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Review of our work on the teaching and learning of functions of two variables

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In this article we present a short review of our research on student understanding of function of two variables. We describe results dealing with basic aspects such as geometrical understanding and understanding of the definition, then we consider results on student understanding of some notions of the differential calculus: plane, tangent plane, partial derivatives, directional derivatives, and the total differential.

Keywords: Teaching and learning of analysis and calculus, teaching and learning of specific topics in university mathematics, functions of two variables, APOS.

INTRODUCTION

It can be expected that natural or human induced phenomena in science, technology, and engineering will commonly involve quantities which are on an invariant (or nearly invariant) relationship with two or more other different quantities. This makes the multivariable calculus an important tool for the modelling and exploration of these phenomena. It is only in the last 10-15 years that research on the didactics of multivariable functions has gathered some momentum. The reason for this is that perhaps for a long time there was the expectation that understanding the didactics of single-variable calculus was enough since generalization to the multivariable context seemed to be non-problematic. However, this was quickly found not to be the case. A common undercurrent to research in multivariable functions is that these functions have their own particularities which make generalizing from the one-variable case challenging for students. In this article we describe research on the teaching and learning of two-variable functions. Because of lack of space, we concentrate on discussing mainly our own research results giving only a brief hint of some research by others. We also try to minimize theoretical considerations and technical nomenclature so that the article may be of value to practitioners, as well as informative to researchers from different perspectives.

Our research question is: What do our research results about the learning of two-variable functions Calculus taken together tell and how can they be used to help promote students' learning in the classroom?

THEORETICAL FRAMEWORK

Since most of the research we have undertaken is APOS-based, we start with a brief summary. In APOS (for more detail see Arnon et al., 2014), an Action is a transformation of a previously constructed Object that the individual perceives as external. It is external in the sense that it is relatively isolated from other mathematical

knowledge and so the individual will not be able to justify the Action. An Action could be the rigid application of an explicitly available or memorized procedure or fact.

When an Action is repeated and the individual reflects on the Action, it may be interiorized into a Process. The Process is perceived as internal in the sense that it has meaningful connections to other mathematical knowledge. This enables the individual to imagine the Process, omit steps, and anticipate results without having to explicitly perform the Process. A Process is thus perceived as dynamic. The meaningful connections also allow the individual to justify the Process and interrelate its different representations. Different Processes may be coordinated to form new Processes.

As new problem situations arise the individual may feel the need to apply Actions on a Process. For this to occur the Process has to be conceived as a totality, an entity in itself. When an individual is able to do Actions on a Process or imagine doing so, it is said that the Process has been encapsulated into an Object.

A Schema is a coherent collection of Actions, Processes, Objects, and other previously constructed Schema dealing with a specific mathematical notion. The Schema is coherent in the sense that its different components are interrelated in a way that allows the individual to determine if a problem situation falls within the scope of the Schema.

In their research, Martínez-Planell and Trigueros used semi-structured interviews with students who had finished a multivariable calculus course and, in some studies, had not used APOS-based didactical strategies. Students are asked several questions and the interviewer interacts with them in order to give them opportunities to show evidence of their constructed structures as defined in the theoretical framework. They analyse students' responses to find evidence showing if they were using Actions, evidenced by students' relying mostly on memorized facts, or in their need to use specific information to do calculations, or if they showed to have interiorized Actions into Processes demonstrated by their capability to generalize their strategies or to skip steps in procedures, for example, or encapsulated them into Objects shown by being able to perform Actions on notions studied to find their properties, for example.

When students work with different problems related to a mathematical notion. they may evidence that they use mainly Actions in their responses. It is then said that those students have constructed an Action conception of that notion. The same can be said about Process and Object conception.

A genetic decomposition (GD) is a conjecture of mental constructions students may do in order to understand a specific mathematical notion. In this article "to understand" a mathematical notion means to attain at least a Process conception of the notion. After a research study where a dialogue between APOS Theory and the Anthropological Theory of Didactics was carried out (Bosch, Gascón & Trigueros, 2017), interesting theoretical constructs emerged. In some researches we have used some of them to include both the cognitive and the institutional aspects that play a role in the teaching and learning of mathematics. Those mentioned in this paper will be discussed when they are needed.

