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# **Professional and academic bases of university mathematics teaching for the 21st century: the anthropological approach to practice-based research**

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*Based on an anthropological approach to the notions of profession and métier within an institution, we show how the label “mathematics” could relate to both notions in the setting of universities. We also propose a finer characterization of segments of the métier. Finally, we revisit some examples of how our own research have addressed different segments so identified. We finally return to the question of how different forms of scholarship interact with the métier and its segments, and formulate a proposal for professionalizing the métier in view of current and future needs.*

*Keywords: university mathematics teaching, ATD, practice-based research*

## **1. INTRODUCTION**

Mathematics as a university discipline has hugely expanded in the 20<sup>th</sup> century, both at the level of research and at the level of education. This development is inseparable from the interaction of mathematics with other – and equally evolving – disciplines, many of which have not only drawn on, but also contributed to the advances of mathematics both as a field of research and as a matter to be taught. It is neither possible nor helpful to try to draw sharp lines between mathematics and other disciplines. This is so not only for research but also in higher education contexts like Engineering and Natural Sciences.

The growth of mathematics from an institutional point of view is visible in the existence and expansion of mathematics departments in virtually any university type institution. One informal, frequently implicit, characterization of a “university mathematician” is certainly an individual who works at such a department. In mathematics departments, we find a great variety of scholarly activities, often with labels such as “pure mathematics”, “applied mathematics”, “statistics” and so on, each with further subdivisions and overlaps; and sometimes also “mathematics education”, “history of mathematics”, “data science” and more. Another informal characterization of a “university mathematician” could be university faculty member with an advanced degree labeled “mathematics”, but in practice, this is quite similar as labels often result from the name of departments where they are obtained. In fact, people whose teaching or research are centered on mathematical contents may have other affiliations. We can think, for instance, of a specialist in mathematical education who teaches mathematics methods courses in an education department, or of a statistics researcher employed at a Medical school to teach statistics to future physicians.

From the point of view of university mathematics education, the notion of a *university mathematics teacher* is more relevant than the departmental categorization. Here,

labels and descriptions of teaching units can be used to clarify more precisely what is being taught by a given university teacher. We shall pursue this delimitation in the next sections. We also return to the important co-existence and interaction of teaching and research in universities in the case of university mathematics teachers.

## 2. THE NOTIONS OF *PROFESSION* AND OF *MÉTIER* IN THE FRAMEWORK OF THE ATD

The English language offers many terms to designate a position within an institution that is characterized by a responsibility to carry out certain types of tasks. We shall define use and two of them, *profession* and *métier*, which both have their roots in Latin (via the French language). Our definitions may not correspond to dictionaries that explain common usage. However, they are, at least to some extent, consistent with distinctions made by Chevallard (2017) and Stigler and Hiebert (1999) to discuss the general status of teachers in Western and Asian societies, respectively.

By a *métier*, we mean simply mean the set of positions within an institution defined by a family of types of tasks, as explained above. Members of the *métier* are individuals holding one of these positions. Naturally, to stay as member of a *métier*, one will normally need minimal capacity to carry out the involved types of task; there may be no further requirements.

A *profession* is a special case of a *métier*, in which the praxis, and knowledge on the praxis, is made explicit and shared among its members. The resulting discourse is a logos  $\Lambda$  describing and justifying the use of certain methods or techniques, using a more or less specialized professional discourse. The combined praxis and logos  $\mathcal{P} = (\Pi, \Lambda)$  is regulated and developed by the members of the profession (possibly by other agents too). Admission to the profession is strongly linked to sharing  $\mathcal{P}$  at some level, obtained through formal training carried out by members of the profession (again, possibly by other agents as well).

In terms of the anthropological theory of the didactic (see e.g. Chevallard, 2019 for details),  $\mathcal{P}$  is a collection of *praxeologies*, including both praxis blocks  $\Pi_i$  (types of tasks, techniques) and logos blocks  $\Lambda_j$  (discourse about techniques, and theory to support and justify the discourse). If  $p$  belongs to the *métier* defined by  $\Pi = (\Pi_i)$ , we can assume some minimal relation  $R_I(p, \Pi)$  of  $p$  within the institution  $I$  at which the *métier* is exercised. Meanwhile for  $p$  to belong to a profession we have further requirements on  $R_I(p, \mathcal{P})$ , including the role of  $p$  in establishing  $R_I(p', \mathcal{P})$  for the position  $p'$  of newcomer to the profession.

Examples of professions, which are well established in most developed societies, include: lawyers, doctors and engineers. Scientific researchers within specific fields also constitute professions – certainly, disciplines of modern science have extensive and explicit logos blocks, and scientists are trained by other scientists of the same discipline and within the same institution. The case of teachers in, say, primary school is less clear – the development of explicit knowledge about teaching, as well as the

training that gives access to it, may only to a very small extent be carried out by primary school teachers.

Considering now the case of university mathematics teachers, many of them *do* belong to professions of research (in a broad range of specialties, as we have seen), although not necessarily, and not necessarily the same. However, one can hardly call the set of university mathematics teachers a profession, as there is little shared and explicit knowledge about how to carry out the types of didactical tasks that characterize this *métier*, and even less training and development of such shared knowledge that is carried out by the members of the *métier*. Of course, individual members of the *métier* have shared their views and principles about the *métier* (e.g. Halmos, 1985; Krantz, 2015), and in some countries, conferences and committees on university mathematics teaching are being organized with and by members of the *métier* (e.g. Burn, Appleby & Maher, 1998; Saxe & Braddy, 2025). Such works are usually written to push new agendas which are not widely shared in the profession, and may remain unknown to most members. On the other hand, the interest and willingness to share knowledge on teaching is growing among university mathematics teachers in many institutions and countries. In 1988, a mathematician noted that

when a mathematician speaks about teaching, colleagues smile tolerantly to one another in the same way family members do when grandpa dribbles his soup down his shirt (Clemens, cited in Krantz, 2015, p. xi).

