



HAL
open science

Tameness and the power of programs over monoids in DA

Nathan Grosshans, Pierre Mckenzie, Luc Segoufin

► **To cite this version:**

Nathan Grosshans, Pierre Mckenzie, Luc Segoufin. Tameness and the power of programs over monoids in DA. 2021. hal-03114304v1

HAL Id: hal-03114304

<https://hal.science/hal-03114304v1>

Preprint submitted on 18 Jan 2021 (v1), last revised 5 May 2022 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

TAMENESS AND THE POWER OF PROGRAMS OVER MONOIDS IN \mathbf{DA}^*

NATHAN GROSSHANS, PIERRE MCKENZIE, AND LUC SEGOUFIN

Universität Kassel, Fachbereich Elektrotechnik/Informatik, Kassel, Germany

e-mail address: nathan.grosshans@polytechnique.edu

URL: <https://nathan.grosshans.me>

DIRO, Université de Montréal, Montréal, Canada

e-mail address: mckenzie@iro.umontreal.ca

Inria, DI ENS, ENS, CNRS, PSL University, Paris, France

e-mail address: luc.segoufin@inria.fr

ABSTRACT. The program-over-monoid model of computation originates with Barrington’s proof that it captures the complexity class \mathbf{NC}^1 . Here we make progress in understanding the subtleties of the model. First, we identify a new tameness condition on a class of monoids that entails a natural characterization of the regular languages recognizable by programs over monoids from the class. Second, we prove that the class known as \mathbf{DA} satisfies tameness and hence that the regular languages recognized by programs over monoids in \mathbf{DA} are precisely those recognizable in the classical sense by morphisms from \mathbf{QDA} . Third, we show by contrast that the well studied class of monoids called \mathbf{J} is not tame. Finally, we exhibit a program-length-based hierarchy within the class of languages recognized by programs over monoids from \mathbf{DA} .

1. INTRODUCTION

A program of range n on alphabet Σ over a finite monoid M is a sequence of pairs (i, f) where $1 \leq i \leq n$ and $f : \Sigma \rightarrow M$ is a function. This program assigns to each word $w_1 w_2 \cdots w_n$ the monoid element obtained by multiplying out in M the elements $f(w_i)$, one per pair (i, f) , in the order of the sequence. When an accepting set $F \subseteq M$ is specified, the program naturally defines the language L_n of words of length n assigned an element in F . A program sequence $(P_n)_{n \in \mathbb{N}}$ then defines the language formed by the union of the L_n .

A flurry of work on programs over monoids was triggered by Barrington’s celebrated discovery [Bar89], in fact extending the scope of an observation made earlier by Maurer and Rhodes [MR65], that polynomial length program sequences over the group S_5 capture the complexity class \mathbf{NC}^1 (of languages accepted by bounded fan-in Boolean circuits of logarithmic depth).

Key words and phrases: Programs over monoids, tameness, \mathbf{DA} , lower bounds.

* Revised and extended version of [GMS17] that includes a more inclusive definition of tameness, thus strengthening the statement that \mathbf{J} is not a tame variety, as explained in Section 3.

After all, a program over M is a mere generalization of a morphism from Σ^* to M and recognition by a morphism equates with acceptance by a finite automaton (M is obtained from the automaton by closing $\{t_a \mid a \in \Sigma \cup \{\varepsilon\}\}$ under composition, where t_a is the transformation induced by a on the state set of the automaton). Given the extensive algebraic automata theory available at the time of Barrington’s discovery [KR65, Eil76, Pin86], it was to be a matter of a few years before the structure of NC^1 got elucidated by algebraic means.

The “optimism period” produced many significant results. The classes $\text{AC}^0 \subset \text{ACC}^0 \subseteq \text{NC}^1$ were characterized by polynomial length programs over the aperiodic, the solvable, and all monoids respectively [Bar89, BT88]. More generally for any variety \mathbf{V} of monoids (a variety being the undisputed best fit with the informal notion of a natural class of monoids) one can define the class $\mathcal{P}(\mathbf{V})$ of languages recognized by polynomial length programs over a monoid drawn from \mathbf{V} . In particular, if \mathbf{A} is the variety of aperiodic monoids, then $\mathcal{P}(\mathbf{A})$ characterizes the complexity class AC^0 [BT88]. It was further observed that in a formal sense, only the regular languages matter for the purpose of understanding much of the structure of NC^1 (see for example [Str94]).

But sadly, the optimism period ended: although partial results in restricted settings were obtained, the holy grail of reproving significant circuit complexity results and forging ahead by recycling the deep theorems afforded by algebraic automata theory never materialized. The test case for the approach was to try to prove, independently from the known combinatorial arguments [Ajt83, FSS84, Hås86] and those based on approximating circuits by polynomials over some finite field [Raz87, Smo87], that $\mathcal{P}(\mathbf{A})$ does not contain the parity language MOD_2 , i.e., that $\text{MOD}_2 \notin \text{AC}^0$. But why to this day has this failed?

The answer of course is that programs are much more complicated than morphisms: programs can read the letter at an input position more than once, in non-left-to-right order, possibly assigning a different monoid element each time. Linear length programs can indeed trivially recognize non-regular languages (though see [BS95]). In the classical theory, any two varieties provably recognize distinct classes of languages [Eil76, Pin86]. In the theory of recognition by polynomial length programs (we will speak then of p -recognition by programs over a monoid or simply by the monoid), distinct varieties can yield the same class, as do, for instance, any two varieties of monoids \mathbf{V} and \mathbf{W} that each contain a simple non-Abelian group, for which $\mathcal{P}(\mathbf{V}) = \mathcal{P}(\mathbf{W}) = \text{NC}^1$ [MPT91, Theorem 4.1].

To further illustrate the subtle behavior of programs, consider the variety of monoids known as \mathbf{J} . This is the variety generated by the syntactic monoids of all languages defined by the presence or absence of certain *subwords*, where u is a subword of v if u can be obtained from v by deleting letters [Sim75]. One deduces that \mathbf{J} is unable to recognize the language defined by the regular expression $(a+b)^*ac^+$. Yet a sequence of programs over \mathbf{J} p -recognizes $(a+b)^*ac^+$ by the following clever trick. Consider the language L of all words having ca as a subword but having as subwords neither cca , caa nor cb . Being defined by the occurrence of subwords, L is recognized by a morphism $\varphi : \{a, b, c\}^* \rightarrow M$ where $M \in \mathbf{J}$, i.e., for this φ there is an $F \subseteq M$ such that $L = \varphi^{-1}(F)$. Here is the trick: the program of range n over M given by the sequence of instructions

$$(2, \varphi), (1, \varphi), (3, \varphi), (2, \varphi), (4, \varphi), (3, \varphi), (5, \varphi), (4, \varphi), \dots, (n, \varphi), (n-1, \varphi),$$

using F as accepting set, defines the set of words of length n in $(a+b)^*ac^+$. For instance, on input $abacc$ the program outputs $\varphi(baabcacc)$ which is in F , while on inputs $abbcc$ and $abacca$ the program outputs respectively $\varphi(babbcbcc)$ and $\varphi(baabcaccac)$ which are not in F . (See [Gro20, Lemma 4.1] for a full proof of the fact that $(a+b)^*ac^+ \in \mathcal{P}(\mathbf{J})$.)

The present work is motivated by the need to better understand such subtle behaviors of polynomial length programs over monoids. Quite a bit of knowledge on such programs has accumulated over nearly thirty years (consider [BST90, Pé190, PST97, MPT00, Tes03] beyond the references already mentioned). Yet, even within the realm of questions that do not hold pretense to major complexity class separations, gaps remain.

One beaming such gap concerns the variety of monoids \mathbf{DA} . The importance of \mathbf{DA} in algebraic automata theory and its connections with other fields are well established (see [TT02] for an eloquent testimony). \mathbf{DA} is a relatively “small” variety, well within the variety of aperiodic monoids. One could anticipate that “small” varieties will be sensitive to duplications and rearrangements in the order in which input letters are read by a program. Presumably in part for that reason, programs over \mathbf{DA} have seemingly not been successfully analyzed prior to our work.

Our main result is a characterization of the regular languages recognized by polynomial length programs over monoids in \mathbf{DA} . We show that $\mathcal{P}(\mathbf{DA}) \cap \mathcal{R}\text{eg}$ is precisely the class $\mathcal{L}(\mathbf{QDA})$ of languages recognized classically by morphisms in quasi- \mathbf{DA} , denoted \mathbf{QDA} . A surjective morphism φ from Σ^* to a finite monoid M is in quasi- \mathbf{DA} if, though M might not be in \mathbf{DA} , its stable monoid induced by φ is in \mathbf{DA} , i.e. there is a number k such that the image by φ of all words over Σ whose length is a multiple of k forms a submonoid of M which is in \mathbf{DA} .

To attach intuition to $\mathcal{P}(\mathbf{DA}) \cap \mathcal{R}\text{eg}$ equating $\mathcal{L}(\mathbf{QDA})$, consider aperiodic monoids. Classically, aperiodic monoids cannot recognize a language if doing so requires keeping a modular count. No monoid from \mathbf{A} can thus recognize the regular language LEN_2 of words $w \in \{a, b\}^*$ of even length. On the other hand, any non-trivial monoid M p -recognizes LEN_2 using the sequence $(P_n)_{n \in \mathbb{N}}$ in which P_n is a single instruction, $(1, f : \{a, b\} \rightarrow M)$, where $f(a) = f(b)$ are set to an accepting element if n is even and to a rejecting element if n is odd. It is known [BCST92] that when $\mathbf{V} = \mathbf{A}$, this haphazard modular counting ability of polynomial length programs over monoids from \mathbf{V} translates algebraically into “quasi- \mathbf{V} power” being necessary and sufficient for morphisms to simulate those programs on regular languages. Our main result shows that the same holds when $\mathbf{V} = \mathbf{DA}$.

Intuitively, the inclusion $\mathcal{P}(\mathbf{DA}) \cap \mathcal{R}\text{eg} \subseteq \mathcal{L}(\mathbf{QDA})$ therefore establishes that $\mathcal{P}(\mathbf{DA})$ contains no more regular languages *than expected*. In order to motivate tameness, we need to elaborate on such an “expectation”.

What *is* to be expected when a program p -recognizes a regular language? For $\mathbf{V} = \mathbf{A}$ and $\mathbf{V} = \mathbf{DA}$, because a program instruction (i, f) operating on a word w is “aware” of i , the expectation is that beyond their ability as classical recognizers, programs over monoids from \mathbf{V} may only make use of constant modular counting on the positions of a letter in w . This can be formalized by considering tagging each letter in w with its position modulo a fixed number. Languages whose sets of words tagged in this manner are recognized classically by a monoid from \mathbf{V} form a well-known class of languages, those recognized by morphisms from the variety $\mathbf{V} * \mathbf{Mod}$. Extending the strategy used to p -recognize LEN_2 above shows that $\mathcal{L}(\mathbf{V} * \mathbf{Mod}) \subseteq \mathcal{P}(\mathbf{V})$ for any \mathbf{V} . The expectation, fulfilled when $\mathbf{V} = \mathbf{A}$, is that no regular language outside $\mathcal{L}(\mathbf{V} * \mathbf{Mod})$ appears in $\mathcal{P}(\mathbf{V})$. That $\mathbf{V} = \mathbf{DA}$ also fulfills that expectation follows from our main result because $\mathbf{QDA} = \mathbf{DA} * \mathbf{Mod}$ [DP13]. Much of the structure of NC^1 would in fact be resolved if a selection of “large” varieties of monoids fulfilled that same expectation (see [MPT91, Corollary 4.13], [Str94, Conjecture IX.3.4]).

But what is there to expect when \mathbf{V} is a “small” variety such as \mathbf{J} ? To be sure, there are ways other than constant modular counting in which programs can supplement their

classical recognition-by-morphism ability. One such way is to treat subsets of positions in a word w arbitrarily differently from the rest of w . Programs can even pick these subsets depending on the length of w . But we are interested in ways that maintain regularity of the languages being p -recognized. So we consider the ability that programs have to treat *bounded-length prefixes* and *suffixes* arbitrarily. Adding or suppressing this ability preserves regularity. When $\mathbf{V} = \mathbf{A}$ or $\mathbf{V} = \mathbf{DA}$, morphisms to monoids from \mathbf{V} happen to possess this ability intrinsically. But morphisms to monoids from $\mathbf{V} = \mathbf{J}$ are limited in their handling of prefixes and suffixes.

We thus suggest to expect the abilities of programs over \mathbf{V} p -recognizing a regular language to be limited to 1) simulating classical recognition, 2) performing the usual counting of letter positions modulo a constant and 3) handling bounded-length prefixes and suffixes. To formalize 3), we introduce the class \mathbf{EV} of morphisms that, modulo the beginning and the end of a word, behave essentially like morphisms into monoids from \mathbf{V} . We find as special cases that $\mathcal{L}(\mathbf{EDA}) = \mathcal{L}(\mathbf{DA})$ and $\mathcal{L}(\mathbf{EJ}) \supset \mathcal{L}(\mathbf{J})$, as it should. To incorporate 2), we define *tameness* of a variety of monoids \mathbf{V} (Definition 3.9). The gist of tameness is to impose that any surjective morphism $\varphi : \Sigma^* \rightarrow M$, whose image $\varphi(\Sigma)$ is appropriately restricted and whose word problems (i.e., whose languages of words over Σ whose image through φ is a given element in M) are p -recognizable by monoids from \mathbf{V} , should belong to \mathbf{EV} .

Put simply then, we expect small varieties to be *tame*.

Having defined tameness, we derive its main property: a variety of monoids \mathbf{V} is tame if and only if $\mathcal{P}(\mathbf{V}) \cap \mathcal{R}\text{eg} \subseteq \mathcal{L}(\mathbf{QEV})$, where \mathbf{QEV} is a class of morphisms defined from \mathbf{EV} in analogy with the definition of \mathbf{QV} from \mathbf{V} . We deduce from this property that

- \mathbf{DA} is tame;
- \mathbf{J} is not tame.

Proving \mathbf{DA} tame is indeed the main technical difficulty behind our main result above characterizing $\mathcal{P}(\mathbf{DA}) \cap \mathcal{R}\text{eg}$. Disproving tameness of \mathbf{J} follows by arguing $(a + b)^*ac^+ \notin \mathcal{L}(\mathbf{QEJ})$. So programs over \mathbf{J} *do* p -recognize more regular languages than expected. Be it the chicken or the egg, the non-tameness of \mathbf{J} then “explains” the surprising power of $\mathcal{P}(\mathbf{J})$, as witnessed by the clever trick in our example above.

Further analysing tameness, we prove that any sp -variety of monoids (concept introduced in [GMS17] as our initial attempt to capture the expected behavior of programs over small varieties) is tame. Being tame is strictly more inclusive than being an sp -variety however: the variety of commutative monoids is shown tame, yet not an sp -variety.

Our notion of a tame variety differs subtly but fundamentally from a similar notion, that of a p -variety, developed for semigroups by Péladeau, Straubing and Thérien [PST97] and also studied in the case of monoids by Péladeau [Pél90] and later Tesson [Tes03] in their respective Ph.D. theses. These authors could show that for any p -variety of the form $\mathbf{V} * \mathbf{D}$ we have $\mathcal{P}(\mathbf{V} * \mathbf{D}) \cap \mathcal{R}\text{eg} = \mathcal{L}(\mathbf{Q}(\mathbf{V} * \mathbf{D}))$. It is possible to show that any p -variety of the form $\mathbf{V} * \mathbf{D}$ is tame. Hence the result of [PST97] mentioned above follows from our result as for varieties of the form $\mathbf{V} * \mathbf{D}$ we have, using a lot of abuse of notation, that $\mathbf{E}(\mathbf{V} * \mathbf{D}) = \mathbf{V} * \mathbf{D}$ and that $\mathcal{L}(\mathbf{Q}(\mathbf{V} * \mathbf{D})) = \mathcal{L}(\mathbf{V} * \mathbf{D} * \mathbf{Mod}) \subseteq \mathcal{P}(\mathbf{V} * \mathbf{D})$. However tame varieties and p -varieties are two different notions as \mathbf{J} is a p -variety but is not tame as we have seen.

Our final result concerns $\mathcal{P}(\mathbf{DA})$. With \mathcal{C}_k the class of languages recognized by programs of length $O(n^k)$ over \mathbf{DA} , we prove that $\mathcal{C}_1 \subset \mathcal{C}_2 \subset \dots \subset \mathcal{C}_k \subset \dots \subset \mathcal{P}(\mathbf{DA})$ forms a strict

hierarchy. We also relate this hierarchy to another algebraic characterization of \mathbf{DA} and exhibit conditions on $M \in \mathbf{DA}$ under which any program over M can be rewritten as an equivalent subprogram (made of a subsequence of the original sequence of instructions) of length $O(n^k)$, refining a result by Tesson and Thérien [TT01].

Organization of the paper. In Section 2 we define programs over varieties of monoids, p -recognition by such programs and the necessary algebraic background. The definition of tameness for a variety \mathbf{V} is given in Section 3 with our first result showing that regular languages in $\mathcal{P}(\mathbf{V})$ are included in $\mathcal{L}(\mathbf{QEV})$ when \mathbf{V} is tame; we also briefly discuss the case of \mathbf{J} , which isn't tame. We show that \mathbf{DA} is tame in Section 4. Finally, Section 5 contains the hierarchy results about $\mathcal{P}(\mathbf{DA})$.

2. PRELIMINARIES

This section is dedicated to the introduction of the mathematical material used throughout this paper. Concerning algebraic automata theory, we only quickly review the basics and refer the reader to the two classical references of the domain by Eilenberg [Eil74, Eil76] and Pin [Pin86].

General notations. Let $i, j \in \mathbb{N}$ be two natural numbers. We shall denote by $\llbracket i, j \rrbracket$ the set of all natural numbers $n \in \mathbb{N}$ verifying $i \leq n \leq j$. We shall also denote by $\llbracket i \rrbracket$ the set $\llbracket 1, i \rrbracket$.

Words and languages. Let Σ be a finite alphabet. We denote by Σ^* the set of all finite words over Σ . We also denote by Σ^+ the set of all finite non empty words over Σ , the empty word being denoted by ε . Given some word $w \in \Sigma^*$, we denote its length by $|w|$ and, for any $a \in \Sigma$, by $|w|_a$ the number of occurrences of the letter a in w . A *language over Σ* is a subset of Σ^* . A language is *regular* if it can be defined using a regular expression. Given a language L , its *syntactic congruence* \sim_L is the relation on Σ^* relating two words u and v whenever for all $x, y \in \Sigma^*$, $xuy \in L$ if and only if $xvy \in L$. It is easy to check that \sim_L is an equivalence relation and a congruence for concatenation. The *syntactic morphism of L* is the mapping sending any word u to its equivalence class in the syntactic congruence.

The *quotient of a language L over Σ relative to the words u and v* is the language, denoted by $u^{-1}Lv^{-1}$, of the words w such that $uwv \in L$.

