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The step from tertiary to secondary education in mathematics. In search of a shared paradigm

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Whereas there is a widely accepted epistemological model of mathematics in the community of mathematicians, there is no such a thing for the teachers of mathematics in secondary education. This makes problematic the transition from tertiary (as students) to secondary education (as teachers). In this work we analyse this didactic phenomenon.

Keywords: teachers' and students' practices at university level; transition to and across university mathematics; epistemological model of mathematics; teaching ends; mathematical praxeology to be taught.

INTRODUCTION

Didactic phenomena concerning the step from secondary education to university level have drawn the interest of researchers for many years. Some thematic working groups of the conferences INDRUM, CERME and ICME, and even some plenary talks, have addressed it in different editions. However, the transition from tertiary to secondary education has been much disregarded. But, first of all, does such a transition exist? Indeed, it is the one experienced by those students at university when they become teachers at secondary education. A possible explanation is that one might naively believe that didactic obstacles only exist when you move forwards in mathematics to meet 'more sophisticated' mathematics, but not when you move backwards, revisiting 'more elementary' mathematics. Another explanation for this lack of attention is that, perhaps, transitions are implicitly assumed not to entail a change of position within the institution, but only a change of institution. It is true that the institutional position changes: they go from the position of students at university to the position of teachers at secondary education. But still, this change of position does not make the transition less problematic and not deserving attention.

Concerning possible obstacles for the transition from the tertiary to the secondary level, there are many differences in the mathematical activity. For instance, in the tertiary level the students typically learn the strongest technique for a given type of task (e.g. Lagrange multipliers for optimisation problems), and this technique cannot be directly translated to the secondary level. In this work we do not try to give a thorough account of all those obstacles. Instead, our goal is to present an analysis of one of the major problems of this transition: the lack of a 'solid' epistemological

model of mathematics to support the mathematics to be taught at secondary level. As we will explain below, the epistemological model of mathematics accepted among the community of mathematicians cannot be directly transferred to secondary education. Hence, future teachers at secondary level need an alternative epistemological model of mathematics, which is not provided in a standard way by our society.

THE ANTHROPOLOGICAL THEORY OF THE DIDACTICS

The theoretical framework of this contribution is the anthropological theory of the didactics (ATD). In this first section we will introduce some basic notions to be used later on. For more information about ATD, the reader is invited to read (Chevallard, 1999, 2007) and (Gascón & Nicolás, 2019).

Praxeologies

According to the ATD, didactics of mathematics is devoted to the analysis of the genesis and diffusion of mathematical knowledge, regarded as an output of intentional actions. ATD has the notion of *praxeology* for the simultaneous analysis of the intentional actions and the resulting pieces of knowledge. Notice that, whenever there is an intentional action, there is, by definition, an *agent*, that is to say, someone trying to carry out this action. A *praxeology* is made of two interrelated components: the *praxis* and the *logos*. In turn, the *praxis* is made of:

- a certain set of *types of tasks* the agent wants to deal with,
- a certain set of *techniques*, which are the ways the agent has in order to deal with those types of tasks.

The *logos* is made of:

- the *technology*, which is devoted to describe the techniques, to show their usefulness, to delimit the scope of validity, and to study their economy (how much effort it takes to use those techniques) and reliability,
- the *theory*, which includes an ontological description of the region of the world involved in the types of tasks, the techniques and the technology (that is, which are the objects or beings under consideration, and which are the relationships between them), but also a normative vision (which should be the goals of my intentional actions, which should be the kind of techniques employed, etc.).

Personal and institutional praxeologies

As we said before, the notion of praxeology helps to describe individual intentional actions. But it is also used to deal with *institutional intentional actions*. Within ATD, the concept of *institution* is understood as a set of constitutive rules that:

- define and determine positions and relationships in a social scheme fixed in a conventional way,
- determine rights and duties, permissions and prohibitions, rewards and penalties.

Some examples of institutions: any regulated game (for instance, chess), matrimony, nationality, procedural law, languages, scientific theories, the teaching of mathematics in a faculty of mathematics, the teaching of mathematics at Secondary Education.

Typically, the members of an institution, due to the fact that they are members of this institution, are the agents of individual intentional actions which share relevant features, and so we can speak of ‘intentional actions’ of the institution. These institutional intentional actions give rise to the so-called *institutional praxeologies*, as opposed to *individual* or personal *praxeologies*. Due to its generality, it is easier to describe institutional praxeologies than personal ones.

