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Introducing group theory with its *raison d'être* for students

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This paper reports results of our sequence of didactic situations for teaching fundamental concepts in group theory—e.g., symmetric group, generator, subgroup, and coset decomposition. In the situations, students in a preservice teacher training course dealt with such concepts, together with card-puzzle problems of a type. And there, we aimed to accompany these concepts with their raisons d'être. Such raisons d'être are substantiated by the dialectic between tasks and techniques in the praxeological perspective of the anthropological theory of the didactic.

Keywords: 2. Teaching and learning of specific topics in university mathematics, 9. Teaching and learning of linear and abstract algebra, Group theory, Raison d'être, Praxeology.

INTRODUCTION AND MAIN THEORETICAL RESOURCES

Abstract algebra is one of the major areas of undergraduate and graduate mathematics. And it has been pointed out by several authors that many students have difficulty in transiting from elementary algebra to abstract algebra (e.g., Dubinsky et al., 1994; Hausberger, 2017, 2018; Bosch et al., 2018). In abstract algebra, we consider the algebraic structures, group, ring, field, etc, which inherit the property of the familiar calculations among numbers or equations as objects of elementary algebra. Among such algebraic structures, group is the simplest but difficult topic. The reason is that a group has only one operation from the first, and non-commutativity is primary in group theory: these properties cannot be observed in standard number systems. Then, we problematize that, in the teaching of abstract algebra, students usually do not experience inquiry where the notion of group with one operation can grow. In fact, Bosch et al. (2018) points out that, regarding the learning of group theory, the *raison d'être* of group theory is seldomly questioned—roughly speaking, a *raison d'être of knowledge* means a problematic situation the study of which naturally produces the knowledge as a significant tool. According to this problematization of the teaching of group theory, let us pose our research question as the follows: *what problems and situations could become raison d'être of group theory for students?* Larsen (2009) seems to be a representative previous work which studies this question. Larsen's approach aimed students' reinvention of the concept of group and isomorphism in geometric context. In this paper, we would like to propose another possible approach with above features. As well as Larsen's approach, we assume teacher's guide, and we expect students to develop the notion of group, subgroup, generators of a group, coset decomposition on their interest and with proactive motivation through this approach. Let us highlight that our approach does not aim the complete abstraction process of the concept of group. Rather, this approach may be the introductory program of the

abstract group theory, which promotes students to notice the notion and to enhance the definition of group on their own consideration.

In the presentation and the analysis of our program, we make use of the *praxeology model* within the framework of the ATD, i.e., the *anthropological theory of the didactic* (cf. Chevallard, 2019). The praxeology in the ATD is a model for describing any bodies of knowledge, e.g., different mathematical domains, i.e. algebra, geometry, and so on. Such domains are based on their own theoretical foundations consisting of axioms, fundamental theorems, problematic questions, basic objects of study, etc. Each theoretical foundation is simply called a *theory* denoted by Θ in the ATD. In turn, a theory Θ describes and justifies many specific statements, particular objects of study, local problems, and so on. The system of such second-level theoretical entities is called the *technology* θ in the ATD. A theory Θ and a technology θ constitute a *logos* part of a given praxeology, i.e. $[\theta / \Theta]$. Let us emphasize here that any *logos* part originally comes from more concrete, specific, even *ad hoc* human actions which are called *praxis* parts of the given praxeology. The *praxis* part is reduced to two subparts of the type of tasks T and the technique τ , that is, $[T / \tau]$. A type of tasks T is any motivation of a given praxeology which is handled by some technique τ . The overall picture of a praxeology is denoted by $[T / \tau / \theta / \Theta]$. The order of emergence of each part of a praxeology depends on the program: a traditional group theory course may start from technology or theory. In our program, we propose to start the program from the extra-mathematical task of card puzzles explained below.

From perspective of the theory of praxeologies, we define the notion of *raison d'être* as any *system of interrelated tasks* (and each task of the system) satisfying the following two conditions: 1) it allows the inquirers to (re)produce a given whole praxeology (and praxeological elements); and 2) it is lively or familiar for the inquirers' viewpoint. Let us emphasize here that *raisons d'être* of praxeologies are *relative* and *changeable*. It depends on performers of the praxeologies like mathematicians and students.

