$  when we consider many nodes in each simplex for a more accurate reconstruction of scalar fields over the mesh.

do the same for  $k > 0$  ?". In other words: "How small  $k$ -simplices,  $0 < k < n$ , should be *thought distributed* in a triangle or a tetrahedron in order to interpolate differential  $k$ -forms by trimmed polynomial ones in a way that remains stable with the growth of the polynomial degree  $r$  ?"

In this work, we provide an answer to the question by generalizing the construction of nonuniform distributions of nodes to that of small  $k$ -simplices, for  $k > 0$ . We investigate how to construct unisolvent and minimal sets of  $k$ -simplices associated with distributions of nodes different from those of the principal lattice of the elements. In particular, we exploit this flexibility to propose new sets of weights on small edges that slow down the growth of the generalised Lebesgue constant introduced in [2] when increasing the polynomial degree. The generalised Lebesgue constant will be used to state the quality of the polynomial interpolation of 1-forms at the new nonuniform distributions of the small 1-simplices in the mesh. For simplicity, we present in detail only the case  $k = 1$ , but the construction is valid for any  $0 < k \leq n$  and coincides with the familiar nodal one for  $k = 0$ .

The paper is organized as follows. In Section 2 we recall few notations and basic notions of polynomial differential forms and recall the role of the Lebesgue constant in the classical interpolation theory. In Section 3 we introduce different sets of 1-simplices and we prove that the associated weights are unisolvent for trimmed polynomial 1-forms. In Section 4, the notions of Vandermonde matrix and Lebesgue constant are presented for  $k > 0$ . In Section 5 we consider two classical choices of interpolation nodes in the interval  $[0, 1]$ , namely the uniform distribution and the Lobatto one, to construct two sets of 1-simplices supporting unisolvent weights for trimmed polynomial 1-forms. We also test two other sets of 1-simplices constructed from the symmetrised Lobatto and warp & blend distributions of nodes in the simplex. We then compare numerically the behaviour of the Lebesgue constant associated with these distributions for dofs in two and three dimensions, when increasing the polynomial degree. The paper ends in Section 6 with some concluding remarks that point out the attained achievements.

## 2. Notation and preliminaries

In this section we explain the notation and some basic notions of polynomial differential forms. Given  $n + 1$  point in general position in  $\mathbb{R}^n$ , the  $n$ -simplex with these vertices is their closed convex hull. Any subset of  $k + 1$  vertices of a  $n$ -simplex defines a face of dimension  $k$ , with  $0 \leq k \leq n$ . Faces are simplices themselves. Let  $\Delta_k(T)$  be the set of faces of dimension  $k$  (or  $k$ -subsimplices) of the  $n$ -simplex  $T$ . The cardinality of  $\Delta_k(T)$  is  $\binom{n+1}{k+1}$  elements.

If we introduce an ordering<sup>2</sup> of the vertices of  $T$ ,  $\{\mathbf{x}_0, \dots, \mathbf{x}_n\}$ , then we can

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<sup>2</sup>We can fix an orientation of the  $n$ -simplex. If  $T = [\mathbf{x}_0, \dots, \mathbf{x}_n] \subset \mathbb{R}^n$  denotes an oriented  $n$ -simplex and  $\rho : \{0, \dots, n\} \rightarrow \{0, \dots, n\}$  is a permutation, we get  $[\mathbf{x}_{\rho(0)}, \dots, \mathbf{x}_{\rho(n)}] = [\mathbf{x}_0, \dots, \mathbf{x}_n]$  if  $\rho$  is an even permutation and  $[\mathbf{x}_{\rho(0)}, \dots, \mathbf{x}_{\rho(n)}] = -[\mathbf{x}_0, \dots, \mathbf{x}_n]$  if  $\rho$  is an odd

identify univocally any  $k$ -face  $F$  of  $T$  with an *increasing* map  $\sigma_F : \{0, \dots, k\} \rightarrow \{0, \dots, n\}$ . The face  $F$  has vertices  $\{\mathbf{x}_{\sigma_F(0)}, \dots, \mathbf{x}_{\sigma_F(k)}\}$ . We denote by  $F - [\mathbf{x}_{\sigma_F(j)}]$  the  $(k-1)$ -face of  $F$  of vertices  $\{\mathbf{x}_{\sigma_F(0)}, \dots, \widehat{\mathbf{x}_{\sigma_F(j)}}, \dots, \mathbf{x}_{\sigma_F(k)}\}$  where the hat is used to indicate a suppressed argument. We extend this notation to  $(k-r)$ -faces of  $F$ :  $F - [\mathbf{x}_{\sigma_F(j_1)}, \dots, \mathbf{x}_{\sigma_F(j_r)}]$ . For instance if  $n = 3$ , the set  $T - [\mathbf{x}_0, \mathbf{x}_2]$  denotes the edge of vertices  $\{\mathbf{x}_1, \mathbf{x}_3\}$ . With each point  $\mathbf{x} \in T$  we may associate a  $(n+1)$ -uple  $(\lambda_0, \lambda_1, \dots, \lambda_n)$  such that  $\mathbf{x} = \sum_{i=0}^n \lambda_i \mathbf{x}_i$ , with the constraints  $\sum_{i=0}^n \lambda_i = 1$  and  $\lambda_i \geq 0$ . We call such functions *barycentric coordinates* for  $\mathbf{x}$  in  $T$ .

For the high-order case, multi-index notations are used. For each  $m \in \mathbb{N}$ ,  $m \geq 0$  and  $q \in \mathbb{N}$  we denote by  $\mathcal{I}(m+1, q)$  the set of  $\binom{m+q}{q}$  multi-indices of length  $m+1$  and weight  $s$ , namely

$$\mathcal{I}(m+1, q) = \{\boldsymbol{\eta} \in \mathbb{N}^{m+1} : \sum_{i=0}^m \eta_i = q\}.$$

Given  $\boldsymbol{\alpha} \in \mathcal{I}(n+1, s)$  we set  $\lambda^\alpha = \prod_{i=0}^n \lambda_i^{\alpha_i}$ . We denote by  $\Lambda^k(T)$  the space of smooth differential  $k$ -forms in  $T$ . We associate with each  $F \in \Delta_k(T)$  the Whitney form  $\omega_F \in \Lambda^k(T)$  defined in the following way (see, e.g. [24]):

$$\omega_F = \sum_{i=0}^k (-1)^i \lambda_{\sigma_F(i)} d\lambda_{\sigma_F(0)} \wedge \dots \wedge \widehat{d\lambda_{\sigma_F(i)}} \wedge \dots \wedge d\lambda_{\sigma_F(k)}.$$

If  $k = 0$  then  $F$  is a vertex  $\mathbf{x}_i$  of  $T$  and  $\omega_{\mathbf{x}_i} = \lambda_i$ . If  $k = n$  then  $F = T$  and  $\omega_T = d\lambda_1 \wedge \dots \wedge d\lambda_n$  is the volume form. We finally denote by

$$\mathcal{P}_{s+1}^- \Lambda^k(T) := \text{Span}\{\lambda^\alpha \omega_F : F \in \Delta_k(T) \text{ and } \boldsymbol{\alpha} \in \mathcal{I}(n+1, s)\}$$

the space of trimmed polynomial  $k$ -forms. Its elements are the Whitney polynomial  $k$ -forms of degree  $s+1$ . When  $k = n$ , we have that  $\mathcal{P}_{s+1}^- \Lambda^n(T) := \text{Span}\{\lambda^\alpha d\lambda_1 \wedge \dots \wedge d\lambda_n : \boldsymbol{\alpha} \in \mathcal{I}(n+1, s)\} = \mathcal{P}_s \Lambda^n(T)$ , the space of polynomial  $n$ -forms of degree  $s$ . On the other hand, for  $k = 0$ , we get  $\mathcal{P}_{s+1}^- \Lambda^0(T) = \mathcal{P}_{s+1} \Lambda^0(T)$ , that coincides with  $\mathbb{P}_{s+1}(T)$ .

