



**HAL**  
open science

# Modulating functions based differentiator of the pseudo-state for a class of fractional order linear systems

Yan-Qiao Wei, Da-Yan Liu, Driss Boutat, Hao-Ran Liu, Chunwan Lv

## ► To cite this version:

Yan-Qiao Wei, Da-Yan Liu, Driss Boutat, Hao-Ran Liu, Chunwan Lv. Modulating functions based differentiator of the pseudo-state for a class of fractional order linear systems. *Journal of Computational and Applied Mathematics*, 2021, 384, pp.113161. 10.1016/j.cam.2020.113161 . hal-03113892

**HAL Id: hal-03113892**

**<https://hal.science/hal-03113892>**

Submitted on 8 Sep 2022

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution - NonCommercial | 4.0 International License

# Modulating functions based differentiator of the pseudo-state for a class of fractional order linear systems

Yan-Qiao Wei<sup>a</sup>, Da-Yan Liu<sup>a,\*</sup>, Driss Boutat<sup>a</sup>, Hao-Ran Liu<sup>b</sup>, Chunwan Lv<sup>c</sup>

<sup>a</sup>*INSA Centre Val de Loire, Université d'Orléans, PRISME EA 4229, Bourges Cedex 18022, France*

<sup>b</sup>*School of Information Science and Engineering, Yanshan University, Qinhuangdao, 066004, Hebei, China*

<sup>c</sup>*School of Mathematics and Big Data, Foshan University, Foshan 528000, China*

---

## Abstract

In this paper, an algebraic and robust fractional order differentiator is designed for a general class of fractional order linear systems with an arbitrary differentiation order in  $]0, 2[$ . It is designed to estimate the fractional derivative of the pseudo-state with an arbitrary differentiation order as well as the one of the output. In particular, it can also estimate the pseudo-state. Different from our previous works, the considered system no longer relies on the matching conditions, which makes the system model be more general. First, the considered system is transformed into a fractional differential equation from the pseudo-state space representation. Second, based on the obtained equation, a series of equations are constructed by applying different fractional derivative operators. Then, the fractional order modulating functions method is introduced to recursively give algebraic integral formulas for a set of fractional derivatives of the output and a set of fractional derivative initial values. These formulas are used to non-asymptotically and robustly estimate the fractional derivatives of the pseudo-state and the output in discrete noisy case. Third, the required modulating functions are designed. After giving the associated estimation algorithm, numerical simulation results are finally given to illustrate the accuracy and robustness of the proposed method.

*Keywords:* Non-asymptotic estimation; Fractional order differentiator; Fractional order modulating functions method; Fractional derivative initial value estimation

---

\*Corresponding author

*Email addresses:* [yanqiao.wei@insa-cvl.fr](mailto:yanqiao.wei@insa-cvl.fr) (Yan-Qiao Wei), [dayan.liu@insa-cvl.fr](mailto:dayan.liu@insa-cvl.fr) (Da-Yan Liu), [driss.boutat@insa-cvl.fr](mailto:driss.boutat@insa-cvl.fr) (Driss Boutat), [liu.haoran@ysu.edu.cn](mailto:liu.haoran@ysu.edu.cn) (Hao-Ran Liu), [lvchunwan@163.com](mailto:lvchunwan@163.com) (Chunwan Lv)

## 1. Introduction

Fractional calculus has a long history of 300 years and has been widely focused in different fields in recent years, such as **theoretical analysis** [1, 2, 3], mechanics [4], viscoelastic materials [5], electrical circuits [6], signal processing [7, 8], etc. In the field of automatic control, fractional order systems and controllers have been widely applied in control design [9, 10, 11, 12, 13, 14, 15]. To obtain fractional order system models and implement fractional order controllers, it is necessary to provide useful system parameters and variables, such as the fractional derivative of the output. However, for economic and technical reasons, these quantities can not always be measured. Moreover, the measurements usually contain noises. Therefore, it is significant to design identification and estimation methods which are robust respect to corrupting noises. In a view of practical on-line applications, non-asymptotic estimation methods with finite-time convergence are sometimes required. Consequently, non-asymptotic and robust estimation algorithms are necessary for fractional order systems and controllers. For this purpose, the modulating functions method was extended to fractional order case: for identification of fractional order systems [16, 17] and fractional order differentiators design [18, 19]. Recall that this method was originally introduced for linear and non-linear identification of continuous-time systems [20]. In recent years, the modulating functions method has been extended to design integer order differentiators [21, 22]. Independently, other two kinds of non-asymptotic integer order differentiators were introduced in [23, 24, 25].

In the last decade, the concept of the pseudo-state becomes more and more recognitory by researchers [26, 27]. **For a fractional order system, its real state is made up by the pseudo-state and an initialization function** [27]. However, for some applications, it is enough to use the **knowledge of the pseudo-state to understand the dynamics of a studied fractional order system** [28]. In the existing works, the fractional order observers methods have been designed to asymptotically estimate the pseudo-state [29]. Recently, by considering the advantages of the modulating functions method, it has been applied to non-asymptotically and robustly estimate the pseudo-state for a class of observable fractional order linear systems with a rational differentiation order  $\alpha = \frac{p}{q}$  ( $p, q \in \mathbb{N}^*$ ) in [30], which fulfill the matching conditions  $(C_2)$ - $(C_3)$ . Later, by introducing a class of fractional order modulating functions, an improved pseudo-state estimator was designed in [31] for the system defined in [30] with  $\alpha = \frac{1}{q}$ .

In control field, fractional order controllers have been investigated by various of researchers to better handle the system dynamics. **The fractional order Proportional-Integral-Derivative (PID) controller was extended from the classical case by using the fractional integral and derivative of the output** [10]. Similarly, the integer order state based controller has also been extended to the

fractional order case, where the fractional integral and derivative of the pseudo-state are needed [32, 33]. Consequently, in order to implement this kind of controllers, the modulating functions method has recently been extended to estimate the fractional derivative of the pseudo-state for the system defined in [19] which is an extension of [30]. It should be mentioned that this estimation can not be done by the fractional order observers.

Though, there are still two drawbacks accompanying the proposed methods in [19, 30, 31]: the matching conditions  $(C_2)$ - $(C_3)$  restrict the class of the studied systems and the dependence of  $q$  from  $\alpha$  on the recursive estimation schemes, which increases the computation cost when  $q$  is large.

Bearing the previous thoughts in mind, this paper aims at designing a non-asymptotic and robust fractional order differentiator for fractional order linear systems. It can be used to estimate the fractional derivative of the pseudo-state with an arbitrary differentiation order as well as the one of the output. In particular, it can also estimate the pseudo-state. **The main contributions of this paper are given as follows:**

- Compared with [19, 30, 31], a wider class of fractional order systems are studied in this paper without imposing the above mentioned matching conditions  $(C_2)$ - $(C_3)$ . Moreover, the differentiation order  $\alpha$  can be an arbitrary real number in  $]0, 2[$ .
- The proposed scheme is unrestricted by a rational differentiation order  $\alpha = \frac{p}{q}$ , which is more suitable than the ones proposed in [19, 30, 31] and can avoid an explosion of computation cost when  $q$  is large.
- In our preliminary work [34], only the case with  $N = 3$  and  $\alpha \in ]0, 1[$  was considered by also imposing  $(C_2)$ - $(C_3)$ . More general case will be studied in this paper. Beside, the construction of the required modulating functions was not given in [34]. This point will also be solved in this work.

The rest part of this paper is organized as follows: some preliminaries are provided in Section 2 and the main problems to be solved are formulated. The details of the proposed fractional order differentiator are presented in Section 3. Numerical results are given in Section 4. Finally, conclusions are outlined in Section 5.

## 2. Preliminaries

### 2.1. Fractional order system with pseudo-state space representation

There are several definitions of fractional derivatives, where the Riemann-Liouville one [32, 35, 36], the Grünwald-Letnikov one [37] and the Caputo one [38, 39] are usually considered. First, the

Grünwald-Letnikov fractional derivative coincides with the Riemann-Liouville one in many cases (see [1] for more details). Second, it was mentioned in [27] that if the Caputo fractional derivative is used to describe a fractional order system, the infinite dimension of the system can be lost. For the previous reasons, the Riemann-Liouville fractional derivative is considered in this paper.

Let  $\alpha \in \mathbb{R}_+$ , the Riemann-Liouville fractional integral is defined as follows:

$$\begin{cases} D_t^0 f(t) & := f(t), \\ D_t^{-\alpha} f(t) & := \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \end{cases} \quad (1)$$

where  $\Gamma(\cdot)$  is the well-known Gamma function [40]. When  $\alpha \notin \mathbb{N}$ , it is defined by an improper integral with an integrable singularity at  $t$ . Let  $l = \lceil \alpha \rceil$  denote the smallest integer greater than or equal to  $\alpha$ . Then, we define the Riemann-Liouville fractional derivative by:

$$D_t^\alpha f(t) := \frac{d^l}{dt^l} \{D_t^{\alpha-l} f(t)\}. \quad (2)$$

Hence, it is also defined by an improper integral.

**Remark 1** *The objective of this paper is to estimate the pseudo-state of the studied fractional order system with the pseudo-state space representation, this is why the previously defined uninitialized Riemann-Liouville fractional derivative is considered. In fact, by considering an initialization function, the initialized Riemann-Liouville fractional derivative can be defined, based on which the fractional order system with the real state can be presented with infinite-dimensional property [41]. There also exists an infinite-dimensional diffusive representation [42, 43], which is equivalent to the one based on the initialization function [41].*

In this paper, the following general fractional order linear system with the pseudo-state space representation is considered:

$$\begin{cases} D_t^\alpha x & = Ax + Bu, \\ y & = Cx + \eta u, \end{cases} \quad (3)$$

on  $I \subset \mathbb{R}_+ \cup \{0\}$ , where  $0 < \alpha < 2$ ,  $A \in \mathbb{R}^{N \times N}$  with  $N \in \mathbb{N}^*$ ,  $B \in \mathbb{R}^{N \times 1}$ ,  $C \in \mathbb{R}^{1 \times N}$ ,  $\eta \in \mathbb{R}$ , the pseudo-state  $x \in \mathbb{R}^N$  is **with unknown initial value**,  $u \in \mathbb{R}$  and  $y \in \mathbb{R}$  are the input and the output, respectively. Besides, the system is assumed to satisfy the observability and the controllability conditions given as follows [9]:

$$(C_1) : \text{rank} \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{N-1} \end{pmatrix} = N,$$

$$(C_4) : \text{rank}(B, AB, \dots, A^{N-1}B) = N.$$

**Remark 2** Compared with the existing works [19, 30, 31], the considered system model in this paper is more general. First, the following matching conditions are no longer considered:

$$(C_2) : CB = CAB = \dots = CA^{N-2}B = 0,$$

$$(C_3) : CA^{N-1}B \neq 0.$$

Although  $(C_2)$ - $(C_3)$  can guarantee the controllability of the considered system, these conditions are stronger than  $(C_4)$  and limit the generality. Second,  $\alpha$  is an arbitrary differentiation order in  $]0, 2[$  instead of a rational number  $\frac{p}{q}$  with  $p, q \in \mathbb{N}^*$ . Third, the input  $u$  is involved in the output  $y$ . It will be shown how to deal with this general model by the proposed method in the sequel.

