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# Continuity of real functions in high school: a teaching sequence based on limits and topology

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*It is well known that students have difficulties with the concept of continuity, specifically on points of discontinuity, and concepts like limits and infinity. In Italian textbooks, the continuity of functions is usually defined using limits, while an intuitive characterization of continuous functions is proposed without providing the students with formal tools to use it, like “the graphs of continuous functions can be drawn without lifting the pencil out of the paper”. Limits are one of the most complex subjects to learn and are usually introduced in an algorithmic way, without a true comprehension of the subject. We argue that introducing the definition of continuous functions using limits is problematic and we designed and tested a teaching sequence to investigate the potentiality of including a topological approach in high school.*

*Keywords: Transition to and across university mathematics, Teaching and learning of analysis and calculus, Topology, Concept image, Intuitive models.*

## INTRODUCTION

Italy has a K13 scholastic system and an introduction to Calculus and Analysis is proposed at the end of high school (18-19 years), in particular in *Liceo Scientifico*. Students are taught continuity, limits and series in different ways in secondary school and university, sometimes with inconsistencies between the two approaches (Trigueros, Bridoux, O’Shea and Branchetti, accepted). Thus there is a typical problem of transition from secondary school to university. It is well known that there are various difficulties with the concept of continuity, specifically on points of discontinuity, limits and infinity. In Italian textbooks, the continuity of functions is usually defined using limits and an intuitive characterization of continuous functions as graphs with no holes, that can be drawn with a pencil without lifting the pencil out of the paper (Bagni, 1994), is proposed in the beginning but never deeply analyzed. On the contrary, after an intuitive approach, traditionally secondary school teaching of continuous functions aims, from the very beginning, to provide the students with the most general and formal conception of continuity, given by a formal epsilon/delta definition. In this approach the intuitive characterization of continuity based on the properties of the graph are considered a crutch to abandon as soon as possible

because it can be a source of future mistakes (Bagni, 1994). We wish to problematize such a tradition.

Moreover, Italian textbooks propose problematic characterization of points of discontinuity, since they are usually ambiguous in linking domain, accumulation points, definition of continuity at a point and global continuity of the function. Furthermore, often continuity is presented as a global, instead of local, property and there is an inconsistency with the formal limit approach that leads some students to say, for instance, that to evaluate the global continuity of a function it is necessary to compute the limit in every point of the domain.

We decided to investigate the following research question: whether and how a different approach, using a mixed topological and analytical approach, could result in a better conceptualization of continuity and limits, which yields more students to correctly identify continuous functions and points of discontinuity? We proposed the following teaching sequence: 1. introduce continuity in a topological way, without using limits; 2. introduce the concept of limit; 3. link the two concepts, both formally and with concrete examples.

In this paper, we present the design of a teaching sequence, a general overview of the learning outcomes of an implementation of the sequence in grade 12 and, finally, a comparison with a class of grade 13 who had been taught with the traditional Italian approach. As we will show, a topological approach might provide the students with useful images and methods to classify functions in many cases. This study had promising outcomes: indeed, in the final tests of the grade 12 compared to the initial test of grade 13, we observed fewer students' misconceptions about continuity and limits and a greater ability to manage the semiotic transformation to keep under control in the solving processes. However the connection between limits and continuity was problematic and a refinement of the teaching sequence is necessary.

## **LITERATURE REVIEW**

Two of the most investigated topics in University Mathematics education are the continuity of functions and the difficulties with limits in the undergraduate courses (Trigueros et al., accepted). Tall and Vinner (1981) showed some typical students' conceptions of continuous functions that cause many undergraduate students' difficulties, showing that the effectiveness of their use of definitions, also in simple tasks, is usually far from the expectations. The formal epsilon/delta definition of continuity is powerful enough to evaluate also "pathological" examples, like the Dirichlet function, but it is very far from the students' concept images (Tall & Vinner, 1981) reported in the literature about continuity (Hanke & Schafer, 2017) and many research results show that limits in many cases do not become conceptual tools that improve students' approach to the evaluation of continuity of functions. Moreover, this formal approach is usually not suitably motivated to the students, since it is used only in tasks which would be solvable with a more intuitive approach.