THE CARD PUZZLE FOR TEACHING GROUP THEORY

Task design

Let us recall the studying process of vector and vector space. Among the algebraic objects taught in secondary mathematics, vectors have exceptionally different algebraic structures from number systems. In secondary mathematics, vectors are introduced in a constructive way based on planer or spatial geometry, not in the axiomatic way, and the focus is on individual vectors and their calculations. It is on this basis that concepts of abstract vector spaces are studied later. Thus, the study on the vectors performs as the previous step towards abstract linear algebra. Likewise, we would like to propose giving opportunities first to experience constructive algebraic structures which have only one non-commutative operation. Studying planer vectors is a nice beginning towards linear algebra because the planer vector space is easy to grasp and possesses typical properties as a vector space. Then, what is the most typical

group? We propose that the symmetric group S_n could be an appropriate candidate. In fact, it is well-known that any finite group is a subgroup of S_n for some large n . Moreover, S_n can be introduced in a constructive way, for example, S_4 can be introduced as permutating operations of 4 numbered cards.

The students in this experimentation belonged to a preservice teacher training course, therefore advanced group theory could not be included. We designed an introductory program dealing with symmetric groups, which would naturally introduce fundamental concepts in group theory without forcing their definitions from the first. This program was conducted in the first term in 2019 for three third-year undergraduate students as a seminar with the first author as a teacher. This seminar started in May and continues for two years; however, we focus on the first seven sessions. As the prerequisite knowledge, they had already studied fundamental set theory including concepts of map, injection, surjection, equivalence relation and quotient set in other courses.

The initial and central type of tasks T designed for the seminar is the following:

We have n cards arranged in a row and each number from 1 to n is written on one of the cards. The objective is to rearrange them in the ascending order using particular available operations only. Less number of operations is preferred.

We call each task $t^{(i)}_T$ of T , which consists of the number of cards and available operations, a *puzzle*. Based on T , several puzzles are proposed to students and regarding each puzzle $t^{(i)}_T$, students struggled not only with analysing them, but also with related questions raised in the analysing processes. Such related tasks will form other derived task types $*T$ and $**T$, which shall be explained in later sections. The first example is Puzzle 1 ($t^{(1)}_T$) indicated in Fig 1. It deals with 3 cards and the available operations are A , switching first and second cards, and B , switching second and third cards. All puzzles proposed to students are indicated in Fig 2.

Puzzle 1: 3 cards with following two operations A and B

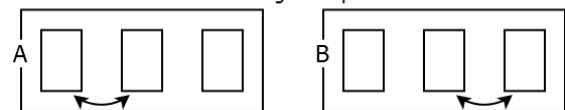
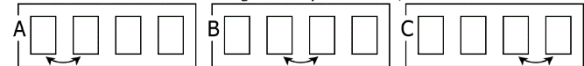


Fig 1: Puzzle 1

Puzzle 2: 4 cards with following three operations A, B and C



Puzzle 3: 3 cards with following two operations A and R



Puzzle 4: 4 cards with following two operations S and T



Puzzle 5: 4 cards with following two operations D and R



Fig 2: Puzzle 2, 3, 4, and 5

First session: From puzzle to non-numerical equational representation

In the first session, T was explained, and Puzzle 1 ($t^{(1)}_T$) was provided for students. Thus, this program started from an *extra-mathematical* context to analyse a puzzle, which was proposed without using any terms of group theory. As the solving method, it was supposed beforehand to draw the Cayley graph, which consists of vertices of all possible orders and edges each of which connects two orders possible to change one to the other by one available operation. Of course, students did not know Cayley graph, but we had expected them to spontaneously use such a graph representation. In fact, a student said:

Student A: Just 6 orders only are possible. Let's list them all.

Then they started to construct Cayley graph without knowing that term (Fig 3). Students produced successfully the method of drawing Cayley graph, which is a technical element $\tau^{(1)}_T$ in their praxeology (in this paper, we use “ τ ” for not only the whole technique but also its elements). This technique enables them to focus on and grasp the whole set of operations or orders. To emphasize that each operation corresponds to an order, the teacher stressed that 123 is the initial order.

