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# **TWG1 report: Calculus and Analysis**

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## **INTRODUCTION**

In the Thematic Working Group about Calculus and Analysis (TWG1) 11 papers were presented in two sessions, followed by discussions in subgroups; between 20 and 27 people participated in each presentation and discussion session. Focusing on the topics and the didactical issues raised by the authors in this group, we observed that both classical and new issues were addressed as compared with the main themes discussed in previous conferences (Trigueros, Bridoux, O'Shea and Branchetti, in press). Thus, we organized the papers into two groups, the first concerning the school-university transition and involving basic concepts of Calculus and the second concerning advanced mathematical topics and new research-based approaches to their teaching in advanced courses, including links with more basic concepts. Within each group we identified two subgroups of papers about similar topics and we separated the participants into two rooms, assigned by us to authors and freely chosen by every virtual participant, so to encourage fruitful discussions about issues of common interest between the authors of the presented papers and the participants.

## **DISCUSSIONS DURING THE CONFERENCES**

### **Group 1**

In a first analysis, two common aspects of the 5 papers concerning basic concepts of Calculus were found. The first subgroup's studies were related to basic key concepts at the transition from secondary school to university (3 papers), in particular: functions, limits and sequences (M. Flores Gonzales et al., F. Khalloufi-Mouha) and different approaches to the definition of continuity, using a topological approach (L. Branchetti et al.). The second subgroup focused on integrals (2 papers), one concerned with Riemann sums (I. Akrouti) and the other with the introduction of a different approach to the teaching of integrals using Klein's plan B (G. Planchon et al.).

The first subgroup faced discussion topics related to the transition to university, including a perspective on advanced courses. Discussion initiated the provoking question: Whether and, in any case, why is it meaningful to make a distinction between functions and sequences, since sequences are functions? An interesting debate about choices in different countries and the importance of discussing this distinction with respect to continuous functions emerged. Then discussion turned to an inquiry about a

study on recursive sequences and how students compute limits using general theorems on sequences and limits and their relation to the teaching of Calculus. Participants considered the topic as pertinent for the community. Attention turned to trigonometric functions, and the differences involved in their presentation when introduced in courses either at the transition to university or at the transition to advanced mathematics, going from angles/arcs to power series and differential equations. The importance of relating both definitions was underlined as well as the need for further studies.

The notion of local routine, such as routines to determine continuity and differentiability at a given point of simple trigonometric functions or their composition was used in one paper in a way that was not convincing for all the participants, so the question: “Is using limits of a function and its algebraic expression to investigate its continuity at a point “local”?” was discussed and the group stressed the need to check these terms carefully to avoid misinterpretations.

Discussion of different theoretical approaches to the introduction of Calculus’ basic concepts, in particular the topological approach, brought about the observation that it allows to exploit the graphical register with a good connection to definitions in some tasks, so students might find it helpful. It was also observed that it would be interesting, but not easy, to find a university course where analytical and topological definitions are mixed. The need to adopt a sort of internal “interdisciplinarity” within the different branches of mathematics and in teaching also emerged and it was recognized that nowadays there are too many barriers between different fields.

In the second subgroup, the discussion focused on the teaching of integrals. The notion of integral and students’ understanding of the fundamental theorem of Analysis and its relation to the notion of area was discussed together with the need of a new possible praxeology for integration. Interesting issues were addressed such as students’ approaches to tasks about integrals and its relation to didactical contract; students’ conceptions of integral, with a particular attention to its meaning (the role of the concept of area and measures in understanding integration) and the need to find ways to relate these ideas to their teaching and to formalize students’ ideas.

The sub-group discussed students’ tendency to conceive “calculate” as “to find the exact value” and to think of it as different from “approximate” in some task. Such an issue was examined in terms of its possibility to lead to different approaches to tasks depending on how they are formulated and also on the didactical contract: either looking for an exact value or expressing the integral as a limit of Riemann sums with an unknown “ $f$ ”. Also, those cases where functions are presented only by their graph and the influence of students’ possibility to use graphing calculators or computers during a test were addressed. A relevant discussion took place about the nature of the four conceptions of integral (primitive, area, sum, approximation), participants posed the question: are they concepts or processes? Another issue discussed in the sub-group concerned the praxeologies to teach integrals that should be presented to prospective

teachers to connect students' ideas to the different theories of integration (Riemann, Lebesgue, etc.) at the school-university transition. Another relevant point was: How can the formalization of the area be concretely handled with university students? As measures (area) are only formalized in a measure theory course at university, an intermediate theory to establish a link with the secondary school integral should be introduced. Should it be focused more on proposing a new didactical engineering or on a strong mathematical basis of integration? In response, it emerged the need to provide students with new theoretically grounded praxeologies starting from their ideas and followed by a reconceptualization of basic concepts to introduce the concept of area, while making the theory behind them explicit.

## **Group 2**

In one room, papers related to the role of representations in the transition to university mathematics and to more advanced mathematical courses were selected (3 papers): the use of local approximation tools to study functions (F. Belaj Amor); the use of diagrams in proving tasks (K. Gallagher & N. Engelke Infante) and a review concerning research on the calculus of two variable functions (R. Martínez-Planell & M. Trigueros). The other room focused on the teaching of advanced mathematical concepts and the role of intuition in this endeavour (3 papers): students' concept images and example spaces concerning continuity and differentiability (E. Lankeit & R. Biehler); the learning potential of including advanced mathematical ideas into more elementary courses (R. Hochmuth) and intuition and discourse in the teaching of the complex path integral (E. Hanke). The main ideas of the discussions follow.