Monoids, semigroups and varieties. A *semigroup* is a non-empty set equipped with an associative law that we will write multiplicatively. A *monoid* is a semigroup with an identity. An example of a semigroup is Σ^+ , the free semigroup over Σ . Similarly Σ^* is the free monoid over Σ . A *morphism φ from a semigroup S to a semigroup T* is a function from S to T such that $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in S$. A morphism of monoids additionally requires that the identity is preserved. Any morphism $\varphi: \Sigma^* \rightarrow M$ for Σ a finite alphabet and M some monoid is uniquely determined by the images of the letters of Σ by φ . A semigroup T is a *subsemigroup* of a semigroup S if T is a subset of S and is equipped with the restricted law of S . Additionally the notion of submonoids requires the presence of the identity. A semigroup T *divides* a semigroup S if T is the image by a semigroup morphism of a subsemigroup of S . Division of monoids is defined in the same way by replacing any occurrence of “semigroup” by “monoid”. The *Cartesian (or direct) product* of two semigroups is simply the semigroup given by the Cartesian product of the two underlying sets equipped with the Cartesian product of their laws.

A language L over Σ is *recognized by a monoid* M if there is a morphism h from Σ^* to M and a subset F of M such that $L = h^{-1}(F)$. We also say that *the morphism h recognizes L* . It is well known that a language is regular if and only if it is recognized by a finite monoid. Actually, as \sim_L is a congruence, the quotient Σ^*/\sim_L is a monoid, called *the syntactic monoid of L* , that recognizes L via the syntactic morphism of L . The syntactic monoid of L is finite if and only if L is regular. The quotient Σ^+/\sim_L is analogously called *the syntactic semigroup of L* .

A *variety of monoids* is a non-empty class of finite monoids closed under Cartesian product and monoid division. A *variety of semigroups* is defined similarly. When dealing with varieties, we consider only finite monoids and semigroups.

An element s of a semigroup is *idempotent* if $ss = s$. For any finite semigroup S there is a positive number (the minimum such number), *the idempotent power of S* , often denoted ω , such that for any element $s \in S$, s^ω is idempotent.

A variety can be defined by means of identities [Rei82]. The variety is then the class of monoids or semigroups such that each of them has all its elements satisfy the identities. For instance, the variety of aperiodic monoids \mathbf{A} can be defined as the class of monoids satisfying the identity $x^\omega = x^{\omega+1}$, where x ranges over the elements of the monoid while ω is the idempotent power of the monoid. The variety of monoids \mathbf{DA} is defined by the identity $(xy)^\omega = (xy)^\omega x (xy)^\omega$. The variety of monoids \mathbf{J} is defined by the identity $(xy)^\omega = (xy)^\omega x = y (xy)^\omega$. One easily deduces that $\mathbf{J} \subseteq \mathbf{DA} \subseteq \mathbf{A}$.

Varieties of languages. A *variety of languages* is a class of languages over arbitrary finite alphabets closed under Boolean operations, quotients and inverses of morphisms (i.e. if L is a language in the class over a finite alphabet Σ , if Γ is some other finite alphabet and $\varphi: \Gamma^* \rightarrow \Sigma^*$ is a morphism of monoids, then $\varphi^{-1}(L)$ is also in the class).

Eilenberg showed [Eil76, Chapter VII, Section 3] that there is a bijective correspondence between varieties of monoids and varieties of languages: to each variety of monoids \mathbf{V} we can bijectively associate $\mathcal{L}(\mathbf{V})$ the variety of languages whose syntactic monoids belong to \mathbf{V} and, conversely, to each variety of languages \mathcal{V} we can bijectively associate $\mathbf{M}(\mathcal{V})$ the variety of monoids generated by the syntactic monoids of the languages of \mathcal{V} , and these correspondences are mutually inverse.

When \mathbf{V} is a variety of semigroups, we will denote by $\mathcal{L}(\mathbf{V})$ the class of languages whose syntactic semigroup belongs to \mathbf{V} . There is also an Eilenberg-type correspondence for an appropriate notion of language varieties, that is *ne-varieties* of languages, but we won't present it here. (The interested reader may have a look at [Str02] as well as [PS05, Lemma 6.3].)

Quasi and locally \mathbf{V} languages, modular counting and predecessor. If S is a semigroup we denote by S^1 the monoid S if S is already a monoid and $S \cup \{1\}$ otherwise.

The following definitions are taken from [PS05, CPS06b]. Let φ be a surjective morphism from Σ^* , for Σ some finite alphabet, to a finite monoid M : such a morphism is called a *stamp*. For all k consider the subset $\varphi(\Sigma^k)$ of M . As M is finite there is a k such that $\varphi(\Sigma^{2k}) = \varphi(\Sigma^k)$. This implies that $\varphi(\Sigma^k)$ is a semigroup. The semigroup given by the smallest such k is called the *stable semigroup of φ* and this k is called the *stability index* of φ . If 1 is the identity of M , then $\varphi(\Sigma^k) \cup \{1\}$ is called the *stable monoid of φ* . If \mathbf{V} is a variety of monoids, then we shall denote by \mathbf{QV} the class of stamps whose stable monoid is in \mathbf{V} and by $\mathcal{L}(\mathbf{QV})$ the class of languages whose syntactic morphism is in \mathbf{QV} .

For \mathbf{V} a variety of monoids, we say that a finite semigroup S is *locally \mathbf{V}* if, for every idempotent e of S , the monoid eSe belongs to \mathbf{V} ; we denote by \mathbf{LV} the class of locally- \mathbf{V} finite semigroups, which happens to be a variety of semigroups.

We now define languages recognized by $\mathbf{V} * \mathbf{Mod}$ and $\mathbf{V} * \mathbf{D}$. We do not use the standard algebraic definition using the wreath product as we won't need it, but directly a characterization of the languages recognized by such algebraic objects [CPS06a, Til87].

Let \mathbf{V} be a variety of monoids. We say that a language over Σ is in $\mathcal{L}(\mathbf{V} * \mathbf{Mod})$ if it is obtained by a finite combination of unions and intersections of languages over Σ for which membership of each word over Σ only depends on its length modulo some integer $k \in \mathbb{N}_{>0}$ and languages L over Σ for which there is a number $k \in \mathbb{N}_{>0}$ and a language L' over $\Sigma \times \{0, \dots, k-1\}$ whose syntactic monoid is in \mathbf{V} , such that L is the set of words w that belong to L' after adding to each letter of w its position modulo k .

Similarly we say that a language over Σ is in $\mathcal{L}(\mathbf{V} * \mathbf{D})$ if it is obtained by a finite combination of unions and intersections of languages over Σ for which membership of each word over Σ only depends on its $k \in \mathbb{N}$ last letters and languages L over Σ for which there is a number $k \in \mathbb{N}$ and a language L' over $\Sigma \times \Sigma^{\leq k}$ (where $\Sigma^{\leq k}$ denotes all words over Σ of length at most k) whose syntactic monoid is in \mathbf{V} , such that L is the set of words w that belong to L' after adding to each letter of w the word composed of the k (or less when near the beginning of w) letters preceding that letter. The variety of semigroups $\mathbf{V} * \mathbf{D}$ can then be defined as the one generated by the syntactic semigroups of the languages in $\mathcal{L}(\mathbf{V} * \mathbf{D})$ as defined above.

A variety \mathbf{V} is said to be *local* if $\mathcal{L}(\mathbf{V} * \mathbf{D}) = \mathcal{L}(\mathbf{LV})$. This is not the usual definition of locality, defined using categories, but it is equivalent to it [Til87, Theorem 17.3]. One of the consequences of locality that we will use is that $\mathcal{L}(\mathbf{V} * \mathbf{Mod}) = \mathcal{L}(\mathbf{QV})$ when \mathbf{V} is local [DP14, Corollary 18], while $\mathcal{L}(\mathbf{V} * \mathbf{Mod}) \subseteq \mathcal{L}(\mathbf{QV})$ in general (see [Dar14, Pap14]).

Programs over varieties of monoids. Programs over monoids form a non-uniform model of computation, first defined by Barrington and Thérien [BT88], extending Barrington's permutation branching program model [Bar89]. Let M be a finite monoid and Σ a finite alphabet. A program P over M is a finite sequence of instructions of the form (i, f) where i is a positive integer and f a function from Σ to M . The *length* of P is the number of its instructions. A program has *range* n if all its instructions use a number less than n . A program P of range n defines a function from Σ^n , the words of length n , to M as follows. On input $w \in \Sigma^n$, each instruction (i, f) outputs the monoid element $f(w_i)$. A sequence of instructions then yields a sequence of elements of M and their product is the output $P(w)$ of the program.

A language L over Σ is *p-recognized* by a sequence of programs $(P_n)_{n \in \mathbb{N}}$ if for each n , P_n has range n and length polynomial in n and recognizes $L \cap \Sigma^n$, that is, there exists a subset F_n of M such that $L \cap \Sigma^n$ is precisely the set of words w of length n such that $P_n(w) \in F_n$. In that case, we also say that L is *p-recognized* by M .

We denote by $\mathcal{P}(M)$ the class of languages *p-recognized* by a sequence of programs $(P_n)_{n \in \mathbb{N}}$ over M . If \mathbf{V} is a variety of monoids we denote by $\mathcal{P}(\mathbf{V})$ the union of all $\mathcal{P}(M)$ for $M \in \mathbf{V}$.

The following is a simple fact about $\mathcal{P}(\mathbf{V})$. Let Σ and Γ be two finite alphabets and $\mu: \Sigma^* \rightarrow \Gamma^*$ be a morphism. We say that μ is *length multiplying*, or that μ is an *lm-morphism*, if there is a constant k such that for all $a \in \Sigma$, the length of $\mu(a)$ is k .

Lemma 2.1. [MPT91, Corollary 3.5] *For \mathbf{V} any variety of monoids, $\mathcal{P}(\mathbf{V})$ is closed under Boolean operations, quotients and inverse images of lm-morphisms.*

Given two range n programs P, P' over some monoid M using the same input alphabet Σ , we shall say that P' is a *subprogram*, a *prefix* or a *suffix* of P whenever P' is, respectively, a subword, a prefix or a suffix of P , looking at P and P' as words over $[n] \times M^\Sigma$.

3. GENERAL RESULTS ABOUT REGULAR LANGUAGES AND PROGRAMS

Let \mathbf{V} be a variety of monoids. By definition any regular language recognized by a monoid in \mathbf{V} is p -recognized by a sequence of programs over a monoid in \mathbf{V} . Actually, since in a program over some monoid in \mathbf{V} , the monoid element output for each instruction can depend on the position of the letter read, hence in particular on its position modulo some fixed number, it is easy to see that any regular language in $\mathcal{L}(\mathbf{V} * \mathbf{Mod})$ is p -recognized by a sequence of programs over some monoid in \mathbf{V} . We will see in Section 3.2 that programs over some monoid in \mathbf{V} can also p -recognize the regular languages that are “essentially \mathbf{V} ” i.e. that differ from a language in $\mathcal{L}(\mathbf{V})$ only on the prefix and suffix of the words.

In this section we characterize those varieties \mathbf{V} such that programs over monoids in \mathbf{V} do not recognize more regular languages than those mentioned above.

We first recall the definitions and results around p -varieties developed by Péladeau, Tesson, Straubing and Thérien and then present the definition of sp -varieties that was inspired by their work and studied in the conference version of the present paper. In order to deal with the limitation of sp -varieties we then define the notion of essentially- \mathbf{V} that will be the last ingredient for our definition of tameness. We then provide an upper bound on the regular languages that can be p -recognized by a sequence of programs over a monoid from a tame variety \mathbf{V} .

3.1. p - and sp -varieties of monoids. We first recall the definition of p -varieties. These seem to have been originally defined by Péladeau in his Ph.D. thesis [Pél90] and later used by Tesson in his own Ph.D. thesis [Tes03]. The notion of a p -variety has also been defined for semigroups by Péladeau, Straubing and Thérien in [PST97].

Let μ be a morphism from Σ^* to a finite monoid M . We denote by $\mathcal{W}(\mu)$ the set of languages L over Σ such that $L = \mu^{-1}(F)$ for some subset F of M . Given a semigroup S there is a unique morphism $\eta_S: S^* \rightarrow S^1$ extending the identity on S , called the *evaluation morphism of S* . We write $\mathcal{W}(S)$ for $\mathcal{W}(\eta_S)$. We define $\mathcal{W}(M)$ similarly for any monoid M . It is easy to see that if $M \in \mathbf{V}$ then $\mathcal{W}(M) \subseteq \mathcal{P}(\mathbf{V})$. The condition to be a p -variety requires a converse of this observation.

Definition 3.1. An p -variety of monoids is a variety \mathbf{V} of monoids such that for any finite monoid M , if $\mathcal{W}(M) \subseteq \mathcal{P}(\mathbf{V})$ then $M \in \mathbf{V}$.

The following result illustrates an important property of p -varieties, when the notion is adapted to varieties of semigroups accordingly.

Proposition 3.2. [PST97] Let $\mathbf{V} * \mathbf{D}$ be a p -variety of semigroups, where \mathbf{V} is a variety of monoids.

Then $\mathcal{P}(\mathbf{V} * \mathbf{D}) \cap \mathcal{Reg} = \mathcal{L}(\mathbf{V} * \mathbf{D} * \mathbf{Mod})$ (where the latter class is defined in the same way as $\mathcal{L}(\mathbf{V} * \mathbf{Mod})$).

It is known that \mathbf{J} is a p -variety of monoids [Tes03] but as we have seen in the introduction, $\mathcal{P}(\mathbf{J})$ contains languages that are more complicated than those in $\mathcal{L}(\mathbf{J} * \mathbf{Mod})$ (see the end of this subsection for a proof). In order to capture those varieties for which programs

are well behaved we need a restriction of p -varieties and this brings us to the following definition.

Definition 3.3. An *sp-variety of monoids* is a variety \mathbf{V} of monoids such that for any finite semigroup S , if $\mathcal{W}(S) \subseteq \mathcal{P}(\mathbf{V})$ then $S^1 \in \mathbf{V}$.

Hence any *sp*-variety of monoids is also a p -variety of monoids, but the converse is not always true as we will see in Proposition 3.6 below that \mathbf{J} is not an *sp*-variety.

An example of an *sp*-variety of monoids is the class of aperiodic monoids \mathbf{A} . This is a consequence of the result that for any number $k > 1$, checking if $|w|_a$ is a multiple of k for $w \in \{a, b\}^*$ cannot be done in $\mathbf{AC}^0 = \mathcal{P}(\mathbf{A})$ [FSS84, Ajt83] (we shall denote the corresponding language over the alphabet $\{0, 1\}$ by MOD_k). Towards a contradiction, assume there would exist a semigroup S such that S^1 is not aperiodic but still $\mathcal{W}(S) \subseteq \mathcal{P}(\mathbf{A})$. Then there is an x in S such that $x^\omega \neq x^{\omega+1}$. Consider the morphism $\mu: \{a, b\}^* \rightarrow S^1$ sending a to $x^{\omega+1}$ and b to x^ω , and the language $L = \mu^{-1}(x^\omega)$. It is easy to see that L is the language of all words with a number of a congruent to 0 modulo k , where k is the smallest number such that $x^{\omega+k} = x^\omega$. As $x^\omega \neq x^{\omega+1}$, $k > 1$. Let $\eta_S: S^* \rightarrow S^1$ be the evaluation morphism of S . The morphism $\varphi: \Sigma^* \rightarrow S^*$ sending each letter $a \in \Sigma$ to $\mu(a)$ verifies that $\mu = \eta_S \circ \varphi$, so that $L = \mu^{-1}(x^\omega) = (\eta_S \circ \varphi)^{-1}(x^\omega) = \varphi^{-1}(\eta_S^{-1}(x^\omega))$. From $\mathcal{W}(S) \subseteq \mathcal{P}(\mathbf{A})$ it follows that $\eta_S^{-1}(x^\omega) \in \mathcal{P}(\mathbf{A})$, hence since φ sends each letter of Σ to a letter of S , it is an *lm*-morphism and as $\mathcal{P}(\mathbf{A})$ is closed under inverses of *lm*-morphisms by Lemma 2.1, we have $L = \varphi^{-1}(\eta_S^{-1}(x^\omega)) \in \mathcal{P}(\mathbf{A})$, a contradiction.

The following is the desired consequence of being an *sp*-variety of monoids.

Proposition 3.4. *Let \mathbf{V} be an sp-variety of monoids. Then $\mathcal{P}(\mathbf{V}) \cap \text{Reg} \subseteq \mathcal{L}(\mathbf{QV})$.*

Proof. Let L be a regular language in $\mathcal{P}(M)$ for some $M \in \mathbf{V}$. Let M_L be the syntactic monoid of L and η_L its syntactic morphism. Let S be the stable semigroup of η_L , in particular $S = \eta_L(\Sigma^k)$ for some k . We wish to show that S^1 is in \mathbf{V} .

We show that $\mathcal{W}(S) \subseteq \mathcal{P}(\mathbf{V})$ and conclude from the fact that \mathbf{V} is an *sp*-variety that $S^1 \in \mathbf{V}$ as desired. Let $\eta_S: S^* \rightarrow S^1$ be the evaluation morphism of S . Consider $m \in S$ and consider $L' = \eta_S^{-1}(m)$. We wish to show that $L' \in \mathcal{P}(\mathbf{V})$. This implies that $\mathcal{W}(S) \subseteq \mathcal{P}(\mathbf{V})$ by closure under union, Lemma 2.1.

Let $L'' = \eta_L^{-1}(m)$. Since m belongs to the syntactic monoid of L and η_L is the syntactic morphism of L , a classical algebraic argument [Pin86, Chapter 2, proof of Lemma 2.6] shows that L'' is a Boolean combination of quotients of L . By Lemma 2.1, we conclude that $L'' \in \mathcal{P}(\mathbf{V})$.

By definition of S , for any element s of S there is a word u_s of length k such that $\eta_L(u_s) = s$. Notice that this is precisely where we need to work with S and not S^1 .

Let $f: S^* \rightarrow \Sigma^*$ be the *lm*-morphism sending s to u_s and notice that $L' = f^{-1}(L'')$. The result follows by closure of $\mathcal{P}(\mathbf{V})$ under inverse images of *lm*-morphisms, Lemma 2.1. \square

We don't know whether it is always true that for *sp*-varieties of monoids \mathbf{V} , $\mathcal{L}(\mathbf{QV})$ is included into $\mathcal{P}(\mathbf{V})$. But we can prove it for local varieties.

Proposition 3.5. *Let \mathbf{V} be a local sp-variety of monoids. Then $\mathcal{P}(\mathbf{V}) \cap \text{Reg} = \mathcal{L}(\mathbf{QV})$.*

Proof. This follows from the fact that for local varieties $\mathcal{L}(\mathbf{QV}) = \mathcal{L}(\mathbf{V} * \mathbf{Mod})$ (see [DP14]). The result can then be derived using Proposition 3.4, as we always have $\mathcal{L}(\mathbf{V} * \mathbf{Mod}) \subseteq \mathcal{P}(\mathbf{V})$. \square

As \mathbf{A} is local [Til87, Example 15.5] and an sp -variety, it follows from Proposition 3.5 that the regular languages in $\mathcal{P}(\mathbf{A})$, hence in \mathbf{AC}^0 , are precisely those in $\mathcal{L}(\mathbf{QA})$, which is the characterization of the regular languages in \mathbf{AC}^0 obtained by Barrington, Compton, Straubing and Thérien [BCST92].