This is certainly not universally true anymore. Still, university mathematics teaching is hardly a profession in the sense defined above. In the rest of this paper, we try to characterize the *métier*, and then consider if and how it might become a profession.

### 3. THE UNIVERSITY MATHEMATICS TEACHER MÉTIER

Mathematical practice and knowledge can itself be modelled in terms of praxeologies  $\omega$ , and within university institutions  $U$ , didactical tasks are all about establishing certain relationships  $R_U(\sigma, \omega)$  for individual in a student position  $\sigma$  within  $U$ . If the university mathematics teacher position is called  $\tau$ , a minimal requirement for  $R_U(\tau, \omega)$  is certainly that  $R_U(\tau, \omega) \supseteq R_U(\sigma, \omega)$ . But more is required:  $\sigma$  must have practical knowledge about the didactical praxis of establishing  $R_U(\sigma, \omega)$ , and if this praxis is called  $\Pi(\omega)$ , there are thus additional requirements for  $R_U(\tau, \Pi(\omega))$  which may be assessed by the extent the praxis of  $\tau$  actually succeeds to establish  $R_U(\sigma, \omega)$ . Indeed, university institutions usually have rather explicit and established ways to assess the latter kind of relation, and the observed student performance is frequently also used to assess  $R_U(\tau, \Pi(\omega))$ . Newcomers to the *métier* will often have to develop  $R_U(\tau, \Pi(\omega))$  more or less through building up personal experience with  $\Pi(\omega)$  and possibly drawing on their own experience as students. However, as the student position  $\sigma$  may be different from positions they have themselves held – for instance, in the case of a background as graduate of pure mathematics, who is faced with the praxis of teaching applied mathematics to populations of some other discipline. In this case, both

$R_U(\tau, \Pi(\omega))$  and  $R_U(\tau, \omega)$  have to be developed based on  $R_U(\tau, \omega')$  where  $\omega'$  consists of mathematical praxeologies somewhat similar to  $\omega$ . While all of this appears at first sight a bit theoretical, it all was eminently concrete to someone who, like the author of this paper, started out in the *métier* with a background as researcher in pure mathematics, faced from day one with the task of teaching mathematics and mathematical biology to future biologists. While the mathematical elements were all very familiar, the full praxeologies to be taught, as well as relevant didactical techniques to do so, were largely to be acquired by the teacher.

One way to characterize the *métier* is thus in terms of the didactical tasks, closely related to the relationship  $R_U(\sigma, \omega)$  to be established, and in particular in terms of the student positions  $\sigma$  and the praxeologies  $\omega$  concerned. Even when teaching students in positions that the teacher has actually occupied,  $R_U(\tau, \omega)$  is evidently of central importance to  $R_U(\tau, \Pi(\omega))$ . In the frequent absence of external support (specific training) to establish  $R_U(\tau, \Pi(\omega))$ , and therefore also of shared logos blocks  $\Lambda(\omega)$ , we see the clear traits of a *métier* in the establishment and function of the position  $\tau$ .

Considering that the set of mathematical praxeologies  $\omega_\sigma$  to be taught to students in position  $\sigma$  depend largely on  $\sigma$ , a first rough “topology” of the *métier* can thus be given in terms of the student populations: a teacher is in position  $\tau_\sigma$  if she must engage in  $\Pi(\omega_\sigma)$ . In brief, the *métier* can be subdivided according to positions  $\tau_\sigma$  for which  $R_U(\tau_\sigma, \Pi(\omega_\sigma))$  must then satisfy some minimal requirements, more or less assessed through  $R_U(\sigma, \omega_\sigma)$ . Still, formal training aiming to support entrance into the position  $\tau_\sigma$  is often generic (see Winsløw, Biehler, Jaworski, Rønning & Wavro, to appear), corresponding to the assumption that  $R_U(\tau_\sigma, \Pi(\omega_\sigma))$  is not only independent of  $\sigma$ , but also that  $\Pi(\omega)$  is independent of  $\omega$ . The techniques from  $\Pi(\omega)$  assumed to be independent of  $\omega$  are basically *pedagogical* and concern, for instance, how to prepare and conduct interactive lectures on a generic praxeology  $\omega$ , relate to a generic student independently of her actual position  $\sigma$ , and so on. While any training on this generic practice  $\Pi$  will then also involve some form of logos block  $\Lambda$ , the relation  $R_U(\tau_\sigma, (\Pi, \Lambda))$  may indeed fail to establish  $R_U(\tau_\sigma, \Pi(\omega_\sigma))$ , even when combined with the relationship  $R_U(\tau_\sigma, \omega')$  that  $\tau_\sigma$  may hold to praxeologies  $\omega'$  that are somewhat similar to  $\omega_\sigma$ , or even include  $\omega_\sigma$ .

The access to the *métier* is, nevertheless, to a large extent based on developing  $R_U(\tau_\sigma, \omega')$  through the *mathematical* training of  $\tau_\sigma$ , which (in the case of researchers) may be assumed to largely guarantee that  $R_U(\tau_\sigma, \omega_\sigma)$  can be established satisfactorily by any person in position  $\tau_\sigma$ , irrespectively of the student position  $\sigma$  concerned. This could seem justified at least in case where  $\tau_\sigma$  has previously developed  $R_U(\sigma, \omega_\sigma)$  successfully. Even in this case, the establishment of  $R_U(\tau_\sigma, \Pi(\omega_\sigma))$  remains, and the assumption that some  $R_U(\tau_\sigma, (\Pi, \Lambda))$  will suffice to complement  $R_U(\sigma, \omega_\sigma)$  is hardly without raising concerns – especially in the frequent case where existing didactical practices  $\Pi(\omega_\sigma)$  appear, for students in position  $\sigma$  at large, in need of improvement in

some sense. Again, taking measures of  $R_U(\sigma, \omega_\sigma)$  as measures of the quality of  $\Pi(\omega_\sigma)$  may at best help to realize such a need, while a logos block  $\Lambda(\omega_\sigma)$  is needed to identify possible shortcomings of  $\Pi(\omega_\sigma)$ , to devise innovations of  $\Pi(\omega_\sigma)$ , and to conduct systematic experiments of these innovations – and finally, to share the results in a way that conveys the position  $\tau_\sigma$  with professional traits.