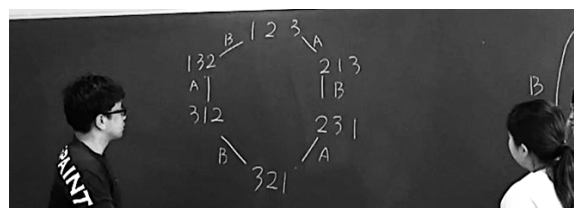


Fig 3: Cayley graph drawn by students

The following are students' remarks regarding this graph representation.

Student A: Operation B once and A twice are needed (to change 321 into 123).

Student B: Operations A and B should be done alternatively. Doing A twice is useless.

At this moment, they did not realize the binary operation among puzzle operations and their representations were based on the natural language. However, being asked how to change 321 into 123, they answered “ BAB or ABA ”; they used the composition operation unconsciously. It was after that time that the definition and the notation of the composition operation was confirmed: for two operations X and Y , we denoted the composition of them, doing X and then Y , by XY . One student said “it's like the multiplication”, and we discussed that $AB \neq BA$. Here, it should be remarked that these discussions were not rigorous because it was not defined what these operations mean mathematically. Strictly speaking, operations should be formulated as maps from $\{1,2,3\}$ into itself, however such theorisation was postponed until some later sessions, that is, the teacher did not intervene in the students' spontaneous praxeology to avoid developing praxeologies without any *raison d'être*.

Also, in this session, based on the above Student B's remark, we tried to express his remark as equations. This is a new derived task $t^{(1)*}_T$. Specifically, being asked what happens when the operation A is repeated, they considered how to express $2n + 1$ times repetition of operation A :

Student A: May be $(2n + 1)A$... oh, it's no good.

Asked by the teacher, they discussed that doing twice and thrice of A are AA and AAA respectively and it was pointed out by students that they look like powers of A . Thus, we negotiated that we note n times repetition of A by A^n and obtained the equation $A^{2n+1} = A$. Then we discussed how to proceed the task $t^{(1)*}_T$, that is, representing their findings that $2n$ times repetition of A is the same as doing nothing:

Student B: Doing nothing is 0 times repetition and it may be written as 0?

Teacher: If it's written as 0, A and doing nothing equals A , then $A0 = A$, right?

Students: Oh, it's 1. 1 is better!

In this way, they faced the new type of tasks $*T$, to express relations between operations and to find new relations between them, and with need in coping with these tasks $t^{(1)}_{*T}$, they developed a technical part $\tau^{(1)}_{*T}$ of representing the relations between puzzle operations in algebraic way, even though they are not algebraic objects for them at this point. This enables them hereafter to use elementary algebraic representations and techniques to express and deal with properties among puzzle operations paying attention to the non-commutativity.

Second and third session: inverse element and order of elements

At the end of the first session, we agreed what puzzle to analyse next, i.e., Puzzle 2 ($t^{(2)}_T$). Thus, at the beginning of the second session, students presented their result (Fig 4). Compared to puzzle 1, the structure of Puzzle 2 is rather complex, however, they engaged rather lively in elaborating the Cayley diagram, because it enables them to see the whole perspective of the permutations in $t^{(2)}_T$: $\tau^{(1)}_T$ functions as a tool to carry out $t^{(2)}_T$. In the process, they naturally began to consider 1234 as the initial order to operate, and to regard operations as permutations as well as in the case of puzzle 1.

After they elaborated this graph, they found many relations like $(AC)^2 = 1$, $(AB)^3 = 1$ and so on. Referring these equations, it was natural to ask the following question: for any operation X , does a natural number n exist, such that $X^n = 1$? This is a new task $t^{(2)}_{*T}$ of $*T$. After some empirical attempts, they could understand that, if we assume that an operation X had no natural number n such that $X^n = 1$, there would be different natural numbers i and j such that $X^i = X^j$. Thus, we focused on the question whether the statement “ $YX = ZX$ implies $Y = Z$ ” is true or not; if it is true, we can have X^{j-i} or $X^{i-j} = 1$. In this context, we defined the inverse of an operation, and confirmed all operations have its inverse. Here, we can point out that the inverse element was introduced not for the axiom of group, but for the solution to the emerged question.