The literature shows that, with the traditional approach, based on intuition on one side and on the formalization based on limits on the other side, the students' concept images about continuous functions are often rooted in specific examples and conflict with the formal definition (Tall & Vinner, 1981). Hanke and Schafer (2017) listed the following seven possible mental images that students use as criteria to discuss the global continuity of a function that are reported in the literature:

I: *"A graph of a continuous function must be connected"*

II: *"The left hand side and right hand side limit at each point must be equal"*

III: *"If you wiggle a bit in  $x$ , the values will only wiggle a bit, too"*

IV: *"Each continuous function is differentiable"*

V: *"A continuous function is given by one term and not defined piecewise"*

VI: *"The function continues at each point and does not stop"*

VII: *"I have to check whether the definition of continuity applies at each point"*.

For what concerns limits, many researchers showed that this is one of the most complex concepts to learn, and that it is usually introduced in an algorithmic way, without a true comprehension of the subject (Trigueros et al., accepted). Several epistemological and cognitive aspects must be considered in order to face the critical issues that characterize their learning. In particular, considering limits of functions, some aspects have been shown as crucial: the potential and actual conceptions of infinity (Tsamir & Tirosh, 1992) and the difficulties caused by metaphors and some uses of the natural language. Dimarakis and Gagatsis (1997) consider the interactions between the mathematical language and the natural language and note how the expressions "tends to the limit", "approaches" and "converges" are mathematically equivalent, but are not in the everyday language: "approximates" and "tends to", often used as a synonym of "approaching", does not suggest situations related to limits but reinforce a dynamic interpretation. In the case of limits of functions, as is the case for the convergence of sequences, the dynamic conceptions are very resistant (Williams, 1991). Teaching should aim at turning the dynamic representations of students into static conceptions, or at least to scaffold the students' approach to limits making them aware of the relationship between the two aspects (Trigueros et al., accepted).

## **RESEARCH FRAMEWORK**

Often mathematics is identified with the precision of rules and a discipline where concepts can be defined in an accurate way so as to build a rigorous theory based on definitions, formal statements and proofs. However, a large amount of research about mathematics learning argues the necessity of an intuitive and informal base for the concepts to be used by the students as thinking tools and support a formal learning; a pre-existing cognitive structure lies in the mind of every person and, when the student is presented a concept, naturally different personal mental images are evoked, before

the formal definition can be accepted. We will use the term *concept image* (Tall & Vinner, 1981) to describe the complete cognitive structure linked to the concept, which includes all mental images as well as related processes and properties. Different stimuli can activate different parts of the concept image; we will call the part of the concept image activated in a precise moment the *evoked concept image*. As the concept image develops, it is not guaranteed it will be coherent; thus when different (and contradictory) parts of it are evoked simultaneously, a sense of confusion emerges. According to Fischbein (1993), while formally there is no difference between accepting a proof and accepting the universality of the assertion, for the pupil the two things usually do not coincide. To pursue the intuitive acceptance of formal reasonings it is necessary to adopt a didactical approach that permits the students to mix and merge different ways of reasoning and make sense of formal statements and proof, connecting them to other kinds of discourses that can activate the intuitive and personal cognition at a different level. According to Lecorre (2016), three types of rationalities are necessary for understanding the learning processes of students in the study of limits:

- *Pragmatic rationality* consists strictly in examining specific cases; there is no attempt to generalize observations.
- *Empirical rationality* is used when a general law is to be obtained; the facts are used to deduce generalizations.
- *Theoretical rationality* begins with theory (theorems, properties, definitions, axioms ...) to establish new properties and theorems.

We relied on this framework to design the activities to introduce limits, encouraging students to connect reasonings of different kinds, bridging empirical and pragmatic rationality with the theoretical one. The notion of representation was also an important reference in the design phase, since representations play a crucial role in the acquisition and the use of the individual's knowledge. As Duval (1995) points out presenting his Theory of Register of Semiotic Representation: "There's no knowledge that can be mobilised by an individual without a representation activity" (p. 15). The main assumptions of the theory are:

1. there are as many different semiotic representations of the same mathematical object, as semiotic registers utilised in mathematics;
2. each different semiotic representation of the same mathematical object does not explicitly state the same properties of the object being represented;
3. the content of semiotic representations must never be confused with the mathematical objects that these represent.

## **RESEARCH METHODOLOGY**

The goal of the research was to check whether and how the didactical approach we designed (including topology as element of the theory, using a methodology oriented

to strengthen the students' reasoning with limits intertwining different rationalities and paying attention to semiotic transformations) could help the students to deal with basic tasks about continuity of functions better than the traditional approach of teachers of *Liceo Scientifico* (Scientific High School) and textbooks in Italy.