Mathematical praxeologies and the epistemological model of mathematics

Mathematical praxeologies are those praxeologies which describe, at once, both the mathematical activity and the output of this activity (the corresponding mathematical works). Since every mathematical praxeology is based on a *theory*, every mathematical praxeology entails (perhaps implicitly) a certain ontological description and a certain normative vision of mathematics. Therefore, we could say that every mathematical praxeology assumes a certain *epistemological model of mathematics*. This terminology is reasonable because, among other things, the *theory* of a mathematical praxeology determines which are the *basics objects* on which mathematics are built, and which are the kind of *accepted arguments* to verify propositions. In other words, the *theory* of a mathematical praxeology provides an account (a *logos*) of how knowledge (*episteme*) is achieved in mathematics.

Mathematical praxeologies to be taught and mathematical praxeologies for teaching

Cirade made in (2006) a distinction between *mathematical praxeologies to be taught* (MPTBT) and *mathematical praxeologies for teaching* (MPFT). Given an educational institution, \mathfrak{S} , the *mathematical praxeologies to be taught* in \mathfrak{S} are those mathematical praxeologies that teachers in \mathfrak{S} plan to teach. Of course, those MPTBT are chosen after considering certain questions, getting certain conclusions, etc. This activity devoted to decide what are going to be the MPTBT constitute by itself a different kind of praxeologies, the so-called *mathematical praxeologies for teaching*.

Let now \mathfrak{S} be a faculty of mathematics. Let us consider a possible MPTBT:

- Type of task: given a certain function, $f : [a, b] \subseteq \mathbb{R} \longrightarrow \mathbb{R}$, and given a real number $y_0 \in \mathbb{R}$, check whether there exists $x_0 \in [a, b]$ such that $f(x_0) = y_0$.
- Technique: If $y_0 = f(a)$ or $y_0 = f(b)$, then we can take $x_0 = a$ or $x_0 = b$, respectively. Otherwise, we can use Bolzano’s theorem. It says that, in the situation above, if f is continuous in $[a, b]$ and y_0 is between $f(a)$ and $f(b)$, then there exists such an x_0 .

- Technology: Among other things, technology is responsible for proving the validity of the techniques employed. Now we prove Bolzano's theorem, underlying the beginning of the main parts of the proof. First notice that we can assume $f(a) < f(b)$ without loss of generality. Now, we are choosing the candidate for our required x_0 . For this, consider the set $X = \{x \in [a, b] \mid f(x) \leq y_0\}$. It is not empty, because $a \in X$. Also, it has an upper bound, for instance b . Then, due to the *least upper bound property* satisfied by the real numbers, there exists a least upper bound x_0 of X . Notice that if we prove that $x_0 \in X$ we are done. Indeed, if $x_0 \in X$ then, by definition of X , we have that $f(x_0) \leq y_0$. In this case, equality is guaranteed. Notice that, if $f(x_0) < y_0$, then there exists a neighbourhood $\mathcal{V} = (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$ of $f(x_0)$ such that $f(x_0) + \varepsilon < y_0$. By continuity of f in x_0 , there exists a neighbourhood $\mathcal{U} = (x_0 - \delta, x_0 + \delta)$ of x_0 which is mapped inside \mathcal{V} by f , which, in turn, implies that x_0 is not an upper bound of X , because there are numbers greater than x_0 (for instance $x_0 + \delta/2$) which still are in X (because $f(x_0 + \delta/2) < f(x_0) + \varepsilon < y_0$). Therefore, it only remains to prove that $x_0 \in X$. For this, we will use a *reductio ad absurdum* argument. If $x_0 \notin X$ then $y_0 < f(x_0)$. Then there exists a neighbourhood \mathcal{V} of $f(x_0)$ 'above' y_0 . By continuity of f in x_0 , there exists a neighbourhood \mathcal{U} of x_0 which is mapped to \mathcal{V} by f , which, in turn, implies that there exists a neighbourhood \mathcal{U} of x_0 such that, for every $x \in \mathcal{U}$, $f(x) > y_0$. Therefore $\mathcal{U} \cap X = \emptyset$. But it is not difficult to prove that the fact that x_0 is a *least upper bound* implies that every neighbourhood of x_0 has no empty intersection with X . Whence the absurdum.