Considering again the variety of student populations that university mathematics teaching may address, and the idea that these somehow provide a structure on the *métier* as a whole, we will draw on an idea initially presented in the thesis of Kim (2015) and further developed by Chevallard (2019). They consider that a human population  $P$  at large can be subdivided as  $P_1 \cup P_2 \cup P_3$ . Here,  $P_1$  consist of people who engage in production of new mathematical knowledge based on a postgraduate degree in some mathematically intense discipline (like physics, pure or applied mathematics, statistics etc.). Population  $P_2$  consist of people who do not engage in production of new mathematics, but whose work is nevertheless crucially based on a postsecondary study of mathematical disciplines (like most secondary level mathematics teachers, but also most engineers, business specialists etc.). Finally,  $P_3$  is “the rest” and certainly the largest portion of  $P$ . While there may be some grey zones left from these somewhat informal definitions, we can nevertheless identify corresponding positions  $\sigma_i$  ( $i = 1, 2, 3$ ) of students at university, who are in some sense preparing for adult life in population  $P_i$ . These are not, in general disjoint, at least in early stages of university studies: especially  $\sigma_1$  and  $\sigma_2$  may be required to build the same relations to basic mathematical praxeologies from, say, linear algebra, and thus be taught together, depending on the institution  $U$ ; and even future members of  $P_3$  may at some point face such requirements. At later stages, specializations may be more common. Nevertheless, university mathematics teachers may be roughly assumed to occupy positions  $\tau_i$  ( $i = 1, 2, 3$ ) corresponding to the students they face, with both overlaps (as mentioned) and with further specializations (e.g.  $\tau_{2,t}$  when the didactical tasks is specialized for the case of future secondary teachers  $t$ ,  $\tau_{1,\mu}$  when the public are future researchers  $\mu$  in a field of pure mathematics, etc.).

We note here that it is common for individuals at  $U$  to occupy certain positions, such as  $\tau_{1,\mu}$  and  $\mu$ , simultaneously. Indeed, the simultaneous occupation of positions as scholar or researcher, and as teacher, is both currently and traditionally considered a hallmark of university institutions – associated with the more radical Humboldtian ideal of *Einheit von Lehre und Forschung* (unity of teaching and scientific research, cf. Madsen & Winsløw, 2009). Before delving further into research based on the subdivisions of university teaching *métier* that were introduced above, we shall dwell on research into how such double occupancy of teaching and research positions may influence and shape the *métier*, and in particular how the first position may lend and draw professional traits from the latter.

#### 4. RESEARCH ON THE TEACHING-RESEARCH NEXUS

The position  $\mu$  as researcher of mathematical sciences (including, for instance, statistics and various fields of applied mathematics), clearly *is* a profession which, from the 19<sup>th</sup> century, has become firmly established at university institutions all over the world. Mathematical knowledge and expertise was of course of great societal importance even before, but was often developed with and under the auspices of other disciplines such as astronomy, geodesy, mechanics and (from the 18<sup>th</sup> century) engineering. However, far into the 20<sup>th</sup> century most university mathematicians were primarily teachers. According to Tucker (2013, p. 699), who focuses on American universities:

While in the early 1950's most faculty at doctoral institutions still saw undergraduate teaching as their primary mission, by 1970 that mission had changed with research becoming the primary focus of these faculty.

This reflects a more general development in many (not only American) university institutions over the 20<sup>th</sup> century (Cuban, 1999): the increasing priority of research over teaching both in the tasks characterizing university faculty positions, and in the selection of individuals to fill those positions. In particular, occupying position  $\tau$  can be a mere corollary of occupying position  $\mu$ .

The position  $\mu$  is clearly what Halmos (1985, p. 400) talks about in the following quote:

I spent most of a lifetime trying to be a mathematician – and what did I learn? What does it take to be one? I think I know the answer: you have to be born right, you must continually strive to become perfect, you must love mathematics more than anything else, you must work at it hard and without stop, and you must never give up.

The role of commitment and personal ability is evident also in similar descriptions of the research profession by other mathematicians. At first sight, it would look like being a mathematician is merely a personal affair. At the same time, the professional character of the research *métier* is evident for mathematics for the same reason as in other research areas (system of training and regulating access to position  $\mu$  are both internal to the *métier*, as is the system for developing new explicit knowledge relative to the *métier*).

Halmos, who was a famous textbook author and an eminent lecturer, feels guilty after a day where he taught well but did not do research (*ibid.*, p. 322):

Despite my great emotional involvement in work, I just hate to start doing it; it's a battle and a wrench every time. Isn't there something I can (must?) do first? Shouldn't I sharpen my pencils, perhaps? (...) Yes, yes. I may not have proved any new theorems today, but at least I explained the law of sines pretty well, and I have earned my keep.

Here, “work” is evidently research, and “sharpening pencils” becomes a metaphor for distractions from  $\mu$ -tasks – including teaching (“the law of sines”). Certainly, filling  $\tau$  well does not make up for failures to accomplish the tasks of  $\mu$ .