The Lebesgue constant is a well known indicator to estimate the quality of a set of nodes for the interpolation in  $\mathbb{P}_r(T)$ , with  $T \subset \mathbb{R}^n$ ,  $n > 0$ , of scalar functions. To have an idea of this effect, let us suppose to have a vector  $\tilde{\mathbf{u}}$  of values  $(\tilde{\mathbf{u}}_j)_{j=1, N_{0,r}}$  that are a perturbation of  $\mathbf{u}$ , the array of data  $(\mathbf{u}_j)_{j=1, N_{0,r}}$  at a set  $X_r^0 = \{\mathbf{x}_j\}_{j=1, N_{0,r}}$  of  $N_{0,r}$  nodes for the polynomial interpolation of degree  $r$  of a continuous scalar function  $f$  in a simplex  $T$ . We recall that for  $f \in \mathcal{C}^0(T)$ , we can define the maximum norm  $\|f\|_\infty = \max_{\mathbf{x} \in T} |f(\mathbf{x})|$ . Being  $\{\mathbf{w}_j\}_j$  the dual basis associated with the values at the set of nodes  $X_r^0$ , the interpolating polynomials of degree  $r$  on the values  $\mathbf{u}_j$  and  $\tilde{\mathbf{u}}_j$  are, respectively,

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permutation. The  $n$ -simplex of  $\mathbb{R}^n$  with vertices  $\{\mathbf{x}_0, \dots, \mathbf{x}_n\}$  is called the support of the oriented  $n$ -simplex  $[\mathbf{x}_{\rho(0)}, \dots, \mathbf{x}_{\rho(n)}]$ .

$\Pi_r^0 f = \sum_{j=1}^{N_{0,r}} \mathbf{u}_j \mathbf{w}_j$  and  $\tilde{\Pi}_r^0 f = \sum_{j=1}^{N_{0,r}} \tilde{\mathbf{u}}_j \mathbf{w}_j$ . We can estimate the maximum norm of the continuous function  $\Pi_r^0 f - \tilde{\Pi}_r^0 f$  and we obtain

$$\begin{aligned} \|\Pi_r^0 f - \tilde{\Pi}_r^0 f\|_\infty &= \max_{\mathbf{x} \in T} \left| \Pi_r^0(\mathbf{x}) - \tilde{\Pi}_r^0(\mathbf{x}) \right| = \max_{\mathbf{x} \in T} \left| \sum_{j=1}^{N_{0,r}} (\mathbf{u}_j - \tilde{\mathbf{u}}_j) \mathbf{w}_j(\mathbf{x}) \right| \\ &\leq \max_{\mathbf{x} \in T} \left( \sum_{j=1}^{N_{0,r}} |\mathbf{w}_j(\mathbf{x})| \right) \max_{j=1, N_{0,r}} |\mathbf{u}_j - \tilde{\mathbf{u}}_j| = \Lambda_{X_r^0}^0 \|\mathbf{u} - \tilde{\mathbf{u}}\|_\infty \end{aligned}$$

where  $\Lambda_{X_r^0}^0 = \max_{\mathbf{x} \in T} \left( \sum_{j=1}^{N_{0,r}} |\mathbf{w}_j(\mathbf{x})| \right)$  is known as the Lebesgue constant. As a consequence, we have that small changes on the data yield small changes on the interpolating polynomial if the Lebesgue constant  $\Lambda_{X_r^0}^0$  is small. This constant thus plays the role of the condition number for the interpolation problem. Moreover, the Lebesgue constant appears when estimating the interpolation error. Let  $f_r^*$  be the best approximation polynomial of degree  $r$  of the scalar function  $f$  in  $T$ , for which  $\|f - f_r^*\|_\infty \leq \|f - z_r\|_\infty$  for any other polynomial  $z_r$  of degree  $r$  defined in  $T$ . Then

$$\begin{aligned} \|f - \Pi_r^0 f\|_\infty &= \|f - \Pi_r^0 f_r^* + \Pi_r^0 f_r^* - \Pi_r^0 f\|_\infty \leq \|f - f_r^*\|_\infty + \|\Pi_r^0 f_r^* - \Pi_r^0 f\|_\infty \\ &\leq (1 + \|\Pi_r^0\|_\infty) \|f - f_r^*\|_\infty, \end{aligned}$$

with  $\|\Pi_r^0\|_\infty = \sup_{f \in \mathcal{C}^0(\bar{T}) \setminus \{0\}} \frac{\|\Pi_r^0 f\|_\infty}{\|f\|_\infty}$ . It can be proven that  $\|\Pi_r^0\|_\infty \leq \Lambda_{X_r^0}^0$ : indeed, for any  $f \in \mathcal{C}^0(T)$  we have

$$\|\Pi_r^0 f\|_\infty = \max_{\mathbf{x} \in T} \left| \sum_{j=1}^{N_{0,r}} f(\mathbf{x}_j) \mathbf{w}_j(\mathbf{x}) \right| \leq \left( \max_{\mathbf{x} \in T} \sum_{j=1}^{N_{0,r}} |\mathbf{w}_j(\mathbf{x})| \right) \max_{j=1, N_{0,r}} |f(\mathbf{x}_j)| \leq \Lambda_{X_r^0}^0 \|f\|_\infty.$$

This bound combined with the previous estimate of  $\|f - \Pi_r^0 f\|_\infty$  yields the following well-known result

$$\|f - \Pi_r^0 f\|_\infty \leq (1 + \Lambda_{X_r^0}^0) \|f - f_r^*\|_\infty, \quad \forall f \in \mathcal{C}^0(T). \quad (1)$$

In (1), the growth of the Lebesgue constant  $\Lambda_{X_r^0}^0$  determines the convergence in the maximum norm. Moreover, the same inequality suggests that, when the Lebesgue constant does not grow too fast, we can have an approximation of  $f$  on  $T$  that is almost as good as the best approximation  $f_r^*$  by taking  $\Pi_r^0 f$ , that is easier to compute than  $f_r^*$ . There is a significative literature for the case  $k = 0$  (see [14, 11, 18] and the references therein), widely dedicated to the problem of selecting a good and easy to be defined set of nodes  $X_r^0$  for the high order polynomial interpolation of continuous functions  $f$  on nontensorial domains as the simplex  $T$ .

Very recently, in [2], the definition of the Lebesgue constant has been generalised to the case of interpolation of differential forms  $\omega \in C^0 \Lambda^k(T)$  by polynomial  $k$ -forms  $\omega \in \mathcal{P}_r^- \Lambda^k(T)$ , for  $k \geq 0$  and a bound as (1) also holds for  $\omega \in C^0 \Lambda^k(T)$ . In Figure 1 we illustrate the interaction among the principal ingredients at play in the polynomial interpolation. The values of the dofs for the elements of a given basis, say  $\{w_j\}_j$ , of the local discrete space constitute the

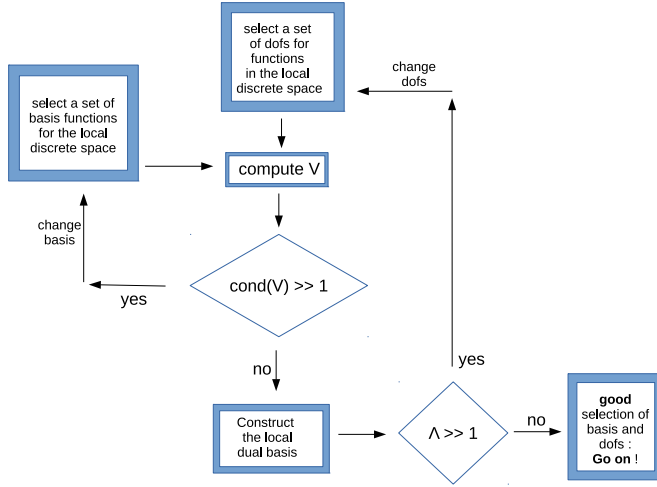


Figure 1: Simplified visualization of the interaction among decisive ingredients or steps for the success of the multivariate high-order polynomial approximation. The conditioning of the Vandermonde matrix  $V$  matters when computing the dual basis and the growth of the Lebesgue constant  $\Lambda$  with the approximation degree  $r$  has to be slow for a stable interpolation. This analysis is done locally, on one mesh element, before performing the approximation over the entire mesh.

entries of the so-called Vandermonde matrix  $V$  (e.g.,  $V_{ij} = w_j(\mathbf{x}_i)$ ,  $i, j = 1, N_{0,r}$ , in the nodal case). This matrix has to be inverted once to construct the local dual basis  $\{w_j\}_j$  associated with the selected dofs and its conditioning  $\text{cond}(V)$  depends on the local basis  $\{w_j\}_j$ . In this work we analyse how it is possible to define different sets of weights, in order to obtain a stable interpolation when the local discrete space is  $\mathcal{P}_r^- \Lambda^k(T)$  with  $k > 0$ , by minimizing the Lebesgue constant  $\Lambda$ . In the spirit of the geometrical construction proposed by Whitney, the question becomes how to construct different distributions of  $k$ -subsimplices,  $k > 0$ , that are unisolvent and minimal for the interpolation in  $\mathcal{P}_r^- \Lambda^k(T)$  of differential  $k$ -forms and compare them in terms of the generalised Lebesgue constant. The basis  $\{w_j\}_j$  we consider for  $\mathcal{P}_r^- \Lambda^k(T)$  yields acceptable values for the conditioning of the Vandermonde matrix when the approximation degree  $r$  increases, and its investigation is thus postponed.