We will see in the next section that \mathbf{DA} is an sp -variety. As it is also local [Alm96], we get from Proposition 3.5 that the regular languages of $\mathcal{P}(\mathbf{DA})$ are precisely those in $\mathcal{L}(\mathbf{QDA})$.

As explained in the introduction, the language $(a+b)^*ac^+$ can be p -recognized by a program over \mathbf{J} . A simple algebraic argument shows that it is not in $\mathcal{L}(\mathbf{QJ})$: just compute the stable monoid of the syntactic morphism of the language, which is equal to the syntactic monoid of the language, that is not in \mathbf{J} . Hence, by Proposition 3.4, we have the following result:

Proposition 3.6. *\mathbf{J} is not an sp -variety of monoids.*

Despite Proposition 3.6 providing some explanation for the unexpected relative strength of programs over monoids in \mathbf{J} , the notion of an sp -variety of monoids isn't entirely satisfactory.

We say that a monoid is *trivial* when its underlying set contains a sole element. The class of all trivial monoids, that we will denote by \mathbf{I} , forms a variety: it is the sole variety containing only trivial monoids, so we may call it the *trivial variety of monoids*.

One observation to be made is that any non-trivial monoid M p -recognizes the language of words over $\{a, b\}$ starting with an a : for the first position in any word, just send a to any element that is not the identity and b to the identity. This means that for any non-trivial variety of monoids \mathbf{V} , we have that $a(a+b)^* \in \mathcal{P}(\mathbf{V})$. But since the stable monoid of the syntactic morphism of $a(a+b)^*$ is equal to the syntactic monoid of this language, it follows that for any non-trivial variety of monoids \mathbf{V} not containing the syntactic monoid of $a(a+b)^*$, we have $\mathcal{P}(\mathbf{V}) \cap \mathcal{R}\text{eg} \not\subseteq \mathcal{L}(\mathbf{QV})$, hence that \mathbf{V} is not an sp -variety of monoids.

Therefore, many varieties of monoids actually aren't sp -varieties of monoids simply because of the built-in capacity of programs over any non-trivial monoid to test the first letter of input words. This is for example true for any non-trivial variety containing only groups and for any non-trivial variety containing only commutative monoids. This built-in capacity, additional to programs' ability to do positional modulo counting that underlies the definition of sp -varieties, should be taken into account in the notion we are looking for to capture "well behavior". In order to define our notion of tameness we first study this extra capacity that is built-in for programs over \mathbf{V} and that we call "essentially- \mathbf{V} ".

3.2. Essentially- \mathbf{V} stamps. It is easy to extend our reasoning above to show that given any non-trivial monoid M and given some $k \in \mathbb{N}_{>0}$, the language of words over $\{a, b\}$ having an a in position k , that is $(a+b)^{k-1}a(a+b)^*$, is p -recognized by M , and the same goes for $(a+b)^*a(a+b)^{k-1}$. By generalizing, we can quickly conclude that given any non-trivial variety of monoids \mathbf{V} , for any finite alphabet Σ and any $x, y \in \Sigma^*$, we have that $x\Sigma^*y \in \mathcal{P}(\mathbf{V})$ by closure of $\mathcal{P}(\mathbf{V})$ under Boolean operations, Lemma 2.1. Put informally, p -recognition by monoids taken from any fixed non-trivial variety of monoids allows to check some constant-length beginning or ending of the input words. Moreover, p -recognition by monoids taken from any fixed non-trivial variety of monoids \mathbf{V} also easily allows to test for membership of words in $\mathcal{L}(\mathbf{V})$ after stripping out some constant-length beginning or ending: that is, languages of the form $\Sigma^{k_1}L\Sigma^{k_2}$ for $k_1, k_2 \in \mathbb{N}$ and $L \in \mathcal{L}(\mathbf{V})$.

This motivates the definition of *essentially- \mathbf{V}* stamps.

Definition 3.7. Let \mathbf{V} be a variety of monoids. Let $\varphi: \Sigma^* \rightarrow M$ be a stamp from a finite alphabet Σ to a finite monoid M . Let s be the stability index of φ .

We say that φ is *essentially- \mathbf{V}* whenever there exists a stamp $\mu: \Sigma^* \rightarrow N$ with $N \in \mathbf{V}$ such that for all $u, v \in \Sigma^*$, we have

$$\mu(u) = \mu(v) \Rightarrow (\varphi(xuy) = \varphi(xvy) \quad \forall x, y \in \Sigma^s) .$$

We will denote by **EV** the class of all essentially- \mathbf{V} stamps¹ and by $\mathcal{L}(\mathbf{EV})$ the class of languages recognized by morphisms in **EV**.

Informally stated, a stamp $\varphi: \Sigma^* \rightarrow M$ is essentially- \mathbf{V} when it behaves like a stamp into a monoid of \mathbf{V} as soon as a sufficiently long beginning and ending of any input word has been fixed. The value for “sufficiently long” depends on φ and is adequately given by the stability index s of φ , as by definition of s , any word w of length at least $2s$ can always be made of length between s and $2s - 1$ without changing the image by φ .

Let us start by giving some examples.

Consider first the language $a(a+b)^*$ over the alphabet $\{a, b\}$. Let’s take $\varphi: \{a, b\}^* \rightarrow M$ to be its syntactic morphism: its stability index is equal to 1 and it has the property that for any $w \in \{a, b\}^*$, we have $\varphi(aw) = \varphi(a)$ and $\varphi(bw) = \varphi(b)$. Hence, if we define $\mu: \{a, b\}^* \rightarrow \{1\}$ to be the obvious stamp into the trivial monoid $\{1\}$, we indeed have that for all $u, v \in \{a, b\}^*$, it holds that

$$\mu(u) = \mu(v) \Rightarrow (\varphi(xuy) = \varphi(xvy) \quad \forall x, y \in \{a, b\}^1) .$$

In conclusion, the stamp φ is essentially- \mathbf{V} for any variety of monoids, in particular $a(a+b)^* \in \mathcal{L}(\mathbf{EI})$.

Let us now consider the language $a(a+b)^*b(a+b)^*a$ over the alphabet $\{a, b\}$ of words starting and ending with an a and containing some b in between. Let $\varphi': \{a, b\}^* \rightarrow M'$ be its syntactic morphism: its stability index is equal to 3 and it has the property that for all $x, y \in \{a, b\}^+$, given any $u, v \in \{a, b\}^*$ verifying that the letter b appears in u if and only if it appears in v , it holds that $\varphi'(xuy) = \varphi'(xvy)$. Hence, if we define $\mu': \{a, b\}^* \rightarrow N'$ to be the syntactic morphism of the language $(a+b)^*b(a+b)^*$, it is direct to see that for all $u, v \in \{a, b\}^*$, it holds that

$$\mu'(u) = \mu'(v) \Rightarrow (\varphi'(xuy) = \varphi'(xvy) \quad \forall x, y \in \{a, b\}^3) .$$

So we can conclude that the stamp φ' is essentially- \mathbf{V} for any variety of monoids containing the syntactic monoid of $(a+b)^*b(a+b)^*$, in particular $a(a+b)^*b(a+b)^*a \in \mathcal{L}(\mathbf{EJ})$. However, note that $\varphi' \notin \mathbf{EI}$ because we have $\varphi'((aaa)a(aaa)) \neq \varphi'((aaa)b(aaa))$.

It is now easy to prove that as long as \mathbf{V} is non-trivial, polynomial-length programs over monoids from \mathbf{V} do have the built-in capacity to recognize any language recognized by an essentially- \mathbf{V} stamp.

Proposition 3.8. *For any non-trivial variety of monoids \mathbf{V} , we have $\mathcal{L}(\mathbf{EV}) \subseteq \mathcal{P}(\mathbf{V})$.*

Proof. Let $\varphi: \Sigma^* \rightarrow M$ for Σ a finite alphabet and M a finite monoid be a stamp in **EV**. By definition, given the stability index s of φ , there exists a stamp $\mu: \Sigma^* \rightarrow N$ with $N \in \mathbf{V}$ such that for all $u, v \in \Sigma^*$, we have

$$\mu(u) = \mu(v) \Rightarrow (\varphi(xuy) = \varphi(xvy) \quad \forall x, y \in \Sigma^s) .$$

¹This class actually is an *ne*-variety of stamps, as defined in [Str02].

Let $F \subseteq M$. By definition of μ , given $m \in N$ and $x, y \in \Sigma^s$, we either have that $x\mu^{-1}(m)y \subseteq \varphi^{-1}(F)$ or that $x\mu^{-1}(m)y \cap \varphi^{-1}(F) = \emptyset$. This entails that there exist $B \subseteq \Sigma^{\leq 2s-1}$ and $I \subseteq \Sigma^s \times N \times \Sigma^s$ such that

$$\varphi^{-1}(F) = B \cup \bigcup_{(x,m,y) \in I} x\mu^{-1}(m)y .$$

We claim that $\{w\} \in \mathcal{P}(\mathbf{V})$ for any $w \in \Sigma^{\leq 2s-1}$ and also that $x\mu^{-1}(m)y \in \mathcal{P}(\mathbf{V})$ for any $x, y \in \Sigma^s$ and $m \in N$. So, by closure of $\mathcal{P}(\mathbf{V})$ under Boolean operations, Lemma 2.1, it follows that $\varphi^{-1}(F) \in \mathcal{P}(\mathbf{V})$. Since this is true for any F , we have that $\mathcal{W}(\varphi) \subseteq \mathcal{P}(\mathbf{V})$ and as this is itself true for all φ , we can conclude that $\mathcal{L}(\mathbf{EV}) \subseteq \mathcal{P}(\mathbf{V})$.

The claim remains to be proven.

Let $k \in \mathbb{N}_{>0}$ and $a \in \Sigma$. Since \mathbf{V} is non-trivial, there exists a non-trivial $N' \in \mathbf{V}$: we shall denote its identity by 1 and by z one of its elements distinct from the identity, chosen arbitrarily. It is easy to see that the language $\Sigma^{k-1}a\Sigma^*$ is p -recognized by the sequence of programs $(P_n)_{n \in \mathbb{N}}$ over N' such that for all $n \in \mathbb{N}$, we have

$$P_n = \begin{cases} (k, f) & \text{if } n \geq k \\ \varepsilon & \text{otherwise} \end{cases}$$

where $f: \Sigma \rightarrow N'$ is defined by $f(b) = \begin{cases} z & \text{if } b = a \\ 1 & \text{otherwise} \end{cases}$ for all $b \in \Sigma$. We prove the same for $\Sigma^*a\Sigma^{k-1}$ symmetrically.

It then follows by closure of $\mathcal{P}(\mathbf{V})$ under Boolean operations, Lemma 2.1, that $\{w\} \in \mathcal{P}(\mathbf{V})$ for any $w \in \Sigma^{\leq 2s-1}$ and that $x\Sigma^*y \in \mathcal{P}(\mathbf{V})$ for any $x, y \in \Sigma^s$.

Finally, let $m \in N$. It is direct to show that there exists $L_m \subseteq \Sigma^*$ in $\mathcal{P}(\mathbf{V})$ verifying that $L_m \cap \Sigma^s\Sigma^*\Sigma^s = \Sigma^s\mu^{-1}(m)\Sigma^s$: just build the sequence of programs $(Q_n)_{n \in \mathbb{N}}$ over N such that for all $n \in \mathbb{N}$, we have

$$Q_n = \begin{cases} (s+1, g)(s+2, g) \cdots (n-s, g) & \text{if } n \geq 2s+1 \\ \varepsilon & \text{otherwise} \end{cases}$$

where $g: \Sigma \rightarrow N$ is defined by $g(b) = \mu(b)$ for all $b \in \Sigma$. We can then conclude that $x\mu^{-1}(m)y \in \mathcal{P}(\mathbf{V})$ for any $x, y \in \Sigma^s$ by closure of $\mathcal{P}(\mathbf{V})$ under Boolean operations, Lemma 2.1, and this holds for any m . \square

3.3. Tameness. We are now ready to define tameness.

We will say that a stamp $\varphi: \Sigma^* \rightarrow M$ for Σ a finite alphabet and M a finite monoid is *stable* whenever $\varphi(\Sigma^2) = \varphi(\Sigma)$, i.e. the stability index of φ is 1.

Definition 3.9. A variety of monoids \mathbf{V} is said to be *tame* whenever for any stable stamp $\varphi: \Sigma^* \rightarrow M$ with Σ a finite alphabet and M a finite monoid, if $\mathcal{W}(\varphi) \subseteq \mathcal{P}(\mathbf{V})$ then $\varphi \in \mathbf{EV}$.

Let us first mention that tameness is a generalization of *sp*-varieties of monoids.

Proposition 3.10. *Any sp-variety of monoids is tame.*

Proof. Let \mathbf{V} be an *sp*-variety of monoids.

Let $\varphi: \Sigma^* \rightarrow M$ with Σ a finite alphabet and M a finite monoid be a stable stamp such that $\mathcal{W}(\varphi) \subseteq \mathcal{P}(\mathbf{V})$.

Let $S = \varphi(\Sigma^+)$: as φ is stable, we have $S = \varphi(\Sigma)$. Let $\rho: S \rightarrow \Sigma$ be an arbitrary mapping from S to Σ such that $\varphi(\rho(s)) = s$. Consider $\eta_S: S^* \rightarrow S^1$ the evaluation morphism of S : the unique morphism $f: S^* \rightarrow \Sigma^*$ sending each letter $s \in S$ to $\rho(s)$ verifies that $\eta_S = \varphi \circ f$. Now, given any $F \subseteq S^1$, we have $\eta_S^{-1}(F) = f^{-1}(\varphi^{-1}(F))$, but since $\varphi^{-1}(F) \in \mathcal{P}(\mathbf{V})$ and as f is an lm -morphism because it sends each letter of S to a letter of Σ , it follows that $\eta_S^{-1}(F) \in \mathcal{P}(\mathbf{V})$ by closure of $\mathcal{P}(\mathbf{V})$ under inverses of lm -morphisms, Lemma 2.1. Therefore, $\mathcal{W}(S) \subseteq \mathcal{P}(\mathbf{V})$.

Since \mathbf{V} is an sp -variety of monoids, this entails that $M = S^1$ belongs to \mathbf{V} , and therefore $\varphi \in \mathbf{EV}$. As this is true for any stable stamp φ such that $\mathcal{W}(\varphi) \subseteq \mathcal{P}(\mathbf{V})$, we can conclude that \mathbf{V} is tame. \square

The notion of essentially- \mathbf{V} stamps can be adapted to varieties of semigroups in a straightforward way. We can then define a notion of tameness for varieties of semigroups accordingly. The exact same proof than the one above then goes through to allow us to show that p -varieties of the form $\mathbf{V} * \mathbf{D}$ are tame.

However there exist varieties of monoids that are tame but not sp -varieties. We give an example of such a variety in Subsection 3.4.

Programs over monoids taken from tame varieties of monoids have the expected behavior as we show next.

Let $\varphi: \Sigma^* \rightarrow M$ be a stamp of stability index s , for Σ a finite alphabet and M a finite monoid. The *stable stamp* of φ is the unique stamp $\varphi': (\Sigma^s)^* \rightarrow M'$ such that $\varphi'(u) = \varphi(u)$ for all $u \in \Sigma^s$ and M' is the stable monoid of φ . For any variety of monoids \mathbf{V} we let \mathbf{QEV} be the class of stamps whose stable stamp is essentially- \mathbf{V} and, accordingly, we define $\mathcal{L}(\mathbf{QEV})$ as the class of languages whose syntactic morphism is in \mathbf{QEV} .

Proposition 3.11. *Any variety of monoids \mathbf{V} is tame if and only if $\mathcal{P}(\mathbf{V}) \cap \mathcal{R}\text{eg} \subseteq \mathcal{L}(\mathbf{QEV})$.*

Proof. Let \mathbf{V} be a variety of monoids.

Left-to-right implication. Assume first that \mathbf{V} is tame. For this direction, the proof follows the same lines as those of Proposition 3.4.

Let $L \in \mathcal{P}(\mathbf{V}) \cap \mathcal{R}\text{eg}$ over some finite alphabet Σ and let $\eta: \Sigma^* \rightarrow M$ be the syntactic morphism of L . For any $m \in M$, a classical algebraic argument [Pin86, Chapter 2, proof of Lemma 2.6] shows that $\eta^{-1}(m)$ is a Boolean combination of quotients of L , so $\eta^{-1}(m) \in \mathcal{P}(\mathbf{V})$ by Lemma 2.1.

Now let s be the stability index of η , let M' be its stable monoid and take $\eta': (\Sigma^s)^* \rightarrow M'$ to be the stable stamp of η . The unique morphism $f: (\Sigma^s)^* \rightarrow \Sigma^*$ such that $f(u) = u$ for all $u \in \Sigma^s$ is an lm -morphism and verifies that $\eta' = \eta \circ f$. Hence, for all $m' \in M'$, we have that $\eta'^{-1}(m') = f^{-1}(\eta^{-1}(m'))$, so that $\eta'^{-1}(m') \in \mathcal{P}(\mathbf{V})$ by closure of $\mathcal{P}(\mathbf{V})$ under inverses of lm -morphisms, Lemma 2.1. Thus, since inverses of monoid morphisms commute with union and $\mathcal{P}(\mathbf{V})$ is closed under unions (Lemma 2.1), we can conclude that $\eta'^{-1}(F) \in \mathcal{P}(\mathbf{V})$ for all $F \subseteq M'$, i.e. $\mathcal{W}(\eta') \subseteq \mathcal{P}(\mathbf{V})$.

But as η' is stable, by tameness of \mathbf{V} , this entails that $\eta' \in \mathbf{EV}$, so that $L \in \mathcal{L}(\mathbf{QEV})$.

Right-to-left implication. Assume now that $\mathcal{P}(\mathbf{V}) \cap \mathcal{R}\text{eg} \subseteq \mathcal{L}(\mathbf{QEV})$. Let $\varphi: \Sigma^* \rightarrow M$ for Σ a finite alphabet and M a finite monoid be a stable stamp verifying $\mathcal{W}(\varphi) \subseteq \mathcal{P}(\mathbf{V})$.

For any $m \in M$, we therefore have $\varphi^{-1}(m) \in \mathcal{L}(\mathbf{QEV})$. Let $\eta_m: \Sigma^* \rightarrow M_m$ be the syntactic morphism of the language $\varphi^{-1}(m)$, we thus have $\eta_m \in \mathbf{QEV}$. We first claim that η_m is a stable stamp. To see this notice first that for all $u, v \in \Sigma^*$ we have $\varphi(u) =$

$\varphi(v) \Rightarrow \eta_m(u) = \eta_m(v)$. Indeed assume that $\varphi(u) = \varphi(v)$, then for all $x, y \in \Sigma^*$ we have $\varphi(xuy) = \varphi(xvy)$ hence we have $xuy \in \varphi^{-1}(m)$ iff $xvy \in \varphi^{-1}(m)$ which entails $\eta_m(u) = \eta_m(v)$ by definition of the syntactic morphism. It follows that $\eta_m(\Sigma^2) = \eta_m(\Sigma)$ as $\varphi(\Sigma^2) = \varphi(\Sigma)$.