Despite this competition for time, it is also a classical idea that teaching and research can somehow resource and inspire each other – in the higher education literature, one speaks of a teaching-research nexus (Neumann, 1992), reflecting that the link between the two is by no means simple or one-way. Of course, we often think about “research based teaching” as teaching that somehow draws on research, but to the extent knowledge is produced by research, this is somewhat trivial. It becomes less trivial if we think of the individual occupant of  $\mu$  and  $\tau$  - so that somehow the concrete research activity of the individual influences the same person’s teaching. In between, one could conceive of how the teaching of a larger or smaller institution (department or university) is affected by its research activity. Finally, there might be influences from teaching on research at both individual and institutional level, such as including students in research activities.

For the case of mathematics, not many studies exist of the teaching-research nexus. Winsløw and Madsen (2009) modeled it in terms of praxeologies of research and teaching,  $\mathcal{P}_R$  and  $\mathcal{P}_T$  for short. Each of these centrally include mathematical praxeologies (used or developed during research and teaching) but not be limited to them – for instance, research praxeologies could be broadly conceived to include practices of publishing, funding, communication, etc. In the study, five researchers in pure mathematics were interviewed on teaching practice, their research practice and connections between them (in that order). Not surprisingly, the links between  $\mathcal{P}_R$  and  $\mathcal{P}_T$  turn out to be rather “indirect”, in the sense that the mathematical praxeologies involved in the two are normally quite distant. As one of the informants says (ibid., p. 756):

How long did it take me before I had an impression of what is going on in the research area that interests me? Well it took 5 years, after I had graduated.... You can’t tell a bachelor student what it is about can you?

In other words, the mathematical praxis and logoi in undergraduate teaching is very far from that involved in the research of the informant.

At the same time, four out of five informants insist with considerable energy and many examples that considerable indirect links exist at the level of types of task. For instance, this involves the didactical task of constructing challenging assignments for students, and also the similarity between the activities aimed at for students (when working on the assignments) and the research task. While solving teaching tasks, one may discover points that later, maybe by association or further development, can be used in research. Many other links are mentioned to illustrate the experience expressed by one informant: “I feel that I can get things forth and back between the two parts.” We can say that the teaching-research nexus is largely implicit and indirect, and it concerns mainly the level of practices which are not the same but somehow similar, while it does reach the level of logoi.



The fact that the experience of students in university mathematics is not always very close to mathematical research was emphasized as a motivation for Burton's (2004, p. 27) interview study with 70 researchers of mathematics:

It was my hope that a gap, between how mathematicians themselves came to know and how they promoted learning in others, if confirmed in the study, would help to explain student disaffection with the discipline, their difficulties in learning it, and the gender bias in those who take it up as a career.

The study itself considers mainly the creative part of research practices, the “enquiry”, through which the individual researchers are supposedly learning (new) mathematics. This activity is found to be quite different from what university students are asked to do, for instance because it involves intuition, room for different “thinking styles” and so on. The study does not investigate the experience of university students but relies, for this part, on the author's personal experience. A main value of this study is the detailed analysis of a large and exciting survey of how  $\mathcal{S}_R$  is described by people in position  $\mu$ . Direct observation of these praxeologies is obviously difficult to arrange, which explains the use of interview methods in both of the above-mentioned studies.

Similar methods were used in Misfeldt's (2006) study of the writing practices of researchers in pure and applied mathematics, also in view of informing didactical practices at university to enhance students' learning in this area. However, in this study, interviews were carried out in the presence of material traces of current research of the informants, in the form of pieces of writing. In particular, three distinct modes of writing were found: exploratory writing (often involving more diagrams and the like, than linear text), “first drafts” (attempts to write out proofs etc. in a linear fashion, often by hand) and “article text” (produced with TeX or similar software).

From all three studies, we can see that possible links between students' and researchers' praxeologies are quite indirect, and maybe even missing (Burton's “gap”). The motivation of all three studies is that when university teachers are also researchers one could hope that those teachers could somehow produce a didactic practice that could make the link closer. In fact, much of recent didactical engineering research on university mathematics is more or less explicitly based on this idea: draw on teacher-researchers' experience with research to create a didactical practice that somehow enable students to do mathematics in ways that are similar to mathematical research. It could also be that students are to engage in mathematical activities close to what is found in another professional field or discipline that they will subsequently encounter. This brings us back to the different teacher positions ( $\tau_{1,\mu}$ , etc.) in the university mathematics teaching *métier*, and research on controlled experiments with corresponding didactical practices.

## 5. THE TEACHING MÉTIER BY STUDENT POPULATIONS

We now return to take a closer look at experimental research involving cases of  $\tau_i$  ( $i = 1,2$ ) and the didactical praxeologies undertaken by teachers in these positions. We note in passing that in some university type institutions, such as liberal arts colleges in the

US, there is a strong tradition for teaching mathematics to “all” students, including those whose study and professional aims do not strictly need postsecondary mathematics. Such teaching often focuses on historic and otherwise humanistic aspects of the discipline (Fried, 2018). At other universities (including, to my knowledge, most European universities) many students have no mathematics courses, and positions  $\sigma_i$  (and hence  $\tau_i$ ) may not exist at all.

