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► **To cite this version:**

Mitsuru Kawazoe. Relation between understandings of linear algebra concepts in the embodied world and in the symbolic world. INDRUM 2020, Université de Carthage, Université de Montpellier, Sep 2020, Cyberspace (virtually from Bizerte), Tunisia. hal-03113855

**HAL Id: hal-03113855**

**<https://hal.science/hal-03113855>**

Submitted on 18 Jan 2021

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# Relation between understandings of linear algebra concepts in the embodied world and in the symbolic world

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*For the use of embodied notions in teaching linear algebra, some studies indicate that it is helpful, but another study indicates that it is sometimes problematic. Hence more study is needed. In this study, linear (in)dependence and basis were focused on, and the relation between understandings of them in the embodied world and in the symbolic world was investigated. The effectiveness of an instruction emphasizing geometric images of them was also investigated. The main results of the study were the following: conceptual understanding of linear dependence of four spatial vectors such that any three of them do not lie on the same plane was positively associated with understanding of basis in the symbolic world; however, understanding of linear dependence of such vectors had not been improved by a geometrical instruction.*

*Keywords: linear algebra, teachers' practices, linear independence, basis.*

## BACKGROUND AND THE PURPOSE OF THE STUDY

It is widely recognized that linear algebra is a difficult subject to learn due to its abstract and formal nature. Dorier and Sierpiska (2001) stated that “linear algebra remains a cognitively and conceptually difficult subject.” It has been a big challenge to overcome the difficulty in teaching linear algebra. Some researchers pointed out that the use of embodied notions, namely the use of visual images, helps students to understand concepts in linear algebra (cf. Stewart & Thomas, 2007; Hannah et al., 2014; Donevska-Todorova, 2018, p. 268). However, there is another study indicating that using visual images is sometimes problematic in teaching linear algebra (Sierpiska, 2000, p. 244). These studies indicate that the use of visual images in teaching linear algebra and its effectiveness should be more investigated. That is a motivation of our research to investigate students' conceptions of linear algebra concepts in the context of geometric vectors.

In our previous studies, we observed the following: (1) there are many students who fail to determine linear dependence of four spatial vectors such that any three of them do not lie on the same plane (Kawazoe et al., 2014); (2) some of those students take a longer time to image that three spatial vectors not lying on the same plane span the whole space (Kawazoe & Okamoto, 2016; Kawazoe, 2018). However, we have not investigated how these observations are related to understanding of concepts and procedures in linear algebra.

In this study, we focused on concepts of linear (in)dependence and basis, and studied the following research questions: (1) Is geometrical understanding of linear (in)dependence in the embodied world related to understanding of linear

(in)dependence and basis in the symbolic world?; (2) Can geometrical understanding of linear (in)dependence in the embodied world including the case of four vectors be improved by an instruction emphasizing a geometric image of linear (in)dependence?

## **THEORETICAL FRAMEWORK**

We use Tall's model of three worlds (Tall, 2013) combined with APOS theory (Arnon et al., 2014) to distinguish students' understanding for linear algebra concepts, following Stewart and Thomas (2007). Tall (2013) described the development of mathematical thinking in terms of three worlds: *embodied* world, *symbolic* world, and *formal* world. Tall stated that "the combination of embodied and symbolic mathematics can be seen as a preliminary stage to the axiomatic formal presentation of mathematics." In linear algebra, the embodied world is a world of geometric vectors (arrows), the symbolic world is a world of numerical vectors, matrices, polynomials, and operations using symbols. APOS theory enables us to distinguish students' conceptions into four levels: *Action*, *Process*, *Object*, and *Schema*. Then, students' conceptions in linear algebra can be described in each of three worlds (cf. Stewart & Thomas, 2007). As for linear (in)dependence, Action-Process-Object conceptions in the embodied world are described as follows. Students having *Action* conception draw a linear combination explicitly in a discussion of linear (in)dependence. Students having *Process* conception can use a set of linear combinations but cannot use a spanned space correctly. Students having *Object* conception can completely understand that any two non-parallel geometric vectors are linear independent and they span a plane, any three geometric vectors not lying on the same plane are linear independent and they span the whole space, and any four geometric vectors are always linearly dependent.

We view some linear algebra concepts from the viewpoint of Lakoff and Núñez (2000). For an example, we regard a role of basis of a vector space as the 'discretization' of a space, following the explanation given by Lakoff and Núñez (ibid., p. 260-261). To give a basis for a vector space is equivalent to give a coordinate for the space. In the embodied world, it means to represent every point in a plane or a space as a pair or a triple of numbers. Moreover, we apply the 'Basic Metaphor of Infinity' (ibid., p. 158) to students' image of spanned space, according to an observation of our previous study (Kawaoze & Okamoto, 2016) that many students image a space spanned by linearly independent three spatial vectors as a 'gradually expanding three-dimensional object' which finally fills the whole space. We used these viewpoints in designing linear algebra lessons in this study.