First of all, we carried out an analysis of Italian high school textbooks. Then we prepared a questionnaire to investigate the students' conceptions in grade 13 (21 students) after a traditional teaching sequence. Then we designed our teaching sequence and, after collecting data about students' conceptions in two classes in grade 12 (38 students), we implemented the teaching sequence with the same students. The design of the teaching sequence was based on: the results from the literature review resumed before, the initial test about students' concept images and the analysis of the audio-recordings lesson by lesson. Finally we carried out a final test, with common tasks in two classes, one in grade 12 and one in grade 13. The questionnaire included open questions and tasks where students were asked to compute limits at the extreme points of the domain and evaluate the continuity of functions, providing explanations. In particular, we collected data about the students' images of continuous functions and limits, their use of concept images of limits and infinity (potential and actual) and the students' ability to manage semiotic transformations in tasks about continuity. Since in grade 13 the students' reasonings had not been clear in some cases, we added in grade 12 one question (task 8, that we discuss later), asking to compute the limits, to state and explain if the function was continuous and then to identify its (possible) points of discontinuity.

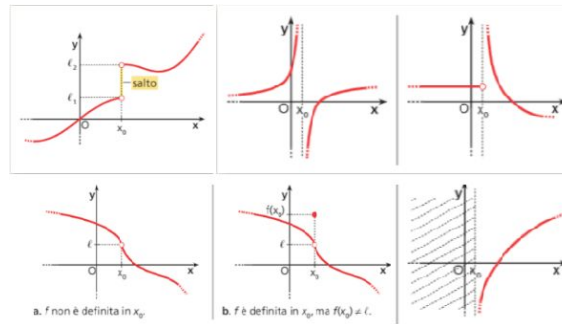
We analysed quantitatively the correct/incorrect answers and we compared the initial test carried out in grade 13 after a traditional teaching with the final test in grade 12 in one class who attended our course, in the same school. The results are not statistically significant but informed us about the potentiality of our approach. Then, we analyzed the whole set of data looking for students' concept images and concept definitions of accumulation points and continuity, comparing them with the students' outcomes, to check the consistency and the efficacy of such images and definitions in the students' solving processes. We also checked the students' abilities to manage semiotic representations in a fruitful way. Finally we explored whether and how the students integrated fruitfully limits and the topological approach to continuity.

## **DIDACTICAL TRANSPOSITION AND ENGINEERING**

In Italian High School textbooks it is usually given the following definition:

*A function  $f$  is said to be continuous in a point  $x_0$  if the limit of  $f(x)$  as  $x$  goes to  $x_0$  coincides with the value  $f(x_0)$ .*

The problems begin with the negation of the aforementioned definition: a function it is said to be discontinuous at a point  $x_0$  if the function is defined but not continuous there or if the point is an accumulation point of the domain but the function is not defined in  $x_0$ , thus calling point of discontinuity also points out of the domain. Textbooks then introduce various types of discontinuity (Fig. 1):



**Fig. 1: Discontinuities represented in an Italian textbook**

In our teaching sequence, we started carrying out 5 interactive frontal lessons (1 hour each) about continuity from a topological point of view. We decided to introduce continuous functions defined on subsets of  $\mathbb{R}$ , using a transposition of the definition of connectedness by arcs and neighbourhoods, coming to this definition:

*A one real variable function (whose domain  $D \subseteq \mathbb{R}$  has a finite number of path-connected components) is continuous if the number of the connected (by paths) components of the domain and of the graph are the same.*