## 2.2. Problem formulation

Recently, some controllers involving the fractional derivative of the pseudo-state have been proposed [32, 33]. Different from the fractional order PID controllers based on the system's output, these controllers are based on the pseudo-state. However, as far as we know, the fractional derivative of the pseudo-state as well as the one of the output can not be measured by sensors. Therefore, the main goal of this paper is to estimate  $D_t^\beta x(t)$  with  $-1 < \beta \in \mathbb{R}$  for the system defined by (3) in noisy environment, where it is assumed the input  $u$  is known and the output  $y$  is measured with noises. Meanwhile, it will be shown that the proposed method can also provide the estimation of the fractional integral and derivative of the output, which can be applied to deal with the problem of output regulation, such as to design fractional order PID controllers. Indeed, if  $\beta \leq -1$ ,  $D_t^\beta x(t)$  is a proper integral which can be estimated by numerical integration methods. If  $\beta > -1$ ,  $D_t^\beta x(t)$  can be estimated by applying the Grünwald-Letnikov scheme to the estimation of  $x(t)$  [9]. However, this scheme is not robust against corrupting noises.

For the above objective, the modulating functions method is applied. In order to better introduce the proposed method, the basic idea is demonstrated in the following example with **zero initial values of the output**. Before doing so, let us recall the right-sided Caputo fractional derivative for

a function  $f$  defined on  $[0, h[$  as follows [44]:  $\forall \alpha \in \mathbb{R}_+, l = \lceil \alpha \rceil$ ,

$${}^C D_{t,h}^\alpha f(t) := \begin{cases} (-1)^l f^{(l)}(t), & \text{if } \alpha = l, \\ \frac{(-1)^l}{\Gamma(l-\alpha)} \int_t^h (\tau-t)^{l-\alpha-1} f^{(l)}(\tau) d\tau, & \text{else.} \end{cases} \quad (4)$$

**Example 1:** Consider a simple model on  $I = [0, h]$  as follows:

$$\begin{cases} D_t^\alpha y + y = \tilde{u}, \\ y(0) = 0, [D_\tau^\alpha y(\tau)]_{\tau=0} = 0, \end{cases} \quad (5)$$

where  $0 < \alpha < 1$  and  $\tilde{u}$  is a known input. The fractional derivative  $D_t^\alpha y(t)$  will be computed by applying the following process.

**Step 1: Construction of equation.** By applying  $D_t^1$  to both sides of (5) and using the additive index law with **zero initial values of the output** [1], we get:

$$D_t^{1+\alpha} y + \dot{y} = \dot{\tilde{u}}. \quad (6)$$

**Step 2: Multiplication and integration.** By multiplying a function  $g$  and taking integration from 0 to  $t$ , we get:

$$\int_0^t g(\tau) D_\tau^{1+\alpha} y(\tau) d\tau + \int_0^t g(\tau) \dot{y}(\tau) d\tau = \int_0^t g(\tau) \dot{\tilde{u}}(\tau) d\tau. \quad (7)$$

**Step 3: Fractional order integration by parts.** By applying the fractional order integration by parts formulas provided in [45], we get:

$$\begin{aligned} \int_0^t {}^C D_{\tau,t}^{1+\alpha} g(\tau) y(\tau) d\tau + \int_0^t {}^C D_{\tau,t}^1 g(\tau) y(\tau) d\tau = \\ \int_0^t g(\tau) \dot{\tilde{u}}(\tau) d\tau - [g(\tau) D_\tau^\alpha y(\tau)]_{\tau=0}^{\tau=t} - [g(\tau) y(\tau)]_{\tau=0}^{\tau=t}. \end{aligned} \quad (8)$$

Hence, if  $g(t) = 1$ , we get:

$$\int_0^t {}^C D_{\tau,t}^{1+\alpha} g(\tau) y(\tau) d\tau + \int_0^t {}^C D_{\tau,t}^1 g(\tau) y(\tau) d\tau = \int_0^t g(\tau) \dot{\tilde{u}}(\tau) d\tau - D_t^\alpha y(t) - y(t). \quad (9)$$

**Step 4: Recursive algorithm.** Since we cannot compute  $D_t^\alpha y(t)$  and  $y(t)$  from only one equation (9), more equations should be constructed. To achieve this, we apply  $D_t^{-j\alpha}$  for  $j = 1, \dots, J$  with  $J \in \mathbb{N}^*$  to (5). Then, we take  $D_t^1$  to the obtained equations as done in Step 1. By following Step 2 and Step 3, we obtain: for  $j = 1, \dots, J$ ,

$$\begin{aligned} \int_0^t {}^C D_{\tau,t}^{1-(j-1)\alpha} g(\tau) y(\tau) d\tau + \int_0^t {}^C D_{\tau,t}^{1-j\alpha} g(\tau) y(\tau) d\tau = \\ \int_0^t g(\tau) D_\tau^{1-j\alpha} \tilde{u}(\tau) d\tau - D_t^{-(j-1)\alpha} y(t) - D_t^{-j\alpha} y(t). \end{aligned} \quad (10)$$

If  $J$  is chosen to fulfill that  $-(J-1)\alpha > -1$  and  $-J\alpha \leq -1$ , then  $D_t^{-J\alpha}y(t)$  is a proper integral and thus the improper integral  $D_t^{-(J-1)\alpha}y(t)$  can be expressed by an algebraic integral formula based on (10). Hence, by using (10),  $D_t^{-(J-1)\alpha}y(t), \dots, y(t)$  can be recursively expressed by algebraic integral formulas. Finally,  $D_t^\alpha y(t)$  can be computed by using (9).

Bearing the above basic thoughts in mind, we intend to estimate  $D_t^\beta x(t)$  for the system defined by (3). For this purpose, with respect to Example 1, there are several problems should be solved.

- **Problem 1: System model.** The applied method in Example 1 is based on a fractional differential equation. However, the considered system in (3) is defined by the pseudo-state space representation.
- **Problem 2: Unknown initial values of the output.** Example 1 is based on the zero initial values of the output. In the general case where the initial values can be non-zero, there will be undesired terms by composing fractional derivatives. Moreover, it will also produce non-zero initial boundary conditions when the fractional order integration by parts formulas are applied.
- **Problem 3:  $\beta$  order fractional derivative of  $x$ .** In Example 1, only the  $\alpha$  order fractional derivative of  $y$  is derived. However, the  $\beta$  order fractional derivative  $D_t^\beta x(t)$  is needed in this paper.

### 3. Main results

In this section, the main results of the proposed modulating functions method are presented, where the above mentioned problems will be solved.

#### 3.1. Transformation of system model

*Problem 1: System model.* The proposed method intends to estimate the fractional derivative of the pseudo-state for the system defined by (3) based on the ideas demonstrated in Example 1. However, the method applied in Example 1 is based on a fractional differential equation. Hence, inspired by [19, 30, 31], the solution is to transform the system model. For this purpose, we need to introduce the Riemann-Liouville fractional sequential derivative at first.



**Definition 1** Let  $k \in \mathbb{Z}$  and  $\alpha \in \mathbb{R}_+$ , the Riemann-Liouville fractional sequential derivative of a function  $f$  on  $[0, t]$  is defined as follows:

$$\mathcal{D}_t^{k\alpha} f(t) := \begin{cases} f(t), & \text{for } k = 0, \\ \mathcal{D}_t^\alpha \left\{ \mathcal{D}_t^{(k-1)\alpha} f(t) \right\}, & \text{for } k \geq 1, \\ \mathcal{D}_t^{-\alpha} \left\{ \mathcal{D}_t^{(k+1)\alpha} f(t) \right\}, & \text{for } k \leq -1. \end{cases} \quad (11)$$

Then, similar as done in [30], the following proposition can be deduced.

**Proposition 1** The system defined in (3) can be transformed into the following form on  $I$ :

$$\sum_{i=0}^N a_i \mathcal{D}_t^{\alpha_i} y(t) = \bar{u}, \quad (12)$$

where

$$\bar{u} = \eta \mathcal{D}_t^{N\alpha} u + \sum_{l=1}^N \lambda_l \mathcal{D}_t^{(N-l)\alpha} u, \quad (13)$$

with

$$\lambda_l = \sum_{i=N-l+1}^N a_i C A^{i-N+l-1} B + \eta a_{N-l}, \quad (14)$$

$a_N = 1$ , and  $a_i \in \mathbb{R}$  for  $i = 0, \dots, N-1$ , are the characteristic polynomial coefficients of  $A$ .

Moreover, we have:

$$x = M_1^{-1} M_2 \begin{pmatrix} y \\ \mathcal{D}_t^\alpha y \\ \vdots \\ \mathcal{D}_t^{(N-1)\alpha} y \end{pmatrix} - M_1^{-1} M_3 \begin{pmatrix} u \\ \mathcal{D}_t^\alpha u \\ \vdots \\ \mathcal{D}_t^{(N-1)\alpha} u \end{pmatrix}, \quad (15)$$

where

$$M_1 = \begin{pmatrix} CA^{N-1} + \sum_{i=1}^{N-1} a_i CA^{i-1} \\ \vdots \\ CA^{N-j} + \sum_{i=j}^{N-1} a_i CA^{i-j} \\ \vdots \\ CA + a_{N-1}C \\ C \end{pmatrix}, \quad (16)$$

$$M_2 = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_{N-1} & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{N-1} & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad (17)$$

$$M_3 = \begin{pmatrix} \lambda_{N-1} & \lambda_{N-2} & \lambda_{N-3} & \cdots & \lambda_1 & \lambda_0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda_1 & \lambda_0 & 0 & \cdots & 0 & 0 \\ \lambda_0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad (18)$$

with  $\lambda_0 = \eta$ .

**Remark 3** In [19, 30, 31], the matching conditions (C<sub>2</sub>)-(C<sub>3</sub>) were considered to avoid involving the fractional derivatives of the input  $u$ . However, the range of the considered systems were constrained. In order to study a wider class of fractional order linear systems in this paper, (C<sub>2</sub>)-(C<sub>3</sub>) are no longer considered. Thus, the fractional derivatives of the input  $u$  are involved in (13).

Finally, the fractional derivative of the pseudo-state  $x$  of the system defined by (3) can be formulated in the following corollary.

**Corollary 1** Let  $-1 < \beta \in \mathbb{R}$ , then  $D_t^\beta x$  can be given as follows:

$$D_t^\beta x = M_1^{-1} M_2 \begin{pmatrix} D_t^\beta y \\ D_t^\beta \{\mathcal{D}_t^\alpha y\} \\ \vdots \\ D_t^\beta \{\mathcal{D}_t^{(N-1)\alpha} y\} \end{pmatrix} - M_1^{-1} M_3 \begin{pmatrix} D_t^\beta u \\ D_t^\beta \{\mathcal{D}_t^\alpha u\} \\ \vdots \\ D_t^\beta \{\mathcal{D}_t^{(N-1)\alpha} u\} \end{pmatrix}, \quad (19)$$

where  $M_1$ ,  $M_2$  and  $M_3$  are defined in Proposition 1.

With a known input  $u$  in (19),  $D_t^\beta \{\mathcal{D}_t^{i\alpha} u(t)\}$  for  $i = 0, \dots, N-1$  can be analytically calculated or successively computed by applying the Grünwald-Letnikov scheme. Therefore, based on Corollary 1, the estimation of  $D_t^\beta x(t)$  can be accomplished via the estimation of the fractional derivatives  $D_t^\beta \{\mathcal{D}_t^{i\alpha} y(t)\}$  for  $i = 0, \dots, N-1$ . Specially, the pseudo-state  $x$  can be obtained by taking  $\beta = 0$ .

### 3.2. Algebraic formulas for the fractional derivative of the pseudo-state

*Problem 2: Unknown initial values of the output.* Thanks to Proposition 1, the considered pseudo-state space representation model is transformed into a fractional differential equation model in (12). However, different from Example 1, the system model given by (12) is based on the

fractional sequential derivative with unknown initial values **of the output**. In order to deal with this, the following additive index law for the fractional sequential derivative is introduced.