- Theory: there are a lot of theoretical elements supporting the previous technology. For instance, there are some properties taken for granted, for example the *least upper bound property* of real numbers, which says that a non-empty set of real numbers with an upper bound always has a least upper bound. There are also definitions (that of upper bound, least upper bound, neighbourhood, function, continuity). Notice that, in the definitions, there must be always a clear separation between the logical terms ("for every", "there exists", "if ... then ...", "and", "or") and the non-logical terms ("real number", "function", "less than or equal"). And, still, the non-logical terms can always be analysed in terms of logical terms and, eventually, only three non-logical terms (i.e., \in , $=$ and \emptyset). But there is another deep feature, which is closely related to the aforementioned logical analysis of definitions. Namely, the arguments used in the technology have to be *deductive*.

Later we will take a closer look at the feature of deductive arguments. But first let us sketch some elements of possible mathematical praxeologies for teaching underlying the previous mathematical praxeology to be taught.

- Tasks: Which proof *should* we choose for Bolzano's theorem?
- Technique: One possible way is the one chosen above, which uses the least upper bound property of real numbers. Another possible technique is to make a different

proof by using a different property of real numbers, for instance Cantor nested intervals, or Dedekind cuts, or the convergence of Cauchy sequences. The choice of one option depends on how the teacher wants to deal with the completeness of real numbers. Concerning the definition of continuity of functions, we have chosen the ε - δ approach. But we could have chosen a different one, based on sequences of real numbers and limits of functions, which would have required a different proof.

- Technology: We may have reasons to prefer one formulation of the completeness of real numbers rather than another one. For example, we might prefer to introduce real numbers axiomatically, and so we would prefer the least upper bound property because it is typically included in the standard axiomatic definition of real numbers. But maybe we prefer to construct the real numbers as an enlargement of the set of rational numbers. This can be done, for instance, by using equivalence classes of Cauchy sequences, and in this case it seems appropriate to define completeness in terms of convergence of Cauchy sequences. But we can also enlarge the set of rational numbers by adding Dedekind cuts, and in this case completeness would be regarded from a different perspective.
- Theory: The theory in this mathematical praxeology for teaching is larger than that of the mathematical praxeology to be taught. For instance, here we consider new properties of real numbers such as the convergence of Cauchy sequences, the property about Cantor nested intervals, Dedekind cuts. But there is an important feature we should remark: this theory is larger but of the same nature because, even if it includes more objects, it embraces the same epistemological model of mathematics. This model says that: first, definitions must be expressed, eventually, in logical terms (“for all”, “there exists”, “no”, “and”, “or”, “implies”, etc.) and three non-logical terms (\in , $=$ and \emptyset); and second, arguments must be deductive.

THE EFFECT OF TEACHING ENDS ON THE EPISTEMOLOGICAL MODEL OF MATHEMATICS

We agree with (Postman, 1996) in pointing out the importance of clarifying and analysing the teaching ends embraced by the different educational institutions. We defended in (Gascón & Nicolás, 2017) that all the scientific activity in didactics of mathematics relies on (typically implicit) assumed teaching, and that only by making those ends explicit rational discussion in didactics of mathematics would be possible.

Here we would like to show that the analysis of teaching ends of educational institutions contributes to explain interesting didactic phenomena. In particular, we would like to explain how the reasons for teaching mathematics assumed by that institution determine to a great extent the epistemological model of mathematics used in that institution. This model, has an influence over the mathematical praxeologies for teaching, which, in turn, shape the mathematical praxeologies to be taught.

Teaching ends of the faculty of mathematics

Let us inspect this idea in more detail with the example of the institution \mathfrak{S} of the teaching of mathematics in a faculty of mathematics. The institution \mathfrak{S} typically embraces, among others (perhaps less taken for granted), the aim of raising future mathematicians. Therefore, faculties of mathematics are typically used to explain to virtual future mathematicians how mathematics is made today, which can be perfectly understandable, and perhaps even desirable. Graduates in mathematics are thus expected to be aware of how propositional knowledge is officially achieved in mathematics nowadays. In particular, graduates should be familiar with the role played by sets (which are typically regarded as the basic objects) and *deductive* arguments (which are the only accepted arguments in official documents) in contemporary mathematics. In other words, graduates should be familiar with the current epistemological model of mathematics prevailing in the community of mathematicians.