BASIC ASPECTS OF FUNCTION OF TWO VARIABLES

Geometric understanding of functions of two variables

The first article we found on the literature of mathematics education that deals with generalization of function of one variable and student understanding of the basic notions of functions of two variables was by Yerushalmy (1997). She studied a group of seven-graders as they generalized notions from single to two-variable functions in a modelling context. Her results, among other things, stress the importance of the interplay between different representations in this generalization. Trigueros and Martínez-Planell (2010) followed with a study of students' geometrical understanding of functions of two variables. It may be argued that the most basic technique for solving problems about functions of two variables is reducing them to problems about functions of one variable; this is done by holding a variable fixed, equivalently, by intersecting a surface with a plane of the form $x=c$, $y=c$, $z=c$ (fundamental plane). However, students tend not to do this on their own. Trigueros and Martínez-Planell (2010) found that intersecting a surface with a fundamental plane and placing the resulting curve in its appropriate place in space needs to be explicitly discussed, since students frequently confuse transversal sections with projections onto a coordinate plane, and need help to construct those Processes needed to relate contours with the graph of a function. Students were found not to easily relate their intuitive knowledge of space with the mathematical notion of \mathbb{R}^3 . This suggests the potential of using concrete manipulatives for students to perform Actions during instruction. Like Yerushalmy (1997), students' constructions were found to be dependent on the specific representation used so that it is important to help students construct Processes to interrelate different representations, including verbal representations, where phrases like "cut the surface", "lift a curve" and so on, are sometimes misunderstood by students. Given the importance of fundamental planes for the study of two-variable functions it may be expected that they would be institutionalized as a technique by explicitly referring to them and exploring their use throughout the course in different problem situations. Follow up studies (Trigueros and Martínez-Planell, 2011, 2015) of institutional conditions (textbook and instruction) using tools from the dialogue between the Anthropological Theory of the Didactic (ATD) and APOS (Bosch, Gascón, and Trigueros, 2017) show that this is not the case. Specifically, the moments of study of the ATD were used to examine the mathematical organization of the Stewart (2006) textbook as it regards the treatment of geometrical aspects of functions of two variables. Results obtained supported APOS based observations.

Understanding of the definition of function of two variables

After studying geometrical representations of functions of two variables, Martínez-Planell and Trigueros (2012) turned their attention to student understanding of the definition of functions of two variables: domain, range, uniqueness of functional values, and the possible arbitrary nature of a functional assignment. They found that generalizing from function of one variable to function of two variables is fraught with difficulties. Students frequently think of the domain of a function of two variables as a

collection of real numbers, and sometimes as the x values with the y values being the range (see also Dorko and Weber, 2014) which implies that students were not offered enough opportunities to construct the notion of domain of two-variable functions through Actions involving ordered pairs and their representation in 3D-space, particularly in cases where the domain is given symbolically and includes a restriction. However, in a later study, students in Şefik and Dost (2019) did not exhibit this difficulty with restricted domains, so institutional issues may play a role here. Other aspects which need specific attention in classrooms include the vertical line test and its use for ascertaining uniqueness of functional value. It was observed that even good students occasionally have difficulties to recognize the uniqueness of functional value and the possible arbitrary nature of a functional assignment. So, to summarize, students tend not to generalize to a notion of function of two variables that allows them to recognize or show behaviour consistent with the modern set theoretic definition. Students need opportunities to reflect on the construction of domain, range, and uniqueness of functional value so they can distinguish functions of two variables from one-variable functions through activities designed to foment student reflection. As before, it was found that student constructions tend to be dependent on specific representations so that treatment Actions and conversion Processes (Duval, 2006) need to be explicitly fostered throughout instruction.

Use of designed activities in the classroom

The Trigueros and Martínez-Planell (2010) and Martínez-Planell and Trigueros (2012) studies were part of a first cycle of APOS research. Study results led to revising the GD where constructions were introduced taking into account results obtained in the first cycle. The new GD was used to design specific activities to help students do the constructions they needed to better understand two-variable functions. After classroom testing the activities for several semesters a second research cycle was undertaken (Martínez-Planell and Trigueros, 2013). It was found that students frequently did not work well with free variables, for example, as in $f(x,y)=x^2$ (missing variable) or as in $f(x,y)=x\sin(y)$ when you set $x=0$ (all variables present). Difficulties associated to both of these types of functions are independent of each other so it was concluded that they need to be explicitly addressed in class. In particular, the use of fundamental planes to graph cylinders (like $z=x^2$), needs to be explicitly considered through reflection on Actions during instruction. It was also found that activities that review function transformations help students construct the Process needed in graphing activities; that students overgeneralize familiar algebraic expressions so they can act as an obstacle; that in order to do the Action of intercepting a surface with a given fundamental plane, some students attempt to first graph the entire surface (consistent with an Action conception), and that some students needed help to interiorize point by point evaluations Actions into a Process (especially in the case of a missing variable). Results of the use of these activities were encouraging since students in the experimental section, who used the designed activities, performed better than those in a lecture-based section. Since it was considered that more reflection opportunities were needed