**Lemma 1** [30] *The Riemann-Liouville fractional integral and derivative of a fractional sequential derivative for a function  $f$  on  $[0, t]$  can be given as follows:  $\forall \beta \in \mathbb{R}, \forall k \in \mathbb{N}^*$ ,*

$$D_t^\beta \{ \mathcal{D}_t^{k\alpha} f(t) \} = D_t^{\beta+k\alpha} f(t) - \phi_{\beta,k,\alpha} \{ f(t) \}, \quad (20)$$

where

$$\phi_{\beta,k,\alpha} \{ f(t) \} := \sum_{j=1}^k \psi_{\beta+(j-1)\alpha,\alpha} \left\{ \mathcal{D}_t^{(k-j)\alpha} f(t) \right\}, \quad (21)$$

with

$$\begin{aligned} \psi_{\beta+(j-1)\alpha,\alpha} \left\{ \mathcal{D}_t^{(k-j)\alpha} f(t) \right\} := \\ \sum_{i=1}^{\lceil \alpha \rceil} c_{\beta+(j-1)\alpha,i} t^{-\beta-(j-1)\alpha-i} \left[ D_t^{\alpha-i} \left\{ \mathcal{D}_t^{(k-j)\alpha} f(t) \right\} \right]_{t=0}, \end{aligned} \quad (22)$$

and

$$c_{\beta+(j-1)\alpha,i} = \begin{cases} 0, & \text{if } \alpha \in \mathbb{R}_- \cup \{0\}, \\ 0, & \text{if } 1 - \beta - (j-1)\alpha - i \in \mathbb{Z}_- \cup \{0\}, \\ \frac{1}{\Gamma(1-\beta-(j-1)\alpha-i)}, & \text{else.} \end{cases} \quad (23)$$

Moreover, we have:  $\forall k \geq 2$ ,

$$\mathcal{D}_t^{k\alpha} f(t) = D_t^{k\alpha} f(t) - \phi_{\alpha,(k-1),\alpha} \{ f(t) \}. \quad (24)$$

**Remark 4** *Based on the additive index law given in Lemma 1, we can get:  $\mathcal{D}_t^{k\alpha} f(t) = D_t^{k\alpha} f(t)$  for any negative integer number  $k$ .*

Then, the idea of constructing equations by applying  $D_t^{-j\alpha}$  in Example 1 will be applied to the differential equation obtained in Lemma 1. For this, the following lemma is useful which is given by applying Lemma 1.

**Lemma 2** *Let  $i \in \{1, \dots, N\}$ ,  $j \in \mathbb{N}^*$ , and  $y$  be the output of system defined by (3), then we have:*

$$D_t^{-j\alpha} \{ \mathcal{D}_t^{i\alpha} y(t) \} = \mathcal{D}_t^{(i-j)\alpha} y(t) - \sum_{j'=1}^{r_{i,j}} \psi_{-j\alpha+(j'-1)\alpha,\alpha} \left\{ \mathcal{D}_t^{(i-j')\alpha} y(t) \right\}, \quad (25)$$

where  $r_{i,j} = \min\{i, j\}$ .

**Proof.** In this proof, two cases are considered as follows:

- If  $j < i$ ,  $D_t^{-j\alpha}\{\mathcal{D}_t^{i\alpha}y(t)\}$  can be rewritten as  $D_t^{-j\alpha}\{\mathcal{D}_t^{j\alpha}\{\mathcal{D}_t^{(i-j)\alpha}y(t)\}\}$ . Then, (25) can be obtained by applying (20) to the operator  $D_t^{-j\alpha}\{\mathcal{D}_t^{j\alpha}\{\cdot\}\}$ .
- If  $j \geq i$ , by using Remark 4, we have  $\mathcal{D}_t^{(i-j)\alpha}f(t) = D_t^{(i-j)\alpha}f(t)$ . Thus, (25) can be obtained by applying (20).

Thus, this proof is completed. ■

*Problem 3:  $\beta$  order fractional derivative of  $x$ .* According to Corollary 1,  $D_t^\beta x$  can be computed by using  $D_t^\beta\{\mathcal{D}_t^{i\alpha}y\}$  for  $i = 0, \dots, N - 1$ . Hence, instead of applying  $D_t^1$  as done in Example 1,  $D_t^{\beta+1}$  will be applied to (12). To do this, the following lemma is useful, which can be obtained by taking  $D_t^{\beta+1}$  to (25).

**Lemma 3** *Let  $i \in \{1, \dots, N\}$ ,  $j \in \mathbb{N}^*$ , and  $\beta > -1$ , then we have:*

$$D_t^{\beta+1}\{D_t^{-j\alpha}\{\mathcal{D}_t^{i\alpha}y(t)\}\} = D_t^{\beta+1}\{\mathcal{D}_t^{(i-j)\alpha}y(t)\} - \Delta_{i,j}^r(t), \quad (26)$$

where

$$\Delta_{i,j}^r(t) = \sum_{j'=1}^{r_{i,j}} D_t^{\beta+1} \left\{ \psi_{-j\alpha+(j'-1)\alpha,\alpha} \left\{ \mathcal{D}_t^{(i-j')\alpha}y(t) \right\} \right\}, \quad (27)$$

and  $r_{i,j}$  is defined in Lemma 2.

As shown in Example 1, the fractional order integration by parts formulas are applied, where the right-sided Caputo fractional derivative is involved. Since Lemma 3 will be applied, the fractional order integration by parts formulas will be successively applied to remove the fractional sequential derivatives. Hence, the right-sided Caputo fractional sequential derivative will be involved, which are defined as follows.

**Definition 2** *Let  $k \in \mathbb{N}$ , the right-sided Caputo fractional sequential derivative is given by:*

$${}^C\mathcal{D}_{t,h}^{k\alpha}f(t) := \begin{cases} f(t), & \text{for } k = 0, \\ {}^C\mathcal{D}_{t,h}^\alpha \left\{ {}^C\mathcal{D}_{t,h}^{(k-1)\alpha}f(t) \right\}, & \text{for } k \geq 1. \end{cases} \quad (28)$$

Before giving the main results of this paper, let us first define the modulating functions.

**Definition 3** [30] *Let  $[a, b] \subset \mathbb{R}$ ,  $n \in \mathbb{N}$ ,  $L \in \mathbb{N}^*$ , and  $g_n$  be a function fulfilling the following properties: for  $k = 0, \dots, L - 1$ ,*

$$(P_1) : g_n \in \mathcal{C}^L([a, b]),$$

$$(P_2) : g_n^{(k)}(a) = 0,$$

$$(P_3) : g_n^{(k)}(b) = 0.$$

Then,  $g_n$  is a  $L^{\text{th}}$  order modulating function on  $[a, b]$ . Besides, if  $g_n$  satisfies  $(P_1)$ - $(P_2)$  and the following property:

$$(P_4) : g_n^{(k)}(b) = 1, \text{ if } k = n, g_n^{(k)}(b) = 0, \text{ else.}$$

Then,  $g_n$  is called a  $(L, n)^{\text{th}}$  order modulating function on  $[a, b]$ .

Then, let  $j \in \mathbb{N}$  and  $\alpha \in ]0, 2[$ , then the following set is defined by:

$$\Psi_{j,\alpha} \subset \mathbb{N}^2 := \{(j', k); 0 \leq j' \leq N - j - 1, 0 \leq k \leq \lceil \alpha \rceil - 1\}. \quad (29)$$

Now, the main results of this paper is ready to be presented.

**Theorem 1** Let  $-1 < \beta \in \mathbb{R}$ ,  $L = \lceil \beta \rceil + 1$ ,  $\alpha \in ]0, 2[$ ,  $N \in \mathbb{N}^*$  and  $J = \lceil \frac{\beta+1}{\alpha} \rceil + N - 1$ . For  $t \in I \setminus \{0\}$  and  $j \in \{1, \dots, J\}$ , let  $g_j$  be a  $(L, 0)^{\text{th}}$  order modulating function on  $[0, t]$  which also fulfills the following property:

$$(P_5) : \forall i \in \{0, \dots, N\}, {}^C \mathcal{D}_{\tau,t}^{(i-j)\alpha} \{ {}^C \mathcal{D}_{\tau,t}^{\beta+1} g_j \} \text{ exists and is integrable on } I,$$

Moreover, for  $j \in \{1, \dots, N - 1\}$ ,  $g_j$  satisfy the following property:

$$(P_6) : \forall (j', k) \in \Psi_{j,\alpha}, \{ {}^C \mathcal{D}_{\tau,t}^{j'\alpha} \{ {}^C \mathcal{D}_{\tau,t}^{\beta+1} g_j(\tau) \} \}^{(k)} = 0 \text{ at } \tau = 0 \text{ and } \tau = t.$$

Then, for the system defined by (3), the required fractional derivatives of the output  $y$  can be recursively given as follows:

- for  $j = J, \dots, 1$ ,

$$\begin{aligned} D_t^\beta \{ \mathcal{D}_t^{(N-j)\alpha} y(t) \} &= \int_0^t g_j(\tau) D_\tau^{\beta-j\alpha+1} \bar{u}(\tau) d\tau + \sum_{i=1}^N a_i \int_0^t g_j(\tau) \Delta_{i,j}^r(\tau) d\tau \\ &\quad - \sum_{i=0}^N a_i \int_0^t {}^C \mathcal{D}_{\tau,t}^{(i-j)\alpha} \{ {}^C \mathcal{D}_{\tau,t}^{\beta+1} g_j(\tau) \} y(\tau) d\tau - \sum_{i=0}^{N-1} a_i D_t^\beta \{ \mathcal{D}_t^{(i-j)\alpha} y(t) \}, \end{aligned} \quad (30)$$

where  $D_t^\beta \{ \mathcal{D}_t^{(i-j)\alpha} y(t) \} = D_t^{\beta+(i-j)\alpha} y(t)$  for  $i = 0, \dots, N - 1$ , which are given by proper integrals.

**Proof.** The proof can be divided into three steps.

**Step 1: Construction of equations.** By applying  $D_t^{\beta+1} \cdot D_t^{-j\alpha}$  to (12) for  $j = 1, \dots, J$  and applying Lemma 3, we get: for  $j = 1, \dots, J$ ,

$$\sum_{i=0}^N a_i D_t^{\beta+1} \{ \mathcal{D}_t^{(i-j)\alpha} y(t) \} = D_t^{\beta-j\alpha+1} \bar{u}(t) + \sum_{i=1}^N a_i \Delta_{i,j}^r(t). \quad (31)$$

**Step 2: Fractional order integration by parts.** By multiplying  $g_j$  to (31) and taking integration on  $[0, t]$ , we get: for  $j = 1, \dots, J$ ,

$$\begin{aligned} \sum_{i=0}^N a_i \int_0^t g_j(\tau) D_{\tau}^{\beta+1} \{ \mathcal{D}_{\tau}^{(i-j)\alpha} y(\tau) \} d\tau = \\ \int_0^t g_j(\tau) D_{\tau}^{\beta-j\alpha+1} \bar{u}(\tau) d\tau + \sum_{i=1}^N a_i \int_0^t g_j(\tau) \Delta_{i,j}^r(\tau) d\tau. \end{aligned} \quad (32)$$

Then, the fractional order integration by parts formulas are applied to remove the fractional derivative  $D_t^{\beta+1}$  from  $y$  and the left side of (32) becomes:

$$\begin{aligned} \sum_{i=0}^N a_i \int_0^t g_j(\tau) D_{\tau}^{\beta+1} \{ \mathcal{D}_{\tau}^{(i-j)\alpha} y(\tau) \} d\tau = \sum_{i=0}^N a_i \int_0^t {}^C D_{\tau,t}^{\beta+1} \{ g_j(\tau) \} \mathcal{D}_{\tau}^{(i-j)\alpha} y(\tau) d\tau \\ + \sum_{i=0}^N \sum_{k=0}^{\lceil \beta+1 \rceil - 1} (-1)^k a_i \left[ g_j^{(k)}(\tau) D_{\tau}^{\beta-k} \{ \mathcal{D}_{\tau}^{(i-j)\alpha} y(\tau) \} \right]_{\tau=0}^{\tau=t}. \end{aligned} \quad (33)$$