Since η_m is equal to its stable stamp and $\eta_m \in \mathbf{QEV}$, it follows that $\eta_m \in \mathbf{EV}$. Therefore there exists a stamp $\mu_m: \Sigma^* \rightarrow N_m$ with $N_m \in \mathbf{V}$ such that for all $u, v \in \Sigma^*$, we have

$$\mu_m(u) = \mu_m(v) \Rightarrow (\eta_m(xuy) = \eta_m(xvy) \quad \forall x, y \in \Sigma) .$$

Now, we define the unique stamp $\mu: \Sigma^* \rightarrow N$ such that $\mu(a) = \prod_{m \in M} \mu_m(a)$ for all $a \in \Sigma$ and N is the submonoid of $\prod_{m \in M} N_m$ generated by the set $\{\prod_{m \in M} \mu_m(a) \mid a \in \Sigma\}$. As \mathbf{V} is a variety, $N \in \mathbf{V}$. Take $u, v \in \Sigma^*$ and assume $\mu(u) = \mu(v)$: this means that $\mu_m(u) = \mu_m(v)$ for all $m \in M$. Let $x, y \in \Sigma$. We then have in particular $\mu_{\varphi(xuy)}(u) = \mu_{\varphi(xuy)}(v)$. This implies by definition of $\mu_{\varphi(xuy)}$ that $\eta_{\varphi(xuy)}(xuy) = \eta_{\varphi(xuy)}(xvy)$. As $\eta_{\varphi(xuy)}$ is the syntactic morphism of $\varphi^{-1}(\varphi(xuy))$, it follows that $\varphi(xvy) = \varphi(xuy)$. And this is true for any $x, y \in \Sigma$.

In conclusion, μ witnesses the fact that φ is essentially- \mathbf{V} . \square

As for the case of *sp*-varieties of monoids, we don't know whether it is always true that for a tame variety of monoids \mathbf{V} , $\mathcal{L}(\mathbf{QEV})$ is included in $\mathcal{P}(\mathbf{V})$. If this were the case then for tame varieties of monoids \mathbf{V} we would have $\mathcal{P}(\mathbf{V}) \cap \mathcal{Reg} = \mathcal{L}(\mathbf{QEV})$. We conjecture this to be at least true for varieties of monoids that are local.

We conclude this subsection by showing that \mathbf{J} , that is not an *sp*-variety of monoids (Proposition 3.6), isn't tame either.

Proposition 3.12. *\mathbf{J} is not tame.*

Proof. To show this, we show that $(a+b)^*ac^+$, that belongs to $\mathcal{P}(\mathbf{J})$, does not belong to $\mathcal{L}(\mathbf{QJ})$.

We first claim that any essentially- \mathbf{J} stamp $\varphi: \Sigma^* \rightarrow M$ of stability index s for Σ a finite alphabet and M a finite monoid verifies that there exists some $k \in \mathbb{N}_{>0}$ such that $\varphi(x(uv)^k y) = \varphi(x(uv)^k u y)$ for all $u, v \in \Sigma^*$ and $x, y \in \Sigma^s$. Indeed, by definition there exists a stamp $\mu: \Sigma^* \rightarrow N$ with $N \in \mathbf{J}$ such that for all $u, v \in \Sigma^*$, we have

$$\mu(u) = \mu(v) \Rightarrow (\varphi(xuy) = \varphi(xvy) \quad \forall x, y \in \Sigma^s) .$$

If we set ω to be the idempotent power of N , we have that for all $u, v \in \Sigma^*$,

$$\mu((uv)^\omega) = (\mu(u)\mu(v))^\omega = (\mu(u)\mu(v))^\omega \mu(u) = \mu((uv)^\omega u)$$

by the identities for \mathbf{J} . Hence, we have that $\varphi(x(uv)^\omega y) = \varphi(x(uv)^\omega u y)$ for all $u, v \in \Sigma^*$ and $x, y \in \Sigma^s$.

Let us now consider the syntactic morphism $\eta: \{a, b, c\}^* \rightarrow M$ of the language $(a+b)^*ac^+$. As already mentioned for Proposition 3.6, the stable monoid of η is equal to the syntactic monoid M . Moreover, the stability index of η is 2. Therefore, the stable stamp of η is the unique stamp $\eta': (\{a, b, c\}^2)^* \rightarrow M$ such that $\eta'(u) = \eta(u)$ for all $u \in \{a, b, c\}^2$. By what we have shown just above, since the stability index of η' is 1, if η' were essentially- \mathbf{J} , there should exist some $k \in \mathbb{N}_{>0}$ such that $\eta'(x(uv)^k y) = \eta'(x(uv)^k u y)$ for all $u, v \in (\{a, b, c\}^2)^*$ and $x, y \in \{a, b, c\}^2$. However, for all $k \in \mathbb{N}_{>0}$, we do have that $(aa)((bb)(aa))^k(cc) \in (a+b)^*ac^+$ while $(aa)((bb)(aa))^k(bb)(cc) \notin (a+b)^*ac^+$, which implies that

$$\eta'((aa)((bb)(aa))^k(cc)) \neq \eta'((aa)((bb)(aa))^k(bb)(cc))$$

for all $k \in \mathbb{N}_{>0}$. Therefore, it follows that the stable stamp η' of η is not essentially-**J**, so we can conclude that $(a+b)^*ac^+ \notin \mathcal{L}(\mathbf{Q EJ})$. \square

3.4. The example of finite commutative monoids. The variety **Com** of finite commutative monoids is defined by the identity $xy = yx$ and $\mathcal{L}(\mathbf{Com})$ is the class of languages that are Boolean combinations of languages of the form $\{w \in \Sigma^* \mid |w|_a \equiv k \pmod{p}\}$ for $k \in \llbracket 0, p-1 \rrbracket$ and p prime or $\{w \in \Sigma^* \mid |w|_a = k\}$ for $k \in \mathbb{N}$ with Σ any finite alphabet and $a \in \Sigma$ (see [Eil76, Chapter VIII, Example 3.5]).

Since the syntactic monoid of the language $a(a+b)^*$ is not commutative, by the discussion at the end of Subsection 3.1, we know that **Com** is not an sp -variety of monoids. It is, however, tame, as we are going to prove now.

We first give a sufficient equational characterisation for any stable stamp φ to be essentially-**Com**.

Lemma 3.13. *Let $\varphi: \Sigma^* \rightarrow M$ for Σ a finite alphabet and M a finite monoid be a stable stamp verifying that for any $x, y, e, f \in \Sigma$ such that $\varphi(e)$ and $\varphi(f)$ are idempotents, we have*

$$\varphi(exyf) = \varphi(eyxf) .$$

Then, $\varphi \in \mathbf{ECom}$.

Proof. Let us define the equivalence relation \sim on Σ^* by $u \sim v$ for $u, v \in \Sigma^*$ whenever $\varphi(xuy) = \varphi(xvy)$ for all $x, y \in \Sigma$. This equivalence relation is actually a congruence, because given $u, v \in \Sigma^*$ verifying $u \sim v$, for all $s, t \in \Sigma^*$ we have $sut \sim svt$ since for any $x, y \in \Sigma$, it holds that

$$\varphi(xsuty) = \varphi(x'uy') = \varphi(x'vy') = \varphi(xsvty)$$

where $x', y' \in \Sigma$ verify $\varphi(xs) = \varphi(x')$ and $\varphi(ty) = \varphi(y')$.

We observe that, since the stability index of φ is equal to 1, φ verifies that $\varphi(euvf) = \varphi(evuf)$ for all $u, v \in \Sigma^*$ and $e, f \in \Sigma$ such that $\varphi(e)$ and $\varphi(f)$ are idempotents. Now take $u, v \in \Sigma^*$ and $x, y \in \Sigma$. Since $\varphi(\Sigma)$ is a finite semigroup and verifies that $\varphi(\Sigma) = \varphi(\Sigma)^2$, by a classical result in finite semigroup theory (see e.g. [Pin86, Chapter 1, Proposition 1.12]), we have that there exist $x_1, e, x_2, y_1, f, y_2 \in \Sigma$ such that $\varphi(x_1ex_2) = \varphi(x)$ and $\varphi(y_1fy_2) = \varphi(y)$ with $\varphi(e)$ and $\varphi(f)$ idempotents. Therefore, it follows that

$$\begin{aligned} \varphi(xuvy) &= \varphi(x_1ex_2uvy_1fy_2) \\ &= \varphi(x_1euvy_1x_2fy_2) \\ &= \varphi(x_1evvy_1x_2ff_2) \\ &= \varphi(x_1ey_1x_2fvvf_2) \\ &= \varphi(x_1ey_1x_2fvvf_2) \\ &= \varphi(x_1evvy_1x_2ff_2) \\ &= \varphi(x_1evvy_1x_2fy_2) \\ &= \varphi(x_1ex_2vuy_1fy_2) \\ &= \varphi(xvuy) . \end{aligned}$$

Thus, we have that $uv \sim vu$ for all $u, v \in \Sigma^*$, implying that $\Sigma^*/\sim \in \mathbf{Com}$. We can eventually conclude that the stamp $\mu: \Sigma^* \rightarrow \Sigma^*/\sim$ defined by $\mu(w) = [w]_{\sim}$ for all $w \in \Sigma^*$ witnesses, by construction, the fact that φ is essentially-**Com**. \square

The following lemma then asserts that any stable stamp φ such that $\mathcal{W}(\varphi) \subseteq \mathcal{P}(\mathbf{Com})$ actually verifies the equation of the previous lemma, which allows us to conclude that \mathbf{Com} is tame by combining those two lemmas.

Lemma 3.14. *Let $\varphi: \Sigma^* \rightarrow M$ for Σ a finite alphabet and M a finite monoid be a stable stamp such that $\mathcal{W}(\varphi) \subseteq \mathcal{P}(\mathbf{Com})$. Then, for any $x, y, e, f \in \Sigma$ such that $\varphi(e)$ and $\varphi(f)$ are idempotents, we have*

$$\varphi(exyf) = \varphi(eyxf) .$$

Proof. Let us first observe that for any program P over some finite commutative monoid N using the input alphabet Σ and of range $n \in \mathbb{N}$, there exist a program P' over N using the same input alphabet and of same range such that $P' = \prod_{i=1}^n (i, h_i)$ verifying $P(w) = P'(w)$ for all $w \in \Sigma^n$ [Tes03, Example 3.4]. We call P' a single-scan program.

The assumption that $\mathcal{W}(\varphi) \subseteq \mathcal{P}(\mathbf{Com})$ thus means that for all $F \subseteq M$, there exists a sequence $(P_{F,n})_{n \in \mathbb{N}}$ of single-scan programs over some $N_F \in \mathbf{Com}$ that recognizes $\varphi^{-1}(F)$.

For all $x, y, e, f, g \in \Sigma$ such that $\varphi(e)$, $\varphi(f)$ and $\varphi(g)$ are idempotents, we claim that

$$\varphi(exfyg) = \varphi(eyfxg) \tag{3.1}$$

$$\varphi(exyf) = \varphi(eyefxf) . \tag{3.2}$$

Assuming the claim, take $x, y, e, f \in \Sigma$ such that $\varphi(e)$ and $\varphi(f)$ are idempotents. We have that

$$\varphi(efef) = \varphi(effef) = \varphi(eefff) = \varphi(ef) ,$$

the middle equality being from (3.1). This implies that $\varphi(ef)$ is an idempotent. As $\varphi(\Sigma^2) = \varphi(\Sigma)$, we have that there exists $g \in \Sigma$ such that $\varphi(ef) = \varphi(g)$, hence

$$\varphi(exyf) = \varphi(eyefxf) = \varphi(eygfx) = \varphi(exgyf) = \varphi(exefyf) = \varphi(eyxf) ,$$

the first and last equalities being from (3.2) and the middle one from (3.1).

Thus, the lemma will be proven once we will have proven that (3.1) and (3.2) hold for all $x, y, e, f, g \in \Sigma$ such that $\varphi(e)$, $\varphi(f)$ and $\varphi(g)$ are idempotents.

(3.1) holds. Let $x, y, e, f, g \in \Sigma$ such that $\varphi(e)$, $\varphi(f)$ and $\varphi(g)$ are idempotents. Set $F = \{\varphi(exfyg)\}$ and $n = 2(|N_F|^2 + 1) + 1$ and assume $P_{F,n} = \prod_{i=1}^n (i, h_i)$. Then, since the function

$$\begin{aligned} \Delta: \llbracket 1, |N_F|^2 + 1 \rrbracket &\rightarrow N_F^2 \\ j &\mapsto (h_{2j}(x), h_{2j}(y)) \end{aligned}$$

cannot be injective, there must exist $j_1, j_2 \in \llbracket 1, |N_F|^2 + 1 \rrbracket$, $j_1 < j_2$ such that $h_{2j_1}(x) = h_{2j_2}(x)$ and $h_{2j_1}(y) = h_{2j_2}(y)$. So

$$P_{F,n}(e^{2j_1-1} x f^{2j_2-1-2j_1} y g^{n-2j_2}) = P_{F,n}(e^{2j_1-1} y f^{2j_2-1-2j_1} x g^{n-2j_2}) ,$$

hence as $e^{2j_1-1} x f^{2j_2-1-2j_1} y g^{n-2j_2} \in \varphi^{-1}(F)$ because

$$\varphi(e^{2j_1-1} x f^{2j_2-1-2j_1} y g^{n-2j_2}) = \varphi(exfyg) ,$$

we must have $e^{2j_1-1} y f^{2j_2-1-2j_1} x g^{n-2j_2} \in \varphi^{-1}(F)$. Thus, we have

$$\varphi(e^{2j_1-1} y f^{2j_2-1-2j_1} x g^{n-2j_2}) = \varphi(eyfxg) = \varphi(exfyg) .$$

So for all $x, y, e, f, g \in \Sigma$ such that $\varphi(e)$, $\varphi(f)$ and $\varphi(g)$ are idempotents, we have that (3.1) holds.

(3.2) holds. Let $x, y, e, f \in \Sigma$ such that $\varphi(e)$ and $\varphi(f)$ are idempotents. Set $F = \{\varphi(axyf)\}$ and $n = 4(2|N_F|^4 + 1)$ and assume $P_{F,n} = \prod_{i=1}^n (i, h_i)$. Then, we have that the function

$$\Delta: \llbracket 1, 2|N_F|^4 + 1 \rrbracket \rightarrow N_F^4$$

$$j \mapsto (h_{4j-2}(x), h_{4j-2}(f), h_{4j-1}(y), h_{4j-1}(e))$$

verifies that there exists $(m_1, m_2, m_3, m_4) \in N_F^4$ such that

$$|\Delta^{-1}((m_1, m_2, m_3, m_4))| \geq 3.$$

Therefore, there exist $j_1, j_2, j_3 \in \llbracket 1, 2|N_F|^4 + 1 \rrbracket, j_1 < j_2 < j_3$ such that $h_{4j_3-2}(x) = h_{4j_2-2}(x)$, $h_{4j_3-2}(f) = h_{4j_2-2}(f)$, $h_{4j_2-1}(y) = h_{4j_1-1}(y)$ and $h_{4j_2-1}(e) = h_{4j_1-1}(e)$. So

$$P_{F,n}(e^{4j_2-3}xyf^{n-4j_2+1}) = P_{F,n}(e^{4j_1-2}ye^{4(j_2-j_1)-2}fef^{4(j_3-j_2)-2}xf^{n-4j_3+2}),$$

hence as $e^{4j_2-3}xyf^{n-4j_2+1} \in \varphi^{-1}(F)$ because

$$\varphi(e^{4j_2-3}xyf^{n-4j_2+1}) = \varphi(axyf),$$

we must have $e^{4j_1-2}ye^{4(j_2-j_1)-2}fef^{4(j_3-j_2)-2}xf^{n-4j_3+2} \in \varphi^{-1}(F)$. Thus, we have

$$\begin{aligned} \varphi(axyf) &= \varphi(e^{4j_1-2}ye^{4(j_2-j_1)-2}fef^{4(j_3-j_2)-2}xf^{n-4j_3+2}) \\ &= \varphi(eyefefxf) \\ &= \varphi(eyefxf). \end{aligned}$$

So for all $x, y, e, f \in \Sigma$ such that $\varphi(e)$ and $\varphi(f)$ are idempotents, we have that (3.2) holds. \square

4. THE CASE OF **DA**

In this section, we prove that **DA** is an *sp*-variety of monoids, which implies that it is tame. Combined with the fact that **DA** is local [Alm96], we obtain the following result by Proposition 3.5.

Theorem 4.1. $\mathcal{P}(\mathbf{DA}) \cap \text{Reg} = \mathcal{L}(\mathbf{QDA})$.

The result follows from the following main technical contribution:

Proposition 4.2. $(c + ab)^*$, $(b + ab)^*$ and $b^*((ab^*)^k)^*$ for any integer $k \geq 2$ are regular languages not in $\mathcal{P}(\mathbf{DA})$.

Before proving the proposition we first show that it implies that **DA** is an *sp*-variety of finite monoids. This implication is a consequence of the following lemma, which is a result inspired by an observation in [TT02] stating that non-membership of a given finite monoid M in **DA** implies non-aperiodicity of M or division of it by (at least) one of two specific finite monoids.

Lemma 4.3. Let S be a finite semigroup such that $S^1 \notin \mathbf{DA}$. Then, one of $(c + ab)^*$, $(b + ab)^*$ or $b^*((ab^*)^k)^*$ for some $k \in \mathbb{N}, k \geq 2$ is recognized by a morphism $\mu: \Sigma^* \rightarrow S^1$, for Σ the appropriate alphabet, such that $\mu(\Sigma^+) \subseteq S$.

Proof. Let $\omega \in \mathbb{N}_{>0}$ be the idempotent power of S^1 .

Aperiodic case. Assume first that S^1 is aperiodic. Then, since $S^1 \notin \mathbf{DA}$, there exist x, y in S such that $(xy)^\omega \neq (xy)^\omega x(xy)^\omega$.

Set $e = (xy)^\omega$, $f = (yx)^\omega$, $s = ex$ and $t = ye$. Our hypothesis says that $exe \neq e$. We now have two cases, depending on whether $fyf = f$ or not.

Subcase $fyf \neq f$. Suppose $fyf \neq f$. In that case, let $\mu: \{a, b, c\}^* \rightarrow S^1$ be the morphism sending a to s , b to t and c to e and consider the language $L = \mu^{-1}(\{1, e\})$. We are now going to show that no word of L can contain aa , bb , ac or cb as a factor.