### 5.1 Research-like situations in undergraduate analysis

A recent survey of undergraduate pure mathematics programmes (Bosch et al., to appear) confirms that their structure offers many similarities, and all programmes analyzed include a calculus-analysis sequence of modules as a central component. The calculus vs. analysis distinction is in general somewhat blurred and contingent upon local conditions. However, a quite common interpretation can be made precise in terms of praxeologies (Winsløw, 2008; Gyöngyösi, Solovej & Winsløw, 2011; Winsløw, 2015; Kondratieva & Winsløw, 2018). Calculus praxeologies  $\omega$  involve types of tasks concerning functions that are given in closed form, including functions of one or more real variables, real and complex valued sequences, and so on; differential and integral calculus are among the central sectors. In calculus courses, the student relationship  $R_U(\sigma, \omega)$  aimed at (and especially, assessed) is often focused on students’ mastery of *techniques* corresponding to a well-defined set of type of tasks. Thus, students will learn techniques to evaluate certain integrals, find extremum points of functions in certain situations, etc.; the logos block is on the other hand more informal, compared to analysis courses. These, on the other hand, focus on *theory*, and first analysis courses may merely complement calculus courses with formal theory, and thus in a sense extend  $R_U(\sigma, \omega)$ ; we call this type of extension a *type I transition*. In more advanced Analysis courses, students face type tasks that are formulated in terms of previous logos blocks (e.g. investigate whether this or that function is a complete metric on  $R^n$ ), thus working with entirely new and generally more “abstract” praxeologies. We call this kind of passage to new praxeologies, in which tasks concern logos from previously developed praxeologies, a *type II transition*.

To support type I transitions, it is evidently crucial to create *links* between familiar praxis from Calculus, and the new theoretical superstructure. One strategy to do so, pursued by Gyöngyösi et al. (2011), is to design student assignments which involve new theoretical material which are explored based on computer supported experiments with objects from Calculus. An example was to “explore the convergence properties of the sequence of functions given by  $f_n(x) = 1/[1+\exp(n(2-x))]$ ”. Based on plots in software like Maple, the students quickly see the pointwise convergence of  $(f_n)$  to a non-continuous function, and infer that the convergence is not uniform. They can also, based on Maple calculations, verify that the limit function has the same integral over any interval as the limit of the integrals of  $f_n$ , and hence that uniform convergence is only a sufficient but not a necessary condition for the interchanging limit and integral. Experiments with such designs in a first-year analysis course showed that a “middle

group” students effectively improved their results when such assignments were included, while it made no difference for high- and low-performing students.

The transition of type II could be attacked with similar ideas for task design, but naturally going further than the mere illustration and application of new theory. Grønbaek and Winsløw (2007) experimented so-called *thematic projects* which are relatively long assignments that proceed from more closed tasks (of the type found at the end of chapters in text books) towards open questions that require students to device and prove a theoretical result, which is naturally supported by the first parts. Instead of an oral exam based on students’ presentation of material directly extracted from a textbook, the students should now present one of their thematic projects. Experimentations over several years with thematic projects in a real analysis course demonstrated a significant increase in students’ work, satisfaction and results, as measured by the standard exam; however, the increased work for the course teachers (both to create new assignments and to provide adequate supervision to students) made the format less viable outside of a funded project.

Very similar experiments were made later in the context of a less advanced real analysis course (Gravesen, Grønbaek and Winsløw, 2017). In a project funded by a University of Copenhagen grant to further the connections between research and undergraduate teaching, we defined a number of *research-like* activities, and constructed a collection of exercises that would engage students in some of these (for each exercise). Of course, mathematical research is not limited to “prove that” activities, while these dominate end-of-chapter exercises in many post-calculus textbooks. Among the activities explicitly focused on in this design, students were to use special cases to investigate an abstract hypothesis or question, to formulate a hypothesis for a given question, to formalize relations between two or more results, to produce or validate  $\varepsilon$ - $\delta$  type definitions, etc.

Another idea for more advanced courses, developed by Kondratieva et al. (2018) but yet to be tested at larger scale, is to link calculus praxis with proofs of major theoretical results in analysis. As an example, a student assignment was developed in which the so-called Basel problem (convergence and value of  $\sum(-1)^n/n^2$ ) is solved by calculus techniques, and then the same sequence of techniques is used to give an elementary proof of Dirichlet’s theorem on Fourier series. The construction of assignments that relate different domains in mathematics, or (as here) basic and more advanced courses in the same domain, is proposed as a strategy for task design research linked to Klein’s idea of “Plan B” (cf. Klein, 2016, p. 83).

Perhaps we can formulate two overall conclusions emerging from these and many other experiments with task design that aims to create “research like” situations for undergraduate students:

- 1 The revised didactic practice  $\Pi(\omega)$  can certainly be realized and as a result, more ambitious aims for  $R_U(\sigma, \omega)$  are in fact realized;

- 2 It is much less straightforward to establish  $R_U(\tau, \Pi(\omega))$  for the position  $\tau$  as such (rather than for an individual teacher in privileged circumstances), like when  $\Pi(\omega)$  requires time-consuming design (e.g. of new assignments).

We close this section by briefly examining a more famous and generic parallel to (1), which offers in some sense also a counterexample to (2): the so-called *Moore method* (described, for instance, by Halmos, 1985, pp. 255 ff.) to teaching theoretical mathematics. Moore was a professor of mathematics at the University of Texas from 1920 to 1969, and a legendary teacher and doctoral supervisor. Over the course of the past century, his methods of teaching expanded and developed several variations in several North American universities. According to the “Moore method” article on Wikipedia (as it looked in December 2020), dozens of professors and departments use some version of it even today.

The core of Moore’s method is to let students (re)construct proofs of given theorems, with no use of books or other sources, but referring only to a handout with Definitions and Theorems carefully prepared by the teacher. The method apparently works for any specific mathematical praxeology or domain, except for the clear focus on formal proof (which is anyway common in almost any post-calculus course in pure mathematics). In that sense the method is a set of pedagogical techniques to teach proof, while the didactical practice  $\Pi(\omega)$  comes with the concrete handout for a given set of praxeologies  $\omega$ . It would be very interesting to investigate the institutional and historic conditions that enabled the success of this approach. It is a rather certain hypothesis that one important condition has been the existence of a well-developed “logos” on the didactical techniques, disseminated in several books written for and by members of the *métier*. The method has not only been transmitted but also further developed by some of these members (see for instance Chalice, 1995). It appears from some of these writings that not only descriptions of the pedagogical techniques, but also examples of handouts for concrete praxeologies  $\omega$ , have been disseminated widely. It remains that the method also shares the challenge of design by the teacher, to the extent  $\Pi(\omega)$  has to be constructed for a given unit of teaching, in view of concrete student populations and specific praxeologies  $\omega$ .