By taking  $N - j$  more times fractional order integration by parts formulas to the first term in the right side of (33) to successively remove the fractional derivatives of  $y$ , we get: for  $j = 1, \dots, J$ ,

- if  $j < N$ ,

$$\begin{aligned} \sum_{i=0}^N a_i \int_0^t {}^C D_{\tau,t}^{\beta+1} \{ g_j(\tau) \} \mathcal{D}_{\tau}^{(i-j)\alpha} y(\tau) d\tau = \sum_{i=0}^N a_i \int_0^t {}^C \mathcal{D}_{\tau,t}^{(i-j)\alpha} \{ {}^C D_{\tau,t}^{\beta+1} g_j(\tau) \} y(\tau) d\tau \\ + \sum_{j'=1}^{N-j} \sum_{i=j'+j}^N \sum_{k=0}^{\lceil \alpha \rceil - 1} (-1)^k a_i \left[ \{ {}^C \mathcal{D}_{\tau,t}^{(j'-1)\alpha} \{ {}^C D_{\tau,t}^{\beta+1} g_j(\tau) \} \}^{(k)} D_{\tau}^{\alpha-1-k} \{ \mathcal{D}_{\tau}^{(i-j-j')\alpha} y(\tau) \} \right]_{\tau=0}^{\tau=t}. \end{aligned} \quad (34)$$

- if  $j \geq N$ ,

$$\sum_{i=0}^N a_i \int_0^t {}^C D_{\tau,t}^{\beta+1} \{ g_j(\tau) \} \mathcal{D}_{\tau}^{(i-j)\alpha} y(\tau) d\tau = \sum_{i=0}^N a_i \int_0^t {}^C \mathcal{D}_{\tau,t}^{(i-j)\alpha} \{ {}^C D_{\tau,t}^{\beta+1} g_j(\tau) \} y(\tau) d\tau. \quad (35)$$

**Step 3: Recursive algorithm.** Due to the properties satisfied by  $g_j$ , using (32) - (35), we get: for  $j = 1, \dots, J$ ,

$$\begin{aligned} D_t^{\beta} \{ \mathcal{D}_t^{(N-j)\alpha} y(t) \} = \int_0^t g_j(\tau) D_{\tau}^{\beta-j\alpha+1} \bar{u}(\tau) d\tau + \sum_{i=1}^N a_i \int_0^t g_j(\tau) \Delta_{i,j}^r(\tau) d\tau \\ - \sum_{i=0}^N a_i \int_0^t {}^C \mathcal{D}_{\tau,t}^{(i-j)\alpha} \{ {}^C D_{\tau,t}^{\beta+1} g_j(\tau) \} y(\tau) d\tau - \sum_{i=0}^{N-1} a_i D_t^{\beta} \{ \mathcal{D}_t^{(i-j)\alpha} y(t) \}. \end{aligned} \quad (36)$$

By choosing a positive integer  $J$  in (36) such that  $\beta + (N-1-J)\alpha \leq -1$  and  $\beta + (N-J)\alpha > -1$ , i.e.  $J = \lceil \frac{\beta+1}{\alpha} \rceil + N - 1$ , we get  $i - J < 0$ ,  $\beta + (i - J)\alpha \leq -1$ , for  $i = 0, \dots, N - 1$ . Hence, according to Lemma 1 and Remark 4, we get:  $D_t^{\beta} \{ \mathcal{D}_t^{(i-J)\alpha} y(t) \} = D_t^{\beta+(i-J)\alpha} y(t)$  are defined by proper

integrals. Thus, using (36),  $D_t^\beta \{\mathcal{D}_t^{(N-J)\alpha} y(t)\}$  can be derived using these proper integrals. Then, by taking  $j = J - 1, \dots, 1$ ,  $D_t^\beta \{\mathcal{D}_t^{(N-j)\alpha} y(t)\}$  can be obtained by using (36) in a recursive way. Thus, this proof is completed. ■

**Remark 5** *It is clearly in the previous proof that only the fractional derivative values produced in (33) from the first integration by parts are kept and the ones produced in (34) are eliminated. Then, the former values are obtained using a recursive way by constructing  $J$  equations. This is why the proposed method can deal with the case with an arbitrary differentiation order. However, in [19, 30, 31] all the fractional derivative values produced from each integration by parts were kept, which were obtained using a recursive way by constructing  $q$  equations with the differentiation order  $\alpha = \frac{p}{q}$  ( $p, q \in \mathbb{N}^*$ ) in [19, 30] and 1 equation in [31], respectively. This yields the methods proposed in [19, 30, 31] are only suitable for the fractional order systems with a rational differentiation order.*

In order to calculate the unknown fractional derivative initial values involved in  $\Delta_{i,j}^r(t)$  for  $i = 1, \dots, N$  and  $j = 1, \dots, J$  of (30) in Theorem 1, the following fractional order modulating functions will be useful.

**Definition 4** *Let  $[a, b] \subset \mathbb{R}$ ,  $N \in \mathbb{N}^*$ ,  $j \in \{0, \dots, N - 1\}$ ,  $n \in \{0, \dots, [\alpha] - 1\}$ , and  $h_{j,n}$  be a function satisfying the following properties:  $\forall (j', k) \in \Psi_{0,\alpha}$ ,*

$$(P_1^\alpha) : {}^C \mathcal{D}_{\tau,b}^{i\alpha} \{h_{j,n}\} \text{ for } i = 0, \dots, N \text{ exist and are integrable on } [a, b],$$

$$(P_2^\alpha) : \left[ {}^C \mathcal{D}_{\tau,b}^{j'\alpha} \{h_{j,n}(\tau)\} \right]_{\tau=a}^{(k)} = 1, \text{ if } (j', k) = (j, n), \left[ {}^C \mathcal{D}_{\tau,b}^{j'\alpha} \{h_{j,n}(\tau)\} \right]_{\tau=a}^{(k)} = 0, \text{ else,}$$

$$(P_3^\alpha) : \left\{ {}^C \mathcal{D}_{\tau,b}^{j'\alpha} \{h_{j,n}(b)\} \right\}^{(k)} = 0.$$

*Then,  $h_{j,n}$  is a  $(\alpha; j, n)^{\text{th}}$  fractional order modulating function on  $[a, b]$ .*

Recall that the right-sided fractional order modulating function defined in [31] is a special case of this definition. Then, the following proposition is introduced.

**Proposition 2** *Let  $\alpha \in ]0, 2[$  and  $N \in \mathbb{N}^*$ . For  $t \in I \setminus \{0\}$ ,  $j \in \{0, \dots, N - 1\}$  and  $n \in \{0, \dots, [\alpha] - 1\}$ , let  $h_{j,n}$  be a  $(\alpha; j, n)^{\text{th}}$  fractional order modulating function on  $[0, t]$ . Then, for the system defined by (3), the fractional derivative initial values of  $y$  can be recursively given as follows:  $\forall n \in \{0, \dots, L - 1\}$ ,*

- if  $j = N - 1$ ,

$$\left[ D_\tau^{\alpha-1-n} y(\tau) \right]_{\tau=0} = (-1)^n \left\{ \sum_{i=0}^N a_i \int_0^t {}^C \mathcal{D}_{\tau,t}^{i\alpha} \{h_{j,n}(\tau)\} y(\tau) d\tau - \int_0^t h_{j,n}(\tau) \bar{u}(t) d\tau \right\}, \quad (37)$$

- if  $j = N - 2, \dots, 0$ ,

$$\begin{aligned} \left[ D_{\tau}^{\alpha-1-n} \{ \mathcal{D}_{\tau}^{(N-j-1)\alpha} y(\tau) \} \right]_{\tau=0} &= (-1)^n \sum_{i=0}^N a_i \int_0^t {}^C \mathcal{D}_{\tau,t}^{i\alpha} \{ h_{j,n}(\tau) \} y(\tau) d\tau \\ &- (-1)^n \int_0^t h_{j,n}(\tau) \bar{u}(t) d\tau - \sum_{i=j+1}^{N-1} a_i \left[ D_{\tau}^{\alpha-1-n} \{ \mathcal{D}_{\tau}^{(i-j-1)\alpha} y(\tau) \} \right]_{\tau=0}. \end{aligned} \quad (38)$$

**Proof.** For  $j = 0, \dots, N-1$  and  $n = 0, \dots, L-1$ , by multiplying  $h_{j,n}$  to (12) and taking integration on  $[0, t]$ , we get:

$$\sum_{i=0}^N a_i \int_0^t h_{j,n}(\tau) \mathcal{D}_{\tau}^{\alpha_i} y(\tau) d\tau = \int_0^t h_{j,n}(\tau) \bar{u}(t) d\tau. \quad (39)$$

By applying the fractional order integration by parts formulas for  $N-1$  times to (39), we get:

$$\begin{aligned} \sum_{i=0}^N a_i \int_0^t {}^C \mathcal{D}_{\tau,t}^{i\alpha} \{ h_{j,n}(\tau) \} y(\tau) d\tau &= \int_0^t h_{j,n}(\tau) \bar{u}(t) d\tau - \sum_{j'=0}^{N-1} \sum_{i=j'+1}^N a_i \\ &\times \sum_{k=0}^{[\alpha]-1} (-1)^k \left[ \{ {}^C \mathcal{D}_{\tau,t}^{j'\alpha} \{ h_{j,n}(\tau) \} \}^{(k)} D_{\tau}^{\alpha-1-k} \{ \mathcal{D}_{\tau}^{(i-j'-1)\alpha} y(\tau) \} \right]_{\tau=0}^{\tau=t}. \end{aligned} \quad (40)$$

Due to the properties fulfilled by  $h_{j,n}$ , (40) becomes:

$$\begin{aligned} \sum_{i=0}^N a_i \int_0^t {}^C \mathcal{D}_{\tau,t}^{i\alpha} \{ h_{j,n}(\tau) \} y(\tau) d\tau &= \int_0^t h_{j,n}(\tau) \bar{u}(t) d\tau \\ &+ \sum_{i=j+1}^N (-1)^n a_i \left[ D_{\tau}^{\alpha-1-n} \{ \mathcal{D}_{\tau}^{(i-j-1)\alpha} y(\tau) \} \right]_{\tau=0}. \end{aligned} \quad (41)$$

Hence, based on (41), (37) and (38) can be derived recursively as  $j = N-1, \dots, 0$ . ■

Moreover, the following corollary can also be introduced to calculate the fractional derivatives of the output of the system defined by (3) by applying Lemma 1, Theorem 1 and Proposition 2.

**Corollary 2** *Under the same hypotheses given in Theorem 1 and Proposition 2, the fractional derivatives of  $y$  of the system defined by (3) can be given as follows: for  $j = 1, \dots, J$ ,*

$$D_t^{\beta+(N-j)\alpha} y(t) = D_t^{\beta} \{ \mathcal{D}_t^{(N-j)\alpha} y(t) \} + \Theta_{\beta, N-j, \alpha} \{ y(t) \}, \quad (42)$$

where  $D_t^{\beta} \{ \mathcal{D}_t^{(N-j)\alpha} y(t) \}$  is given in Theorem 1, and  $\Theta_{\beta, N-j, \alpha} \{ y(t) \}$  is defined by:  $\Theta_{\beta, N-j, \alpha} \{ y(t) \} = \phi_{\beta, N-j, \alpha} \{ y(t) \}$  if  $N-j > 0$ ,  $\Theta_{\beta, N-j, \alpha} \{ y(t) \} = 0$  else,  $\phi_{\beta, N-j, \alpha} \{ y(t) \}$  is defined in Lemma 1, and can be given by Proposition 2.