- Assume that L contains a word w with two consecutive a 's. Then $w = w_1 a a w_2$ with $w_1, w_2 \in \{a, b, c\}^*$ and as $w \in L$, either $e = \mu(w_1) e x e x \mu(w_2)$ or $1 = \mu(w_1) e x e x \mu(w_2)$. In both cases $e = u_1 e x e u_2$ for some suitable values of u_1 and u_2 taken from S . This implies that

$$e = u_1 e (x e u_2) = u_1^2 e (x e u_2)^2 = u_1^3 e (x e u_2)^3 = \dots = u_1^\omega e (x e u_2)^\omega$$

and, similarly, that $e = (u_1 e x)^\omega e u_2^\omega$. Because S^1 is aperiodic, this in turn entails

$$e x e u_2 = u_1^\omega e (x e u_2)^\omega (x e u_2) = u_1^\omega e (x e u_2)^\omega = e$$

and

$$e u_2 = (u_1 e x)^\omega e u_2^\omega u_2 = (u_1 e x)^\omega e u_2^\omega = e .$$

Hence $exe = exe u_2 = e$, contradicting the fact that $exe \neq e$. So L does not contain any word with two consecutive a 's.

- Assume that L contains a word w with the factor ac . Then $w = w_1 a c w_2$ with $w_1, w_2 \in \{a, b, c\}^*$ and as $w \in L$, either $e = \mu(w_1) e x e \mu(w_2)$ or $1 = \mu(w_1) e x e \mu(w_2)$. So, as just before, in both cases $e = u_1 e x e u_2$ for some suitable values of u_1 and u_2 taken from S , which entails $exe = e$, contradicting the fact $exe \neq e$. So L does not contain any word with the factor ac .
- Assume that L contains a word w with two consecutive b 's. Then $w = w_1 b b w_2$ with $w_1, w_2 \in \{a, b, c\}^*$ and as $w \in L$, either $e = \mu(w_1) f y f y \mu(w_2)$ or $1 = \mu(w_1) f y f y \mu(w_2)$, as $ye = y(xy)^\omega = (yx)^\omega y = fy$. In both cases $f = u_1 f y f u_2$ for some suitable values of u_1 and u_2 taken from S , because, by aperiodicity of S^1 , we have $yex = y(xy)^\omega x = (yx)^{\omega+1} = (yx)^\omega = f$. Similarly to what we did for the factor aa , this implies that $f = u_1^\omega f (y f u_2)^\omega = (u_1 f y)^\omega f u_2^\omega$, which in turn entails $f = f y f u_2 = f u_2$. Hence $fyf = f y f u_2 = f$, contradicting the fact that $fyf \neq f$. So L does not contain any word with two consecutive b 's.
- Assume that L contains a word w with the factor cb . Then $w = w_1 c b w_2$ with $w_1, w_2 \in \{a, b, c\}^*$ and as $w \in L$, either $e = \mu(w_1) e y e \mu(w_2)$ or $1 = \mu(w_1) e y e \mu(w_2)$. So, similarly to what we did for the factor aa , in both cases $e = u_1 e y e u_2$ for some suitable values of u_1 and u_2 taken from S , which entails $eye = e$. Now this means that

$$\begin{aligned} eye &= e \\ yeye &= ye \\ f y f y &= f y && \text{since } ye = fy \\ f y f y x &= f y x \\ f y f &= f && \text{as } f y x = (y x)^\omega y x = (y x)^{\omega+1} = (y x)^\omega = f , \end{aligned}$$

contradicting the fact $fyf \neq f$. So L does not contain any word with the factor cb .

Because L is a language over the alphabet $\{a, b, c\}$, any word w in it is of the form $u_0 v_1 u_1 \cdots u_{k-1} v_k u_k$ where $k \in \mathbb{N}$, $v_1, \dots, v_k \in c^+$ and $u_0, \dots, u_k \in (a+b)^*$. As w does not contain aa nor bb as a factor, we have that $u_0, \dots, u_k \in (b+\varepsilon)(ab)^*(a+\varepsilon)$. When $k \geq 1$, as moreover w does not contain ac nor cb as a factor, it follows that $u_1, \dots, u_{k-1} \in (ab)^*$, $u_0 \in (b+\varepsilon)(ab)^*$ and $u_k \in (ab)^*(a+\varepsilon)$; u_0 can therefore be written as $\beta u'_0$ where $u'_0 \in (ab)^*$ and β is b if $u_0 \in b(ab)^*$ and the empty word otherwise, and u_k can be written as $u'_k \alpha$ where $u'_k \in (ab)^*$ and α is a if $u_1 \in (ab)^* a$ and the empty word otherwise. We now observe that $\mu(ab) = exye = (xy)^{2\omega+1} = (xy)^\omega = e$ by aperiodicity and we consider four different cases.

- $\beta = \alpha = \varepsilon$.

Then, $\mu(w) = \mu(u'_0 v_1 u_1 \cdots u_{k-1} v_k u'_k) = e$.

- $\beta = b$ and $\alpha = \varepsilon$.

Then $\mu(w) = \mu(b)\mu(u'_0 v_1 u_1 \cdots u_{k-1} v_k u'_k) = yee = ye$ that does not belong to $\{1, e\}$, otherwise we would have $eye = e$ which would entail $fyf = f$, as shown in the previous paragraph. But this contradicts the fact that $w \in L$, so this case cannot occur.

- $\beta = \varepsilon$ and $\alpha = a$.

Then $\mu(w) = \mu(u'_0 v_1 u_1 \cdots u_{k-1} v_k u'_k)\mu(a) = eex = ex$ that does not belong to $\{1, e\}$, otherwise we would have $exe = e$. But this contradicts the fact that $w \in L$, so this case cannot occur.

- $\beta = b$ and $\alpha = a$.

Then $\mu(w) = \mu(b)\mu(u'_0 v_1 u_1 \cdots u_{k-1} v_k u'_k)\mu(a) = yeex = yex = f$ by aperiodicity. We have that f does not belong to $\{1, e\}$. Indeed, suppose for the sake of contradiction that it does: there are two cases to examine. Either $(yx)^\omega = f = e = (xy)^\omega$, and then $exe = exf = (xy)^\omega x (yx)^\omega = (xy)^\omega (xy)^\omega x = (xy)^\omega x = ex$. But $ex = (xy)^\omega x = x(yx)^\omega = x(xy)^\omega xy = xexy$ by aperiodicity, so $ex = x^\omega exy^\omega$. Hence $exy = x^\omega exy^\omega y = x^\omega exy^\omega = ex$ by aperiodicity, while $exy = (xy)^\omega xy = (xy)^\omega = e$ by aperiodicity again. So $exe = ex = e$, contradicting the fact $exe \neq e$. Or $(yx)^\omega = f = 1$, and then $fyf = y$. But $y = y(yx)^\omega = y(yx)^\omega yx = yyx$ by aperiodicity, so $y = y^\omega yx^\omega$. Hence $yx = y^\omega yx^\omega x = y^\omega yx^\omega = y$ by aperiodicity, while $yx = yx(yx)^\omega = (yx)^\omega = f$ by aperiodicity again. So $fyf = y = f$, contradicting the fact $fyf \neq f$. Therefore, $\mu(w)$ does not belong to $\{1, e\}$, contradicting the fact that $w \in L$, so this case cannot occur either.

This means that, necessarily, $\alpha = \beta = \varepsilon$, so that $u_0, u_k \in (ab)^*$. And for the same reasons, $u_0 \in (ab)^*$ when $k = 0$. Therefore, we have $w \in (c+ab)^*$ and since it is true for any $w \in L$, it follows that $L \subseteq (c+ab)^*$. Combined with the fact that $\mu((c+ab)^*) = \{1, e\}$, we can conclude that $\mu^{-1}(\{1, e\}) = L = (c+ab)^*$, showing $(c+ab)^*$ is recognized by μ verifying $\mu(\{a, b, c\}^+) \subseteq S$.

Subcase $fyf = f$. Suppose now $fyf = f$. In that case, let $\mu: \{a, b\}^* \rightarrow S^1$ be the morphism sending a to s and b to t and consider the language $L = \mu^{-1}(\{1, e, t\})$. Assume that L contains a word w with two consecutive a 's. Then $w = w_1 a a w_2$ with $w_1, w_2 \in \{a, b\}^*$ and as $w \in L$, we have that $\mu(w_1) e x e x \mu(w_2)$ is equal to t, e or 1 . Since $xt = xye = xy(xy)^\omega = (xy)^\omega = e$ by aperiodicity, in all cases $e = u_1 e x e u_2$ for some suitable values of u_1 and u_2 taken from S , which, as for the subcase $fyf \neq f$, implies $exe = e$, contradicting the fact $exe \neq e$. So L does not contain any word with two consecutive a 's.

This means that any word in L belongs to $(b+ab)^*(a+\varepsilon)$ so that any word w in L is of the form $u\alpha$ where $u \in (b+ab)^*$ and α is a if $w \in (b+ab)^* a$ and the empty word otherwise. Since, as for the previous case, $\mu(ab) = e$, but also $\mu(bb) = tt = yeye = fyfy = fy = ye = t = \mu(b)$,

$te = yee = ye = t$ and $et = eye = efy = xfyfy = xfy = e$ (by aperiodicity), we have that $\mu(u) \in \{1, e, t\}$. Assume now that $\alpha = a$. There are three different cases.

- $\mu(u) = 1$.

Then $\mu(w) = 1\mu(a) = ex$ that does not belong to $\{1, e, t\}$, otherwise we would have $exe = e$, because $ete = et = e$ by the equalities proved just above. But this contradicts the fact that $w \in L$, so this case cannot occur.

- $\mu(u) = e$.

Then $\mu(w) = e\mu(a) = eex = ex$ that does not belong to $\{1, e, t\}$ (see the previous case). But this contradicts the fact that $w \in L$, so this case cannot occur.

- $\mu(u) = t$.

Then $\mu(w) = t\mu(a) = yeex = yex = f$ by aperiodicity. We have that f does not belong to $\{1, e, t\}$. Indeed, suppose for the sake of contradiction that it does: there are three cases to examine. Either $(yx)^\omega = f = t = ye = y(xy)^\omega$, and then $exe = (xy)^\omega x(xy)^\omega = x(yx)^\omega (xy)^\omega = xy(xy)^\omega (xy)^\omega = (xy)^\omega = e$ by aperiodicity, contradicting the fact $exe \neq e$. Or $(yx)^\omega = f = e = (xy)^\omega$, and then we have $exe = (xy)^\omega x(xy)^\omega = (xy)^\omega x(yx)^\omega = (xy)^\omega x(yx)^\omega y(yx)^\omega = (xy)^\omega (xy)^{\omega+1} (xy)^\omega = (xy)^\omega = e$ by aperiodicity and since $fyf = f$, contradicting the fact $exe \neq e$. Or $(yx)^\omega = f = 1$, and then $y = fyf = f = 1$. But $e = (xy)^\omega = x^\omega$, so $exe = x^\omega x x^\omega = x^\omega = e$, contradicting the fact $exe \neq e$. Therefore, $\mu(w)$ does not belong to $\{1, e, t\}$, contradicting the fact that $w \in L$, so this case cannot occur either.

This means that, necessarily, $\alpha = \varepsilon$, so that $w \in (b + ab)^*$ and since it is true for any $w \in L$, it follows that $L \subseteq (b + ab)^*$. Combined with the fact that $\mu((b + ab)^*) = \{1, e, t\}$, we can conclude that $\mu^{-1}(\{1, e, t\}) = L = (b + ab)^*$, showing $(b + ab)^*$ is recognized by μ verifying $\mu(\{a, b\}^+) \subseteq S$.

Non-aperiodic case. Assume now that S^1 is not aperiodic. Then there is an x in S such that $x^\omega \neq x^{\omega+1}$. Consider the morphism $\mu: \{a, b\}^* \rightarrow S^1$ sending a to $x^{\omega+1}$ and b to x^ω , and the language $L = \mu^{-1}(x^\omega)$. Let $k \in \mathbb{N}, k \geq 2$ be the smallest positive integer such that $x^{\omega+k} = x^\omega$, that cannot be 1 because $x^\omega \neq x^{\omega+1}$. Using this, for all $w \in \{a, b\}^*$, we have

$$\mu(w) = x^{|w|\omega + |w|_a} = x^{\omega + (|w|_a \bmod k)},$$

where $|w|$ indicates the length of w and $|w|_a$ the number of a 's it contains, so that w belongs to L if and only if $|w|_a = 0 \bmod k$. Hence, L is the language of all words with a number of a 's divisible by k , $b^*((ab^*)^k)^*$. In conclusion, $b^*((ab^*)^k)^*$ is recognized by μ verifying $\mu(\{a, b\}^+) \subseteq S$. \square

Let now S be any finite semigroup such that $\mathcal{W}(S) \subseteq \mathcal{P}(\mathbf{DA})$. Let $\eta_S: S^* \rightarrow S^1$ be the evaluation morphism of S . To show that S^1 is in \mathbf{DA} , we assume for the sake of contradiction that it is not the case. Then Lemma 4.3 tells us that one of $(c + ab)^*$, $(b + ab)^*$ or $b^*((ab^*)^k)^*$ for some $k \in \mathbb{N}, k \geq 2$ is recognized by a morphism $\mu: \Sigma^* \rightarrow S^1$, for Σ the appropriate alphabet, such that $\mu(\Sigma^+) \subseteq S$.

In all cases, we thus have a language $L \subseteq \Sigma^*$ equal to $\mu^{-1}(Q)$ for some subset Q of S^1 with the morphism μ sending letters of Σ to elements of S . Consider then the morphism $\varphi: \Sigma^* \rightarrow S^*$ sending each letter $a \in \Sigma$ to $\mu(a)$, a letter of S : we have $\mu = \eta_S \circ \varphi$, so that $L = \varphi^{-1}(\eta_S^{-1}(Q))$. As $\mathcal{W}(S) \subseteq \mathcal{P}(\mathbf{DA})$, we have that $\eta_S^{-1}(Q) \in \mathcal{P}(\mathbf{DA})$, hence since φ is an lm -morphism and $\mathcal{P}(\mathbf{DA})$ is closed under inverses of lm -morphisms by Lemma 2.1, we have $L = \varphi^{-1}(\eta_S^{-1}(Q)) \in \mathcal{P}(\mathbf{DA})$: a contradiction to Proposition 4.2.

In the remaining part of this section we prove Proposition 4.2.

Proof of Proposition 4.2. The idea of the proof is the following. We work by contradiction and assume that we have a sequence of programs over some monoid M of \mathbf{DA} deciding one of the targeted language. Let n be much larger than the size of M , and let P_n be the program running on words of length n . Consider a language of the form Δ^* for some finite set Δ of words (for instance assume $\Delta = \{c, ab\}$, $\Delta = \{b, ab\}$, \dots). We will show that we can fix a constant (depending on M and Δ but not on n) number of entries to P_n such that P_n always outputs the same value and there are completions of the entries in and out of Δ^* . Hence, if Δ was chosen so that there is actually a completion of the fixed entries in the targeted language and one outside of it, P_n cannot recognize Δ^* . We cannot prove this for all Δ , in particular it will not work for $\Delta = \{ab\}$ and indeed $(ab)^*$ is in $\mathcal{P}(\mathbf{DA})$. The key property of our Δ is that after fixing any letter at any position, except maybe for a constant number of positions, one can still complete the word into one within Δ^* . This is not true for $\Delta = \{ab\}$ because after fixing a b in an odd position all completions fall outside of $(ab)^*$.

We now spell out the technical details.

Let Δ be a finite non-empty set of non-empty words. Let Σ be the corresponding finite alphabet and let \perp be a letter not in Σ . A *mask* is a word over $\Sigma \cup \{\perp\}$. The positions of a mask carrying a \perp are called *free* while the positions carrying a letter in Σ are called *fixed*. A mask λ' is a *submask* of a mask λ if it is formed from λ by replacing some occurrences (possibly zero) of \perp by a letter in Σ .

A *completion* of a mask λ is a word w over Σ that is built from λ by replacing all occurrences of \perp by a letter in Σ . Notice that all completions of a mask have the same length as the mask itself. A mask λ is Δ -*compatible* if it has a completion in Δ^* (it will always be possible and easy to find a completion of λ outside of Δ^*).

The *dangerous* positions of a mask λ are the positions within distance $2l - 2$ of the fixed positions or within distance $l - 1$ of the beginning or the end of the mask, where l is the maximal length of a word in Δ . A position that is not dangerous is said to be *safe* and is necessarily free.

We say that Δ is *safe* if the following holds. Let λ be a Δ -compatible mask. Let i be any free position of λ that is not dangerous. Let a be any letter in Σ . Then the submask of λ constructed by fixing a at position i is Δ -compatible. We have already seen that $\Delta = \{ab\}$ is not safe. However our targeted Δ , $\Delta = \{c, ab\}$, $\Delta = \{b, ab\}$, $\Delta = \{a, b\}$, are safe. We always consider Δ to be safe in the following.

Note that it is important in the definition of safe for Δ that we fix only safe positions, i.e. positions far apart and far from the beginning and the end of the mask. Indeed, depending on the chosen Δ , there might be words that never appear as factors in any word of Δ^* , such as bb when $\Delta = \{c, ab\}$ or aa when $\Delta = \{b, ab\}$, so that fixing a position near an already fixed position to an arbitrary letter in a Δ -compatible mask may result in a mask that has no completion in Δ^* . This is why we make sure that safe positions are far from those already fixed and from the beginning and the end of the mask, where far depends on the length of the words of Δ .

Finally, we say that a completion w of a mask λ is *safe* if w is a completion of λ belonging to Δ^* or is constructed from a completion of λ in Δ^* by modifying only letters at safe positions of λ , the dangerous positions remaining unchanged.

Let M be a monoid in \mathbf{DA} whose identity we will denote by 1.

We define a version of Green's relations for decomposing monoids that will be used, as often in this setting, to prove the main technical lemma in the current proof. Given two elements u, u' of M we say that $u \leq_J u'$ if there are elements v, v' of M such that $u' = vuv'$. We write $u \sim_J u'$ if $u \leq_J u'$ and $u' \leq_J u$. We write $u <_J u'$ if $u \leq_J u'$ and $u' \not\leq_J u$. Given two elements u, u' of M we say that $u \leq_R u'$ if there is an element v of M such that $u' = uv$. We write $u \sim_R u'$ if $u \leq_R u'$ and $u' \leq_R u$. We write $u <_R u'$ if $u \leq_R u'$ and $u' \not\leq_R u$. Given two elements u, u' of M we say that $u \leq_L u'$ if there is an element v of M such that $u' = vu$. We write $u \sim_L u'$ if $u \leq_L u'$ and $u' \leq_L u$. We write $u <_L u'$ if $u \leq_L u'$ and $u' \not\leq_L u$. Finally, given two elements u, u' of M , we write $u \sim_H u'$ if $u \sim_R u'$ and $u \sim_L u'$.

We shall use the following well-known fact about these preorders and equivalence relations (see [Pin86, Chapter 3, Proposition 1.4]).