to encourage the construction of a Process conception, it was decided that a third cycle of APOS research would be needed. So, the GD was further refined and the activity sets were improved to reflect the results of the second cycle. After classroom implementation of the activity sets, a third cycle of student interviews was undertaken.

Comparing understanding in two groups

Results of the third research cycle and their comparison with the main results of the first two cycles and the evolution of GD and activity sets through the three cycles showed that the activity sets were adequate in terms of helping students to interiorize their Actions into Processes (Martínez-Planell and Trigueros, 2019). Students from the sections using the activity sets were found to be more likely to show they had constructed a Process conception of function of two variables (8 of 12) than students not using the activity sets (0 of 7). Students who did not construct Processes did not show to have reflected on their Actions while working on the activity sets.

DIFFERENTIAL CALCULUS OF FUNCTIONS OF TWO VARIABLES

Vertical change on a plane and the genetic decomposition

Tall (1992) proposed a way to interpret the total differential $dz = (\partial z / \partial x)dx + (\partial z / \partial y)dy$ as $dz = (dz_x / dx)dx + (dz_y / dy)dy$ where dz_x , dz_y , dx , and dy are lengths of segments in the tangent plane (see Figure 1) and so may be cancelled. This makes more sense than the two ∂z 's occurring in the original equation, where they mean different things in each of the two terms. We adapted this idea using the notion of vertical change on a plane to model the point-slopes equation of a plane, the tangent plane, the directional derivative, and the total derivative as suggested in Figures 2 and 3, thus aiming to bring coherence to students' differential calculus Schema (see Arnon et al., 2014, for a discussion of Schemas and Schema coherence).

A genetic decomposition was proposed in terms of vertical change on a plane (Martínez-Planell, Trigueros, and McGee, 2015). The vertical change on a plane dz from an initial base point (a, b, c) to a final generic point (x, y, z) is the sum of the vertical changes in the x direction (dz_x) and the y direction (dz_y) as one moves from the initial to the final point. But each one of these vertical changes can be expressed in terms of slope: dz_x is the slope in the x direction times the horizontal change in the x direction, $(dz_x / dx)dx$ and similarly for dz_y . From here one may obtain the point-slopes equation of a plane (Figure 2) $z - f(a, b) = m_x(x - a) + m_y(y - b)$, the tangent plane $z - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b)$, and the total differential $df(a, b) = f_x(a, b)dx + f_y(a, b)dy$. One may also obtain the directional derivative in direction of a given vector (Figure 3) $D_{\langle \Delta x, \Delta y \rangle} f(a, b) = (f_x(a, b)\Delta x + f_y(a, b)\Delta y) / \|\langle \Delta x, \Delta y \rangle\|$.

The original genetic decomposition is detailed in Martínez-Planell, Trigueros, and McGee (2015), a revised version in Martínez-Planell, Trigueros, and McGee (2017), and further refinements for the total differential in Trigueros, Martínez-Planell, and McGee (2018). These three papers will be discussed in the following subsections.