Some features of the current epistemological model of mathematics of 'professional' mathematicians

Here we do not aim to give a thorough account of what is this model. We will rather emphasise one key feature concerning the current standards of achievement of knowledge. This feature is about the kind of arguments allowed nowadays in the community of mathematicians. At the end of the nineteenth century and the beginning of the twentieth century, the existence of alarming contradictions and paradoxes led many mathematicians to look for sound foundations for mathematical knowledge (Kline, 1972). Finally, Hilbert's proposal was gradually adopted and it is today a standard commonly accepted (Hintikka, 1996). At the center of this proposal one finds the notion of deductive argument.

First of all, let us see what an *argument* is. It is a speech act with which the speaker attempts to make someone else (or perhaps to herself) agree that a certain statement, the *conclusion*, is supported by a certain set of statements, the *premises*. An argument is *correct* if it really shows that the conclusion does receive support from the premises. Notice that, so far, we have not referred to the idea of truth. Now it is the right moment. An argument is *successful* if it is correct and the truth-value of the premises is justified. In this case, the truth-value of the conclusion would also be justified.

One of the characteristic properties of mathematical arguments is that they are intended to be not only successful, but also *deductive*. An argument is *deductive* if the speaker claims that nobody could believe that the premises are true without believing that the conclusion is also true. A classical example of deductive argument is the following:

- Premises: {All men are mortal, Socrates is a man}.
- Conclusion: Socrates is mortal.

The discipline which studies deductive arguments is *deductive logic*. To get a better understanding of what deductive arguments are, let us review some basic notions of a part of deductive logic called *first order logic*, strongly related to mathematics. All the known mathematics nowadays is virtually expressible in a *first order-language*, that is to say, in a language made of:

- logical symbols: variables, parenthesis, connectives (\vee for the disjunction, \wedge for the conjunction, \neg for the negation, \rightarrow for the implication), quantifiers (the existential \exists and the universal \forall).
- non-logical symbols: constants, n-ary predicates (with $n \geq 1$).

For different purposes we use different first-order languages, distinguished one from the other by the non-logical symbols. For instance, in the deductive argument above, we use the constant s for Socrates, the 1-ary predicate H for the property “being a man”, and the predicate M for the property “being mortal”. In this first-order language, the argument would be as follows:

- Premises: $\{\forall x (Hx \rightarrow Mx), Hs\}$
- Conclusion: Ms

To provide the sentences in a first-order logic with a meaning, we need a *model*, which is a way to link the non-logical symbols with parts of the world. In the model underlying the previous argument, the constant s maps to the man Socrates, the predicate H maps to the set of all men, and the predicate M maps to the set of all mortal things.

When a sentence φ , written in a first-order language, is *true* in a model \mathcal{M} of such a language we write $\mathcal{M} \models \varphi$. We say that a sentence φ is a *logical consequence* of a set Γ of sentences, written $\Gamma \models \varphi$, if we have $\mathcal{M} \models \varphi$ for every model \mathcal{M} which satisfies $\mathcal{M} \models \Gamma$. Now we can give a more precise definition of *deductive argument*: it is an argument in which the speaker claims that the conclusion is a logical consequence of the premises.

Notice that, not being the idea of deductive argument relative to a precise fixed model, the property of being deductive is independent of any model. In other words, the fact of being deductive relies uniquely in the logical form of the argument, (the syntax), not in the interpretation of the non-logical terms (the semantic).

Apparently, by the very definition, in order to present a deductive argument, we would need to consider all the possible models for our language and to check that they do not make the premises true without making the conclusion true. Fortunately, this is not the case. Instead, we can use a certain collection of *deductive rules* which allow us to derive the conclusion from the premises. One of these rules used in the

proof of Bolzano's theorem is the *reductio ad absurdum*: if from a set of premises $\Gamma \cup \{\alpha\}$ one can deduce both β and $\neg\beta$, then from Γ we can deduce $\neg\alpha$. Another rule, also used in the proof above is: if from a set of premises Γ one can deduce both α and β , then one can deduce $\alpha \wedge \beta$. In our proof we have also implicitly used a rule which tells you when it is allowed to deduce a statement involving the universal quantifier. The interested reader can find more information about first order logic in (Smullyan, 1968).