According to the literature referred to, the Moore method is found to offer an excellent experience for students in position  $\sigma_1$ , as considered in this section. However, at the undergraduate level, such students usually mix with students in position  $\sigma_2$ , for whom training to prove theorems may not be as important. Apparently, the method works best with advanced courses and hardworking students, who are more or less clearly in position  $\sigma_1$ . It is still remarkable as a case of sustained, explicit development of shared didactical practice by the *métier* itself, which moreover connects clearly to an important aspect of mathematical research, the construction of formal proofs. It is a fair hypothesis that this implicit or even explicit link between  $\mathcal{P}_R$  and  $\mathcal{P}_T$  contributed to the success of Moore’s way to organize  $\mathcal{P}_T$ .

We now turn to studies of the *métier* of teaching mathematics to students in position  $\sigma_2$ , focusing on two variants of this type that are both more common than  $\sigma_1$  in most university institutions.

## 5.2 Klein's second discontinuity: the case of teacher students

We first consider the case of  $\sigma_{2,t}$  or, for short,  $\sigma_t$ : university students who prepare to become mathematics teachers outside of the university, generally at primary or secondary level. In some universities, the position  $\sigma_t$  is found in specialized programs apart from  $\sigma_1$ . In other universities – including many universities in Europe, particularly when it comes to future secondary mathematics teachers – the two positions are indiscriminate at least in the first years of study. In other words, the same requirements are made for  $R_U(\sigma_t, \omega)$  and  $R_U(\sigma_1, \omega)$ , for a good deal of the mathematical praxeologies  $\omega$  that are taught at  $U$ . In this situation, which appears common in many universities even today, Klein (2016, p. 1) identified a major problem as early as 1908:

The young university student finds himself, at the outset, confronted with problems, which do not remember, in any particular, the things with which he had been concerned at school. Naturally he forgets all these things quickly and thoroughly. When, after finishing his course of study, he becomes a teacher, he suddenly finds himself expected to teach the traditional elementary mathematics according to school practice; and, since he will be scarcely able, unaided, to discern any connection between this task and his university mathematics, he will soon fall in with the time honoured way of teaching, and his university studies remain only a more or less pleasant memory which has no influence upon his teaching.

Klein identifies these two phases of “forgetting previous mathematics” as the first and second discontinuity. The first is a general problem of transition *from school to university* that has been the subject of much (if not most!) research on university mathematics education, given the struggles in which many students  $\sigma$  find themselves in (irrespectively of future orientation). The second discontinuity, *from university to school*, is specific to  $\sigma_t$ , or rather to the passage

$$R_U(\sigma_t, \omega) \rightarrow R_S(t, o)$$

where  $S$  is naturally the school institution and  $t$  the position as teacher and where  $o$  designates school praxeologies. The change of praxeologies correspond to the fact that what Klein calls “traditional elementary mathematics according to school practice” is at best somehow related to mathematical praxeologies  $\omega$  met at  $U$ , “his university mathematics”. In fact, Klein also emphasizes that the position  $t$  requires not only a relation to  $o$  but also the “task” to “teach” it “according to school practice”; a more accurate representation of this passage is thus

$$R_U(\sigma_t, \omega) \rightarrow R_S(t, \Pi(o)).$$

If we represent the full story of Klein's unfortunate character, who starts out in the position as school student  $s$ , we then get

$$R_S(s, o) \rightarrow R_U(\sigma_t, \omega) \rightarrow R_S(t, \Pi(o))$$

under the assumption that  $o$ , “elementary mathematics according to school practice” does not change too much while our friend is at university (which may, in fact, be somewhat incorrect in times of curriculum change).

The point of Klein’s book is that universities need to take more responsibility when it comes to enrich  $R_U(\sigma_t, \omega)$  with explicit links between  $\omega$  and  $o$ . In fact, most of the text consists of revisiting elements of  $o$  – especially within the domains of arithmetic, analysis and geometry – from the “higher standpoint” (as the title says) of  $\omega$ . The text, indeed, resulted from Klein’s own lectures to future teachers during the preceding decades, following his inauguration as professor at the University of Erlangen in 1872. We could represent this effort as an attempt to “smoothen” the second discontinuity by adding a relation to be developed (cf. Barquero and Winsløw, in preparation):

$$R_S(s, o) \rightarrow R_U(\sigma_t, \omega) \rightarrow R_U(\sigma_t, \omega \cup o) \rightarrow R_S(t, \Pi(o))$$

where the subject matter of the “Klein course” is naturally not supposed to be a disjoint union  $\omega \cup o$ , but to emphasize links and overlaps.

The emergence of Didactics of Mathematics (or mathematics education research, in Anglophone countries) as a scientific discipline, both results and departs from this project, particularly from the sixties onwards. On the one hand, Klein type courses were established at many universities (in Germany, often specialized in domains, labeled Didactics of Analysis and so on; in USA as so-called “capstone courses” which are also offered at the end of several other professional university degrees). Still, the last passage  $R_U(\sigma_t, \omega \cup o) \rightarrow R_S(t, \Pi(o))$ , may remain somewhat discontinuous, given that  $\Pi(o)$  is more than  $o$ . In many countries, official systems of “induction” into the teaching *métier* are offered (see e.g. Britton, Paine & Pimm, 2003) to take care of the passage to the praxis  $\Pi(o)$ , with more or (often) less attention to the specificity of  $o$ . This, in fact, means, that yet another relationship is added to smoothen the second discontinuity, between  $R_U(\sigma_t, \omega \cup o)$  and  $R_S(t, \Pi(o))$ . This may involve both university course units, given by specialists of Didactics of Mathematics or Pedagogy, who may introduce more or less subject specific elements of logos  $\Lambda(o)$  related to elements of practice  $\Pi(o)$  in school. One could then pose the complete model of mathematics teacher education that exists today, with local variations (such as leaving out entire relations aimed at):

$$R_S(t, o) \rightarrow R_U(\sigma_t, \omega) \rightarrow R_U(\sigma_t, \omega \cup o) \rightarrow R_{U/S}(\sigma_t/t, (\Lambda(o), \Pi(o))) \rightarrow R_S(t, \Pi(o)).$$