### 3. Distributions of unisolvent and minimal 1-simplices

For simplicity, we present the construction for  $k = 1$ . We stress that the same construction can be repeated for any value of  $k > 0$ . We consider a symmetric<sup>3</sup> distribution  $Q_s$  of  $s$  points  $q_i$  in the interval  $(0, 1) \subset \mathbb{R}$ , such that

<sup>3</sup>The term *symmetric* is here used to denote a symmetry of the nodes' distribution with respect to the midpoint of the interval  $(0, 1)$ . If  $s = 2\ell + 1$ , then  $q_{\ell+1} = \frac{1}{2}$ ,  $q_1 = 1 - q_s$ ,

$0 < q_1 < q_2 < \dots < q_s < 1$ . For instance, we can have  $q_i = \frac{i}{s+1}$ , if we wish a *uniform distribution*, or  $q_i = (1 + \cos(\frac{i\pi}{s+1}))/2$ , if we prefer a *nonuniform* one. For each  $n$ -simplex  $T$  on  $\mathbb{R}^n$  with vertices  $\{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n\}$  we denote

$$\mathcal{Z}(T, Q_s) := \{\mathbf{x} \in T : \lambda_0(\mathbf{x}) \neq 0 \text{ and } \lambda_i(\mathbf{x}) \in \{0, q_1, \dots, q_s, 1\}, \forall i \in \{1, \dots, n\}\},$$

being  $\{\lambda_i(\mathbf{x})\}_{i=0}^{k_0}$  the barycentric coordinates on  $T$ . The set  $\mathcal{Z}(T, Q_s)$  consists of  $\binom{n+s}{s}$  points of  $T$  and the only polynomial of degree lower or equal than  $s$  in  $n$  variables that is equal to zero on each point of  $\mathcal{Z}(T, Q_s)$  is the zero polynomial.

For each  $\boldsymbol{\xi} \in \mathbb{R}^n$  we consider the map obtained as the composition of the homothety  $\mathbf{x} \mapsto \mathbf{x}_0 + \lambda_0(\boldsymbol{\xi})(\mathbf{x} - \mathbf{x}_0)$  and the translation  $\mathbf{x} \mapsto \mathbf{x} + (\boldsymbol{\xi} - \mathbf{x}_0)$ , namely

$$\tau_{\boldsymbol{\xi}} : \mathbf{x} \mapsto \tau_{\boldsymbol{\xi}}(\mathbf{x}) = \mathbf{x}_0 + \lambda_0(\boldsymbol{\xi})(\mathbf{x} - \mathbf{x}_0) + (\boldsymbol{\xi} - \mathbf{x}_0).$$

We thus have

$$\tau_{\boldsymbol{\xi}}(\mathbf{x}) = \boldsymbol{\xi} + \lambda_0(\boldsymbol{\xi})(\mathbf{x} - \mathbf{x}_0) = \sum_{i=0}^n \lambda_i(\boldsymbol{\xi})\mathbf{x}_i + \lambda_0(\boldsymbol{\xi})(\mathbf{x} - \mathbf{x}_0) = \lambda_0(\boldsymbol{\xi})\mathbf{x} + \sum_{i=1}^n \lambda_i(\boldsymbol{\xi})\mathbf{x}_i.$$

We denote  $T_{\boldsymbol{\xi}} := \tau_{\boldsymbol{\xi}}(T)$ . If  $\lambda_0(\boldsymbol{\xi}) \neq 0$  then  $T_{\boldsymbol{\xi}}$  is a  $n$ -simplex, similar to  $T$  with  $n$  vertices in the  $(n-1)$ -dimensional affine subspace  $\pi_T$  containing  $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ . Otherwise  $T_{\boldsymbol{\xi}} = \boldsymbol{\xi}$ . (See Figure 2.)

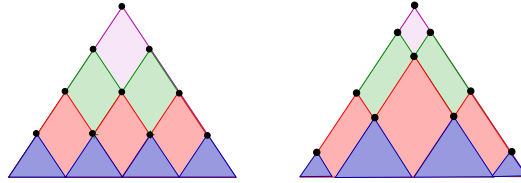


Figure 2: The dots represent the set  $\mathcal{Z}(T, Q_3)$  for a 2-simplex  $T$  in  $\mathbb{R}^2$  and two different choices of  $Q_3$ : on the left the uniform distribution  $\{q_i = i/4 \text{ with } i \in \{1, 2, 3\}\}$ , on the right a non uniform distribution. The coloured 2-simplex are  $T_{\boldsymbol{\xi}}$  for  $\boldsymbol{\xi} \in \mathcal{Z}(T, Q_3)$ .

For each  $k$  such that  $0 \leq k \leq n$  we denote

$$\Sigma^k(T, Q_s) = \{\tau_{\boldsymbol{\xi}}(F) : F \in \Delta_k(T), \boldsymbol{\xi} \in \mathcal{Z}(T, Q_s)\}.$$

**Remark 1.** If  $k > 0$  the total number of elements of  $\Sigma^k(T, Q_s)$  is  $\binom{n+1}{k+1} \binom{n+s}{s}$ , namely, the product of the number of  $k$ -faces of an  $n$ -simplex by the number of points in the set  $\mathcal{Z}(T, Q_s)$ . This because each element of  $\Sigma^k(T, Q_s)$  belongs to exactly one  $n$ -simplex  $T_{\boldsymbol{\xi}}$ . On the other hand, the number of elements of  $\Sigma^0(T, Q_s)$  is equal to  $\binom{n+s+1}{s+1}$  because some elements of  $\Sigma^0(T, Q_s)$  belong to two or more  $n$ -simplex  $T_{\boldsymbol{\xi}}$ . Similarly, if  $0 < k < n$  the number of elements of  $\Sigma^k(T, Q_s)$  contained in the  $(n-1)$ -face opposite to  $\mathbf{x}_0$  is  $\binom{n}{k+1} \binom{n+s}{s}$  because

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$q_2 = 1 - q_{s-1}$ , and so on.



each  $n$ -simplex  $T_{\xi}$  has exactly  $\binom{n}{k+1}$   $k$ -faces in the  $(n-1)$ -face of  $T$  opposite to  $\mathbf{x}_0$ . On the other hand the number of points of  $\Sigma^0(T, Q_s)$  contained in any  $k_0$ -face  $F$  of  $T$  is  $\binom{k_0+s+1}{s+1}$ .

For each  $k_0 \in \mathbb{N}$  with  $0 \leq k_0 \leq n$  and each  $F \in \Delta_{k_0}(T)$  we can define analogously

$$\mathcal{Z}(F, Q_s) := \{\mathbf{x} \in F : \lambda_0^F(\mathbf{x}) \neq 0 \text{ and } \lambda_i^F(\mathbf{x}) \in \{0, q_1, \dots, q_s, 1\}, \forall i \in \{1, \dots, k_0\}\},$$

being  $\lambda_i^F(\mathbf{x})$  the barycentric coordinate on  $F$ , namely  $\lambda_i^F(\mathbf{x}) = \lambda_{\sigma_F(i)|F}$ , for  $i \in \{0, 1, \dots, k_0\}$ , and

$$\Sigma^k(F, Q_s) = \{\tau_{\xi}(f) : f \in \Delta_k(F), \xi \in \mathcal{Z}(F, Q_s)\}, \quad 0 \leq k \leq k_0.$$

The set of elements of  $\Sigma^k(T, Q_s)$  supported on  $F \in \Delta_{k_0}(T)$  contains  $\Sigma^k(F, Q_s)$  for each  $k \in \mathbb{N}$  with  $0 \leq k \leq k_0$ . They coincide if and only if  $\mathbf{x}_0 \in F$ . (See Figure 3.)

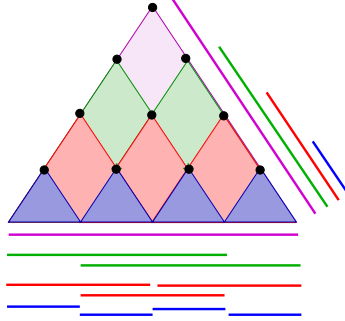


Figure 3: The elements of the set  $\Sigma^1(T, Q_3)$  supported on two different edges of  $T$ : on the right the edge  $[\mathbf{x}_0, \mathbf{x}_2]$ , on the bottom the edge  $[\mathbf{x}_1, \mathbf{x}_2]$ . Notice that the set on the left coincides with  $\Sigma^1([\mathbf{x}_0, \mathbf{x}_2], Q_3)$  so it has  $\binom{1+1}{1+1} \binom{1+3}{3} = 4$  elements while the set on the bottom has  $\binom{2}{1+1} \binom{2+3}{3} = 10$  elements. In this case  $Q_3$  is the uniform distribution, namely,  $\{q_i = i/4 \text{ with } i \in \{1, 2, 3\}\}$ .