By applying the algebraic integral formulas given by Theorem 1 and Proposition 2 to (19),  $D_t^{\beta} x(t)$  can also be given by using an algebraic integral formula. If the functions  $g_j$  and  $h_{j,n}$  are



appropriately chosen, these formulas can all be given by proper integrals. For this, the modulating functions  $g_j$  and  $h_{j,n}$  will be constructed in the next subsection. It is worth noting that these algebraic integral formulas do not have any source of errors in continuous noise free case, and non-asymptotic estimations can be provided in finite-time. Besides, thanks to the role of low-pass filter of the involved integrals, the noise effect of the output can be mitigated [46]. Similar conclusions are suitable for estimating the fractional derivatives of the output  $D_t^{\beta+(N-j)\alpha}y(t)$  for  $j = 1, \dots, J$  by using Corollary 2, where  $-1 < \beta + (N - j)\alpha \leq \beta + (N - 1)\alpha$ .

### 3.3. Construction of modulating functions

In this subsection, the construction of the required modulating functions in Theorem 1 and Proposition 2 is provided. These modulating functions are constructed based on the fractional order polynomial form to make them easy to handle by the fractional integral and derivative. For this purpose, the following two lemmas concerning the right-sided Caputo fractional derivative of a fractional order polynomial are necessary.

**Lemma 4** [44] *Let  $\gamma \in \mathbb{R}$  and the function  $f(\tau) = (t - \tau)^\beta$  be defined on  $[0, t]$  with  $\max\{-1, \lceil \gamma \rceil - 1\} < \beta \in \mathbb{R}$ , then we have:*

$${}^C D_{\tau,t}^\gamma f(\tau) = \frac{\Gamma(\beta + 1)}{\Gamma(\beta + 1 - \gamma)} (t - \tau)^{\beta - \gamma}. \quad (43)$$

Then, the following proposition provides the details to construct the modulating functions required in Theorem 1.

**Proposition 3** *Let  $-1 < \beta \in \mathbb{R}$ ,  $L = \lceil \beta \rceil + 1$ ,  $\alpha \in ]0, 2[$ ,  $N \in \mathbb{N}^*$  and  $J = \lceil \frac{\beta+1}{\alpha} \rceil + N - 1$ . For  $t \in I \setminus \{0\}$  and  $j \in \{1, \dots, J\}$ , let  $L - 1 < m_2 \in \mathbb{R}_+$ ,  $N_j, M_j \in \mathbb{N}$  with  $M_j \geq 2$  and  $g_j$  be a function defined on  $[0, t]$  as follows:*

$$g_j(\tau) = c_0 + \sum_{i=0}^{M_j-2} c_{i+1} \frac{(t - \tau)^{(i+N_j)\alpha + \beta + 1 + m_2}}{t^{(i+N_j)\alpha + \beta + 1 + m_2}}, \quad (44)$$

the coefficient vector  $c = (c_0, c_1, \dots, c_{M_j-1})^T$  is the solution of the following system:

$$B_m c = d, \quad (45)$$

where  $B_m \in \mathbb{R}^{M_j \times M_j}$  and  $d \in \mathbb{R}^{M_j \times 1}$  are defined as follows:

- if  $N \leq j \leq J$ , let  $M_j = L + 1$ , then for  $k', l' = 0, \dots, L$ ,

$$B_m(k', l') = \begin{cases} 1, & \text{if } (k', l') = (0, 0) \text{ or } (L, 0), \\ (-1)^{k'} \frac{\Gamma((l' + N_j - 1)\alpha + \beta + m_2 + 2)}{\Gamma((l' + N_j - 1)\alpha + \beta + m_2 + 2 - k')}, & \text{if } k' \leq L - 1 \text{ and } 1 \leq l', \\ 0, & \text{else,} \end{cases} \quad (46)$$

and  $d(k') = 1$  if  $k' = L$ ,  $d(k') = 0$  else. Then,  $g_j$  is a  $(L, 0)^{th}$  order modulating function on  $[0, t]$  satisfying  $(P_5)$  required in Theorem 1.

- if  $1 \leq j \leq N - 1$ , let  $(N - j - 1) + \frac{1}{\alpha}([\alpha] - 1 - m_2) < N_j \in \mathbb{N}^*$ ,  $M_j = N' + L + 1$  with  $N' = \text{card}(\Psi_{j,\alpha})$ , then for  $k', l' = 0, \dots, N' - 1$ , and for  $(j', k) \in \Psi_{j,\alpha}$ ,

$$B_m(k', l') = \begin{cases} 1, & \text{if } (k', l') = (0, 0) \text{ or } (L, 0), \\ (-1)^{k'} \frac{\Gamma((l' + N_j - 1)\alpha + \beta + m_2 + 2)}{\Gamma((l' + N_j - 1)\alpha + \beta + m_2 + 2 - k')}, & \text{if } k' \leq L - 1 \text{ and } 1 \leq l', \\ (-1)^k \frac{\Gamma((l' + N_j - 1)\alpha + \beta + m_2 + 2)}{\Gamma((l' + N_j - 1 - j')\alpha + m_2 + 1 - k)}, & \text{if } L < k' \leq N' + L \text{ and } 1 \leq l', \\ 0, & \text{else,} \end{cases} \quad (47)$$

and  $d(k') = 1$  if  $k' = L$ ,  $d(k') = 0$  else. Then,  $g_j$  is a  $(L, 0)^{th}$  order modulating function on  $[0, t]$  satisfying  $(P_5)$ - $(P_6)$  required in Theorem 1.

**Proof.** For  $j = 1, \dots, J$ , on the one hand, by taking  $N_j \in \mathbb{N}$ ,  $L - 1 \leq m_2 \in \mathbb{R}_+$ ,  $(P_1)$  is fulfilled by  $g_j$ , and the  $(k')^{th}$  order derivative of  $g_j$  is given by: for  $k' = 0, \dots, L - 1$ ,

$$g_j^{(k')}(\tau) = c_0^{(k')} + (-1)^{k'} \sum_{i=0}^{M_j-2} c_{i+1} \frac{\Gamma((i + N_j)\alpha + \beta + m_2 + 2)}{\Gamma((i + N_j)\alpha + \beta + m_2 + 2 - k')} \frac{(t - \tau)^{(i + N_j)\alpha + \beta + 1 + m_2 - k'}}{t^{(i + N_j)\alpha + \beta + 1 + m_2}}, \quad (48)$$

where  $c_0^{(k')} = c_0$  if  $k' = 0$ ,  $c_0^{(k')} = 0$  else. Then, by taking  $\tau = 0$  in (48), we get:

$$g_j^{(k')}(0) = c_0^{(k')} + (-1)^{k'} t^{-k'} \sum_{i=0}^{M_j-2} c_{i+1} \frac{\Gamma((i + N_j)\alpha + \beta + m_2 + 2)}{\Gamma((i + N_j)\alpha + \beta + m_2 + 2 - k')}. \quad (49)$$

Since  $\forall t > 0$ ,  $t^{-k'} = 1$  for  $k' = 0$ , and  $c_0^{(k')} = 0$  for  $k' \neq 0$ , the coefficient vector  $c$  can be chosen independently of the term  $t^{-k'}$  such that  $g_j^{(k')}(0) = 0$  i.e.  $(P_2)$  is fulfilled by  $g_j$ . In (48), since  $(i + N_j)\alpha + \beta + 1 + m_2 - k' > 0$  with  $\beta > -1$ , we get:  $g_j^{(k')}(t) = c_0$  if  $k' = 0$ ,  $g_j^{(k')}(t) = 0$  else. Hence, by taking  $c_0 = 1$ ,  $(P_4)$  is fulfilled by  $g_j$ .

On the other hand, according to Lemma 4 since  $N_j\alpha + \beta + 1 + m_2 > 0$ ,  $(P_5)$  is fulfilled by  $g_j$ , for  $j = N, \dots, J$ . Moreover, for  $j = 1, \dots, N - 1$ , by taking  $(N_j - (N - j - 1))\alpha + m_2 - ([\alpha] - 1) > 0$ ,

i.e.  $N_j > \frac{1}{\alpha}((N-j-1)\alpha + (\lceil \alpha \rceil - 1) - m_2)$ ,  $(P_5)$  is fulfilled by  $g_j$ , and by successively applying Lemma 4, we get: for  $j' = -j, \dots, N-j$ ,

$${}^C \mathcal{D}_{\tau,t}^{j'\alpha} \{ {}^C \mathcal{D}_{\tau,t}^{\beta+1} g_j(\tau) \} = \sum_{i=0}^{M_j-2} c_{i+1} \frac{\Gamma((i+N_j)\alpha + \beta + m_2 + 2)}{\Gamma((i+N_j-j')\alpha + m_2 + 1)} \frac{(t-\tau)^{(i+N_j-j')\alpha + m_2}}{t^{(i+N_j)\alpha + \beta + 1 + m_2}}. \quad (50)$$

Then, for  $j = 1, \dots, N-1$ , we get: for  $(j', k) \in \Psi_{j,\alpha}$ ,

$$\begin{aligned} & \{ {}^C \mathcal{D}_{\tau,t}^{j'\alpha} \{ {}^C \mathcal{D}_{\tau,t}^{\beta+1} g_j(\tau) \} \}^{(k)} = \\ & (-1)^k \sum_{i=0}^{M_j-2} c_{i+1} \frac{\Gamma((i+N_j)\alpha + \beta + m_2 + 2)}{\Gamma((i+N_j-j')\alpha + m_2 + 1 - k)} \frac{(t-\tau)^{(i+N_j-j')\alpha + m_2 - k}}{t^{(i+N_j)\alpha + \beta + 1 + m_2}}. \end{aligned} \quad (51)$$

If  $\tau = 0$ , (51) becomes: for  $(j', k) \in \Psi_{j,\alpha}$ ,

$$\left[ \{ {}^C \mathcal{D}_{\tau,t}^{j'\alpha} \{ {}^C \mathcal{D}_{\tau,t}^{\beta+1} g_j(\tau) \} \}^{(k)} \right]_{\tau=0} = (-1)^k \sum_{i=0}^{M_j-2} c_{i+1} t^{-j'\alpha - k} \frac{\Gamma((i+N_j)\alpha + \beta + m_2 + 2)}{\Gamma((i+N_j-j')\alpha + m_2 + 1 - k)}. \quad (52)$$

Since  $(N_j - (N-j-1))\alpha + m_2 - (\lceil \alpha \rceil - 1) > 0$ , we obtain  $\{ {}^C \mathcal{D}_{\tau,t}^{j'\alpha} \{ {}^C \mathcal{D}_{\tau,t}^{\beta+1} g_j(t) \} \}^{(k)} = 0$ . Thus, the coefficient vector  $c$  can be chosen independently of the term  $t^{-j'\alpha - k}$  such that  $\left[ \{ {}^C \mathcal{D}_{\tau,t}^{j'\alpha} \{ {}^C \mathcal{D}_{\tau,t}^{\beta+1} g_j(\tau) \} \}^{(k)} \right]_{\tau=0}$  equals 0. Thus,  $(P_6)$  is fulfilled by  $g_j$ .

Consequently, the corresponding matrices  $B_m$  and  $d$  can be defined based on this proof. ■

In order to compute the fractional derivative initial values, the construction of the fractional order modulating functions required in Proposition 2 is provided in the following proposition.