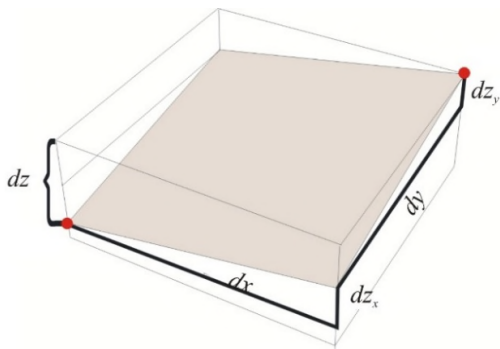


Figure 1: Tall's suggestion

$$dz = dz_x + dz_y = (dz_x / dx)dx + (dz_y / dy)dy$$

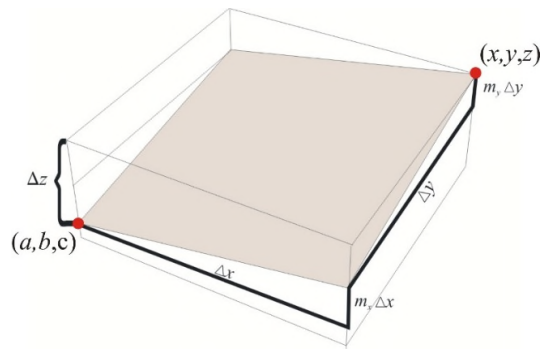


Figure 2: Vertical change on a plane

$\Delta z = m_x \Delta x + m_y \Delta y$ may be used to model point-slopes $z - c = m_x(x - a) + m_y(y - b)$, tangent plane $z - f(a,b) = f_x(a,b)(x - a) + f_y(a,b)(y - b)$, and total differential $df(a,b) = f_x(a,b)dx + f_y(a,b)dy$

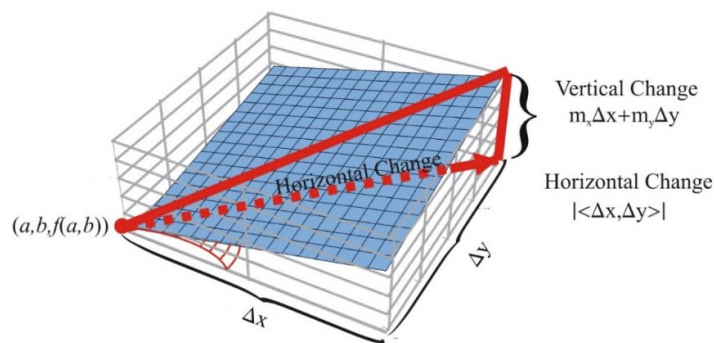


Figure 3: Directional derivative at (a,b) in the direction $\langle \Delta x, \Delta y \rangle$,

$$D_{\langle \Delta x, \Delta y \rangle} f(a,b) = (f_x(a,b)\Delta x + f_y(a,b)\Delta y) / \|\langle \Delta x, \Delta y \rangle\|$$

Slope and other pre-requisites to the differential calculus

One of the most important pre-requisite ideas necessary for the differential calculus is the notion of slope. It was found that even students that showed behaviour consistent with a Process conception of slope in two dimensions sometimes were restrained to working with slope as an Action when dealing with lines in three dimensions. Moore-Russo, Conner, and Rugg (2011) had previously observed student difficulty in generalizing slope to three dimensions. They observed that students would have difficulty realizing that in 3D slope must be directed slope. Indeed, in Martínez-Planell et al. (2017) these results were confirmed when discussing directional derivatives even in the case of some of the best interviewed students. Results showed that students need to have constructed an Object conception of slope in 3D to be able to think of a Process of slope as a totality upon which Actions can be performed. The notion of slope is central in the construction of a Process of vertical change which is the main concept giving coherence to a differential calculus Schema that includes the notions of point-

slopes equation of a plane, tangent plane, directional derivative, and total differential. Indeed, McGee and Moore-Russo (2015) and McGee, Moore-Russo, and Martínez-Planell (2015) showed that students who explicitly consider 3D slopes during instruction seem to develop a better understanding of the differential calculus than students who do not.

Other basic pre-requisite ideas also play a role in students' possible construction of the differential calculus (Martínez-Planell et al., 2015): Some students do not realize that slope in the x and y directions are invariants of planes that do not depend on a base point, or that the total vertical change Δz is the sum of the vertical changes in the x and y directions, $\Delta z_x + \Delta z_y$. This behaviour is reminiscent of the "two-change problem" as discussed in Weber (2015). Some students need to develop flexibility in their notion of variable and still consider slope to be $\Delta y / \Delta x$, similar to the before-mentioned observation of Dorko and Weber (2014).