Lemma 4.4. *For all elements u and v of M , if $u \leq_R v$ and $u \sim_J v$, then $u \sim_R v$. Similarly, if $u \leq_L v$ and $u \sim_J v$, then $u \sim_L v$.*

From the definition it follows that for all elements u, v, r of M , we have $u \leq_R ur$ and $v \leq_L rv$. When the inequality is strict in the first case, i.e. $u <_R ur$, we say that r is R -bad for u . Similarly r is L -bad for v if $v <_L rv$. It follows from $M \in \mathbf{DA}$ that being R -bad or L -bad only depends on the \sim_R or \sim_L class, respectively. This is formalized in the following lemma, that is folklore and used at least implicitly in many proofs involving \mathbf{DA} (see for instance [TT02, proof of Theorem 3]). Since we didn't manage to find the lemma stated and proven in the form below, we include a proof for completeness.

Lemma 4.5. *If M is in \mathbf{DA} , then $u \sim_R u'$ and $ur \sim_R u$ implies $u'r \sim_R u$. Similarly $u \sim_L u'$ and $ru \sim_L u$ implies $ru' \sim_L u$.*

Proof. Let $u, u', r \in M$ such that $u \sim_R u'$ and $ur \sim_R u$. This means that there exist $v, v', s \in M$ such that $u = u'v'$, $u' = uv$ and $urs = u$.

This implies that

$$u' = uv = ursv = u'v'rsv = u'(v'rsv)^2 = \dots = u'(v'rsv)^\omega$$

where ω is the idempotent power of M . Hence, we have that

$$u'r(v'rsv)^\omega v' = u'(v'rsv)^\omega r(v'rsv)^\omega v'.$$

But, by [TT02, Theorem 2], since $M \in \mathbf{DA}$, we have that $(xyz)^\omega y(xyz)^\omega = (xyz)^\omega$ for all $x, y, z \in M$, so that

$$u'r(v'rsv)^\omega v' = u'(v'rsv)^\omega v' = u'v' = u.$$

Therefore, we have $u'r \leq_R u$ and since $uvr = u'r$, we also have $u \leq_R u'r$, so that $u'r \sim_R u$ as claimed.

The proof goes through symmetrically for \sim_L . □

Let Δ be a finite set of words and Σ be the corresponding finite alphabet, Δ being safe, and let $n \in \mathbb{N}$. We are now going to prove the main technical lemma that allows us to assert that after fixing a constant number of positions in the input of a program over M , it can still be completed into a word of Δ^* , but the program cannot make the difference between any two possible completions anymore. To prove the lemma, we define a relation \prec on the set of quadruplets (λ, P, u, v) where λ is a mask of length n , P is a program over M for words of length n and u and v are two elements of M . We will say that an element $(\lambda_1, P_1, u_1, v_1)$ is strictly smaller than $(\lambda_2, P_2, u_2, v_2)$, written $(\lambda_1, P_1, u_1, v_1) \prec (\lambda_2, P_2, u_2, v_2)$, if and only if λ_1 is a submask of λ_2 , P_1 is a subprogram of P_2 and one of the following cases occurs:

- (1) $u_2 <_R u_1$ and $v_1 = v_2$ and P_1 is a suffix of P_2 and $u_1 P_1(w)v_1 = u_2 P_2(w)v_2$ for all safe completions w of λ_1 ;
- (2) $v_2 <_L v_1$ and $u_1 = u_2$ and P_1 is a prefix of P_2 and $u_1 P_1(w)v_1 = u_2 P_2(w)v_2$ for all safe completions w of λ_1 ;
- (3) $u_2 = u_1$ and $v_1 = 1$ and P_1 is a prefix of P_2 and $u_1 P_1(w)v_1 <_J u_2 P_2(w)v_2$ for all safe completions w of λ_1 ;
- (4) $v_2 = v_1$ and $u_1 = 1$ and P_1 is a suffix of P_2 and $u_1 P_1(w)v_1 <_J u_2 P_2(w)v_2$ for all safe completions w of λ_1 .

Note that, since M is finite, this relation is well-founded (that is, it has no infinite decreasing chain, an infinite sequence of quadruplets $\mu_0, \mu_1, \mu_2, \dots$ such that $\mu_{i+1} \prec \mu_i$ for all $i \in \mathbb{N}$) and the maximal length of any decreasing chain can be upper bounded by $2 \cdot |M|^2$, that does only depend on M . For a given quadruplet μ , we shall also call its height the biggest $i \in \mathbb{N}$ such that there exists a decreasing chain $\mu_i \prec \mu_{i-1} \prec \dots \prec \mu_0 = \mu$.

The following lemma is the key to the proof. It shows that modulo fixing a few entries, one can fix the output: to count the number of fixed positions for a given mask λ , we denote by $|\lambda|_\Sigma$ the number of letters in λ belonging to Σ , that is to say, the number of fixed positions in λ .

Lemma 4.6. *Let λ be a Δ -compatible mask of length n , let P be a program over M of range n , let u and v be elements of M such that (λ, P, u, v) is of height h . Then there is an element t of M and a Δ -compatible submask λ' of λ verifying $|\lambda'|_\Sigma \leq (2^h 6l)^{2^h} \cdot \max\{|\lambda|_\Sigma, 1\}$ such that any safe completion w of λ' verifies $uP(w)v = t$.*

Proof. The proof goes by induction on the height h .

Let λ be a Δ -compatible mask of length n , let P be a program over M for words of length n , let u and v be elements of M such that (λ, P, u, v) is of height h , and assume that for any quadruplet (λ', P', u', v') strictly smaller than (λ, P, u, v) , the lemma is verified. Consider the following conditions concerning the quadruplet (λ, P, u, v) :

- (a) there does not exist any instruction (x, f) of P such that for some letter a the submask λ' of λ formed by setting position x to a is Δ -compatible and $f(a)$ is R -bad for u ;
- (b) v is not R -bad for u ;
- (c) there does not exist any instruction (x, f) of P such that for some letter a the submask λ' of λ formed by setting position x to a is Δ -compatible and $f(a)$ is L -bad for v ;
- (d) u is not L -bad for v .

We will now do a case analysis based on which of these conditions are violated or not.

Case 1: condition (a) is violated. So there exists some instruction (x, f) of P such that for some letter a the submask λ' of λ formed by setting position x to a (if it wasn't already the case) is Δ -compatible and $f(a)$ is R -bad for u . Let i be the smallest number of such an instruction.

Let P' be the subprogram of P until, and including, instruction $i - 1$. Let w be a safe completion of λ . For any instruction (y, g) of P' , as $y < i$, $g(w_y)$ cannot be R -bad for u , so $u \sim_R ug(w_y)$. Hence, by Lemma 4.5, $u \sim_R uP'(w)$ for all safe completions w of λ .

So, because $f(a)$ is R -bad for u , any safe completion w of λ' , which is also a safe completion of λ , is such that $u \sim_R uP'(w) <_R uP'(w)f(a) \leq_R uP(w)v$ by Lemma 4.5, hence $uP'(w) <_J uP(w)v$ by Lemma 4.4. So $(\lambda', P', u, 1) \prec (\lambda, P, u, v)$, therefore, by induction we get a Δ -compatible submask λ_1 of λ' and a monoid element t_1 such that $uP'(w) = t_1$ for all safe completions w of λ_1 .

Let P'' be the subprogram of P starting from instruction $i+1$. Notice that, since $u \sim_R t_1$ (by what we have proven just above), $u <_R t_1 f(a)$ (by Lemma 4.5) and $t_1 f(a) P''(w) v = u P'(w) f(a) P''(w) v = u P(w) v$ for all safe completions w of λ_1 . Hence, $(\lambda_1, P'', t_1 f(a), v)$ is strictly smaller than (λ, P, u, v) and by induction we get a Δ -compatible submask λ_2 of λ_1 and a monoid element t such that $t_1 f(a) P''(w) v = t$ for all safe completions w of λ_2 .

Thus, any safe completion w of λ_2 is such that

$$u P(w) v = u P'(w) f(a) P''(w) v = t_1 f(a) P''(w) v = t .$$

Therefore λ_2 and t form the desired couple of a Δ -compatible submask of λ and an element of M . We still have to show that $|\lambda_2|_\Sigma$ satisfies the desired upper bound.

By induction, since $(\lambda', P', u, 1)$ is of height $h' \leq h - 1$, we have

$$|\lambda_1|_\Sigma \leq (2^{h'} 6l)^{2^{h'}} \cdot |\lambda'|_\Sigma \leq (2^{h-1} 6l)^{2^{h-1}} \cdot |\lambda'|_\Sigma .$$

Consequently, by induction again, as $(\lambda_1, P'', t_1 f(a), v)$ is of height $h'' \leq h - 1$, we have

$$\begin{aligned} |\lambda_2|_\Sigma &\leq (2^{h''} 6l)^{2^{h''}} \cdot |\lambda_1|_\Sigma \\ &\leq (2^{h-1} 6l)^{2^{h-1}} \cdot |\lambda_1|_\Sigma \\ &\leq (2^{h-1} 6l)^{2^{h-1}} \cdot (2^{h-1} 6l)^{2^{h-1}} \cdot |\lambda'|_\Sigma \\ &= (2^{h-1} 6l)^{2^h} \cdot |\lambda'|_\Sigma . \end{aligned}$$

Moreover, it holds that $|\lambda'|_\Sigma \leq |\lambda|_\Sigma + 1$, so that

$$\begin{aligned} |\lambda_2|_\Sigma &\leq (2^{h-1} 6l)^{2^h} \cdot (|\lambda|_\Sigma + 1) \\ &\leq (2^{h-1} 6l)^{2^h} \cdot 2^{2^h} \cdot \max\{|\lambda|_\Sigma, 1\} \\ &= (2^h 6l)^{2^h} \cdot \max\{|\lambda|_\Sigma, 1\} . \end{aligned}$$

Case 2: condition (a) is verified but condition (b) is violated, so v is R -bad for u and Case 1 does not apply.

Let w be a safe completion of λ : for any instruction (x, f) of P , as the submask λ' of λ formed by setting position x to w_x is Δ -compatible (by the fact that Δ is safe and w is a safe completion of λ), $f(w_x)$ cannot be R -bad for u , otherwise condition (a) would be violated, so $u \sim_R u f(w_x)$. Hence, by Lemma 4.5, $u \sim_R u P(w)$ for all safe completions w of λ . Notice then that $u \sim_R u P(w) <_R u P(w) v$ (by Lemma 4.5), hence $u P(w) <_J u P(w) v$ (by Lemma 4.4) for all safe completions w of λ . So $(\lambda, P, u, 1) \prec (\lambda, P, u, v)$, therefore we obtain by induction a monoid element t_1 and a Δ -compatible submask λ' of λ such that $u P(w) = t_1$ for all completions w of λ' . If we set $t = t_1 v$, we get that any safe completion w of λ' is such that $u P(w) v = t_1 v = t$. Therefore λ' and t form the desired couple of a Δ -compatible submask of λ and an element of M .

Moreover, by induction, since $(\lambda, P, u, 1)$ is of height $h' \leq h - 1$, we have

$$|\lambda'|_\Sigma \leq (2^{h'} 6l)^{2^{h'}} \cdot \max\{|\lambda|_\Sigma, 1\} \leq (2^h 6l)^{2^h} \cdot \max\{|\lambda|_\Sigma, 1\} ,$$

the desired upper bound.

Case 3: condition (c) is violated. So there exists some instruction (x, f) of P such that for some letter a the submask λ' of λ formed by setting position x to a (if it wasn't already the case) is Δ -compatible and $f(a)$ is L -bad for v .

We proceed as for Case 1 by symmetry.

Case 4: condition (c) is verified but condition (d) is violated, so u is L -bad for v and Case 3 does not apply.

We proceed as for Case 2 by symmetry.

Case 5: conditions (a), (b), (c) and (d) are verified.

As it was in Case 2 and Case 4, using Lemma 4.5, the fact that condition (a) and condition (c) are verified implies that $u \sim_R uP'(w)$ and $v \sim_L P''(w)v$ for any prefix P' of P , any suffix P'' of P and all safe completions w of λ . Moreover, since condition (b) and condition (d) are verified, by Lemma 4.5, we get that $uP(w)v \sim_R u$ and $uP(w)v \sim_L v$ for all safe completions w of λ . This implies that (λ, P, u, v) is minimal for \prec and that $h = 0$.

Let w_0 be a completion of λ that is in Δ^* . Let λ' be the submask of λ fixing all free dangerous positions of λ using w_0 and let $t = uP(w_0)v$. Then, for any completion w of λ' , which is a safe completion of λ by construction, we have that $uP(w)v \sim_R u \sim_R t$ and $uP(w)v \sim_L v \sim_L t$. Thus, $uP(w)v \sim_H t$ for any completion w of λ' . As M is aperiodic, this implies that $uP(w)v = t$ for all completions w of λ' (see [Pin86, Chapter 3, Proposition 4.2]). Therefore λ' and t form the desired couple of a Δ -compatible submask of λ and an element of M .

Now, since the number of free positions of λ fixed in λ' , i.e. $|\lambda'|_\Sigma - |\lambda|_\Sigma$, is exactly the number of free dangerous positions in λ , and as a position in λ is dangerous if it is within distance $2l - 2$ of a fixed position or within distance $l - 1$ of the beginning or the end of λ , we have

$$\begin{aligned} |\lambda'|_\Sigma &\leq 2 \cdot |\lambda|_\Sigma \cdot (2l - 2) + 2 \cdot (l - 1) + |\lambda|_\Sigma = |\lambda|_\Sigma \cdot (4l - 3) + 2l - 2 \\ &\leq (2^0 6l)^{2^0} \cdot \max\{|\lambda|_\Sigma, 1\} , \end{aligned}$$

the desired upper bound.

This concludes the proof of the lemma. \square

Setting $\Delta = \{c, ab\}$ or $\Delta = \{b, ab\}$ with Σ the associated alphabet, when applying Lemma 4.6 with the trivial Δ -compatible mask λ of length n containing only free positions, with P some program over M of range n and with u and v equal to 1, the resulting mask λ' has the property that we have an element t of M such that $P(w) = t$ for any safe completion w of λ' . Since the mask λ' is Δ -compatible and has a number of fixed positions upper-bounded by $(2^h 6l)^{2^h}$ where h is the height of (λ, P, u, v) , itself upper-bounded by $2 \cdot |M|^2$, as long as n is big enough, we have a safe completion $w_0 \in \Delta^*$ and a safe completion $w_1 \notin \Delta^*$. Hence, P cannot be part of any sequence of programs p -recognizing Δ^* . This implies that $(c + ab)^* \notin \mathcal{P}(M)$ and $(b + ab)^* \notin \mathcal{P}(M)$. Finally, for any $k \in \mathbb{N}, k \geq 2$, we can prove that $b^*((ab^*)^k)^* \notin \mathcal{P}(M)$ by setting $\Delta = \{a, b\}$ and completing the mask given by the lemma by setting the letters in such a way that we have the right number of a modulo k in one case and not in the other case.

This concludes the proof of Proposition 4.2 because the argument above holds for any monoid in **DA**.

5. A FINE HIERARCHY IN $\mathcal{P}(\mathbf{DA})$

The definition of p -recognition by a sequence of programs over a monoid given in Section 2 requires that for each n , the program reading the entries of length n has a length polynomial in n . In the case of $\mathcal{P}(\mathbf{DA})$, the polynomial-length restriction is without loss of generality: any program over a monoid in **DA** is equivalent to one of polynomial length over the same

monoid [TT01] (in the sense that they recognize the same languages). In this section, we show that this does not collapse further: in the case of \mathbf{DA} , programs of length $O(n^{k+1})$ express strictly more than those of length $O(n^k)$.

Following [GT03], we use an alternative definition of the languages recognized by a monoid in \mathbf{DA} . We define by induction a hierarchy of classes of languages SUM_k , where SUM stands for *strongly unambiguous monomial*. A language L is in SUM_0 if it is of the form A^* for some finite alphabet A . A language L is in SUM_k for $k \in \mathbb{N}_{>0}$ if it is in SUM_{k-1} or $L = L_1 a L_2$ for some languages $L_1 \in SUM_i$ and $L_2 \in SUM_j$ and some letter a with $i + j = k - 1$ such that no word of L_1 contains the letter a or no word of L_2 contains the letter a .

Gavaldà and Thérien stated without proof that a language L is recognized by a monoid in \mathbf{DA} iff there is a $k \in \mathbb{N}$ such that L is a Boolean combination of languages in SUM_k [GT03] (see [Gro18, Theorem 4.1.9] for a proof). For each $k \in \mathbb{N}$, we denote by \mathbf{DA}_k the variety of monoids generated by the syntactic monoids of the Boolean combinations of languages in SUM_k . It can be checked that, for each k , \mathbf{DA}_k forms a variety of monoids recognizing precisely Boolean combinations of languages in SUM_k : this is what we do in the first subsection.

In the two following subsections, we then give a fine program-length-based hierarchy within $\mathcal{P}(\mathbf{DA})$ for this parametrization of \mathbf{DA} .

5.1. A parametrization of \mathbf{DA} . For each $k \in \mathbb{N}$, we denote by SUL_k the class of regular languages that are Boolean combinations of languages in SUM_k ; it is a variety of languages as shown just below. But as \mathbf{DA}_k is the variety of monoids generated by the syntactic monoids of the languages in SUL_k , by Eilenberg's theorem, we know that, conversely, all the regular languages whose syntactic monoids lie in \mathbf{DA}_k are in SUL_k .

Back to the fact that SUL_k is a variety of languages for any $k \in \mathbb{N}$. Closure under Boolean operations is obvious by construction. Closure under quotients and inverses of morphisms is respectively given by the following two lemmas and by the fact that both quotients and inverses of morphisms commute with Boolean operations.

Given a word u over a given finite alphabet Σ , we will denote by $\text{alph}(u)$ the set of letters of Σ that appear in u .

Lemma 5.1. *For all $k \in \mathbb{N}$, for all $L \in SUM_k$ over a finite alphabet Σ and $u \in \Sigma^*$, $u^{-1}L$ and Lu^{-1} both are unions of languages in SUM_k over Σ .*

Proof. We prove it by induction on k .

Base case: $k = 0$. Let $L \in SUM_0$ over a finite alphabet Σ and $u \in \Sigma^*$. This means that $L = A^*$ for some $A \subseteq \Sigma$. We have two cases: either $\text{alph}(u) \not\subseteq A$ and then $u^{-1}L = Lu^{-1} = \emptyset$; or $\text{alph}(u) \subseteq A$ and then $u^{-1}L = Lu^{-1} = A^* = L$. So $u^{-1}L$ and Lu^{-1} both are unions of languages in SUM_0 over Σ . The base case is hence proved.

Inductive step. Let $k \in \mathbb{N}_{>0}$ and assume that the lemma is true for all $k' \in \mathbb{N}, k' < k$.

Let $L \in SUM_k$ over a finite alphabet Σ and $u \in \Sigma^*$. This means that either L is in SUM_{k-1} and the lemma is proved by applying the inductive hypothesis directly for L and u , or $L = L_1 a L_2$ for some languages $L_1 \in SUM_i$ and $L_2 \in SUM_j$ and some letter $a \in \Sigma$ with $i + j = k - 1$ and, either no word of L_1 contains the letter a or no word of L_2 contains

the letter a . We shall only treat the case in which a does not appear in any of the words of L_1 ; the other case is treated symmetrically.