## **CONTEXT: THE COURSE, STUDENTS, DESIGN OF LESSONS AND TASKS**

The study was conducted in a linear algebra course aiming at engineering students at our university, but in a special class for students who failed to pass it when they were in the first-year. The course consists of a spring semester class and a fall semester class. The former is a 2-credit class, meeting for 90 minutes each week for 15 weeks. The latter is a 4-credit class, meeting for 180 minutes each week for 15 weeks. Each of

them is followed by an examination period. The course covers usual linear algebra topics: matrix, gaussian elimination, system of linear equations, and determinant, etc. in the spring semester; formal vector space, spanned space, linear (in)dependence, basis, dimension, linear map, inner product, orthogonal basis, eigenvalue, eigenvector, and diagonalization, etc. in the fall semester. This study was conducted during the first five weeks in the fall semester class. In these weeks, students learned formal vector space, spanned space, linear (in)dependence, basis, and dimension.

### **Design of lessons**

Each lesson consisted of a lecture part and an exercise part. Lectures were given in the first half, and exercises were given in the second half. The lecture part was designed as to emphasize geometric images of linear algebra concepts especially by using the image of a spanned space in the embodied world. In the lecture part, the teacher introduced linear algebra concepts in the following way.

First, the notions of linear combination and spanned space were introduced. A space spanned by three linearly independent spatial vectors was shown to students by using teacher's fingers, and it was emphasized that linear combinations with negative coefficients were contained in the spanned space. The teacher stressed the importance of imaging a part of the space consisting of linear combinations with some (or all) coefficients being negative in order to grasp the correct image of the spanned space.

The notions of linear independence and dependence were introduced by using usual formal definitions, but the meaning of linear independence and dependence of vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  in a vector space were explained in terms of spanned space as follows:

Vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  are linearly dependent if and only if one of the  $n$  vectors can be represented by a linear combination of the other  $n-1$  vectors, that is, one of the  $n$  vectors is contained in the space spanned by the other  $n-1$  vectors.

Vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  are linearly independent if and only if none of the  $n$  vectors can be represented by a linear combination of the other  $n-1$  vectors, that is, none of the  $n$  vectors is contained in the space spanned by the other  $n-1$  vectors.

It was also explained that linearly independent vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  give an ascending sequence of vector spaces  $V_1 \subsetneq V_2 \subsetneq \dots \subsetneq V_n$  where  $V_k$  ( $k=1, 2, \dots, n$ ) is the space spanned by  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ .

Then, the notion of basis was introduced by using a usual formal definition:

Vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  in a vector space  $V$  is a basis of  $V$  if and only if they are linearly independent and any vector in  $V$  can be represented as a linear combination of them.

It was explained that the second condition is equivalent to that  $V$  is spanned by  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ . In the introduction of basis, the role of basis was explained as to give a coordinate system, and a basis was explained as a set of 'axes.' It was explained that the second condition means that it contains a sufficient number of axes to represent the whole space and that the first condition means that there is no extra axis in the set.

In the exercise part, students worked on paper-based exercises given by the teacher. Exercises mainly consisted of questions in the symbolic world and some of them can be viewed as questions in the embodied world: determining linear (in)dependence of vectors in  $\mathbb{R}^n$  ( $n=2,3,4$ ) or in polynomial spaces, determining whether a given set of vectors in  $\mathbb{R}^n$  ( $n=2,3,4$ ) or in polynomial spaces is a basis or not, finding a basis and the dimension of given subspaces in  $\mathbb{R}^n$  ( $n=2,3,4$ ) or in polynomial spaces, etc. Many of the questions were computational ones. Some of them were related to the geometric instruction given in the lecture part and they can be answered with geometrical reasoning.