We start by proving that the set of  $k$ -simplices  $\Sigma^k(T; Q_s)$  is unisolvent for  $\mathcal{P}_{s+1}^- \Lambda^k(T)$ . We proceed as in [8]. The proof is based on two lemmas.

**Lemma 1.** *Let  $T \subset \mathbb{R}^n$  be an  $n$ -simplex, and let  $\omega \in \mathcal{P}_s \Lambda^n(T)$  be such that  $\int_{T_{\xi}} \omega = 0$  for all  $\xi \in \mathbb{R}^n$  with  $\lambda_0(\xi) \neq 0$ . Then  $\omega = 0$ .*

**Proof of Lemma 1.** *Identify  $\omega$  with its coefficient  $p(\mathbf{x})$ . Since  $s < \infty$ , the graph of  $p(\mathbf{x})$  defines a finite number of connected regions in which  $p(\mathbf{x}) \neq 0$  and there exists  $\xi \in \mathbb{R}^n$  such that the interior of  $T_{\xi}$  (that is different from the empty set, since  $\lambda_0(\xi) \neq 0$ ) is contained in one of such regions. Then it follows that  $\omega \equiv 0$  in  $T_{\xi}$  but then, being  $\omega \in \mathcal{P}_s \Lambda^n(\mathbb{R}^n)$ , we have  $\omega = 0$ .  $\square$*

Note that Lemma 1 is valid also for any  $k_0$ -face  $F \subset \mathbb{R}^n$ : as soon as  $\omega \in \mathcal{P}_s \Lambda^{k_0}(F)$  (with  $k_0 = \dim F$ ) is such that  $\int_{F_\xi} \omega = 0$  for all  $\xi \in \mathbb{R}^n$  with  $\lambda_0^F(\xi) \neq 0$ , we can conclude that  $\omega = 0$ .

In the following  $\Pi_T^k$  denotes the interpolation operator onto Whitney  $k$ -forms of lowest order determined by the simplex  $T$ . If  $\omega \in \Lambda^k(\mathbb{R}^n)$  then  $\tau_\xi \omega \in \Lambda^k(\mathbb{R}^n)$  denotes the pullback of  $\omega$  by  $\tau_\xi$ , namely

$$(\tau_\xi \omega)(\mathbf{x}) = \omega(\xi + \lambda_0(\xi)(\mathbf{x} - \mathbf{x}_0)).$$

**Lemma 2.** *Let  $T \subset \mathbb{R}^n$  be an  $n$ -simplex with vertices  $\{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n\}$ , and let  $\omega \in \mathcal{P}_s \Lambda^k(\mathbb{R}^n)$ , with  $0 < k < n$ , be such that  $\Pi_T^k(\tau_\xi \omega) = 0$  for all  $\xi \in \mathbb{R}^n$  with  $\lambda_0(\xi) \neq 0$ . Then  $\omega = 0$ .*

**Proof of Lemma 2.** *The proof is similar to that of Lemma 3.13 in [8]. If  $\Pi_T^k(\tau_\xi \omega) = 0$  for all  $\xi \in \mathbb{R}^n$  with  $\lambda_0(\xi) \neq 0$ , then by Lemma 1 the pullback of  $\omega$  to any affine  $k$ -dimensional subspace of  $\mathbb{R}^n$  parallel to a  $k$ -face of  $T$  not contained in  $T - [\mathbf{x}_0]$  is 0. We show that this is enough to conclude that  $\omega = 0$ .*

*For each  $\mathbf{x} \in \mathbb{R}^n$  let us denote  $\mathbf{u}_\ell = \tau_\mathbf{x}(\mathbf{x}_0 - \mathbf{x}_\ell)$ , with  $\ell \in \{1, \dots, n\}$ , and  $h_F$  the affine  $k$ -space of  $\mathbb{R}^n$  parallel to  $F \in \Delta_k(T)$  passing through  $\mathbf{x}$ .*

*Each  $h_F$  with  $F \not\subset T - [\mathbf{x}_0]$  is generated by a choice of  $k$  different elements of  $\{\mathbf{u}_\ell\}_{\ell=1}^n$  and each choice of  $k$  different elements of  $\{\mathbf{u}_\ell\}_{\ell=1}^n$  generates  $h_F$  for some  $F \in \Delta_k(T)$ ,  $F \not\subset T - [\mathbf{x}_0]$ .*

*From Lemma 1, if  $\mathbf{v}_1, \dots, \mathbf{v}_k \in h_F$  for some  $F \in \Delta_k(T)$ , such that  $F \not\subset T - [\mathbf{x}_0]$ , then  $\omega_\mathbf{x}(\mathbf{v}_1, \dots, \mathbf{v}_k) = 0$ .*

*If  $\widehat{\mathbf{v}}_1, \dots, \widehat{\mathbf{v}}_k$  are  $k$  generic vectors of  $\mathbb{R}^n$  then*

$$\omega_\mathbf{x}(\widehat{\mathbf{v}}_1, \dots, \widehat{\mathbf{v}}_k) = \omega_\mathbf{x} \left( \sum_{\ell=1}^n c_\ell(\widehat{\mathbf{v}}_1) \mathbf{u}_\ell, \dots, \sum_{\ell=1}^n c_\ell(\widehat{\mathbf{v}}_k) \mathbf{u}_\ell \right) = \sum_{\sigma \in \Sigma_{n,k}} C_\sigma \omega_\mathbf{x}(\mathbf{u}_{\sigma(1)}, \dots, \mathbf{u}_{\sigma(k)})$$

*being  $\Sigma_{n,k}$  the set of combinations of  $k$  elements taken from the  $\{1, \dots, n\}$ . It is equal zero because  $\mathbf{u}_{\sigma(1)}, \dots, \mathbf{u}_{\sigma(k)}$  are in  $h_F$  for some  $F \in \Delta_k(T)$ ,  $F \not\subset T - [\mathbf{x}_0]$ .  $\square$*

**Proposition 1.** *If  $\omega \in \mathcal{P}_{s+1}^- \Lambda^k(T)$  is such that  $\int_\sigma \omega = 0$  for all  $\sigma \in \Sigma^k(T, Q_s)$  then  $\omega = 0$ .*

**Proof of Proposition 1.** *First we recall that if  $\omega \in \mathcal{P}_{s+1}^- \Lambda^n(T)$  then  $\omega = p(\mathbf{x}) \omega_T$  with  $p \in \mathbb{P}_s$ . Hence the map*

$$\xi \in \mathbb{R}^n \rightarrow q(\xi) = \int_T p(\xi + \lambda_0(\xi)(\mathbf{x} - \mathbf{x}_0)) \omega_T$$

*is a polynomial of degree  $s$  in  $\xi$ . By assumption  $q$  is zero at each point of  $\mathcal{Z}(T, Q_s)$  and then  $q \equiv 0$ . From Lemma 1, it follows that  $\omega = 0$ . We fix  $k_0 \in \mathbb{Z}$  such that  $0 \leq k_0 < n$  and we assume that the result holds true for each  $k' \in \mathbb{Z}$  such that  $k_0 < k' \leq n$ . We thus prove that the result holds true for  $k = k_0$ .*

We have to state that, if  $\omega \in \mathcal{P}_{s+1}^- \Lambda^{k_0}(T)$  is such that  $\int_{\tau_{\xi}(F)} \omega = 0$  for each  $F \in \Delta_{k_0}(T)$  and  $\xi \in \mathcal{Z}(T, Q_s)$ , then  $\omega = 0$ .

For such  $\omega$  we have  $d\omega \in \mathcal{P}_{s+1}^- \Lambda^{k_0+1}(T)$  and

$$\int_{K_{\xi}} d\omega = \int_{\partial K_{\xi}} \omega = 0$$

for each  $K \in \Delta_{k_0+1}(T)$  and  $\xi \in \mathcal{Z}_s(T)$ . Then by the inductive hypothesis  $d\omega = 0$  and hence  $\omega \in \mathcal{P}_s \Lambda^{k_0}(T)$ . Denoting  $\Pi_T^{k_0}$  the interpolation operator onto Whitney  $k_0$  forms of lowest order we have that  $\Pi_T^{k_0}(\tau_{\xi}\omega)$  is a polynomial of degree  $s$  in  $\xi$  which is zero at each point of  $\mathcal{Z}_s(T)$  hence it is zero. Then, from Lemma 2,  $\omega = 0$ .  $\square$

In particular in  $\mathbb{R}^2$  it holds that the set

$$\Sigma_{min}^1(T, Q_s) := \{e \in \Sigma^1(T, Q_s) : e \not\subset T - [\mathbf{x}_0]\} \cup \Sigma^1(T - [\mathbf{x}_0], Q_s)$$

is unisolvent for  $\mathcal{P}_{s+1}^- \Lambda^1(T)$ , namely it is such that

$$\left( \omega \in \mathcal{P}_{s+1}^- \Lambda^1(T) \quad \text{and} \quad \int_{\sigma} \omega = 0, \quad \forall \sigma \in \Sigma_{min}^1(T, Q_s) \right) \implies \omega = 0.$$

In fact, when it occurs that  $\int_{\sigma} \omega = 0$  for all  $\sigma \in \Sigma_{min}^1(T, Q_s)$ , we have in particular  $\int_{\sigma} \omega = 0$  for all  $\sigma$  in  $\Sigma^1(T - [\mathbf{x}_0], Q_s)$  hence the restriction of  $\omega$  to  $T - [\mathbf{x}_0]$  is zero and  $\int_{\sigma} \omega = 0$  for all  $\sigma$  in  $\Sigma^1(T, Q_s)$ . Then  $\omega = 0$  results from Proposition 1.