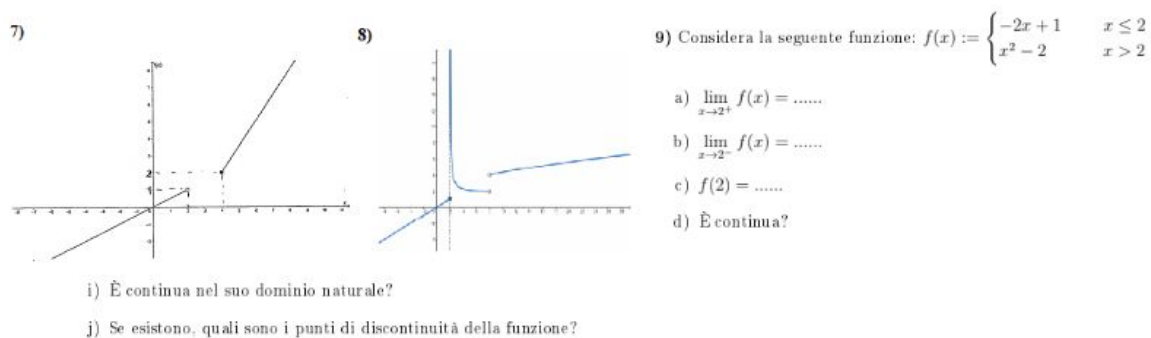
This definition (path-connected function) is equivalent to that of continuous function in  $\mathbb{R}$  whenever the domain of the function is locally path-connected and simply-connected (Hanke, 2018). The difference between the two is that this definition, from a didactical point of view, allow to use the graphical representation of the function to evaluate the continuity and to find the possible points of discontinuity (by using a local version of the definition in a neighbourhood of the point) and should help not to confuse a point outside of the domain with a point of discontinuity. Furthermore in this case the negation of continuity is more straightforward. We hypothesized that this approach could reduce the students' misconceptions about continuity and points of discontinuity and provide them with a powerful resource to use facing tasks about continuity of real functions.

Then, we carried out 5 lessons (1 hour each) about infinity, limits and sequences, using a didactical methodology based on the intertwining of rationalities (Lecorre, 2016). We included several examples and definitions from the history, to promote gradually an intuitive acceptance of the formal definitions: Archimedes' *Measure of the circle* and the proof by exhaustion, the Paradox of Achilles and the tortoise, periodic numbers, a graphical representation of the geometric series, an original piece from Cauchy's *Cours d'Analyse* (1821, first definition of limit). To gradually connect a formal definition of limit with the students' concept images of functions, we introduced and discussed with the students some examples of sequences and we showed to the students the limit for  $n$  which tends to infinity from the dynamic point of view, using the graphic representation of functions and the numerical representation. Then we moved to a more static approach through the concept of accumulation point.

In the final 2 hours, we matched the two different approaches and we linked the topological idea of continuity with the traditional definition based on limits, discussing with the students about continuity and points of discontinuity using limits, to show to them that the two approaches lead to the same conclusions.

## DATA ANALYSIS

Analysing the questionnaires only in terms of correct/incorrect answers, the comparison between the tests in grade 13 and in grade 12 showed that the students in grade 12 had in general better outcomes. The most common error of the 13-grade students was to classify continuous functions as discontinuous. For instance, in grade 13, no one classified the equilateral hyperbola function  $f(x) = 1/x$  continuous in its natural domain, while many students in grade 12 answered correctly. The following three tasks concerning continuous functions are good examples of the questions we asked in the questionnaire; 7 and 9 were asked in grade 12 and 13, while 8 was only in the grade 12 test. In task 7 the function is continuous in its domain, while 8 and 9 have points of discontinuity. In task 7 and 8 we also asked to infer the limits at the extreme points of the “natural domains” from the graph.



**Fig. 2: Questions about continuity and points of discontinuity**



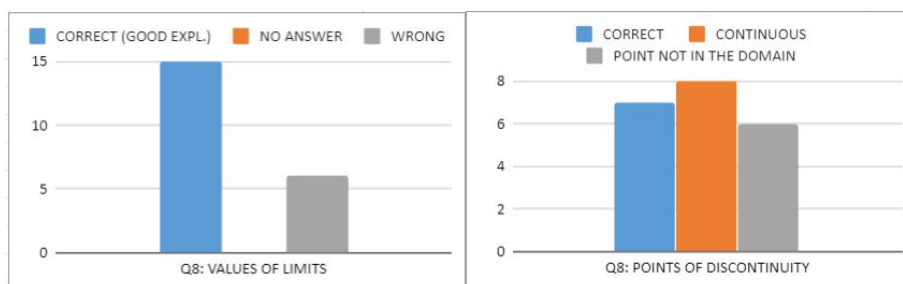
**Fig. 3: Students' outcomes in the tasks about continuity in grade 12 (left) and 13**

Comparing the answers to tasks 7 and 9 (19 students, grade 13; 21 students, grade 12), it emerged more students in grade 12 answered correctly that a function was continuous (task 7, graphical) rather than discontinuous (task 9, analytical), while the opposite happened in grade 13. Only 2 on 19 in grade 13 recognized the first as a continuous function, while 18 on 21 in grade 12 did it. In task 9 the trend is the



opposite, even if the difference is much smaller; 13-grade students who were taught to classify continuity with limits in many cases answered without computing limits.