In the first group participants discussed a difficulty found in the transition to university, namely the fact that when students are taking the first Calculus courses at university, they tend to continue using what they learnt in secondary school instead of new methods. Participants agreed that this may be possibly due to the need to provide opportunities for students to reflect on relations and differences between methods learnt before and the new ones, since this is not frequently addressed in most university courses. Students' difficulties with Taylor series were described as due to a rupture occurring between the ideas of derivative as the slope of the tangent at a point and derivative as a way to find a linear approximation to a curve around a point. The fact that these two ideas remain compartmentalized may be a major cause of students' difficulties. It emerged the need to clarify the role of Taylor series in problem solving when it is introduced.

When confronted with new abstract mathematical concepts, university students may face a new transition. According to research findings, the transition from one variable to two variables calculus involves a reconstruction on the part of the students of their previous knowledge that is new to them to take into account changes and similarities involved when learning new types of functions. Two ideas were brought forward firstly this transition could be approached using different registers of semiotic representation,

and secondly, pedagogical means such as teamwork might be useful to facilitate this transition. A question was raised about the need of reconstructing previous knowledge when introducing multi-variable functions, or other functions where mathematical objects are impossible to draw or imagine. It led to a general discussion about knowledge reconstruction. From this discussion it emerged that “very simple” ideas, such as the use of the same symbolism for different mathematical objects, need to be deeply discussed to grasp the meaning of those abstract concepts. It was also commented that, for example, discussing with students how to define the slope could be a way for them to understand that “direction” plays a role in the construction of derivatives of multi-variable functions. Discussion on how some specific notions change when considered in different contexts led to pointing out the need to give students opportunities to represent them in different registers, or to imagine them as a way to give meaning to those objects.

Another important topic was the design of new strategies to teach advanced mathematics: the participants discussed what choices could be more helpful to introduce students to advanced mathematical concepts. The development of students’ intuition and its possible use in proofs presented by an author was considered very interesting by participants. The use of diagrams, letting students manipulate and discuss them with their peers, was debated and questions about how to trace students’ gestures as they work emerged together with its relation to the development of intuition. Responses to this question lead to a teaching strategy based on these ideas which offers new possibilities to discover how students imagine things and develop their ideas even when they are asked to work on difficult topology problems and proofs. Students’ possibility to explore and discuss while they draw and interact with diagrams received a lot of attention and helped to understand its role in developing key ideas as ideas that convince the prover that the statement is true, but which the prover must also be able to translate and use into a written proof. The importance of letting students try, and possibly fail, if topics are not easily represented using diagrams, was also discussed. Participants considered that it might help in gaining valuable information about mathematical practices, such as why certain objects, processes and relationships cannot be represented well in a drawing.

In the second subgroup, students’ repertoire of studied functions was considered to be generally limited to certain classes, so students develop from their courses a concept image about function which includes mostly smooth functions. Approaching other functions, such as those that are “locally flat” was considered to provide opportunities to foster a better approach to functions’ properties, particularly their differentiability. Questions such as: What happens if the function studied is not that regular? What can we tell about continuity and differentiability in those cases? were raised and suggestions to use some ideas coming from historical studies about partial differentiability implying differentiability, once the latter is defined in a non-standard way, were found helpful in developing students’ intuition about difficult results.

As mathematical concepts become increasingly abstract, they become difficult to visualize and mathematical intuition is not readily available. Discussion about what is intuitive in courses such as complex analysis was considered by participants worthy of attention on the part of researchers. Approaches to intuition like that of Fischbein were recalled, but other perspectives, as the commognitive discursive approach, were suggested as useful to both investigate the nature of intuition and provide tools for teachers to introduce students to difficult ideas. An interesting question arose about how to draw the line between rigorous and intuitive discourses. The use of intuitive arguments to make sense of mathematics was pointed out but comments about the difficulty to define an intuitive discourse also appeared. Participants considered the need to use a variety of approaches to abstract concepts so that students can develop meaningful ideas based on visualization, intuition and creativity. A way proposed to foster students' development of intuition for advanced mathematical ideas consisted in designing tasks that could give students a glimpse of advanced mathematics in basic courses. The group discussed what type of tasks could be used, and how they could be presented to students. Approximation was suggested as a good topic for this purpose since it can be related to questions about the smoothness of a function or be used to discuss continuity for functions where continuity is a real issue.

## **FINAL REFLECTION**

Papers presented during the conference were original in terms of the topics studied. They provided interesting aspects related to the teaching and learning of Calculus and Analysis that opened rich discussions on new topics that need more attention from researchers. Common topics in discussions covered the need to relate basic and advanced concepts through a reconceptualization of basic ideas when introducing advanced topics and to present advanced ideas in basic courses to anticipate meaningful concepts. This may require innovative approaches and the use of novel representations, such as diagrams, that can offer new ways to foster a diversity of teaching strategies to develop a deeper understanding of Analysis and abstract mathematical thinking. Moreover, even if the formal aspects were taken into account, as usual, by several authors, in this conference the issues of visualization, intuition and representation and the need to start from the students' ideas became crucial and were addressed during all the group discussions. It was considered that developing students' creativity and intuition is important also in the learning of advanced topics.

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