**Proposition 4** *Let  $\alpha \in ]0, 2[$ ,  $N, \tilde{N} \in \mathbb{N}^*$  with  $\tilde{N} > \frac{1}{\alpha}((N-1)\alpha + (\lceil \alpha \rceil - 1) - m_1)$ ,  $m_1 \in \mathbb{R}$  with  $m_1 \geq \lceil \alpha \rceil - 1$ . For  $t \in I \setminus \{0\}$ ,  $j \in \{0, \dots, N-1\}$  and  $n \in \{0, \dots, \lceil \alpha \rceil - 1\}$ , let  $h_{j,n}$  be a function defined on  $[0, t]$  as follows:*

$$h_{j,n}(\tau) = \sum_{i=0}^{N'-1} c_i \frac{(t-\tau)^{(i+\tilde{N})\alpha + m_1}}{t^{(i+\tilde{N}-j)\alpha + m_1 - n}}, \quad (53)$$

where  $N' = \text{card}(\Psi_{0,\alpha})$ . The coefficient vector  $\tilde{c} = (c_0, c_1, \dots, c_{N'-1})^T$  is the solution of the following system:

$$B_m \tilde{c} = d, \quad (54)$$

where  $B_m \in \mathbb{R}^{N' \times N'}$  and  $d \in \mathbb{R}^{N' \times 1}$  are defined as follows: for  $i', l' = 0, \dots, N'-1$  and  $(j', k) \in \Psi_{0,\alpha}$ ,

$$B_m(i', l') = (-1)^{k'} \frac{\Gamma((l' + \tilde{N})\alpha + m_1 + 1)}{\Gamma((l' + \tilde{N} - j')\alpha + m_1 + 1 - k)}, \quad (55)$$

and  $d(i') = 1$  if  $(j', k) = (j, n)$ ,  $d(i') = 0$  else. Then,  $h_{j,n}$  is a  $(\alpha; j, n)^{\text{th}}$  fractional order modulating function on  $[0, t]$  required in Proposition 2.

**Proof.** For  $j = 0, \dots, N-1$  and  $n = 0, \dots, \lceil \alpha \rceil - 1$ , according to Lemma 4 and by taking  $(\tilde{N} - (N-1))\alpha + m_1 > \lceil \alpha \rceil - 1$ , i.e.,  $\tilde{N} > \frac{1}{\alpha}((N-1)\alpha + (\lceil \alpha \rceil - 1) - m_1)$ ,  $(P_1^\alpha)$  is fulfilled by  $h_{j,n}$ . Then, by successively applying Lemma 4, the fractional derivative  $\{{}^C\mathcal{D}_{\tau,t}^{j'\alpha}\{h_{j,n}(\tau)\}\}^{(k)}$  with  $(j', k) \in \Psi_{0,\alpha}$  is given as follows:

$$\{{}^C\mathcal{D}_{\tau,t}^{j'\alpha}\{h_{j,n}(\tau)\}\}^{(k)} = (-1)^k \sum_{i=0}^{N'-1} c_i \frac{\Gamma((i+\tilde{N})\alpha + m_1 + 1)t^{-i\alpha}}{\Gamma((i+\tilde{N}-j')\alpha + m_1 + 1 - k)} \frac{(t-\tau)^{(i+\tilde{N}-j')\alpha + m_1 - k}}{t^{(i+\tilde{N}-j)\alpha + m_1 - n}}. \quad (56)$$

If  $\tau = 0$ , we have:

$$\left[\{{}^C\mathcal{D}_{\tau,t}^{j'\alpha}\{h_{j,n}(\tau)\}\}^{(k)}\right]_{\tau=0} = (-1)^k \frac{t^{(\tilde{N}-j')\alpha + m_1 - k}}{t^{(\tilde{N}-j)\alpha + m_1 - n}} \sum_{i=0}^{N'-1} c_i \frac{\Gamma((i+\tilde{N})\alpha + m_1 + 1)}{\Gamma((i+\tilde{N}-j')\alpha + m_1 + 1 - k)}. \quad (57)$$

Thus, the coefficient vector  $c$  can be chosen independently of  $t$  such that  $h_{j,n}$  fulfills  $(P_2^\alpha)$ . Moreover, since  $(\tilde{N} - (N-1))\alpha + m_1 - (\lceil \alpha \rceil - 1) > 0$ , we get  $\{{}^C\mathcal{D}_{\tau,t}^{j'\alpha}\{h_{j,n}(t)\}\}^{(k)} = 0$ . Thus,  $(P_3^\alpha)$  is fulfilled by  $h_{j,n}$ . Accordingly, the elements of matrices  $B_m$  and  $d$  can be defined such that  $h_{j,n}$  fulfills  $(P_2^\alpha)$ . ■

**Remark 6** *Since  $t$  is not involved in the systems (45) and (54), then the above modulating functions can be constructed independently of  $t$ . Thus, their coefficients can be computed in an off-line way.*

### 3.4. Algorithm in discrete noisy environment

In this subsection, the proposed method is provided in discrete noisy environment. Consider the following discrete noisy observation  $y^\varpi$  of the output  $y$  on  $I = [0, h]$ :

$$y^\varpi(t_i) = y(t_i) + \varpi(t_i), \quad (58)$$

where  $t_i = iT_s$  for  $i = 0, 1, \dots, M$  with  $T_s = \frac{h}{M}$ , and  $\{\varpi(t), t \in I\}$  is a noise. In this noisy case, a numerical integration method is applied to approximate the integrals involved in the algebraic formulas obtained in Theorem 1 and Proposition 2. **Thus, the estimation error produced by the proposed method comes from two sources: the numerical errors caused by the numerical integration and the noisy errors caused by the corrupting noises.** On the one hand, it is well known that the numerical errors convergent to 0 as  $T_s \rightarrow 0$  [47]. On the other hand, using a similar technique as done in [30], it can be shown that if the noise  $\{\varpi(t), t \in I\}$  is an independent continuous stochastic process with a zero mean value function and a bounded variance function, then the noisy errors convergent to 0 in mean square as  $T_s \rightarrow 0$ . Similar result can also be obtained using a non-standard analysis as done in [46]. Consequently, both the two sources of errors of the proposed method can be reduced by decreasing the sampling period  $T_s$ . For a given  $T_s$ , noise error bounds can be

obtained using the Bienaymé-Chebyshev inequality as done in [30], which depend on the design parameters  $m_1$  and  $m_2$  of the proposed modulating functions. Hence, the error bounds can help us to choose appropriate parameters in order to reduce the noise error contributions. More details can be found in [30], which are omitted in this paper for the reason of space.

The main processes of the proposed method in discrete noise free case can be summarized in the following algorithm:

- Step 1.** Let  $-1 < \beta \in \mathbb{R}$ ,  $I = [0, h]$ ,  $T_s = \frac{h}{M}$  with  $M \in \mathbb{N}^*$ . For  $s = 1, \dots, M$ , let  $t_s = sT_s$ .
- Step 2.** Compute the values of  $D_{t_s}^\beta \{\mathcal{D}_{t_s}^{i\alpha} \bar{u}(t_s)\}$  for  $i = 0, \dots, N-1$  by applying the Grünwald-Letnikov scheme to  $\bar{u}$  defined in Lemma 1.
- Step 3.** Compute the proper integrals  $D_{t_s}^{\beta+(i-J)\alpha} y^\varpi(t_s)$  with  $J = \lceil \frac{\beta+1}{\alpha} \rceil + N - 1$ , for  $i = 0, \dots, N-1$  by applying a numerical integration method.
- Step 4.** Compute the coefficients of the constructed modulating functions  $g_j$  for  $j = 1, \dots, J$  and  $h_{j,n}$  for all  $(j, n) \in \Psi_{0,\alpha}$  on  $[0, t_s]$  by using Proposition 3 and Proposition 4, respectively. Then, compute  ${}^C \mathcal{D}_{\tau, t_s}^{(i-j)\alpha} \{ {}^C \mathcal{D}_{\tau, t_s}^{\beta+1} g_j(\tau) \}$  and  ${}^C \mathcal{D}_{\tau, t_s}^{i\alpha} \{ h_{j,n}(\tau) \}$  for  $\tau \in [0, t_s]$  and  $i = 0, \dots, N$  by successively applying Lemma 4.
- Step 5.** Compute  $[D_\tau^{\alpha-1-n} y(\tau)]_{\tau=0}$  for  $j = N-1$  and  $n = 0, \dots, \lceil \alpha \rceil - 1$  by (37) using a numerical integration method, where the modulating functions  $h_{j,n}$  and their fractional derivatives are constructed in Step 4. Then, compute  $[D_\tau^{\alpha-1-n} \{ \mathcal{D}_\tau^{(N-j-1)\alpha} y^\varpi(\tau) \}]_{\tau=0}$  for  $j = 0, \dots, N-2$  and  $n = 0, \dots, \lceil \alpha \rceil - 1$  in a recursive way by (38).
- Step 6.** Compute  $\Delta_{i,j}^r(t_s)$  for  $i = 1, \dots, N$ ,  $j = 1, \dots, J$  by using (27), where the involved  $[D_\tau^{\alpha-1-n} \{ \mathcal{D}_\tau^{(N-j-1)\alpha} y^\varpi(\tau) \}]_{\tau=0}$  are calculated in Step 5.
- Step 7.** Compute  $D_{t_s}^\beta \{ \mathcal{D}_{t_s}^{(N-J)\alpha} y^\varpi(t_s) \}$  by (30) using the proper integrals calculated in Step 3. Recursively compute  $D_{t_s}^\beta \{ \mathcal{D}_{t_s}^{(N-j)\alpha} y^\varpi(t_s) \}$  by taking  $j$  from  $J-1$  to 1 in (30), where the modulating functions  $g_j$  are provided in Step 4.
- Step 8.** Compute  $D_{t_s}^\beta x(t_s)$  by (19), where  $D_{t_s}^\beta \{ \mathcal{D}_{t_s}^{(N-j)\alpha} y^\varpi(t_s) \}$  for  $j = 1, \dots, N$  are calculated in Step 7.

By following a similar algorithm, the fractional derivatives of the output  $y$  can be estimated by using Corollary 2.

#### 4. Numerical results

In this section, numerical simulation results are provided to demonstrate the accuracy and robustness of the proposed differentiator. In the following examples, the numerical algorithm used to simulate the studied fractional order linear systems is the Grünwald-Letnikov scheme [9].

**Example 2:** Let us take the following parameters in (3):  $N = 3$ ,  $\alpha = \frac{5}{8}$ ,  $\eta = 0$ ,  $u = \sin(5t)$ ,  $C = (-1, 0, 1)$ ,  $B = (0, 1, 0)^\top$ , and

$$A = \begin{pmatrix} 11 & 1 & -11 \\ 6 & 0 & -6 \\ 18 & 1 & -17 \end{pmatrix}. \quad (59)$$

Then, it can be verified that the system satisfies conditions  $(C_1) - (C_3)$ . We intend to estimate the pseudo-state of this system from the noisy output observation by applying the methods proposed in [30] and this paper. The sampling period is chosen as  $T_s = 0.001$ .

First, as mentioned in Remark 1, the real state of a fractional order system is made up of the pseudo-state and an initialization function which is also called history function [27, 48]. The history function can be used to guarantee the uniqueness of the real state. Therefore, it is reasonable to consider the initialization problem. The considered fractional order system is initialized as done in [29, 36]:

- the system is rest on  $] -\infty, -2[$ ,
- the imposed input on  $[-2, 0]$  is  $u = \sin(5t)$ ,
- the initialization function of  $x$  is obtained on  $[-2, 0]$  with  $x(-2) = (0, 0, 0)^T$ .

Thus, we get the pseudo-state initial value:  $x(0) = (-0.1297, -0.3234, -0.1470)^T$ . Then, the pseudo-state  $x$  can be numerically obtained and then the output  $y$  can be obtained using  $y = Cx$ .

Let  $y^\varpi(t_i) = y(t_i) + \sigma\varpi(t_i)$  be the discrete noisy output, where  $t_i = iT_s$  for  $i = 0, 1, \dots, 4000$ ,  $\{\varpi(t_i)\}$  is a zero-mean white Gaussian noise with the adjusted value  $\sigma$  such that the signal-to-noise ratio  $\text{SNR} = 10 \log_{10} \left( \frac{\sum |y^\varpi(t_i)|^2}{\sum |\sigma\varpi(t_i)|^2} \right)$  is equal to 25 dB [49]. The original and noisy output are depicted in Fig. 1.