There are again two cases to consider, depending on whether a does appear in u or not.

If $a \notin \text{alph}(u)$, then it is straightforward to check that $u^{-1}L = (u^{-1}L_1)aL_2$ and $Lu^{-1} = L_1a(L_2u^{-1})$. By the inductive hypothesis, we get that $u^{-1}L_1$ is a union of languages in \mathcal{SUM}_i over Σ and that L_2u^{-1} is a union of languages in \mathcal{SUM}_j over Σ . Moreover, it is direct to see that no word of $u^{-1}L_1$ contains the letter a . By distributivity of concatenation over union, we finally get that $u^{-1}L$ and Lu^{-1} both are unions of languages in \mathcal{SUM}_k over Σ .

If $a \in \text{alph}(u)$, then let $u = u_1au_2$ with $u_1, u_2 \in \Sigma^*$ and $a \notin \text{alph}(u_1)$. It is again straightforward to see that

$$u^{-1}L = \begin{cases} u_2^{-1}L_2 & \text{if } u_1 \in L_1 \\ \emptyset & \text{otherwise} \end{cases}$$

and

$$Lu^{-1} = L_1a(L_2u^{-1}) \cup \begin{cases} L_1u_1^{-1} & \text{if } u_2 \in L_2 \\ \emptyset & \text{otherwise} \end{cases}.$$

As before, by the inductive hypothesis, we get that $L_1u_1^{-1}$ is a union of languages in \mathcal{SUM}_i over Σ and that both $u_2^{-1}L_2$ and L_2u^{-1} are unions of languages in \mathcal{SUM}_j over Σ . And, again, by distributivity of concatenation over union, we get that $u^{-1}L$ and Lu^{-1} both are a union of languages in \mathcal{SUM}_k over Σ .

This concludes the inductive step and therefore the proof of the lemma. \square

Lemma 5.2. *For all $k \in \mathbb{N}$, for all $L \in \mathcal{SUM}_k$ over a finite alphabet Σ and $\varphi: \Gamma^* \rightarrow \Sigma^*$ a morphism of monoids where Γ is another finite alphabet, $\varphi^{-1}(L)$ is a union of languages in \mathcal{SUM}_k over Γ .*

Proof. We prove it by induction on k .

Base case: $k = 0$. Let $L \in \mathcal{SUM}_0$ over a finite alphabet Σ and $\varphi: \Gamma^* \rightarrow \Sigma^*$ a morphism of monoids where Γ is another finite alphabet. This means that $L = A^*$ for some $A \subseteq \Sigma$. It is straightforward to check that $\varphi^{-1}(L) = B^*$ where $B = \{b \in \Gamma \mid \varphi(b) \in A\}$. B^* is certainly a union of languages in \mathcal{SUM}_0 over Σ . The base case is hence proved.

Inductive step. Let $k \in \mathbb{N}_{>0}$ and assume that the lemma is true for all $k' \in \mathbb{N}, k' < k$.

Let $L \in \mathcal{SUM}_k$ over a finite alphabet Σ and $\varphi: \Gamma^* \rightarrow \Sigma^*$ a morphism of monoids where Γ is another finite alphabet. This means that either L is in \mathcal{SUM}_{k-1} and the lemma is proved by applying the inductive hypothesis directly for L and φ , or $L = L_1aL_2$ for some languages $L_1 \in \mathcal{SUM}_i$ and $L_2 \in \mathcal{SUM}_j$ and some letter $a \in \Sigma$ with $i + j = k - 1$ and, either no word of L_1 contains the letter a or no word of L_2 contains the letter a . We shall only treat the case in which a does not appear in any of the words of L_1 ; the other case is treated symmetrically.

Let us define $B = \{b \in \Gamma \mid a \in \text{alph}(\varphi(b))\}$ as the set of letters of Γ whose image word by φ contains the letter a . For each $b \in B$, we shall also let $\varphi(b) = u_{b,1}au_{b,2}$ with $u_{b,1}, u_{b,2} \in \Sigma^*$ and $a \notin \text{alph}(u_{b,1})$. It is not too difficult to see that we then have

$$\varphi^{-1}(L) = \bigcup_{b \in B} \varphi^{-1}(L_1u_{b,1}^{-1})b\varphi^{-1}(u_{b,2}^{-1}L_2).$$

By the inductive hypothesis, by Lemma 5.1 and by the fact that inverses of morphisms commute with unions, we get that $\varphi^{-1}(L_1 u_{b,1}^{-1})$ is a union of languages in \mathcal{SUM}_i over Γ and that $\varphi^{-1}(u_{b,2}^{-1} L_2)$ is a union of languages in \mathcal{SUM}_j over Γ . Moreover, it is direct to see that no word of $\varphi^{-1}(L_1 u_{b,1}^{-1})$ contains the letter b for all $b \in B$. By distributivity of concatenation over union, we finally get that $\varphi^{-1}(L)$ is a union of languages in \mathcal{SUM}_k over Γ .

This concludes the inductive step and therefore the proof of the lemma. \square

5.2. Strict hierarchy. For each k we exhibit a language $L_k \subseteq \{0, 1\}^*$ that can be recognized by a sequence of programs of length $O(n^k)$ over a monoid M_k in \mathbf{DA}_k but cannot be recognized by any sequence of programs of length $O(n^{k-1})$ over any monoid in \mathbf{DA} .

For a given $k \in \mathbb{N}_{>0}$, the language L_k expresses a property of the first k occurrences of 1 in the input word. To define L_k we say that S is a k -set over n for some $n \in \mathbb{N}$ if S is a set where each element is an ordered tuple of k distinct elements of $[n]$. For any sequence $\Delta = (S_n)_{n \in \mathbb{N}}$ of k -sets over n , we set $L_\Delta = \bigcup_{n \in \mathbb{N}} K_{n, S_n}$, where for each $n \in \mathbb{N}$, K_{n, S_n} is the set of words over $\{0, 1\}$ of length n such that for each of them, it contains at least k occurrences of 1 and the ordered k -tuple of the positions of the first k occurrences of 1 belongs to S_n .

On the one hand, we show that for all k there is a monoid M_k in \mathbf{DA}_k such that for all Δ the language L_Δ is recognized by a sequence of programs over M_k of length $O(n^k)$. The proof is done by an inductive argument on k .

On the other hand, we show that for all k there is a Δ such that for any finite monoid M and any sequence of programs $(P_n)_{n \in \mathbb{N}}$ over M of length $O(n^{k-1})$, L_Δ is not recognized by $(P_n)_{n \in \mathbb{N}}$. This is done using a counting argument: for some monoid size i , for n big enough, the number of languages in $\{0, 1\}^n$ recognized by a program over some monoid of size i of length at most $\alpha \cdot n^{k-1}$ for α some constant is upper-bounded by a number that turns out to be asymptotically smaller than the number of different possible K_{n, S_n} .

Upper bound. We start with the upper bound. Notice that for some $k \in \mathbb{N}_{>0}$ and $\Delta = (S_n)_{n \in \mathbb{N}}$, the language of words of length n of L_Δ is exactly K_{n, S_n} . Hence the fact that L_Δ can be recognized by a sequence of programs over a monoid in \mathbf{DA}_k of length $O(n^k)$ is a consequence of the following proposition.

Proposition 5.3. *For all $k \in \mathbb{N}_{>0}$ there is a monoid $M_k \in \mathbf{DA}_k$ such that for all $n \in \mathbb{N}$ and all k -sets S_n over n , the language K_{n, S_n} is recognized by a program over M_k of length at most $4n^k$.*

Proof. We first define by induction on k a family of languages Z_k over the alphabet $Y_k = \{\perp_l, \top_l \mid 1 \leq l \leq k\}$. For $k = 0$, Z_0 is $\{\varepsilon\}$. For $k > 0$, Z_k is the set of words containing \top_k and such that the first occurrence of \top_k has no \perp_k to its left, and the sequence between the first occurrence of \top_k and the first occurrence of \perp_k or \top_k to its right, or the end of the word if there is no such letter, belongs to Z_{k-1} . A simple induction on k shows that Z_k is defined by the following expression

$$Y_{k-1}^* \top_k Y_{k-2}^* \top_{k-1} \cdots Y_1^* \top_2 \top_1 Y_k^*$$

and therefore it is in \mathcal{SUM}_k and its syntactic monoid M_k is in \mathbf{DA}_k .

Fix n . If $n = 0$, the proposition follows trivially, otherwise, we define by induction on k a program $P_k(i, S)$ for every k -set S over n and every $1 \leq i \leq n+1$ that will for the moment output elements of $Y_k \cup \{\varepsilon\}$ instead of outputting elements of M_k .

For any $k > 0$, $1 \leq j \leq n$ and S a k -set over n , let $f_{j,S}$ be the function with $f_{j,S}(0) = \varepsilon$ and $f_{j,S}(1) = \top_k$ if j is the first element of some ordered k -tuple of S , $f_{j,S}(1) = \perp_k$ otherwise. We also let g_k be the function with $g_k(0) = \varepsilon$ and $g_k(1) = \perp_k$. If S is a k -set over n and $1 \leq j \leq n$ then $S|j$ denotes the $(k-1)$ -set over n containing the ordered $(k-1)$ -tuples \bar{t} such that $(j, \bar{t}) \in S$.

For $k > 0$, $1 \leq i \leq n+1$ and S a k -set over n , the program $P_k(i, S)$ is the following sequence of instructions:

$$(i, f_{i,S})P_{k-1}(i+1, S|i)(i, g_k) \cdots (n, f_{n,S})P_{k-1}(n+1, S|n)(n, g_k).$$

In other words, the program guesses the first occurrence $j \geq i$ of 1, returns \perp_k or \top_k depending on whether it is the first element of an ordered k -tuple in S , and then proceeds for the next occurrences of 1 by induction.

For $k = 0$, $1 \leq i \leq n+1$ and S a 0-set over n (that is empty or contains ε , the only ordered 0-tuple of elements of $[n]$), the program $P_0(i, S)$ is the empty program ε .

A simple computation shows that for any $k \in \mathbb{N}_{>0}$, $1 \leq i \leq n+1$ and S a k -set over n , the number of instructions in $P_k(i, S)$ is at most $4n^k$.

A simple induction on k shows that when running on a word $w \in \{0, 1\}^n$, for any $k \in \mathbb{N}_{>0}$, $1 \leq i \leq n+1$ and S a k -set over n , $P_k(i, S)$ returns a word in Z_k iff the ordered k -tuple of the positions of the first k occurrences of 1 starting at position i in w exists and is an element of S .

For any $k > 0$ and S_n a k -set over n , it remains to apply the syntactic morphism of Z_k to the output of the functions in the instructions of $P_k(1, S_n)$ to get a program over M_k of length at most $4n^k$ recognizing K_{n, S_n} . \square

Lower bound. The following claim is a simple counting argument.

Claim 5.4. *For all $i \in \mathbb{N}_{>0}$ and $n \in \mathbb{N}$, the number of languages in $\{0, 1\}^n$ recognized by programs over a monoid of size i , reading inputs of length n over the alphabet $\{0, 1\}$, with at most $l \in \mathbb{N}$ instructions, is bounded by $i^{i^2} 2^i \cdot (n \cdot i^2)^l$.*

Proof. Fix a monoid M of size i . Since a program over M of range n with less than l instructions can always be completed into such a program with exactly l instructions recognizing the same languages in $\{0, 1\}^n$ (using the identity of M), we only consider programs with exactly l instructions. As $\Sigma = \{0, 1\}$, there are $n \cdot i^2$ choices for each of the l instructions of a range n program over M reading inputs in $\{0, 1\}^*$. Such a program can recognize at most 2^i different languages in $\{0, 1\}^n$. Hence, the number of languages in $\{0, 1\}^n$ recognized by programs over M of length at most l is at most $2^i \cdot (n \cdot i^2)^l$. The result follows from the facts that there are at most i^{i^2} isomorphism classes of monoids of size i and that two isomorphic monoids allow to recognize the same languages in $\{0, 1\}^n$ through programs. \square

If for some $k \in \mathbb{N}_{>0}$ and $1 \leq i \leq \alpha$, $\alpha \in \mathbb{N}_{>0}$, we apply Claim 5.4 for all $n \in \mathbb{N}$, $l = \alpha \cdot n^{k-1}$, we get a number $\mu_i(n)$ of languages upper-bounded by $n^{O(n^{k-1})}$, which is asymptotically strictly smaller than the number of distinct K_{n, S_n} , which is $2^{\binom{n}{k}}$, i.e. $\mu_i(n)$ is in $o(2^{\binom{n}{k}})$.

Hence, for all $j \in \mathbb{N}_{>0}$, there exist an $n_j \in \mathbb{N}$ and T_j a k -set over n_j such that no program over a monoid of size $1 \leq i \leq j$, of range n_j and of length at most $j \cdot n^{k-1}$ recognizes K_{n_j, T_j} . Moreover, we can assume without loss of generality that the sequence $(n_j)_{j \in \mathbb{N}_{>0}}$ is increasing. Let $\Delta = (S_n)_{n \in \mathbb{N}}$ be such that $S_{n_j} = T_j$ for all $j \in \mathbb{N}_{>0}$ and $S_n = \emptyset$ for any $n \in \mathbb{N}$ verifying that it is not equal to any n_j for $j \in \mathbb{N}_{>0}$. We show that no sequence of programs over a finite monoid of length $O(n^{k-1})$ can recognize L_Δ . If this were the case, then let i be the size of the monoid. Let $j \geq i$ be such that for any $n \in \mathbb{N}$, the n -th program has length at most $j \cdot n^{k-1}$. But, by construction, we know that there does not exist any such program of range n_j recognizing K_{n_j, T_j} , a contradiction.

This implies the following hierarchy, where $\mathcal{P}(\mathbf{V}, s(n))$ for some variety of monoids \mathbf{V} and a function $s: \mathbb{N} \rightarrow \mathbb{N}$ denotes the class of languages recognizable by a sequence of programs of length $O(s(n))$:

Proposition 5.5. *For all $k \in \mathbb{N}$, $\mathcal{P}(\mathbf{DA}, n^k) \subsetneq \mathcal{P}(\mathbf{DA}, n^{k+1})$. More precisely, for all $k \in \mathbb{N}$ and $d \in \mathbb{N}$, $d \leq \max\{k-1, 0\}$, $\mathcal{P}(\mathbf{DA}_k, n^d) \subsetneq \mathcal{P}(\mathbf{DA}_k, n^{d+1})$.*

To prove this proposition, we use two facts. First, that for all $k \in \mathbb{N}$ and all $d \in \mathbb{N}$, $d \leq \max\{k-1, 0\}$, any monoid from \mathbf{DA}_d is also a monoid from \mathbf{DA}_k . And second, that $a^* \in \mathcal{P}(\mathbf{DA}_0, n) \setminus \mathcal{P}(\mathbf{DA}_0, 1)$ simply because any program over some finite monoid of range n for $n \in \mathbb{N}$ recognizing a^n must have at least n instructions, one for each input letter.

5.3. Collapse. Tesson and Thérien showed that any program over a monoid M in \mathbf{DA} is equivalent to one of polynomial length [TT01]. We now show that if we further assume that M is in \mathbf{DA}_k then the length can be assumed to be $O(n^{\max\{k, 1\}})$.

Proposition 5.6. *Let $k \geq 0$. Let $M \in \mathbf{DA}_k$. Then any program over M is equivalent to a program over M of length $O(n^{\max\{k, 1\}})$.*

The equivalent program of length $O(n^{\max\{k, 1\}})$ is actually a subprogram of the initial one. For each possible acceptance set, an input word to the program is accepted if and only if the word over the alphabet M produced by the program belongs to some fixed Boolean combination of languages in SUM_k . The idea is then just to keep enough instructions so that membership of the produced word over M in each of these languages does not change.

Recall that if P is a program over some monoid M of range n , then $P(w)$ denotes the element of M resulting from the execution of the program P on w . It will be convenient here to also work with the word over M resulting from the sequence of executions of each instruction of P on w . We denote this word by $EP(w)$.

The result is a consequence of the following lemma and the fact that for any acceptance set $F \subseteq M$, a word $w \in \Sigma^n$ (where Σ is the input alphabet) is accepted iff $EP(w) \in L$ where L is a language in SUL_k , a Boolean combination of languages in SUM_k .

Lemma 5.7. *Let Σ be a finite alphabet, M a finite monoid, and n, k natural numbers.*

For any program P over M of range n and any language K over M in SUM_k , there exists a subprogram Q of P of length $O(n^{\max\{k, 1\}})$ such that for any subprogram Q' of P that has Q as a subprogram, we have for all words w over Σ of length n :

$$EP(w) \in K \Leftrightarrow EQ'(w) \in K .$$

Proof. A program P over M of range n is a finite sequence (p_i, f_i) of instructions where each p_i is a positive natural number which is at most n and each f_i is a function from Σ to M .

We denote by l the number of instructions of P . For each set $I \subseteq [l]$ we denote by $P[I]$ the subprogram of P consisting of the subsequence of instructions of P obtained after removing all instructions whose index is not in I . In particular, $P[1, m]$ denotes the initial sequence of instructions of P , until instruction number m .

We prove the lemma by induction on k .

The intuition behind the proof for a program P on inputs of length n and some $K_1\gamma K_2 \in \mathit{SUM}_k$ when $k \geq 2$ is as follows. We assume that K_1 does not contain any word with the letter γ , the other case is done symmetrically. Consider the subset of all indices $I_\gamma \subseteq [l]$ that correspond, for a fixed letter a and a fixed position p in the input, to the first instruction of P that would output the element γ when reading a at position p . We then have that, given some w as input, $EP(w) \in K_1\gamma K_2$ if and only if there exists $i \in I_\gamma$ verifying that the element at position i of $EP(w)$ is γ , $EP[1, i-1](w) \in K_1$ and $EP[i+1, l](w) \in K_2$. The idea is then that if we set I to contain I_γ as well as all indices obtained by induction for $P[1, i-1]$ and K_1 and for $P[i+1, l]$ and K_2 , we would have that for all w , $EP(w) \in K_1\gamma K_2$ if and only if $EP[I](w) \in K_1\gamma K_2$, that is $EP(w)$ where only the elements at indices in I have been kept.

The intuition behind the proof when $k < 2$ is essentially the same, but without induction.

We now spell out the details of the proof, starting with the inductive step.

Inductive step. Let $k \geq 2$ and assume the lemma proved for all $k' < k$. Let n be a natural number, P a program over M of range n and length l and any language K over M in SUM_k . If $K \in \mathit{SUM}_{k-1}$, by the inductive hypothesis, we are done. Otherwise, by definition, $K = K_1\gamma K_2$ for $\gamma \in M$ and some languages $K_1 \in \mathit{SUM}_{k_1}$ and $K_2 \in \mathit{SUM}_{k_2}$ over M with $k_1 + k_2 = k - 1$. Moreover either γ does not occur in any of the words of K_1 or it does not occur in any of the words of K_2 . We only treat the case where γ does not appear in any of the words in K_1 . The other case is treated similarly by symmetry.