Now it is easy to check that in  $\mathbb{R}^3$  the set of 1-simplices

$$\begin{aligned} \Sigma_{min}^1(T, Q_s) &:= \{e \in \Sigma^1(T, Q_s) : e \not\subset T - [\mathbf{x}_0]\} \\ &\quad \cup \{e \in \Sigma^1(T - [\mathbf{x}_0], Q_s) : e \not\subset T - [\mathbf{x}_0, \mathbf{x}_1]\} \\ &\quad \cup \Sigma^1(T - [\mathbf{x}_0, \mathbf{x}_1], Q_s) \end{aligned}$$

that can be rewritten as

$$\Sigma_{min}^1(T, Q_s) = \{e \in \Sigma^1(T, Q_s) : e \not\subset T - [\mathbf{x}_0]\} \cup \Sigma_{min}^1(T - [\mathbf{x}_0], Q_s)$$

has the same property of unisolvence for  $\mathcal{P}_{s+1}^- \Lambda^1(T)$ .

If  $\int_{\sigma} \omega = 0$  for all  $\sigma$  in  $\Sigma_{min}^1(T, Q_s)$  then in particular  $\int_{\sigma} \omega = 0$  for all  $\sigma \in \Sigma_{min}^1(T - [\mathbf{x}_0], Q_s)$ . Hence the restriction of  $\omega$  to  $T - [\mathbf{x}_0]$  is zero and  $\int_{\sigma} \omega = 0$  for all  $\sigma$  in  $\Sigma^1(T, Q_s)$ . Proposition 1 yields  $\omega = 0$ .

Moreover, since  $\binom{k_0+1}{2} - \binom{k_0}{2} = k_0$  for all  $k_0$ , from Remark 1 the number of elements of  $\Sigma_{min}^1(T, Q_s)$  is equal to

$$3 \binom{3+s}{s} + 2 \binom{2+s}{s} + s + 1 = \dim(\mathcal{P}_{s+1}^- \Lambda^1(T)) = (s+1) \binom{s+4}{2},$$

hence it is minimal in the sense that the proper subsets of  $\Sigma_{min}^1(T, Q_s)$  are not unisolvent.

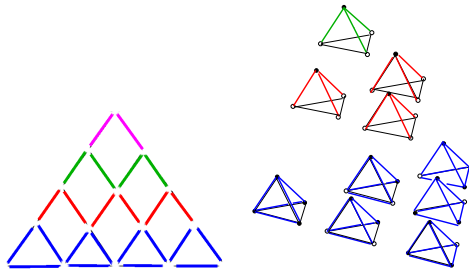


Figure 4: The sets  $X^1(F; Q_3)$  for a 2-simplex  $F$  (on the left) and  $X^1(T, Q_2)$  for a 3-simplex  $T$  (on the right) for a uniform distribution of points.

**Remark 2.** *In order to avoid an overlapping between portions of different 1-simplices, it is convenient to chop the 1-simplices of the set  $\Sigma_{min}^1(T, Q_s)$  at their intersection points (compare Figure 3 to Figure 4, left). We denote by  $X^1(T, Q_s)$  the set of 1-simplices obtained in this way. In the case of a uniform distribution of the points  $q_i$  in the interval  $(0, 1)$ , one thus obtains a set of small edges, a particular subset of those defined in [20]. (See Figure 4 for a visualization in two and three dimensions.) For  $k = 0$  we have  $\Sigma^0(T, Q_s) = \Sigma_{min}^0(T, Q_s) =: X^0(T, Q_s)$ .*

#### 4. Polynomial interpolation of differential forms and the Lebesgue constant

In order to introduce the definition of the *generalized* Lebesgue constant, we recall few essential concepts and refer to [2] for more details. The mass  $|\sigma|_0$  of a  $k$ -simplex  $\sigma \subset \mathbb{R}^n$  is its  $k$ -dimensional Hausdorff's measure. In particular, the mass of a point is 1. A simplicial  $k$ -chain  $c$  is a formal (finite) sum of  $k$ -simplices with real coefficients. The mass  $|c|_0$  of a simplicial chain  $c = \sum_{i=1}^I a_i \sigma_i$  is defined as  $|c|_0 = \sum_{i=1}^I |a_i| |\sigma_i|_0$ . We denote by  $\mathcal{C}_k$  (resp.  $\mathcal{C}_k(T)$ ) the space of simplicial  $k$ -chains in  $\mathbb{R}^n$  (resp., supported in  $T$ ). Given a set  $X_r^k = \{\sigma_1, \dots, \sigma_{N_{k,r}}\}$  of  $N_{k,r} = \dim(\mathcal{P}_r^- \Lambda^k(T))$  distinct  $k$ -simplices in  $T$  (not necessarily subsimplices of  $T$ ), the interpolation problem for a given differential  $k$ -form  $\omega \in C^0 \Lambda^k(T)$  consists in finding a Whitney differential  $k$ -form,  $\Pi_r^k \omega \in \mathcal{P}_r^- \Lambda_k(T)$ , such that  $\int_{\sigma_i} \Pi_r^k \omega = \int_{\sigma_i} \omega$  for all  $\sigma_i \in X_r^k$ . As in the case of Lagrange's interpolation, chosen a basis  $\mathcal{B} = \{w_j\}_{j=1}^{N_{k,r}}$  for the space  $\mathcal{P}_r^- \Lambda^k(T)$  the interpolation problem is well-defined if and only if the generalized Vandermonde matrix  $V_{X_r^k, \mathcal{B}}$  with entries  $(V_{X_r^k, \mathcal{B}})_{i,j} = \int_{\sigma_i} w_j$ ,  $i, j = 1, \dots, N_{k,r}$ , is invertible.

We say that the set  $X_r^k$  of  $k$ -simplices supported in  $T$  is unisolvent for the space  $\mathcal{P}_r^- \Lambda^k(\mathbb{R}^n)$  if the interpolation problem is well-defined. Note that this means that  $N_{k,r}$ , the number of elements in the set  $X_r^k$ , coincides with the dimension of  $\mathcal{P}_r^- \Lambda^k(\mathbb{R}^n)$ . Moreover, if  $\omega \in \mathcal{P}_r^- \Lambda^k(\mathbb{R}^n)$  is such that  $\int_{\sigma_j} \omega = 0$  for all  $\sigma_j \in X_r^k$  then  $\omega = 0$ . In fact, being  $\mathcal{B} = \{w_j\}_{j=1}^{N_{k,r}}$  a basis of  $\mathcal{P}_r^- \Lambda^k(T)$ , we can

write  $\omega = \sum_j a_j w_j$ . Then,  $\int_{\sigma_j} \omega = 0$  for all  $\sigma_j \in X_r^k$  means that  $V_{X_r^k, \mathcal{B}} \mathbf{a} = \mathbf{0}$ . If  $V_{X_r^k, \mathcal{B}}$  is invertible, it follows that the vector  $\mathbf{a}$ , and thus the  $k$ -form  $\omega$ , is zero. In particular, in Section 3, we have proved that the set  $X_r^1 = X^1(T, Q_{r-1})$  is unisolvent for the space  $\mathcal{P}_r^- \Lambda^k(T)$ .

With any unisolvent set  $X_r^k$ , it is possible to associated a dual basis  $\{\mathbf{w}_j^{X_r^k}\}_{j=1}^{N_{k,r}}$  of  $\mathcal{P}_r^- \Lambda^k(\mathbb{R}^n)$  (namely, such that  $\int_{\sigma_i} \mathbf{w}_j^{X_r^k} = \delta_{i,j}$ ). Definition 1 introduces the generalised Lebesgue constant as it firstly appeared in [2].