According to the method proposed in [30], in order to estimate the pseudo-state  $x(t)$ , the estimations of  $\mathcal{D}_t^{i\alpha}y(t)$  for  $i = 0, 1, 2$  are required, which can also be obtained by the proposed method with  $\beta = 0$ . On the one hand, the fractional derivative initial values  $[\mathcal{D}_\tau^{\alpha-1}\{\mathcal{D}_\tau^{i\alpha}y(\tau)\}]_{\tau=0}$  for  $i = 0, 1, 2$  are estimated by both two methods, where the method of this paper does not need to construct additional equations and has less undesired terms to be eliminated. On the other hand,

the fractional derivatives of the output  $y$  are needed. By applying the method of [30], in order to obtain  $D_t^{i\alpha}y(t)$  for  $i = 0, 1, 2$ ,  $D_t^{\frac{j}{8}}y(t)$  for  $j = -7, -6, \dots, 10$  need to be computed. However, by using the method of this paper,  $\mathcal{D}_t^{i\alpha}y(t)$  for  $i = -1, 0, 1, 2$  can be directly recursively computed. Consequently, the proposed method can greatly reduce the computation cost. The parameters of the proposed modulating functions are chosen as follows:  $\tilde{N} = 2$  and  $m_1 = 1$  to estimate the fractional derivative initial values and  $N_j = 1, 0$  for  $j = 1, 2$ ,  $N_j = 1$  for  $j = 3, 4$  and  $m_2 = 1$  to estimate the fractional derivatives of the output  $y$ . In [30], the parameter  $m = 3$  is used for both two kinds of modulating functions. The numerical estimation and errors of  $x_1$  are provided in Fig. 2 and Fig. 3, respectively. **Accordingly, the results obtained by the proposed method provide similar accuracy as given in [30]. Meanwhile, the good robustness against noises and fast convergence of the proposed method are also shown. Therefore, the proposed method can efficiently estimate the pseudo-state as done in [30]. Besides, the proposed method can also provide more estimations for more general models as shown in the following example.**

**Example 3:** Let us take in (3)  $N = 3$ ,  $\alpha = \frac{\sqrt{5}}{5}$ ,  $u = \sin(t)\sin(3t)\sin(5t)$ ,  $C = (2, 0, 1)$ ,  $B = (1, -1, -2)^T$ ,  $\eta = \frac{1}{2}$  and

$$A = \begin{pmatrix} -1.25 & -0.25 & 0.35 \\ -0.25 & -1.25 & 0.25 \\ 0 & 0 & -0.9 \end{pmatrix}, \quad (60)$$

which satisfy the conditions  $(C_1)$  and  $(C_4)$ . The objective is to estimate the  $\beta$  order fractional derivative of the pseudo-state  $x$  on  $I = [0, 8]$  in discrete noisy case with  $T_s = 0.0001$ .

The considered fractional order system is initialized as follows:

- the system is rest on  $]-\infty, -3[$ ,
- the imposed input on  $[-3, 0]$  is  $u = t + \sin(t)$ ,
- the initialization function of  $x$  is given on  $[-3, 0]$  with  $x(-3) = (0.1, 0.2, 0.3)^T$ .

Thus, we get **the pseudo-state initial value:**  $x(0) = (-0.1997, 0.7657, 1.2341)^T$ . Then, the pseudo-state  $x$  can be numerically obtained and then the output  $y$  can be obtained using  $y = Cx + \eta u$ .

Let  $y^\varpi(t_i) = y(t_i) + \sigma\varpi(t_i)$  on  $I$ , where  $t_i = iT_s$  for  $i = 0, 1, \dots, 80000$ ,  $\{\sigma\varpi(t_i)\}$  is a zero-mean white Gaussian noise with SNR = 20 dB. The original and noisy signals are given in Fig. 4.

By following the steps of the algorithm given in Section 3 and applying the algebraic integral formulas given in Theorem 1 and Proposition 2,  $D_t^\beta x(t)$  can be estimated by (19). The parameter  $m$  involved in the constructed modulating functions is chosen as follows: we take  $\tilde{N} = 1$  and

$m_1 = 1$  to estimate the fractional derivative initial values and  $N_j = 1$  for each  $j$  and  $m_2 = \lceil \beta \rceil + 1$  to estimate the fractional derivatives of the output  $y$ . Then, by taking  $\beta$  as  $\frac{\sqrt[3]{7}}{5}$ ,  $-\frac{\sqrt{3}}{3}$  and 0 respectively, the estimation results of the fractional derivative  $D_t^{\frac{\sqrt[3]{7}}{5}} x_1(t)$ , the fractional integral  $D_t^{-\frac{\sqrt{3}}{3}} x_1(t)$  and the pseudo-state  $x_1(t)$  are obtained. These results are provided in Fig. 5 - Fig. 7, where the one obtained by the Grünwald-Letnikov scheme is calculated in noise free case to verify the accuracy of the proposed modulating functions method. Moreover, for  $\beta = -\frac{\sqrt{3}}{3}$ , the estimation result of  $D_t^{\beta+(N-1)\alpha} y(t)$  obtained by applying Corollary 2 is provided in Fig. 8. **The given results show that the proposed method can not only estimate the pseudo-state  $x(t)$ , but also the fractional integral and derivative of both the pseudo-state  $x$  and the output  $y$ . It can be seen that all these estimation results efficiently coincide the ones obtained by the Grünwald-Letnikov scheme in noise-free case, which verifies the robustness, accuracy, fast convergence and generality of the proposed method.**

## 5. Conclusions

In this paper, a modulating functions based fractional order differentiator for a class of fractional order linear systems was designed. By using the proposed fractional order differentiator, the fractional integral and derivative of the pseudo-state as well as the ones of the output with an arbitrary differentiation order order can be estimated. In particular, it can also estimate the pseudo-state. Different from the existing works [19, 30, 31], the considered system was more general. First, the order of the system was considered as an arbitrary differentiation order in  $]0, 2[$  instead of only a rational number. Second, the input  $u$  was involved in the output  $y$ . Third, only the observability condition  $(C_1)$  and the controllability condition  $(C_4)$  were required by the considered system, instead of fulfilling the matching conditions  $(C_2)$ - $(C_3)$ . To deal with the new features, the proposed method was designed based on the basic ideas of Example 1. First, the considered system was transformed into a fractional differential equation from the pseudo-state space representation and the main goal of this paper was demonstrated to rely on the estimation of the fractional sequential derivatives of the output. Then, the corresponding algebraic integral formulas were given in Theorem 1 in continuous time case, where the involved fractional derivative initial values were also provided by algebraic integral formulas in Proposition 2. Moreover, the used modulating functions were constructed. Hence, the fractional derivatives of the pseudo-state and the output of the considered system can be obtained. Finally, numerical simulation results illustrated the accuracy and robustness of the proposed method. **In our future work, in order to better understand and control a fractional order system, the real state will be studied by estimating**



both the initialization function and the pseudo-state.

## Acknowledgements

This work was supported by the China Scholarship Council, the National Natural Science Foundation of China (Grant No. 11801077), and the Key Technology Research and Application Demonstration Project of Regional Network Collaborative Manufacturing of High-efficiency and Precision CNC Machine Tools (Project No. : 2019YFB1707300).

## References

- [1] Podlubny, Igor, *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, Vol. 198, Academic press, 1998.
- [2] Sousa, J Vanterler da C and de Oliveira, E Capelas, Leibniz type rule:  $\psi$ -Hilfer fractional operator, *Communications in Nonlinear Science and Numerical Simulation* 77 (2019) 305–311.
- [3] Sousa, J Vanterler da C and de Oliveira, E Capelas, On the  $\psi$ -Hilfer fractional derivative, *Communications in Nonlinear Science and Numerical Simulation* 60 (2018) 72–91.
- [4] D. Baleanu, S. I. Vacaru, Fractional analogous models in mechanics and gravity theories, in: *Fractional Dynamics and Control*, Springer, 2012, pp. 199–207.
- [5] M. Di Paola, V. Fiore, F. P. Pinnola, A. Valenza, On the influence of the initial ramp for a correct definition of the parameters of fractional viscoelastic materials, *Mechanics of Materials* 69 (1) (2014) 63–70.
- [6] T. Kaczorek, K. Rogowski, *Fractional linear systems and electrical circuits*, Springer, 2015.
- [7] D.-Y. Liu, Y. Tian, D. Boutat, T.-M. Laleg-Kirati, An algebraic fractional order differentiator for a class of signals satisfying a linear differential equation, *Signal Processing* 116 (2015) 78–90.
- [8] X.-L. Li, Y.-M. Chen, D.-Y. Liu, Y.-Q. Wei, D. Boutat, Model-free fractional order differentiator based on fractional order jacobi orthonormal functions, *Digital Signal Processing* 71 (2017) 69–82.

- [9] C. A. Monje, Y. Chen, B. M. Vinagre, D. Xue, V. Feliu-Batlle, *Fractional-order systems and controls: fundamentals and applications*, Springer Science & Business Media, 2010.
- [10] I. Podlubny, Fractional-order systems and  $\pi^{\lambda}d^{\mu}$ -controllers, *IEEE Transactions on automatic control* 44 (1) (1999) 208–214.
- [11] C. Yin, Y. Chen, S.-m. Zhong, Fractional-order sliding mode based extremum seeking control of a class of nonlinear systems, *Automatica* 50 (12) (2014) 3173–3181.
- [12] D.-Y. Liu, O. GIBARU, W. Perruquetti, T.-M. Laleg-Kirati, Fractional order differentiation by integration and error analysis in noisy environment, *IEEE Transactions on Automatic Control* 60 (11) (2015) 2945–2960.
- [13] D.-Y. Liu, G. Zheng, D. Boutat, H.-R. Liu, Non-asymptotic fractional order differentiator for a class of fractional order linear systems, *Automatica* 78 (2017) 61–71.
- [14] W. W. Hager, J. Liu, S. Mohapatra, A. V. Rao, X.-S. Wang, Convergence rate for a gauss collocation method applied to constrained optimal control, *SIAM Journal on Control and Optimization* 56 (2) (2018) 1386–1411.
- [15] H. Antil, E. Otárola, A. J. Salgado, A space-time fractional optimal control problem: analysis and discretization, *SIAM Journal on Control and Optimization* 54 (3) (2016) 1295–1328.
- [16] D.-Y. Liu, T.-M. Laleg-Kirati, O. GIBARU, W. Perruquetti, Identification of fractional order systems using modulating functions method, in: *2013 American Control Conference, IEEE*, 2013, pp. 1679–1684.
- [17] A. Aldoghaither, D.-Y. Liu, T.-M. Laleg-Kirati, Modulating functions based algorithm for the estimation of the coefficients and differentiation order for a space-fractional advection-dispersion equation, *SIAM Journal on Scientific Computing* 37 (6) (2015) A2813–A2839.
- [18] D.-Y. Liu, T.-M. Laleg-Kirati, Robust fractional order differentiators using generalized modulating functions method, *Signal Processing* 107 (2015) 395–406.
- [19] Y.-Q. Wei, D.-Y. Liu, D. Boutat, Innovative fractional derivative estimation of the pseudo-state for a class of fractional order linear systems, *Automatica* 99 (2019) 157–166.
- [20] T. B. Co, S. Ungarala, Batch scheme recursive parameter estimation of continuous-time systems using the modulating functions method, *Automatica* 33 (6) (1997) 1185–1191.