Looking at the sheets, we observed that while in grade 13 no students made a semiotic transformation from the analytical to the graphical representation in task 9, 13 students drew the graph of the function and 12 of them answered correctly. Even if we did not find evidence of reasonings about limits and continuity, in grade 12 the same students that had problems with discontinuities also made mistakes in the computation of limits (even if there are correct answers and errors with limits).



**Fig. 4: Students' outcomes in the tasks about discontinuities in grade 12**

Task 8, proposed in grade 12, was the most problematic: it had a discontinuity point in 2 and a “false” discontinuity in 8, that was not an element of the domain; 6 students considered 8 a discontinuity point, and 4 of them correctly classified the function in task 9. In these last 4 cases, we are not sure if students carried out a correct reasoning, since they could have classified the function discontinuous even if the point was not in the domain. 8 students considered the function continuous.

We also asked the students the question: “How would you explain in your own words what is a continuous function?”. Looking at their answers we identified two further criteria that were not included in the list by Hanke and Schafer (2017):

*“A continuous function has  $R$  as domain.”*

*“The number of the components of the domain is equal to the number of the components of the graph”.*

The last criterion, that is the one proposed in the lessons, was expressed by 11 (out of 21) students of grade 12 out and was used effectively to discriminate continuity.

In order to check what kind of conceptualization of limits the same students had developed, before analysing their link to continuity of functions, we analysed the audio-recording of lessons looking bottom-up for emerging relevant students' concept images. A basic recurrent image was that of the asymptote: the limit that “approaches but does not reach a point” was often linked to the concept of asymptote, known to students from grade 11 thanks to the study of hyperbole and homographic functions in analytic geometry. Also the images of the “full and empty cue ball” was often used by the students, for example when they explained what is difference between  $0, \bar{9}$  and 1, or to say if a point belonged (full) or not (empty) to the graph of

a function. Moreover this image was used to explain if it made sense or not to compute the limit at a point (if the cue ball is empty it makes sense to compute a limit). In both cases the students' tried to recall previous images and use them to create connect limits and functions before our final lesson about the connection limits-continuity. When students were required to connect the concept of limit and that of point of accumulation, inconsistencies appeared between the students' images; many of them did not understand that it did not make sense to calculate the limit for  $x$  tending to  $x_0$  if it is not a point of accumulation; moreover, in most cases they said it did not make sense to compute limits at a point in the domain.

Such images did not prevent the students from answering correctly questions about the continuity in task 7, and some students identified correctly discontinuities using the graphs even making errors with limits (1 in task 8, 6 in task 9), but, even if there are no explicit reasonings carried out by the students using limits to identify discontinuities, they seemed quantitatively to affect their answers to determination of the points of discontinuity, in particular where the students had to reason on the graph about the asymptote and the empty-full cue balls. Moreover some students in the questions about limits wrote that the limit of the functions does not exist *since* the function is discontinuous at the point. This point should be explored more.

## **DISCUSSION AND CONCLUSIONS**

As regards the continuity of functions, the comparison between grade 12 and 13 showed that the topological approach was useful to improve the students' use of graphical representations to state the continuity of functions that was at the basis of the better outcomes in grade 12 rather than grade 13. This result encourages us to carry out further experimentations refining the didactical engineering and the data collection in order to grasp in a more accurate way the students' reasonings. However, some students had still problems determining discontinuities; in particular, some students considered points of discontinuities also points not in the domain and in task 8 and 9 classified discontinuous functions as continuous, contrary to the usual trend observed with the traditional approach. These errors seemed to be due to difficulties to visualize the components of the domain, linked to the images of accumulation points, and to classical concept images stressed by Tall & Vinner (1981), like "a jump in the functions implies the function to be not continuous". This result made an important aspect arise: the topological approach is useful if the students are able to count the components of the domain, but in the most relevant cases it requires a good image of accumulation points that is demanding for students; however, this is also a problem of the traditional approach.

Students in general overcame the classical dichotomy potential/actual infinity in most cases and had good outcomes in the questions about limits. However, only in a few cases we observed an explicit connection between limits and points of discontinuity and we were not able to reach clear results about it. It seems that the values of limits

are not used by the students, but concept images of limits of functions, built spontaneously by the students recalling previous practices, could have influenced them to face the tasks about continuity, since quantitative data showed a correlation between the answers about limits and the correct identification of discontinuities. The connection between topological continuity and limits should be deeply improved in further experimentations.

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