**Definition 1.** *Given a set  $X_r^k = \{\sigma_1, \dots, \sigma_{N_{k,r}}\}$  of  $N_{k,r}$   $k$ -simplices supported in the  $n$ -simplex  $T$  that is unisolvent for the space  $\mathcal{P}_r^- \Lambda^k(T)$ , the Lebesgue function  $\mathcal{L}_{X_r^k} : \mathcal{C}_k(T) \rightarrow \mathbb{R}^+$  is defined as*

$$\mathcal{L}_{X_r^k}^k(c) = \sum_{j=1}^{N_{k,r}} |\sigma_j|_0 \left| \int_c \mathbf{w}_j^{X_r^k} \right|$$

with  $\{\mathbf{w}_j^{X_r^k}\}_{j=1}^{N_{k,r}} \subset \mathcal{P}_r^- \Lambda^k(T)$  the dual basis associated with  $X_r^k$ . The Lebesgue constant is then  $\Lambda_{X_r^k}^k = \sup_{c \in \mathcal{C}_k(T)} \frac{\mathcal{L}_{X_r^k}^k(c)}{|\mathcal{C}|_0}$ .

Note that the supremum is taken on the set of all  $k$ -chains in  $T$ . In order to estimate  $\Lambda_{X_r^k}^k$  this supremum is taken on the  $k$ -simplices of an additional mesh in  $T$  that is much finer than the one of small  $k$ -simplices adopted to compute the dual basis  $\{w_j\}_j$  of  $\mathcal{P}_r^- \Lambda^k(T)$ . The following pseudo-algorithm allows to estimate the Lebesgue's constant introduced in Definition 1, given a degree  $r \geq 1$  and a value  $k > 0$ .

Preliminary steps :

1. Choose an unisolvent set  $X_r^k = \{\sigma_1, \dots, \sigma_{N_{k,r}}\}$  of  $N_{k,r}$  distinct  $k$ -simplices in  $T$ .
2. Choose a basis  $\mathcal{B} = \{w_j\}_{j=1, N_{k,r}}$  for the space  $\mathcal{P}_r^- \Lambda^k(T)$ .
3. Construct the generalized Vandermonde matrix  $\mathbf{V}$  with entries  $(\mathbf{V})_{i,j} = \int_{\sigma_i} w_j$ ,  $i, j = 1, \dots, N_{k,r}$ .
4. Compute the inverse  $\mathbf{W}$  of  $\mathbf{V}$  by solving the linear system  $\mathbf{W} = \mathbf{V} \setminus \mathbf{I}$  with  $\mathbf{I}$  the identity matrix of size  $N_{k,r}$ .
5. Define a fine mesh  $\tau$  in  $T$  and the set  $Y_k = \{c_\ell\}_{\ell=1, M_k}$  containing the  $k$ -simplices of  $\tau$ .

```

Numerical evaluation of the Lebesgue constant :
Lebfunc = zeros(M_k,1)
for l = 1:M_k
  compute |c_l|_0
  for j = 1:N_{k,r}
    compute |\sigma_j|_0
    compute the dual functions
    w_j^{X_r^k} = \sum_{i=1}^{N_{k,r}}(w)_{ij} w_i at the quadrature nodes
    compute val = \int_{c_l} w_j^{X_r^k} by the quadrature rule
    Lebfunc(l) = Lebfunc(l) + |\sigma_j|_0 |val|
  end % j
  Lebfunc(l) = Lebfunc(l)/|c_l|_0
end % l
Lebconst = norm(Lebfunc, 'inf')

```

## 5. Estimation of the Lebesgue constant for some families of 1-simplices

Let us start recalling some well-known results in the case  $k = 0$ . The term *uniform* recalls the fact that the  $N_{0,r} = \dim \mathcal{P}_r^- \Lambda^0(T)$  nodes are those of the principal lattice of  $T$ . Any other distribution that does not fulfill this requirement will be referred to as *nonuniform*. If  $T$  is the interval  $[-1, 1]$  a classical nonuniform distribution of nodes is the one corresponding with the zeros of the Lobatto polynomials provided we add the interval extremities  $\pm 1$ . It is well known that it is optimal for the scalar interpolation in  $[-1, 1]$  (see [10]). The Lobatto polynomial of degree  $s$  is defined as  $Lo_s(t) = L'_{s+1}(t)$ , where  $L'_{s+1}(t)$  is the first derivative of the Legendre polynomial of degree  $s + 1$  in  $t \in (-1, 1)$ . Therefore, Lobatto nodes  $\{t_i\}_{i=0,r}$  associated with a degree  $r = s + 1$  in  $[-1, 1]$  are the zeros of  $(1-t^2) L'_{s+1}(t) = (1-t)^2 Lo_s(t)$ . The nonuniform Lobatto distribution of nodes in the unit simplex  $T$  is defined, starting from the corresponding values of  $t_j$  in  $[-1, 1]$ , by

$$\begin{cases} v_i = (1 + t_i)/2, \\ i = 0, \dots, r \end{cases} \quad \begin{cases} (v_i, v_j), \\ i = 0, \dots, r, \\ j = 0, \dots, r + 1 - i \end{cases} \quad \begin{cases} (v_i, v_j, v_\ell), \\ i = 0, \dots, r, \\ j = 0, \dots, r + 1 - i, \\ \ell = 0, \dots, r + 2 - i - j. \end{cases}$$

in the unit interval  $[0, 1]$ ,      in the unit triangle  $T$ ,      in the unit tetrahedron  $T$ .

These are the points of the set  $X_r^0 = X^0(T, Q_{r-1})$  when  $Q_{r-1} = \{(1 + t_i)/2 : t_i \text{ are the roots of } Lo_{r-1}(t), i \in \{1, \dots, r - 1\}\}$ .

In high order interpolation of scalar functions over a simplex  $T$ , the nodes are generally nonuniformly distributed and endowed of a rotational symmetry. In this way, the Runge phenomenon is minimized (thus spectral convergence of the interpolation error is ensured for smooth scalar functions) and the complexity of the generation algorithm is reduced. The existing literature deals with the complex problem of generating such points in a precise and efficient

way, starting from the Lobatto distribution along the edges  $E \in \Delta_1(T)$  (see [12, 13, 21, 18] and the references therein). Among all the possibilities, we consider the Lobatto, the symmetrised Lobatto (Lbs) and the warp & blend (WB) nodes, see [5, 22]. They have been chosen because of their attractive features with respect to convergence and also because they are given through an explicit formula, which is of great practical interest, especially in three dimensions where the optimization procedures involved in the definitions of other nonuniform distributions become quite complicated. These nonuniform distributions of nodes in a simplex  $T$  are obtained by a modification of the uniform one through a suitable mapping, therefore the incidence matrices of the small  $k$ -simplices, namely the small edge-to-small node, small face-to-small edge and small tetra-to-small face tables, for the nonuniform case are the same as those of the uniform case as they don't encode informations associated with spatial coordinates. The generation of the warp & blend node distribution in a triangle or tetrahedron  $T$  has been performed by running the `Matlab` software available in [22] for the triangle and in [14] for the tetrahedron.

In the case  $k = 1$ , we consider four different unisolvent sets  $X_r^1$ . Two of them are 1-simplices parallel to the edges  $E$  of  $T$ . They are two instances of  $X^1(T, Q_{r-1})$  as defined in Remark 2.

- The uniform distribution of 1-simplices, namely,  $X_r^1 = X^1(T, Q_{r-1})$  with  $Q_{r-1} = \{q_i = i/r \text{ with } 1 \leq i \leq r-1\}$ .

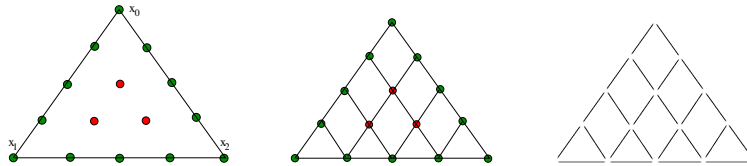


Figure 5: Construction, in a triangle, of a uniform and parallel distribution of small edges with ending points in the nodes of the principal lattice (here drawn for the scalar interpolation of degree  $r = 4$ ). On the left, the set of points  $X^0(T, Q_{r-1})$ ; on the right, the set of edges  $X^1(T, Q_{r-1})$ .

Once the uniform distribution of (green) nodes in Figure 5 is defined on the edges of  $T$  (left), connect these points by segments parallel to the  $n$  edges  $E \in \Delta_1(T)$  that have one ending point in  $\mathbf{x}_0$ . Chop these segments at the intersection (red) points in  $T$  (center). The red points together with the green points on the boundary of  $T$  constitute a unisolvent set of nodes for the interpolation of 0-forms, which is indeed the principal lattice of order  $r$  in  $T$ . A unisolvent and minimal set of small edges is defined by all the small segments between two points among the green or the red ones (right).