- [21] D.-Y. Liu, T.-M. Laleg-Kirati, W. Perruquetti, O. Gibaru, Non-asymptotic state estimation for a class of linear time-varying systems with unknown inputs, *IFAC Proceedings Volumes* 47 (3) (2014) 3732–3738.
- [22] Y. Tian, Y.-Q. Wei, D.-Y. Liu, D. Boutat, Fast and robust estimation for positions and velocities from noisy accelerations using generalized modulating functions method, *Mechanical Systems and Signal Processing* 133 (2019) 106270.
- [23] G. Pin, B. Chen, T. Parisini, Robust deadbeat continuous-time observer design based on modulation integrals, *Automatica* 107 (2019) 95–102.
- [24] G. Pin, M. Lovera, A. Assalone, T. Parisini, Kernel-based non-asymptotic state estimation for linear continuous-time systems, in: *2013 American Control Conference*, IEEE, 2013, pp. 3123–3128.
- [25] P. Li, G. Pin, G. Fedele, T. Parisini, Non-asymptotic numerical differentiation: a kernel-based approach, *International Journal of Control* 91 (9) (2018) 2090–2099.
- [26] C. Farges, M. Moze, J. Sabatier, Pseudo state feedback stabilization of commensurate fractional order systems, in: *2009 European Control Conference (ECC)*, IEEE, 2009, pp. 3395–3400.
- [27] J. Sabatier, C. Farges, J.-C. Trigeassou, Fractional systems state space description: some wrong ideas and proposed solutions, *Journal of Vibration and Control* 20 (7) (2014) 1076–1084.
- [28] J. Sabatier, C. Farges, M. Merveillaut, L. Feneteau, On observability and pseudo state estimation of fractional order systems, *European journal of control* 18 (3) (2012) 260–271.
- [29] Z. Belkhatir, T. M. Laleg-Kirati, High-order sliding mode observer for fractional commensurate linear systems with unknown input, *Automatica* 82 (2017) 209–217.
- [30] X. Wei, D.-Y. Liu, D. Boutat, Non-asymptotic pseudo-state estimation for a class of fractional order linear systems, *IEEE Transactions on Automatic Control* 62 (3) (2017) 1150–1164.
- [31] Y.-Q. Wei, D.-Y. Liu, D. Boutat, Y.-M. Chen, An improved pseudo-state estimator for a class of commensurate fractional order linear systems based on fractional order modulating functions, *Systems & Control Letters* 118 (2018) 29–34.

- [32] S. Djennoune, M. Bettayeb, Optimal synergetic control for fractional-order systems, *Automatica* 49 (7) (2013) 2243–2249.
- [33] C. Yin, X. Huang, Y. Chen, S. Dadras, S.-m. Zhong, Y. Cheng, Fractional-order exponential switching technique to enhance sliding mode control, *Applied Mathematical Modelling* 44 (2017) 705–726.
- [34] Y.-Q. Wei, D.-Y. Liu, D. Boutat, H.-R. Liu, Non-asymptotic fractional pseudo-state differentiator for a class of fractional order linear systems, in: 2019 Chinese Control Conference (CCC), IEEE, 2019, pp. 2347–2352.
- [35] S. Victor, P. Melchior, J. Lévine, A. Oustaloup, Flatness for linear fractional systems with application to a thermal system, *Automatica* 57 (2015) 213–221.
- [36] S. Ibrir, M. Bettayeb, New sufficient conditions for observer-based control of fractional-order uncertain systems, *Automatica* 59 (2015) 216–223.
- [37] S. Abrashov, R. Malti, M. Moze, X. Moreau, F. Aioun, F. Guillemard, Simple and robust experiment design for system identification using fractional models, *IEEE Transactions on Automatic Control*.
- [38] X. Tang, Y. Shi, L.-L. Wang, A new framework for solving fractional optimal control problems using fractional pseudospectral methods, *Automatica* 78 (2017) 333–340.
- [39] W. Yu, Y. Li, G. Wen, X. Yu, J. Cao, Observer design for tracking consensus in second-order multi-agent systems: Fractional order less than two, *IEEE Transactions on Automatic Control* 62 (2) (2017) 894–900.
- [40] Abramowitz, Milton and Stegun, Irene A and others, *Handbook of mathematical functions, Applied mathematics series 55 (62) (1966) 39.*
- [41] Hartley, Tom T and Lorenzo, Carl F and Trigeassou, Jean-Claude and Maamri, Nezha, Equivalence of history-function based and infinite-dimensional-state initializations for fractional-order operators, *Journal of Computational and Nonlinear Dynamics* 8 (4) (2013) 041014.
- [42] Trigeassou, Jean-Claude and Maamri, Nezha and Sabatier, Jocelyn and Oustaloup, Alain, State variables and transients of fractional order differential systems, *Computers & Mathematics with Applications* 64 (10) (2012) 3117–3140.

- [43] Trigeassou, Jean-Claude and Maamri, Nezha and Sabatier, Jocelyn and Oustaloup, Alain, Transients of fractional-order integrator and derivatives, *Signal, Image and Video Processing* 6 (3) (2012) 359–372.
- [44] A. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and applications of fractional differential equations, Vol. 204, Elsevier Science Limited, 2006.
- [45] I. Podlubny, Y. Chen, Adjoint fractional differential expressions and operators, in: Proceedings of the ASME 2007 International Design Engineering Technical Conferences & Computers and Information in Engineering Conference IDETC/CIE, 2007, pp. 4–7.
- [46] M. Fliess, Analyse non standard du bruit, *Comptes Rendus Mathematique* 342 (10) (2006) 797–802.
- [47] Ralston, Anthony and Rabinowitz, Philip, *A first course in numerical analysis*, Courier Corporation, 2001.
- [48] T. T. Hartley, C. F. Lorenzo, J.-C. Trigeassou, N. Maamri, Equivalence of history-function based and infinite-dimensional-state initializations for fractional-order operators, *Journal of Computational and Nonlinear Dynamics* 8 (4) (2013) 041014.
- [49] S. Haykin, B. Van Veen, *Signals and systems*, John Wiley & Sons, 2007.

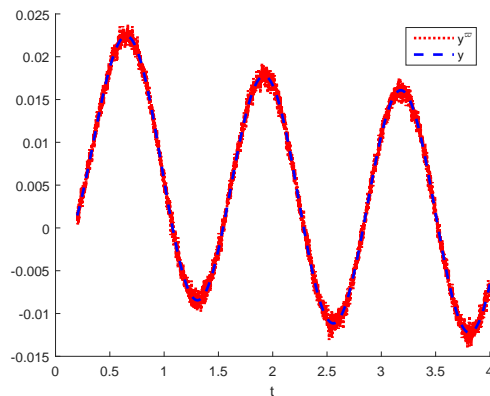


Figure 1: Example 2: Original signal and noisy signal.

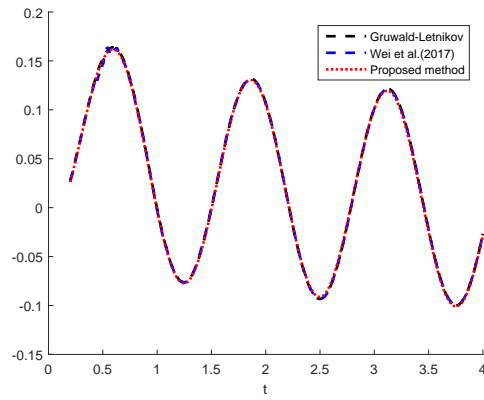


Figure 2: Example 2: Estimation of  $x_1(t)$ .

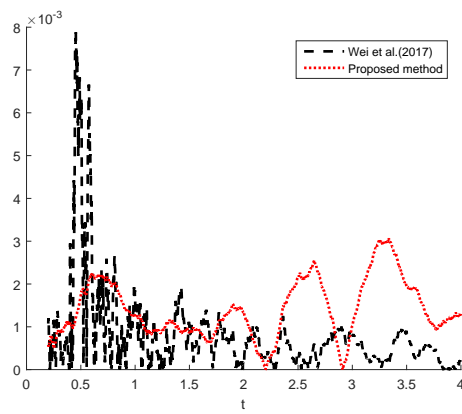


Figure 3: Example 2: Estimation errors of  $x_1(t)$ .

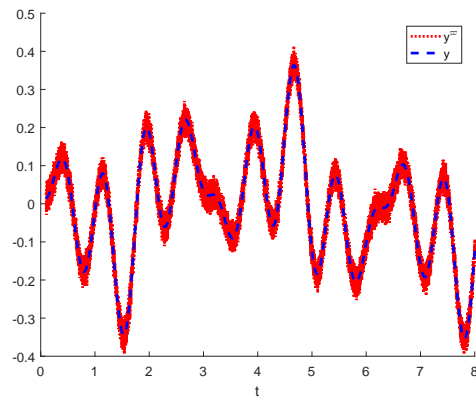


Figure 4: Example 3: Original signal and noisy signal.

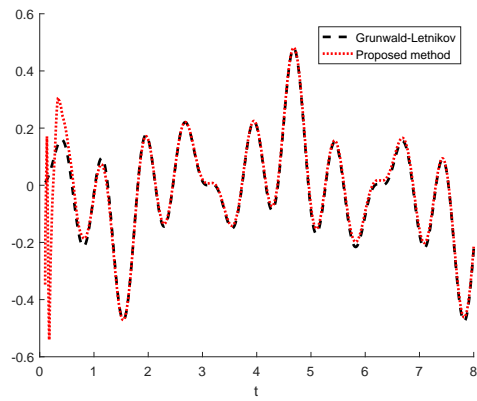


Figure 5: Example 3: Estimation of  $D_t^{\frac{\sqrt{7}}{5}} x_1(t)$ .

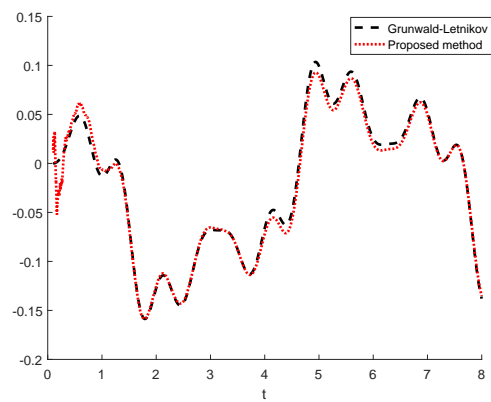


Figure 6: Example 3: Estimation of  $D_t^{-\frac{\sqrt{3}}{3}} x_1(t)$ .

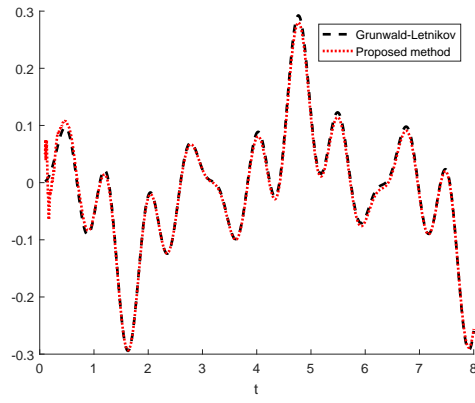


Figure 7: Example 3: Estimation of  $x_1(t)$ .

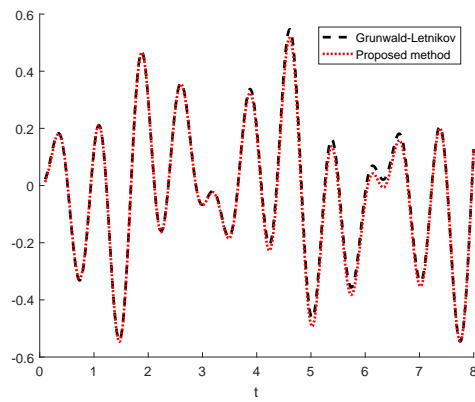


Figure 8: Example 3: Estimation of  $D_t^{-\frac{\sqrt{3}}{3} + \frac{2\sqrt{5}}{5}} y(t)$ .