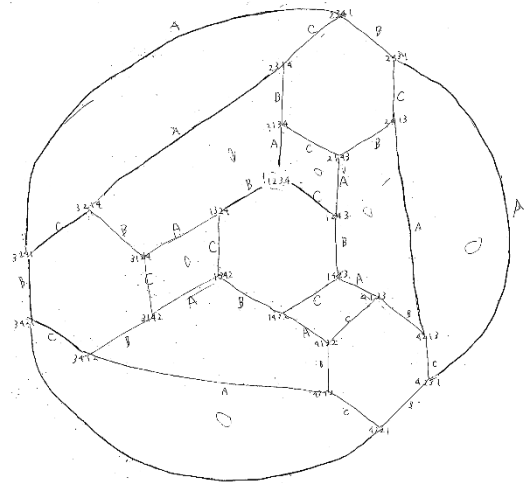


Fig 4: Cayley graph of puzzle 2 drawn by students.

In the third session, we dealt Puzzle 3 ($t^{(3)}_T$), which includes operation R whose inverse R^{-1} is different from the original. Thus, the notion of inverse became more clarified. Puzzle 3 was selected by the teacher for this purpose.

Through carrying out $t^{(2)}_{*T}$ and $t^{(3)}_T$ in these sessions, the students elaborated the concept of inverse elements ($\tau^{(2)}_{*T}$). In more detail, the task $t^{(2)}_{*T}$ required the technique $\tau^{(2)}_{*T}$ and this technique enabled them to carry out $t^{(3)}_T$. Besides this, the students became to use algebraic expressions frequently and naturally in their discourse. It seemed that permutations had become algebraic elements, which can be dealt in algebraic ways through the composition operation.

Fourth session: Symmetric group and its generators

Until the third session, every analysed puzzle involved all permutations, however Puzzle 4 ($t^{(4)}_T$) had been assigned at the end of the third session and they had found that is not always true. Puzzle 4 is also selected by the teacher to highlight this phenomenon. In the fourth session they drew the Cayley graph of Puzzle 4 from the initial order 1234 as usual and found it involves only 12 permutations. Moreover, students, as it was expected, added an optional graph which involves permutations that do not appear in the original graph (Fig 5).