- The nonuniform distribution of 1-simplices associated with the Lobatto nodes on  $[0, 1]$ . In this case,  $X_r^1 = X^1(T, Q_{r-1})$  with  $Q_{r-1} = \{(1+t_i)/2 : t_i \text{ are the roots of } Lo_{r-1}(t) = L'_r(t)\}$ .

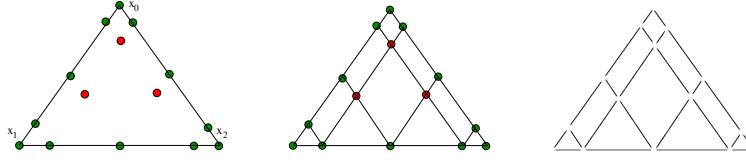


Figure 6: Construction, in a triangle, of a nonuniform distribution of small edges that are  $\parallel$  to  $E \in \Delta_1(T)$ , with ending points in the Lobatto nodes (here drawn for the scalar interpolation of degree  $r = 4$ ). On the left, the set of points  $X^0(T, Q_{r-1})$ ; on the right, the set of edges  $X^1(T, Q_{r-1})$ .

To improve the interpolation accuracy, in [6] the uniform distribution was cleverly modified by expanding or contracting the intervals between neighboring nodes in the simplex, in order to coincide with the Lobatto (Lb) distribution along the edges of the simplex. Once the Lobatto (green) nodes in Figure 6 are defined on the edges of  $T$  (left), connect these points by segments parallel to the  $n$  edges  $E \in \Delta_1(T)$  that have one ending point in  $\mathbf{x}_0$  (center). Chop these segments at the intersection (red) points in  $T$ . The red points together with the green points on the boundary of  $T$  constitute a unisolvent set of nodes for the interpolation of 0-forms, which is indeed the Lobatto distribution of degree  $r$  in  $T$ . A unisolvent and minimal set of small edges is defined by all the small segments between two points among the green or the red ones (right).

- The other two configurations  $X_r^1$  have 1-simplices that are not parallel to the edges  $E$  of  $T$ . Hence they are not of the form  $X^1(T, Q_{r-1})$ . The proof of unisolvence presented in Section 3 does not cover this situation. However the Vandermonde matrix associated with any of the considered distributions is not singular. It can thus be inverted, and this yields the construction of the dual basis  $\{\mathbf{w}_j^{X_r^1}\}_j$  associated with  $X_r^1$ , basis involved in Definition 1.

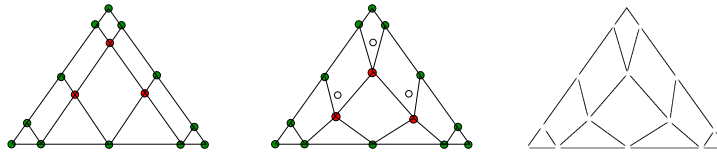


Figure 7: Construction, in a triangle, of a nonuniform distribution of small edges that are  $\parallel$  to  $E \in \Delta_1(T)$ , with ending points in the symmetrised Lobatto or warp & blend nodes (here drawn for the scalar interpolation of degree  $r = 4$ ). On the left, the set of points  $X^0(T, Q_{r-1})$  and edges  $X^1(T, Q_{r-1})$ ; on the center and on the right, a possible set of nodes and edges (the latter, not yet covered by the theory of the previous sections but for which the construction of an invertible Vandermonde matrix is possible).

Once the Lobatto (green) nodes in Figure 7 are defined on the edges of  $T$ , connect these points by segments parallel to the  $n$  edges  $E \in \Delta_1(T)$  that



have one ending point in  $\mathbf{x}_0$  (left). Chop these segments at the intersection (red) points in  $T$ . Finally, move the red points at the interior of  $T$  in the position corresponding with the symmetrised Lobatto or warp & blend ones (center). The small segments at the interior, between two points among the green or the red ones, which follow the node movement, are thus stretched and their direction is no more parallel to the edges of  $T$  (right).

We present some numerical results on the Lebesgue constant associated with these distributions of  $k$ -simplices in a simplex  $T$ , for  $k = 1$ , in two and three dimensions. The case  $k = 0$  for high-order interpolations of scalar fields on a simplex  $T$  has been widely studied in the literature and constitutes a reference to comment on the new case  $k = 1$ . To construct the Vandermonde matrix, and thus the dual basis, we consider the basis  $\{w_j\}_j$  of the space  $\mathcal{P}_r^- \Lambda^1(T)$  with elements  $w_j$  that are products between Bernstein polynomials of degree  $(r - 1)$  and Whitney 1-forms of polynomial degree 1 (see for example [8]). With this choice of local basis, the conditioning of the Vandermonde matrix varies in a acceptable range of values when the approximation degree  $r$  increases. It is known that, for  $r$  large, the classical Lagrange interpolation of a scalar function at uniformly distributed nodes can become unstable whereas that for the same function at particular families of not uniformly distributed nodes is stable [4]. This is due to the way the Lebesgue constant increases with  $r$ . A theoretical proof of this behavior in the scalar case exists in one dimension, see [12] for example. In higher dimensions and for  $k > 0$ , the generalised Lebesgue function is evaluated over a fine mesh of  $k$ -simplices and the largest result selected as Lebesgue constant. We refer to [18] (resp., [19]) for results on the accuracy of the proposed interpolation in two dimensions, for  $k = 0$  (resp.,  $k = 1$  and the uniform distribution).

### 5.1. In 2D

For the Lebesgue constant given in Definition 1, the computed values in two dimensions for  $k = 1$  are given in Table 2 and are compared with those of Table 1 for  $k = 0$  taken from [5, 22]. The results for  $k = 1$  are obtained by considering the supremum on an independent test mesh in  $T$ , that is much finer than the one corresponding with degree 12 and not obtained as a refinement of those associated with the analysed degrees. It has to be said that by modifying the test mesh, the computed values can have slight changes in decimals but their magnitude order does not change. These values are visualized in Figures 8 (for the same  $k$ ) and in Figure 9 (for the same type of distribution) in semi-log scale with respect to the polynomial degree  $r$  of the  $k$ -forms to be interpolated, with  $k = 0, 1$ .

By looking at Figure 8, we remark that the difference of behavior in the Lebesgue constant for the uniform and nonuniform distributions, which is well-known for  $k = 0$ , holds for  $k = 1$  too. For  $k = 0$ , the warp & blend (WB) distribution is known to perform, in terms of the Lebesgue constant growth, as one of the best, among those that are nonuniform with rotational symmetry, thus

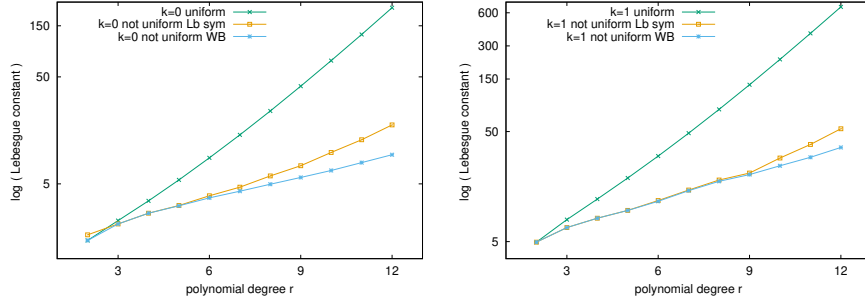


Figure 8: The Lebesgue constant in semi-log scale as a function of the polynomial degree  $r \geq 2$  of the  $k$ -form in a triangle  $T$ , with  $k = 0$  (left) and  $k = 1$  (right), respectively. The two sets of small  $k$ -simplices, the uniform and the not uniform ones, support unisolvent degrees of freedom (weights) for the space  $\mathcal{P}_r^- \Lambda^k$ , for  $k = 0, 1$ .