When we can derive a sentence φ from a set Γ of sentences by using those deductive rules, we write $\Gamma \vdash \varphi$ and we say that φ is *deducible* from Γ . We do not only have that $\Gamma \vdash \varphi$ implies $\Gamma \models \varphi$ (which means that the relationship of deducibility is right: that is, if you use the deductive rules you produce a correct deductive argument), but also that $\Gamma \models \varphi$ implies $\Gamma \vdash \varphi$ (which means that the relationship of deducibility is complete: every correct deductive argument can be expressed by using the deductive rules).

Of course, we do not claim that the only kind of arguments mathematicians take under consideration in their *everyday* activity are the deductive ones. Concerning this (Thurston, 1994) and (Brousseau, 1995) are interesting reading. Neither do we say that mathematicians point out *explicitly* all the used deductive rules when they deal with deductive arguments. But it would be too naïve not to admit that deductive arguments are essential in the contemporary epistemological model of mathematics. According to this model, the only *institutional* knowledge is the one produced by deductive arguments. Indeed, if someone shows that a theorem has not been proved with a deductive argument, then this theorem will be immediately removed from the realm of the official knowledge. In other words, the community of mathematicians accepts in practice arguments which do not present explicitly the deductive rules used, but only because: first, those arguments are believed to be theoretically expressible in terms of deductive rules, and second, it would be extremely tedious to write down all the deductive rules used.

Evidence of the key role played by deductive arguments is that, for them to be theoretically possible, one needs to use definitions expressed in the first order language. This entails a deep logical analysis of intuitive notions to make them ready to play a role in deductive games. For instance, the notion of continuity of a function, or the very notion of function, in the eighteenth century was used according to an intuitive meaning, rooted in pragmatic considerations related to whether a function could be written down by using a single analytic expression or not (Kline, 1972). But, as deductive arguments started to become more and more important in the epistemological model of mathematics, a logical analysis of function or continuity was needed. This is how we ended up with the ε - δ definition which expresses continuity in terms of quantifiers, implications, real numbers and inequalities.

It is important to notice that the current epistemological model of mathematics shapes not only the mathematical praxeologies to be taught in the faculties of mathematics, but also the mathematical praxeologies for teaching. Indeed, these last praxeologies consider different forms of reconstructing certain works of mathematics, but they never question the epistemological model.

Teaching ends of mathematics in secondary education and the missing of a corresponding epistemological model of mathematics

Primary education aims to provide the very basics of our culture and our knowledge of the world, for the students to begin their path towards the status of autonomous and suitable citizens and to be able to continue further in their studies. Then, secondary education aims to provide with more specialised culture and knowledge for the students to become entirely autonomous and suitable citizens, and for them to be ready to get involved in some profession or to get into the deep study of some disciplines, for instance mathematics.

But, of course, the institution of the teaching of mathematics in secondary education does not intend to raise future mathematicians, and so there is no point in explaining to the students of secondary education how mathematical knowledge is officially achieved today among professional mathematicians. This entails that teachers in secondary education should not frame their teaching within the current epistemological model of mathematics of the faculties of mathematics. In particular, there is no point in putting the logical analysis of notions and deductive arguments at the center of the teaching. But then, how does one explain those notions? What kind of arguments should be used? For instance, if there is no point in using set theory (ordered pairs, Cartesian products) to define what a function is, or the ε - δ statement to define what continuity is, how to explain what a continuous function is? And if we do not use the logical analysis for the definitions, then we cannot use deductive arguments for the theorems. Hence, how to make a non-deductive argument for Bolzano's theorem? Is it possible? Is it needed? These are crucial questions of mathematical praxeologies for teaching mathematics in secondary education.

CONCLUSIONS

The change from the institution of teaching in the faculty of mathematics to the institution of teaching mathematics in secondary education entails a change of teaching ends. This, in turn, forces the future teachers of secondary education to look for a new epistemological model of mathematics. The one transmitted at the faculty of mathematics is not only deeply rooted in those future teachers, but also it is received as if it were the *faithful* account of *real* mathematics. On the contrast, there is no official alternative epistemological model of mathematics at hand.

This problem could be tackled at university, by postgraduate masters' degree on teacher training in secondary education, but it is far from being the case. On the contrary, it seems to be widely assumed that, on the side of mathematics, teachers at

secondary education should not find any problem at all, as they already are sufficiently well-informed in this discipline.

The lack of a genuine and coherent alternative epistemological model of mathematics for secondary education means a huge field of open problems for the community of researchers in didactics of mathematics.

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