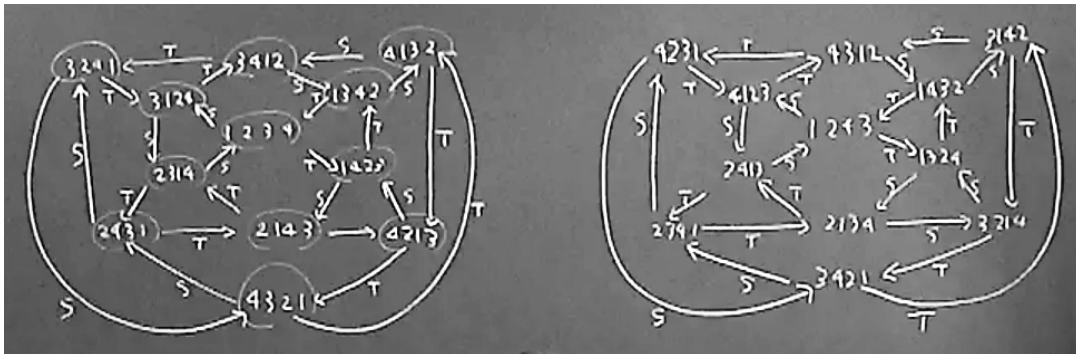


Fig 5: Cayley graph of puzzle 4

Then, they faced the question: why this diagram split into two parts, what is the difference from the previous puzzles? This is a new task $t^{(1)**T}$ belongs to the new type of tasks $**T$ involving the structure of the group. To discuss them, it was necessary to formulate what operations are mathematically and to make the discussion more rigorous. On this account, we negotiated the notation representing a bijection f from $\{1,2,3,4\}$ to itself: f is indicated by the result (written in framed numbers) of the corresponding permutations of 1,2,3,4. (This is different from the standard notation of permutation, however, their recognition is based on the permutation of 4 cards, this notation was rather acceptable for them.) For example, $\boxed{1234}$ means the identity permutation and $\boxed{1243}$ means the transposition of 3 and 4. And here the term “symmetric group of degree n ” and the notation S_n was introduced by the teacher. Also, it was confirmed that the vertices of the graphs we have elaborated correspond to the elements of S_3 or S_4 . Thus these formulation technique $\tau^{(1)**T}$ was carried out for the need to proceed the consideration on $t^{(1)**T}$.

Next, students were asked to explain the difference between the situation of Puzzle 4 and those of previous ones, for example Puzzle 2. They tried to express it:

Student B: In the case of puzzle 2, all elements of S_4 can be *made* by A, B, C , however it isn't in this case.

Teacher: What do you mean by “can be *made*”? Please explain more precisely.

Finally, they could not express it clearly, however, they entirely agreed on the description given by the teacher: “in the case of puzzle 2, all elements of S_4 can be expressed as a finite composition of A, B, C , and their inverses in some order”. It was this moment that the definition of generators is provided. Here we remark that, though

the definitive expression was finally given by the teacher, their cognitive process was rather different from the usual studying process which starts from providing definitions. In fact, they *could* consider how to express it, since they could distinguish the case of being generated from otherwise. These notions of generators and property of being generated should be the new technique to cope with the task $t^{(1)**T}$ and at the same time they could be a technological element $\theta^{(1)}_T$ within group theory, relative to the type of tasks T (we also use “ θ ” for the technological elements like in the case of τ).

Fifth and sixth session: Coset decomposition and subgroup, at the same time.

In the previous session, we realized the difference between Puzzle 2 and 4, however, it remained mysterious that in the case of Puzzle 4, whole elements of S_4 split into two *congruent* graphs, that is, $t^{(1)**T}$ was not solved completely. To consider its reason, they were suggested by the teacher to understand this splitting phenomenon as a quotient set by an equivalence relation: what equivalence relation is behind this splitting? This is the new emerged task $t^{(2)**T}$, based on the application of some notions within set theory ($\theta^{(2)}_T$). Since it was difficult to consider from one example only, they were encouraged to analyse one more case, Puzzle 5 (Fig 6).

Considering these cases, students noticed that, in the case of Puzzle 4, two permutations P and P' are in the same connected graph, if and only if “ P can become P' using S and T ” (this is student’s exact description). Again, the student’s description was not accurate. Then, with teacher’s help, they managed to elaborate the

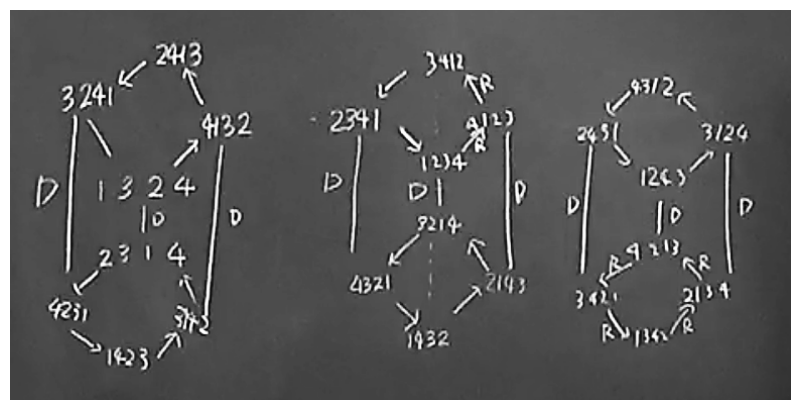


Fig 6: Cayley graph of puzzle 5 (the edges connecting the top and bottom are omitted)

description: “it is if and only if P' can be expressed by the composition of P and a finite composition of S, T, S^{-1} , and T^{-1} .” Here, we denoted this binary relation as $P \sim P'$. This is the basic concept of coset decomposition included in $\theta^{(1)}_T$.