Partial derivatives

The construction of partial derivatives is conjectured as the coordination of Processes of fundamental planes and derivative of a one-variable function (Martínez-Planell et al., 2015), thus being restricted to an Action conception of either notion will not allow the construction of a Process of partial derivative or any of the other notions that depend on this. Given observed student difficulty with derivative of function of one variable, this suggests that the basic ideas in the construction of derivative of a one-variable function should be repeated in the discussion of partial derivatives, but now inserted in corresponding fundamental planes. As observed in Asiala et al. (1997), the notion of derivative needs the construction of graphical and symbolic trajectories that then must be coordinated. The same care should be taken by explicitly considering conversions between different representations in the construction of the ideas of the differential calculus (Martínez-Planell et al., 2015, 2017).

Directional derivative

The directional derivative is considered in detail in Martínez-Planell, Trigueros, and McGee (2017). One unexpected finding was that even some of the best performing students may have difficulty representing or imagining the direction vector $\langle a, b \rangle$ in space as $\langle a, b, 0 \rangle$. This has implications in their geometrical interpretation of directional derivative. These students can only perform Actions when doing symbolic computations and are not always sure of what they are doing when looking for directional derivatives. Students who have constructed an Action conception of vertical change may be expected to have difficulty to interpret the usual formula found in textbooks (Stewart, 2006) $D_{\langle u_1, u_2 \rangle} f(a, b) = f_x(a, b)u_1 + f_y(a, b)u_2$ where $\langle u_1, u_2 \rangle$ is a unit vector, or the formula proposed in the genetic decomposition $D_{\langle \Delta x, \Delta y \rangle} f(a, b) = (f_x(a, b)\Delta x + f_y(a, b)\Delta y) / \|\langle \Delta x, \Delta y \rangle\|$. So again, vertical change plays a key role. Also, as mentioned before, some students did not realize that slopes in 3D must be directed slopes. This was a difficult construction for students; only one of 26 interviewed students managed to construct a Process conception of directional derivative. Our

findings suggest that the construction of geometric meaning for the directional derivative should be carefully worked with students during instruction, thus helping them find a solution to the “two change problem” discussed by Weber (2015).

The total differential

Trigueros, Martínez-Planell and McGee (2018) considered student understanding of the total differential. They found this to be a very hard construction for students as none of 26 interviewed students managed to construct a Process conception of the total differential, only one of the 26 was judged to be in transition to such a conception, six were found to have an Action conception, and 19 of the 26 students showed to have no knowledge or recollection of the total differential. As with the previously described concepts, it is fundamental to develop a Process conception of vertical change on a plane in order for students to construct the geometric interpretation of the total differential. Most students do not think of the total differential as a function of independent variables dx and dy . Those students showing an Action conception were able to do mechanical computations where dx and dy were thought of as “very small numbers” which was not enough to develop a geometric insight that would help justify their solution procedures. Some students showed to have constructed the notions of function and tangent plane as isolated concepts thus had difficulties to explain the analytical and graphical consequence of the relation between them that is formalized in the total differential. These results led to revising the genetic decomposition and the activity sets so as to help students be able to coordinate Processes of tangent plane to a surface and of two-variable function into a Process where they can recognize that the tangent plane to a point of the function can be considered as the local linear approximation of the function, both in analytical and graphical representations.

The Trigueros et al. (2018) study included an analysis of some institutional issues (textbook, activities, and instruction) that constraint student learning of the total differential and provide supporting evidence for the APOS-based observations in the paper. The model of the moments of study of the ATD was used together with another tool from the ATD-APOS dialogue (Bosch et al., 2017): the institutional classification of techniques as action-technique (memorized, rigid, or applied to isolated tasks), process-technique (supported by a technological-theoretical discourse, presenting variations, connected to other techniques), and object technique (taken as an object of study). It was found that the moments of study were unbalanced (i.e. some moments and or ideas were not well represented) and may not foster a deep understanding of the total differential and its relation to the tangent plane. The analysis of the different types of techniques indicates that techniques introduced in the textbook are constrained to action-techniques which may limit the understanding of the concepts introduced and their relations while process-techniques are needed to foster students’ understanding.

SUMMARY AND CONCLUSION

In this short paper we have briefly summarized our research on functions of two variables, the basic aspects, and the differential calculus. The results of this research

give ideas that may influence the design of better teaching strategies. All results show the role that construction of three-dimensional space and different representations play in students' understanding of the differential calculus of two-variable functions and how students' reflection on detailed activities and class discussion can help them to construct Processes and Objects needed in deepening their understanding of the main ideas of this discipline. Results obtained also open the door for further research on students' understanding of the differential calculus of functions of two variables.

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