More can be said about this last extension, and especially of the frequent absence of logos in the last relationship (see Miyakawa and Winsløw, 2019). However, from the point of university mathematics education, which is assumed in this paper, the second passage  $R_U(\sigma_t, \omega) \rightarrow R_U(\sigma_t, \omega \cup o)$  is of special interest, as it concerns university teaching of mathematics. It is still important to bear in mind that this passage is very often followed by training more directly related to  $\Pi(o)$ .

In this vein, let us first recall the considerable body of research which, beginning with Begle's (1972) first demonstrations that  $R_U(\sigma_t, \omega) \rightarrow R_S(t, \Pi(o))$  does not succeed better (in terms of performance of the students of  $t$ ) simply because  $\omega$  (measured as numbers of advanced courses taken) was larger. Later studies refined his results and nuanced the view both from mere volume to a closer look at contents. Without going into details that are better explained elsewhere, the following recommendation seems still to be of current, consensual value, at least in the United States:

Prospective high school teachers of mathematics should be required to complete the equivalent of an undergraduate major in mathematics that include three courses with a primary focus on high school mathematics from an advanced viewpoint (CBMS, 2012, p. 18).

We are thus faced essentially with the proposal of Klein, when it comes to the university responsibility to prepare  $R_S(t, \Pi(o))$ , in the case of upper secondary school  $S$ : establish  $R_U(\sigma_t, \omega \cup o)$  with a "primary focus" on  $o$ , but linking it to the "advanced standpoint" of  $\omega$ . It is an important challenge for the university mathematics *métier* to identify what  $R_U(\sigma_t, \omega \cup o)$  could best function as stepping stone towards  $R_S(t, \Pi(o))$ , and to implement didactical practices that can establish such  $R_U(\sigma_t, \omega \cup o)$ . The complexity of this task is evident, and probably more acute than in the time of Klein, where very small minorities reached the position  $s$ .

To solve this task evidently requires a teacher relation  $R_U(\tau, \Pi(\omega \cup o))$  which is not immediately derived from  $R_U(\tau, \omega)$ , although it *also* involves this relation. But in addition to that, to design  $\Pi(\omega \cup o)$ , requires a relation  $R_U(\tau, R_S(t, \Pi(o)))$ , where the complexity is even more evident. Such expertise, on the other hand, is in principle held by the faction of the university mathematics *métier* who engage in *empirical research* not only on mathematics teaching in secondary school, but also on *secondary mathematics teacher knowledge*. This field, of course, is currently under development, and is only slowly getting specialized enough to capture specific praxeologies  $o$ . At any rate, we can summarize this theoretical discussion by agreeing that devising and adjusting courses (three, perhaps?) is an excellent opportunity to combine and mix expertise from both teaching and research in mathematics including but also beyond the classical domains of mathematics. We return to this in Section 6.

As an example of evidence from newer "Klein type" courses, Winsløw and Grønbaek (2014) conducted an analysis of student challenges in such a course at the University of Copenhagen, based on praxeological analysis along the lines outlined above. One of the striking observations was that even  $R_U(\sigma_t, \omega)$  may have to be developed in such a course. Within the same context, Barquero et al. (in preparation) will delve further into specific challenges when it comes to students' perception on and challenges with praxeologies  $\omega \cup o$  related to the real number system.

The education of future school mathematics teachers may be of special interest to scholars in university mathematics education, as they are often also teacher educators.

However, when it comes to the *métier* of university mathematics teaching – our object of research – there are other target *métiers*, if not professions, which are equally important, if not more so. We now turn to a major example: future engineers.

### 5.3 Authentic Problems of Engineering in first year mathematics

As for teacher education, the role of mathematics in engineering education has been the subject of numerous policy papers. Naturally engineering programs include different specialties and academic levels, whose mathematical needs vary significantly. An engineering student  $\sigma_e$  encounters mathematical praxis and logos in many different settings of the university study, but “mathematics courses” (given by members of the university mathematics *métier*) appears mostly in the first year or two of undergraduate studies. Whatever mathematical praxeologies  $\omega$  that  $\sigma_e$  studies then, the aim for  $R_U(\sigma_e, \omega)$  is to prepare and facilitate the establishment of relationships of type  $R_U(\sigma_e, \varepsilon)$  where  $\varepsilon$  is some praxeology from engineering courses at large, in which mathematical practices or logos related to those of  $\omega$  appear.

A main problem for university mathematics education in this context is that the transition (or knowledge transfer) from  $R_U(\sigma_e, \omega)$  to  $R_U(\sigma_e, \varepsilon)$  is not automatic, even when  $\omega$  and  $\varepsilon$  are actually bridged by the expert (or teacher) of both. For instance, in the context of a signal theory course, Hochmuth, Biehler and Schreiber (2015) investigated specific ruptures between the mathematical model of “Dirac impulse” treated (and calculated with) in this course, and the technology associated to functions, limits, and distributions in mathematics courses. The techniques required to solve associated problems in the signal theory course, which involve operating with functions that assume the value  $\infty$  at isolated points, “do not fit with higher mathematics discourses (technologies)”. The students have somehow to learn that they should neglect specific aspects from those discourses” (ibid., p. 696).