Then, the above notion led them to the task $t^{(3)**T}$ of considering whether this binary relation is an equivalence relation or not and its reason. We checked the three properties: reflexivity, symmetry, and transitivity. The following is the result of our discussion.

- Reflexivity holds, since 1 is one of finite compositions of S, T, S^{-1} , and T^{-1} .
- Symmetry holds, since the inverse of a finite composition of S, T, S^{-1} , and T^{-1} is also a finite composition of the same elements.
- Transitivity holds, since the composition of two finite compositions of S, T, S^{-1} , and T^{-1} is also a finite composition of S, T, S^{-1} , and T^{-1} .

Through this consideration, we negotiated to denote the set of finite compositions of S , T , S^{-1} , and T^{-1} by $\langle S, T \rangle$, and realized that $\langle S, T \rangle$ has nice properties: it is these properties that make the corresponding binary relation \sim to be an equivalence relation. Also, we reflected that what happens in cases of $\langle D, R \rangle$ in S_4 (Puzzle 5) and $\langle A, B, C \rangle$ in S_4 (Puzzle 2), and found that the corresponding relation of $\langle D, R \rangle$ classifies S_4 into 3 classes and that of $\langle A, B, C \rangle$ classifies S_4 into just one class.

Then, it was this moment that the definition of subgroup of S_n was provided: a subset K of S_n is a subgroup if and only if K includes 1 and is closed under the composition operation and the inversion. This definition was not forced from above, but rather naturally raised from the discussion to understand the splitting of Cayley graphs, and would be acceptable for students. These notions of subgroup, coset decomposition, are certainly elements of $\theta^{(1)}_T$ in their praxeology. These were endowed with their *raison d'être* from the first: the properties in the definition of subgroup were raised to support the concept of coset decomposition which was observed through the type of tasks $**T$.

Seventh session: Coset and Lagrange theorem

It was natural for students to focus on the question “Is there any subgroup whose corresponding coset decomposition consists of 4 classes or 5 classes?” In this manner we investigated the structure of a coset and found that each coset has the same number of elements (Lagrange theorem in symmetric groups).

DISCUSSION

The following is the list of praxeological elements at stake in the whole program.

T : To find properties regarding puzzle operations
$t^{(1)}_T, t^{(2)}_T, t^{(3)}_T, \dots$: Puzzle 1, Puzzle 2, Puzzle 3, ...
$*T$: To express relations between operations and to find new relations between them
$t^{(1)*}_T$: To express the relation found in the first session
$t^{(2)*}_T$: To consider whether every operation has a finite order
$**T$: To consider and (or) explain the principle regarding the structure of the group
$t^{(1)**}_T$: To consider the principle of the splitting of Cayley graphs
$t^{(2)**}_T$: To consider what equivalence relation is behind the splitting
$t^{(3)**}_T$: To consider if the found binary relation is equivalence relation or not, why and what is behind it
$\tau^{(1)}_T$: To draw Cayley graphs
$\tau^{(1)*}_T$: To represent the relations between puzzle operations in algebraic way
$\tau^{(2)*}_T$: To deal with the inverse operations
$\tau^{(1)**}_T$: To formulate symmetric groups using set theoretical terms
$\theta^{(1)}_T$: Symmetric group theory—especially, notions of S_3 , S_4 , generator, subgroup, coset
$\theta^{(2)}_T$: Elementary set theory —maps, equivalence relation, and quotient set
$(\Theta)_T$: General group theory)

Table 1: Praxeological elements

Up to this point, we have described the study process in the program, where different technical and technological entities in group theory emerged from a sequence of lively tasks. The bundle of these tasks constructs a *raison d'être* of group theory for the students. Let us clarify here the relationship between such tasks and technical-and-technological elements, as the first property of our experiment (Fig. 7). On the one hand, each task brings about a derived technical-and-technological element. On the other hand, such an element produces a certain new task and is applied to some tasks. In Fig 7, the arrows indicate these deriving, producing, and applying relations. This *dialectical interplay between tasks and techniques* is the first property of our implementation. Note that our analysis relies on a basic assumption of the ATD, that is, the *postulate of the relativity of praxeological entities*. $\theta^{(1)}_T$ and $\theta^{(2)}_T$ are *technological for T*—which is the motivation of this praxeology—, but *technical for $**T$* . Type of tasks, techniques, technologies, and theories are not the natures of praxeological elements but their functions.