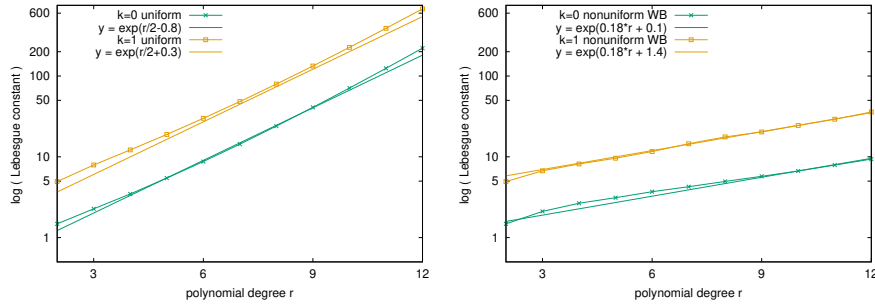


Figure 9: The Lebesgue constant in semi-log scale as a function of the polynomial degree  $r \geq 2$  of the  $k$ -form in a triangle  $T$ , uniform case (left) and nonuniform case (right), respectively.

better than the symmetrised Lobatto (Lb sym). In Table 2, we see that the order of performance for the nonuniform distributions of small edges is maintained. Among the three considered distributions, the best ones in terms of Lebesgue constant for  $k = 1$  are characterised by nonuniform small edges not parallel to edges  $E \in \Delta_1(T)$ , namely the symmetrised Lobatto or WB distributions. Whitney 1-forms of degree  $r$  are products of a polynomial of degree  $s = r - 1$ , which largely benefits from a nonuniform distribution of the nodes (the red and the green), and the Whitney 1-forms of degree 1. The small edges interior to  $T$  and associated with the WB or the symmetrised Lobatto nodes, are not all parallel to edges of  $T$ . Intermediate values are given for small edges insisting in the symmetrised Lobatto nodes. In the left picture of Figure 9, we can see an exponential fit for both  $k = 0, 1$ , as soon as the distribution of the small simplices over  $T$  is uniform. In the right picture of Figure 9, we have an exponential fit also when the distribution of the small  $k$ -simplices ( $k = 0, 1$ ) over  $T$  is nonuniform but with a smaller coefficient. We remark that  $\Lambda_{X_r^1}^1 > \Lambda_{X_r^0}^0$  for the considered degrees  $q$

and small-simplex distributions.

### 5.2. In 3D

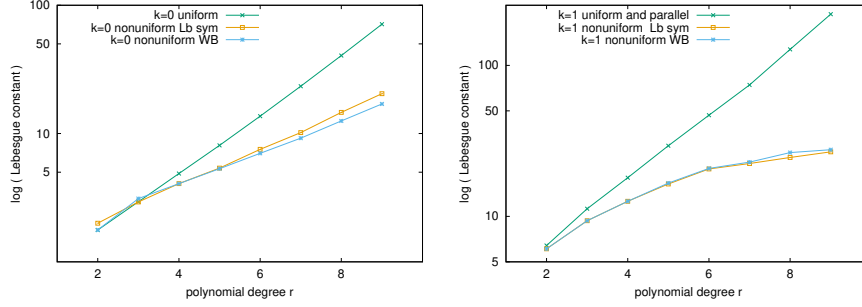


Figure 10: Comparison of the Lebesgue constant in semi-log scale as a function of the polynomial degree  $r \geq 2$  of the  $k$ -form in three dimensions (up to degree 9, only), with  $k = 0$  (left) and  $k = 1$  (right), respectively.

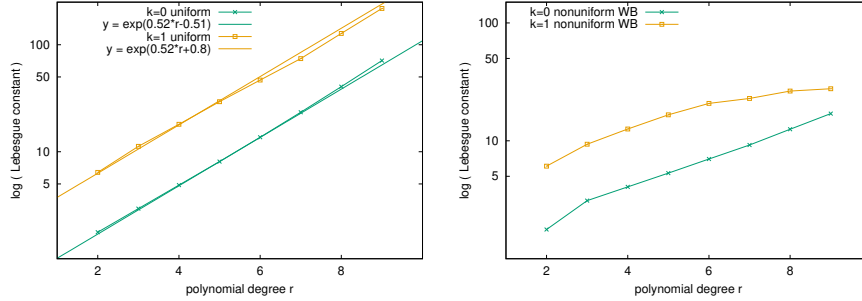


Figure 11: The Lebesgue constant in semi-log scale as a function of the polynomial degree  $r \geq 2$  of the  $k$ -form in a triangle  $T$ , uniform case (left) and WB nonuniform case (right), respectively.

Computed values for the Lebesgue constant in three dimensions for  $k = 1$  are given in Table 4 and are compared with those of Table 3 for  $k = 0$  taken from [5, 22]. Again, the results for  $k = 1$  are obtained by estimating the supremum on an independent test mesh in  $T$ , much finer than the one corresponding with degree 9 and not obtained as a refinement of those associated with the analysed degrees. These values are visualized in Figures 10 (for the same  $k$ ) and in Figure 11 (for the same type of distribution) in semi-log scale with respect to the polynomial degree  $r$  of the  $k$ -forms to be interpolated, with  $k = 0, 1$ . By looking at Figure 10, we remark that the behavior of the Lebesgue constant for the uniform and nonuniform distribution in two dimensions, also holds in three dimensions. The unisolvent and minimal configuration of small edges associated

with either symmetrised Lobatto or WB nodes are again the best ones in term of Lebesgue values for  $k = 1$ .

## 6. Conclusions

We have proposed a flexible rule to select a unisolvent set of small edges to interpolate a differential 1-form  $\omega$  using high order Whitney finite elements. The interpolating polynomial differential form has the same weights (integrals on the small edges) that  $\omega$ . Weights are alternative degrees of freedom to moments for high order trimmed polynomial spaces. We have tried different choices of weight's supports and we have studied the growth of the generalized Lebesgue constant when increasing the polynomial degree

Numerical results are in agreement with the fact that to have a stable polynomial interpolation, a nonuniform distribution of the geometrical supports for dofs is better than a uniform one. They have confirmed that better Lebesgue constants correspond to edges that do not remain parallel to those of the element  $T$ . Indeed, nonuniformity for  $k = 1$  has to be intended in terms of lengths and directions of the small edges supporting the degrees.

A deeper investigation is in progress concerning in particular the very natural extension of this approach to differential  $k$ -forms for  $0 < k < n$  being  $n$  the ambient dimension.

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$k = 0$ $r$	uniform in 2D	nonuniform in 2D	
	$\Lambda_{Un}$	$\Lambda_{Lb\ sym}$	$\Lambda_{WB}$
2	1.48	1.67	1.48
3	2.27	2.11	2.11
4	3.47	2.66	2.66
5	5.45	3.14	3.12
6	8.75	3.87	3.70
7	14.35	4.66	4.27
8	24.01	5.93	4.96
9	40.92	7.39	5.74
10	70.89	9.83	6.67
11	124.53	12.92	7.90
12	221.41	17.78	9.36

Table 1: Lebesgue constants associated with a uniform and not uniform (symmetrised Lobatto and warp & blend) distribution of nodes in a *triangle*  $T$  for different polynomial degrees  $r \geq 2$ , as computed in [5, 22].

$k = 1$ $r$	uniform	nonuniform in 2D		
	in 2D and $\parallel$	and $\parallel$ with Lb	and $\parallel$ with Lb sym	and $\parallel$ with WB
2	4.94	4.94	4.94	4.94
3	7.92	6.67	6.71	6.71
4	12.17	9.17	8.16	8.16
5	18.92	14.51	9.61	9.60
6	29.95	23.49	11.80	11.62
7	48.31	41.55	14.71	14.51
8	79.45	77.15	18.13	17.65
9	133.03	154.18	20.99	20.32
10	226.20	327.36	28.74	24.44
11	389.59	827.80	38.15	29.19
12	678.10	2142.45	52.97	35.85

Table 2: Lebesgue constants in a *triangle*  $T$ , associated with uniform and nonuniform distributions of small edges for different polynomial degrees  $r \geq 2$ . The ending points of the small edges are either in the uniform or in the nonuniform (Lobatto, symmetrised Lobatto or warp & blend) sets.

$k = 0$ $r$	uniform in 3D	nonuniform in 3D	
	$\Lambda_{Un}$	$\Lambda_{Lb\ sym}$	$\Lambda_{WB}$
2	1.77	2.00	1.77
3	2.94	2.93	3.11
4	4.88	4.07	4.07
5	8.09	5.38	5.32
6	13.66	7.53	7.01
7	23.38	10.17	9.21
8	40.55	14.63	12.54
9	71.15	20.46	17.02

Table 3: Lebesgue constants associated with a uniform and not uniform (symmetrised Lobatto and warp & blend) distribution of small nodes in a *tetrahedron*  $T$  for different polynomial degrees  $r \geq 2$ , as computed in [5, 22].

$k = 1$ $r$	uniform	nonuniform in 3D		
	in 3D and $\parallel$	and $\parallel$ with Lb	and $\parallel$ with Lb sym	and $\parallel$ with WB
2	6.41	6.41	6.10	6.10
3	11.23	11.40	9.36	9.36
4	18.04	22.38	12.59	12.59
5	29.37	69.45	16.38	16.61
6	46.76	274.58	20.61	20.77
7	74.19	-	22.38	22.87
8	127.53	-	24.55	26.44
9	218.19	-	26.75	27.64

Table 4: Lebesgue constants in a *tetrahedron*  $T$ , associated with uniform and nonuniform distributions of small edges for different polynomial degrees  $r \geq 2$ . The ending points of the small edges are either in the uniform or in the nonuniform (Lobatto, symmetrised Lobatto or warp & blend) sets.