Another, related problem, concern the specialized *métier*  $\tau_e$  of teaching mathematics to students in position  $\sigma_e$ . The classical solution is that the mathematical praxeologies  $\omega$  to be taught are simply some subset of what is taught to  $\sigma_1$  (including extensive work on formal logos with proofs etc.). While this model still exists in some countries, it seems to disappear in many places due to problems with students’ motivation, attrition and transfer (Pohjolainen et al., 2018). Hernandez-Gomes & González-Martín (2016, 2020) found that teachers’ relationship  $R_U(\tau_e, \omega)$  with basic calculus topics depended significantly on their scholarly background (which included pure mathematics, mathematics education, mechanical engineering and electrical engineering), and on the teachers’ corresponding experiences as undergraduate students. For instance, only the university mathematics teachers with an engineering background had precise ideas about how specific mathematical techniques appear (or do not appear) in the engineering program. In the context of supervising capstone projects, professional experience from engineering institutions outside of the university is also of considerable importance, even when it comes to the ways in which teachers assist students with mathematical techniques. These case studies mainly suggest that



different institutional backgrounds offer somewhat different qualities to the position of  $\tau_e$ .

To improve students' motivation to develop, apart from exam requirements, and also to prepare the transfer of type  $R_U(\sigma_e, \omega) \rightarrow R_U(\sigma_e, \varepsilon)$ , it is an interesting strategy to integrate some concrete tasks from  $\varepsilon$  in the mathematics course, which can be used to show the relevance to engineering of techniques and logos from  $\omega$ . In his doctoral thesis, Wolf (2017) carried out an ambitious project on designing and experimenting *application-oriented exercises* in a first-year mathematics course for students of mechanical engineering. The applications were “authentic” in the sense that problems and data were taken from professional contexts of machine construction. The authenticity was ensured by collaboration with university teachers of engineering.

Schmidt and Winsløw (to appear) investigate a similar, but more longitudinal and entirely spontaneous collaboration pattern, focused on designing *authentic problems of engineering assignments* for a first-year mathematics course with more than 1100 students every year. They describe the explicit principles that have developed, through practice but also from leadership in the position  $\tau_e$  to facilitate collaboration with scholars of engineering fields, who often produce a first draft of the assignment, which is subsequently revised and implemented by (mostly) mathematics faculty in position  $\tau_e$ . The assignments appear in a first-year course on calculus, linear algebra and differential equations, and the main challenges for students are thus to be of a mathematical nature, combining several praxeologies taught in the course. Still, the mathematical model is built up from an authentic problem of engineering. Here, “authentic” in that it comes from recent publication in scholarly engineering. To organize the systematic collaboration with institutions of scholarly engineering, informal didactical logos was developed from the position  $\tau_e$ , adding a professional trait to this specific form of the university mathematics métier.

## 6. A PROPOSAL FOR THE FUTURE

In the preceding two sections, we have analyzed instances of the variety of positions and knowledge bases that the university mathematics métier is currently based on. Both scholarly knowledge, coming from engaging in research praxeologies, and knowledge built from didactical praxis, contribute to this knowledge basis. In some cases, didactical practice is supplied with more or less strongly developed logos, which often takes on relatively generic forms. Experiments initiated by scholars specializing in university mathematics education research appear to be mostly punctual, while we have also identified instances of more sustained and explicitly framed efforts to develop the métier from positions  $\tau_1$  and  $\tau_2$ . As is the case for mathematics teaching in other institutions, the development of professional – shared, explicit and practice-specific knowledge – remains quite limited and local. Professional journals focusing on university mathematics teaching do exist in some countries like the USA, but even then, there seems to be a considerable distance, in terms of logos and readership,

between these and scholarly publications in the field of university mathematics education research (such as the present volume).

Indeed, the university mathematics teaching *métier* remains, to a large extent, a secondary occupation of various professions of scholarly research in the mathematical sciences, interesting new forms of collaborations emerge especially in the position of type  $\tau_2$ . Still the formal preparation for occupying such positions seems to be mainly pedagogical, as a complement to the more substantial training for a scholarly profession. Reactions to external reform requirements often take the form of more or less minimal reconstruction of external didactical transpositions at least when it comes to undergraduate programs in pure mathematics (Bosch et al., to appear). In the parts of the *métier* catering to students in positions  $\sigma_2$ , more significant developments appear, while in all cases, massive challenges with attrition and failure remain evident and perhaps even growing in many universities.

As a result, the *impact problem* for research in university mathematics education is pointed out in several recent syntheses (e.g. Winsløw, Gueudet, Hochmuth and Nardi, 2018, p. 71). The current institutional model separates, largely, such research from the university mathematics teaching *métier*. To seek impact of an external scholarly field on a *métier* of teaching implies two risks that are very well known from the teaching *métiers* at primary and secondary levels in many countries. The first is to continue to fail. The second is to succeed, at least to some extent, but to have merely “robbed teachers of the opportunity to participate in the development of new knowledge about teaching” (Stigler et al., 1999, p. 174).

To avoid these risks, a new nexus between teaching and research seems necessary. Of course, various blends of scholars and teachers have appeared spontaneously under current institutional conditions, as has now and then also appeared in some of the efforts outlined here. However, the vast majority of the professional training of PhDs in mathematical sciences remains totally disjoint from the preparation of PhDs in Didactics of Mathematics, including those focusing on university mathematics education. Why would the latter not include some level of further mathematical education and experience with research in some mathematical domain? And why not include elements of education and research in the didactical domain? Should future teachers of university mathematics not be prepared to engage in (rather than, hypothetically, be mute consumers of) didactical research on university mathematics? What are the institutional and intellectual conditions under which it would be realistic to establish mixed doctoral programs (of various compositions) that could prepare for shared and fruitful professional development of the *métier* in all its different forms?

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