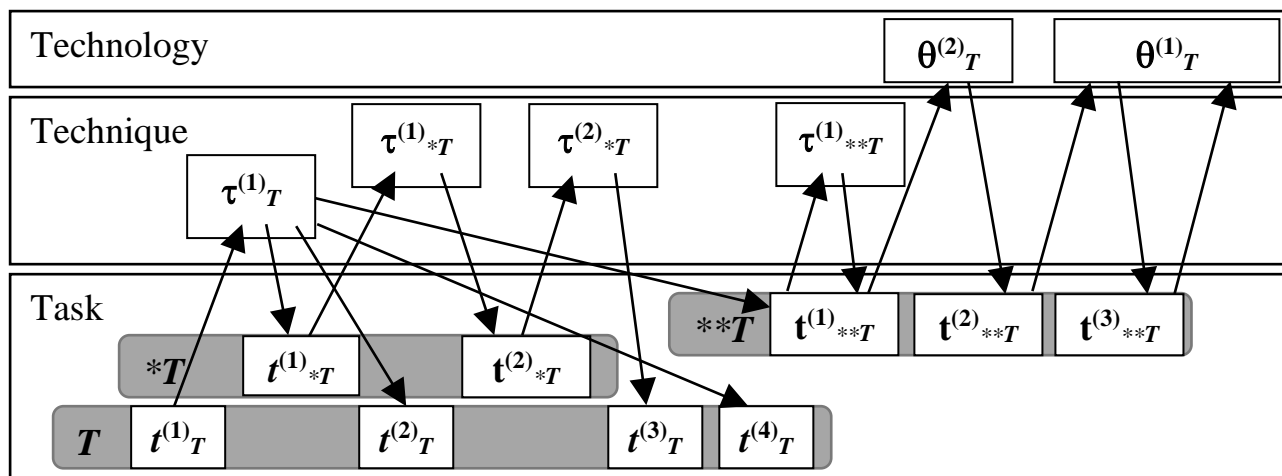


Fig 7: The dialectic between tasks and techniques

The second property is related to the *incompleteness of theoretical elements* in the program. The praxeology has the algebraic nature but excessively focuses on symmetric groups, that is, not get access to the general group theory. This is a main reason why we call this seminar an *introductory* program of abstract algebra. However, the praxeology at stake could reach to standard structural results in symmetric group such as coset decomposition and Lagrange theorem, which can be easily extended to general groups. This potentially extendable structural results might be a gate for proceeding to general group theory with *raison d'être*.

The third property related to the second property is the *implicitness of associativity*. The definition of group involves three conditions: associativity, existence of an identity element, and existence of inverse elements. In our program, the identity element was required to make equational representation of found relations ($\tau^{(1)*_T}$), and inverse elements also emerged to accomplish the task $t^{(2)*_T}$. However, in this program all objects in consideration were maps and associativity was always satisfied from the first. Students *knew* that associativity holds but never focused on it. It was used with no special consciousness. Associativity seems often trivial and tends to be transparent for

students. Thus, the teacher had no chance to highlight associativity regarding group theory with *raison d'être*.

FINAL REMARKS

Winsløw et al. (2014) points out the two types of institutional transitions in university mathematical praxeologies. The first type is the transition where a *praxis* part of a praxeology without the *logos* part gets its *logos* part. On the other hand, in the second type, a *logos* part of a mathematical praxeology changes to a *praxis* part of another advanced praxeology. Our program should fit as the first type transition. Taking above three properties of our program into consideration, we might be able to proceed to a further development of didactic situation for teaching group theory as follows: one is a situation where their understanding of potentially extendable technology elements derives the general group theory. Such situation would require group-like objects in entirely other context, to which the existing technology elements can be applied. Also, to focus on associativity group-like objects, in which associativity is not trivially satisfied, would be needed.

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