Observe that when $n = 0$, we necessarily have $P = \varepsilon$, so that the lemma is trivially proven in that case. So we now assume $n > 0$.

For each $1 \leq p \leq n$ and each $a \in \Sigma$ consider within the sequence of instructions of P the first instruction of the form (p, f) with $f(a) = \gamma$, if it exists. We let I_γ be the set of indices of these instructions for all a and p . Notice that the size of I_γ is in $\mathcal{O}(n)$.

For all $i \in I_\gamma$, we let $J_{i,1}$ be the set of indices of the instructions within $P[1, i-1]$ appearing in its subprogram obtained by induction for $P[1, i-1]$ and K_1 , and $J_{i,2}$ be the same for $P[i+1, l]$ and K_2 .

We now let I be the union of I_γ and $J_{i,1}$ and $J'_{i,2} = \{j+i \mid j \in J_{i,2}\}$ for all $i \in I_\gamma$. We claim that $Q = P[I]$ has the desired properties.

First notice that by induction the sizes of $J_{i,1}$ and $J'_{i,2}$ for all $i \in I_\gamma$ are in $\mathcal{O}(n^{\max\{k-1, 1\}}) = \mathcal{O}(n^{k-1})$ and because the size of I_γ is linear in n , the size of I is in $\mathcal{O}(n^k) = \mathcal{O}(n^{\max\{k, 1\}})$ as required.

Let Q' be a subprogram of P that has Q as a subprogram: it means that there exists some set $I' \subseteq [l]$ containing I such that $Q' = P[I']$.

Now take $w \in \Sigma^n$.

Assume now that $EP(w) \in K$. Let i be the position in $EP(w)$ of label γ witnessing the membership in K . Let (p_i, f_i) be the corresponding instruction of P . In particular we have that $f_i(w_{p_i}) = \gamma$. Because γ does not occur in any word of K_1 , for all $j < i$ such that $p_j = p_i$ we cannot have $f_j(w_{p_j}) = \gamma$. Hence $i \in I_\gamma$. By induction we have that $EP[1, i-1][J](w) \in K_1$ for any set $J \subseteq [i-1]$ containing $J_{i,1}$ and $EP[i+1, l][J](w) \in K_2$

for any set $J \subseteq [l - i]$ containing $J_{i,2}$. Hence, if we set $I'_1 = \{j \in I' \mid j < i\}$ as the subset of I' of elements less than i and $I'_2 = \{j - i \in I' \mid j > i\}$ as the subset of I' of elements greater than i translated by $-i$, we have

$$EP[I'](w) = EP[1, i - 1][I'_1](w) \gamma EP[i + 1, l][I'_2](w) \in K_1 \gamma K_2 = K$$

as desired.

Assume finally that $EP[I'](w) \in K$. Let i be the index in I' whose instruction provides the letter γ witnessing the fact that $EP[I'](w) \in K$. This means that if we set $I'_1 = \{j \in I' \mid j < i\}$ as the subset of I' of elements less than i and $I'_2 = \{j - i \in I' \mid j > i\}$ as the subset of I' of elements greater than i translated by $-i$, we have $EP[I'](w) = EP[1, i - 1][I'_1](w) \gamma EP[i + 1, l][I'_2](w)$ with $EP[1, i - 1][I'_1](w) \in K_1$ and $EP[i + 1, l][I'_2](w) \in K_2$. If $i \in I_\gamma$, then it means that $I'_1 \subseteq [i - 1]$ contains $J_{i,1}$ and that $I'_2 \subseteq [l - i]$ contains $J_{i,2}$ by construction, so that, by induction,

$$EP(w) = EP[1, i - 1](w) \gamma EP[i + 1, l](w) \in K_1 \gamma K_2 = K .$$

If not this shows that there is an instruction (p_j, f_j) with $j < i$, $j \in I'$, $p_j = p_i$ and $f_j(w_{p_j}) = \gamma$. But that would contradict the fact that γ cannot occur in K_1 . So we have $EP(w) \in K$ as desired.

Base case. There are two subcases to consider.

Subcase $k = 1$. Let n be a natural number, P a program over M of range n and length l and any language K over M in SUM_1 .

If $K \in SUM_0$, we can conclude by referring to the subcase $k = 0$.

Otherwise $K = A_1^* \gamma A_2^*$ for $\gamma \in M$ and some finite alphabets $A_1 \subseteq M$ and $A_2 \subseteq M$. Moreover either $\gamma \notin A_1$ or $\gamma \notin A_2$. We only treat the case where γ does not belong to A_1 , the other case is treated similarly by symmetry.

We use the same idea as in the inductive step.

Observe that when $n = 0$, we necessarily have $P = \varepsilon$, so that the lemma is trivially proven in that case. So we now assume $n > 0$.

For each $1 \leq p \leq n$, each $\alpha \in M$ and $a \in \Sigma$ consider within the sequence of instructions of P the first and last instruction of the form (p, f) with $f(a) = \alpha$, if they exist. We let I be the set of indices of these instructions for all a, α and p . Notice that the size of I is in $O(n) = O(n^{\max\{k, 1\}})$.

We claim that $Q = P[I]$ has the desired properties. We just showed that it has the required length.

Let Q' be a subprogram of P that has Q as a subprogram: it means that there exists some set $I' \subseteq [l]$ containing I such that $Q' = P[I']$.

Take $w \in \Sigma^n$.

Assume now that $EP(w) \in K$. Let i be the position in $EP(w)$ of label γ witnessing the membership in K . Let (p_i, f_i) be the corresponding instruction of P . In particular we have that $f_i(w_{p_i}) = \gamma$ and this is the γ witnessing the membership in K . Because $\gamma \notin A_1$, for all $j < i$ such that $p_j = p_i$ we cannot have $f_j(w_{p_j}) = \gamma$. Hence $i \in I \subseteq I'$. From $EP[1, i - 1](w) \in A_1^*$ and $EP[i + 1, l](w) \in A_2^*$ it follows that $EP[I' \cap [1, i - 1]](w) \in A_1^*$ and $EP[I' \cap [i + 1, l]](w) \in A_2^*$, showing that $EP[I'](w) = EP[I' \cap [1, i - 1]](w) \gamma EP[I' \cap [i + 1, l]](w) \in K$ as desired.

Assume finally that $EP[I'](w) \in K$. Let i be the index in I' whose instruction provides the letter γ witnessing the fact that $EP[I'](w) \in K$. This means that $EP[I' \cap [1, i - 1]](w) \in$

A_1^* and $EP[I' \cap \llbracket i+1, l \rrbracket](w) \in A_2^*$. If there is an instruction (p_j, f_j) , with $j < i$ and $f_j(w_{p_j}) \notin A_1$ then either $j \in I'$ and we get a direct contradiction with the fact that $EP[I' \cap \llbracket 1, i-1 \rrbracket](w) \in A_1^*$, or $j \notin I'$ and we get a smaller $j' \in I \subseteq I'$ with the same property, contradicting again the fact that $EP[I' \cap \llbracket 1, i-1 \rrbracket](w) \in A_1^*$. Hence for all $j < i$, $f_j(w_{p_j}) \in A_1$. By symmetry we have that for all $j > i$, $f_j(w_{p_j}) \in A_2$, showing that $EP(w) \in A_1^* \gamma A_2^* = K$ as desired.

Subcase $k = 0$. Let n be a natural number, P a program over M of range n and length l and any language K over M in SUM_0 .

Then $K = A^*$ for some finite alphabet $A \subseteq M$.

We again use the same idea as before.

Observe that when $n = 0$, we necessarily have $P = \varepsilon$, so that the lemma is trivially proven in that case. So we now assume $n > 0$.

For each $1 \leq p \leq n$, each $\alpha \in M$ and $a \in \Sigma$ consider within the sequence of instructions of P the first instruction of the form (p, f) with $f(a) = \alpha$, if it exists. We let I be the set of indices of these instructions for all a, α and p . Notice that the size of I is in $O(n) = O(n^{\max\{k, 1\}})$.

We claim that $Q = P[I]$ has the desired properties. We just showed that it has the required length.

Let Q' be a subprogram of P that has Q as a subprogram: it means that there exists some set $I' \subseteq [l]$ containing I such that $Q' = P[I']$.

Take $w \in \Sigma^n$.

Assume now that $EP(w) \in K$. As $EP[I'](w)$ is a subword of $EP(w)$, it follows directly that $EP[I'](w) \in A^* = K$ as desired.

Assume finally that $EP[I'](w) \in K$. If there is an instruction (p_j, f_j) , with $j \in [l]$ and $f_j(w_{p_j}) \notin A$ then either $j \in I'$ and we get a direct contradiction with the fact that $EP[I'](w) \in A^* = K$, or $j \notin I'$ and we get a smaller $j' \in I \subseteq I'$ with the same property, contradicting again the fact that $EP[I'](w) \in A^* = K$. Hence for all $j \in [l]$, $f_j(w_{p_j}) \in A$, showing that $EP(w) \in A^* = K$ as desired. \square

6. CONCLUSION

We introduced a notion of tameness, particularly relevant to the analysis of programs over monoids from “small” varieties. The main source of interest in tameness is Proposition 3.11, stating that a variety of monoids \mathbf{V} is tame if and only if the class of regular languages p -recognized by programs over monoids from \mathbf{V} is included in the class $\mathcal{L}(\mathbf{QEV})$. A first question that arises is for which \mathbf{V} those two classes of regular languages are equal. We could not rule out the possibility that for some tame \mathbf{V} , $\mathcal{L}(\mathbf{QEV}) \setminus \mathcal{P}(\mathbf{V}) \neq \emptyset$. We conjecture that if \mathbf{V} is local, abusing notation, $\mathbf{QEV} = \mathbf{EV} * \mathbf{Mod}$, by analogy with \mathbf{QV} equating $\mathbf{V} * \mathbf{Mod}$ in that case; as $\mathcal{L}(\mathbf{EV} * \mathbf{Mod}) \subseteq \mathcal{P}(\mathbf{V})$ holds unconditionally, under our conjecture $\mathcal{P}(\mathbf{V}) \cap \mathbf{Reg} = \mathcal{L}(\mathbf{QEV})$ would hold for tame local varieties \mathbf{V} .

Concretely, we have obtained the technical result that **DA** is a tame variety. We have given **A** and **Com** as further examples of tame varieties. Our proof that **A** is tame needed the fact that $\mathbf{MOD}_m \notin \mathbf{AC}^0$ for all $m \geq 2$, so it would be interesting to prove **A** tame purely algebraically. But tameness of **A** implies $\mathbf{MOD}_2 \notin \mathbf{AC}^0$ by Proposition 3.11, confronting us again with the holy grail alluded to in the introduction, certainly a challenging barrier to overcome.

By contrast, we have shown that \mathbf{J} is not tame. So programs over monoids from \mathbf{J} p -recognize “more regular languages than expected”. A natural question to ask is what these regular languages in $\mathcal{P}(\mathbf{J})$ are. Partial results in that direction were obtained in [Gro20].

To conclude we should add, in fairness, that the progress reported here does not in any obvious way bring us closer to major NC^1 complexity subclasses separations. Our concrete contributions here largely concern $\mathcal{P}(\mathbf{DA})$ and $\mathcal{P}(\mathbf{J})$, classes that are well within AC^0 . But this work does uncover new ways in which a program can or cannot circumvent the limitations imposed by the underlying monoid algebraic structure available to it.

REFERENCES

- [Ajt83] Miklós Ajtai. Σ_1^1 -formulae on finite structures. In *Ann. Pure and Appl. Logic*, volume 24, pages 1–48, 1983.
- [Alm96] Jorge Almeida. A syntactical proof of locality of DA. *Int. J. of Algebra and Computation (IJAC)*, 6(2):165–178, 1996.
- [Bar89] David A. Mix Barrington. Bounded-width polynomial-size branching programs recognize exactly those languages in NC^1 . *J. Comput. Syst. Sci.*, 38(1):150–164, 1989.
- [BCST92] David A. Mix Barrington, Kevin J. Compton, Howard Straubing, and Denis Thérien. Regular languages in NC^1 . *J. Comput. Syst. Sci.*, 44(3):478–499, 1992.
- [BS95] David A. Mix Barrington and Howard Straubing. Superlinear lower bounds for bounded-width branching programs. *J. Comput. Syst. Sci.*, 50(3):374–381, 1995.
- [BST90] David A. Mix Barrington, Howard Straubing, and Denis Thérien. Non-uniform automata over groups. *Inf. Comput.*, 89(2):109–132, 1990.
- [BT88] David A. Mix Barrington and Denis Thérien. Finite monoids and the fine structure of NC^1 . *J. ACM*, 35(4):941–952, 1988.
- [CPS06a] Laura Chaubard, Jean-Éric Pin, and Howard Straubing. Actions, wreath products of C -varieties and concatenation product. *Theoretical Computer Science*, 356(1-2):73–89, 2006.
- [CPS06b] Laura Chaubard, Jean-Eric Pin, and Howard Straubing. First order formulas with modular predicates. In *21th IEEE Symposium on Logic in Computer Science (LICS 2006), 12-15 August 2006, Seattle, WA, USA, Proceedings*, pages 211–220, 2006.
- [Dar14] Luc Dartois. *Méthodes algébriques pour la théorie des automates*. PhD thesis, Université Paris Diderot, Paris, 2014.
- [DP13] Luc Dartois and Charles Paperman. Two-variable first order logic with modular predicates over words. In *30th International Symposium on Theoretical Aspects of Computer Science, STACS 2013, February 27 - March 2, 2013, Kiel, Germany*, pages 329–340, 2013.
- [DP14] Luc Dartois and Charles Paperman. Adding modular predicates. *CoRR*, abs/1401.6576, 2014.
- [Eil74] Samuel Eilenberg. *Automata, Languages, and Machines*, volume A. Academic Press, New York, 1974.
- [Eil76] Samuel Eilenberg. *Automata, Languages, and Machines*, volume B. Academic Press, New York, 1976.
- [FSS84] Merrick L. Furst, James B. Saxe, and Michael Sipser. Parity, circuits, and the polynomial-time hierarchy. *Mathematical Systems Theory*, 17(1):13–27, 1984.
- [GMS17] Nathan Grosshans, Pierre McKenzie, and Luc Segoufin. The power of programs over monoids in DA. In *42nd International Symposium on Mathematical Foundations of Computer Science, MFCS 2017, August 21-25, 2017 - Aalborg, Denmark*, pages 2:1–2:20, 2017.
- [Gro18] Nathan Grosshans. *The limits of Nečiporuk’s method and the power of programs over monoids taken from small varieties of finite monoids*. PhD thesis, University of Paris-Saclay, France, 2018.
- [Gro20] Nathan Grosshans. The power of programs over monoids in \mathbf{J} . In *Language and Automata Theory and Applications - 14th International Conference, LATA 2020, Milan, Italy, March 4-6, 2020, Proceedings*, pages 315–327, 2020.
- [GT03] Ricard Gavaldà and Denis Thérien. Algebraic characterizations of small classes of boolean functions. In *STACS 2003, 20th Annual Symposium on Theoretical Aspects of Computer Science, Berlin, Germany, February 27 - March 1, 2003, Proceedings*, pages 331–342, 2003.

- [Häs86] Johan Håstad. Almost optimal lower bounds for small depth circuits. In *Proceedings of the 18th Annual ACM Symposium on Theory of Computing, May 28-30, 1986, Berkeley, California, USA*, pages 6–20, 1986.
- [KR65] Kenneth Krohn and John L. Rhodes. Algebraic theory of machines. I. Prime decomposition theorem for finite semigroups and machines. In *Trans. Amer. Math. Soc.*, volume 116, pages 450–464, 1965.
- [MPT91] Pierre McKenzie, Pierre Péladeau, and Denis Thérien. NC^1 : The automata-theoretic viewpoint. *Computational Complexity*, 1:330–359, 1991.
- [MPT00] Alexis Maciel, Pierre Péladeau, and Denis Thérien. Programs over semigroups of dot-depth one. *Theor. Comput. Sci.*, 245(1):135–148, 2000.
- [MR65] Ward D. Maurer and John L. Rhodes. A property of finite simple non-abelian groups. In *Amer. Math. Soc.*, volume 16, pages 552–554, 1965.
- [Pap14] Charles Paperman. *Circuits booléens, prédicats modulaires et langages réguliers*. PhD thesis, Université Paris Diderot, Paris, 2014.
- [Pél90] Pierre Péladeau. *Classes de circuits booléens et variétés de monoïdes*. PhD thesis, Université Pierre-et-Marie-Curie (Paris-VI), Paris, France, 1990.
- [Pin86] Jean-Éric Pin. *Varieties of formal languages*. North Oxford, London and Plenum, New-York, 1986. (Traduction de Variétés de langages formels).
- [PS05] Jean-Éric Pin and Howard Straubing. Some results on C -varieties. *RAIRO-Theor. Inf. Appl.*, 39(1):239–262, 2005.
- [PST97] Pierre Péladeau, Howard Straubing, and Denis Thérien. Finite semigroup varieties defined by programs. *Theor. Comput. Sci.*, 180(1-2):325–339, 1997.
- [Raz87] Alexander A. Razborov. Lower bounds on the size of bounded depth circuits over a complete basis with logical addition. *Mathematical Notes*, 41(4):333–338, 1987.
- [Rei82] Jan Reiterman. The Birkhoff theorem for finite algebras. *Algebra Universalis*, 14(1):1–10, 1982.
- [Sim75] Imre Simon. Piecewise testable events. In *Automata Theory and Formal Languages, 2nd GI Conference, Kaiserslautern, May 20-23, 1975*, pages 214–222, 1975.
- [Smo87] Roman Smolensky. Algebraic methods in the theory of lower bounds for Boolean circuit complexity. In *Proceedings of the 19th Annual ACM Symposium on Theory of Computing, 1987, New York, New York, USA*, pages 77–82, 1987.
- [Str94] Howard Straubing. *Finite Automata, Formal Logic, and Circuit Complexity*. Birkhauser, Boston, 1994.
- [Str02] Howard Straubing. On logical descriptions of regular languages. In *LATIN 2002: Theoretical Informatics, 5th Latin American Symposium, Cancun, Mexico, April 3-6, 2002, Proceedings*, pages 528–538, 2002.
- [Tes03] Pascal Tesson. *Computational Complexity Questions Related to Finite Monoids and Semigroups*. PhD thesis, McGill University, Montreal, Canada, 2003.
- [Til87] Bret Tilson. Categories as algebra: An essential ingredient in the theory of monoids. *J. of Pure and Applied Algebra*, 48(1-2), 1987.
- [TT01] Pascal Tesson and Denis Thérien. The computing power of programs over finite monoids. *J. Autom. Lang. Comb.*, 7(2):247–258, November 2001.
- [TT02] Pascal Tesson and Denis Thérien. Diamonds are forever: the variety DA. *Semigroups, algorithms, automata and languages*, 1:475–500, 2002.