### Design of tasks

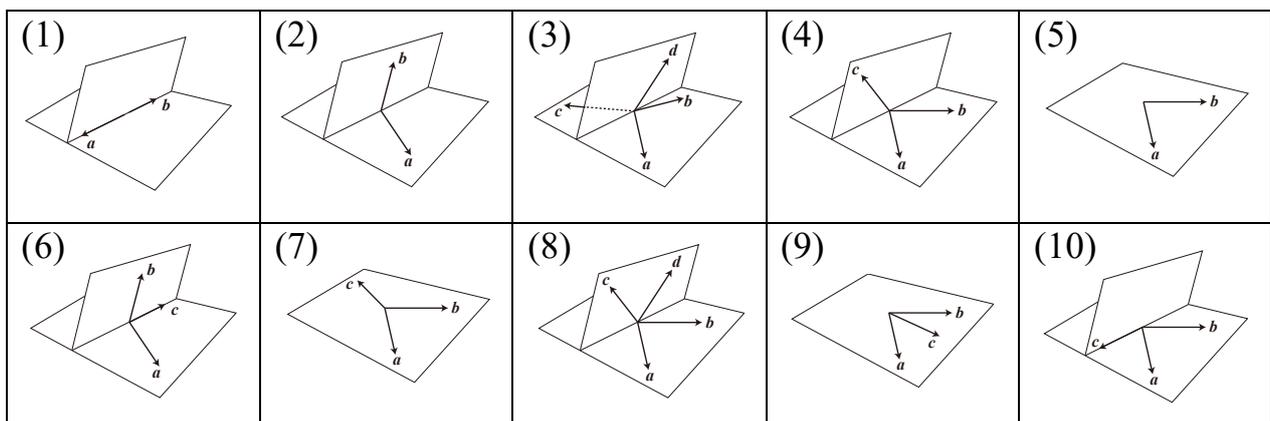
The following four tasks, which were translated from Japanese, were designed in order to investigate students' understanding of dimension, linear (in)dependence and basis.

*Task 1:* Answer the following questions. If you do not know (or if you have not learned), write your answer as "I don't know."

(1) Describe your image of an example of one dimension, two dimension, and three dimension, respectively, using figures and words freely.

(2) For vectors  $v_1, v_2, \dots, v_n, v_{n+1}$ , assume that vectors  $v_1, v_2, \dots, v_n$  span an  $n$ -dimensional space  $V$ , and that  $v_1, v_2, \dots, v_n, v_{n+1}$  span an  $(n+1)$ -dimensional space  $W$ . When you draw a picture showing this situation, what kind of picture do you draw? Draw a picture of your image.

*Task 2:* Determine whether spatial vectors given in each picture are linearly independent or not. Note that each vector lies on a line or a plane shown in the picture. (If there are multiple planes, each vector lies on one of them.)



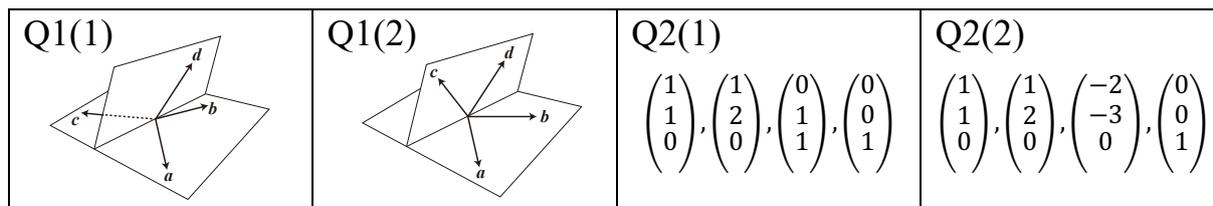
**Figure 1: Test items in Task 2**

*Task 3:* (Q1) For vectors  $v_1, v_2, \dots, v_n$  in a vector space  $V$  over  $K$ , describe two conditions (in the definition of basis) for  $v_1, v_2, \dots, v_n$  to be a basis of  $V$ . Write your answer in the answer columns (A) and (B). (Q2) Determine whether the following set of vectors is a basis or not. If it is not a basis, answer which condition that you described in Q1 is not satisfied. In the latter case, write your answer by using the symbol A or B, and write "A, B" in both cases. (Vector spaces are as follows: (1)  $\mathbb{R}^4$ , (2)  $\mathbb{R}^3$ , (3)  $\mathbb{R}^2$ , (4)  $\mathbb{R}^3$ , (5)

the space of polynomials  $f(x)$  with degree less than 3 whose coefficients are in  $\mathbb{R}$ , (6) the space of polynomials  $f(x)$  with degree less than 2 whose coefficients are in  $\mathbb{R}$ .)

- (1)  $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$  (2)  $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ -1 \end{pmatrix}$  (3)  $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$  (4)  $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -5 \\ -5 \end{pmatrix}$   
 (5)  $x + 1, x^2$  (6)  $x - 1, x + 1$

*Task 4:* (Q1) Determine whether spatial vectors given in each picture are linearly independent or not, and describe the reason. (Q2) Determine whether the given vectors in  $\mathbb{R}^3$  are linearly independent or not, and describe the reason.



**Figure 2: Test items in Task 4**

### A priori analysis of tasks

Task 1 and Task 2 are pre-tests conducted at the beginning of the semester. Task 1(1) can be answered as ‘line’, ‘plane’, and ‘space’. Task 1(2) is a non-routine task to examine whether students have an image that  $V$  is contained in  $W$ , or  $W$  extends outside of  $V$  as a space. Task 2 includes all of the important cases of less than or equal to four spatial vectors regarding linear (in)dependence. Task 2 is the same one that we used in our previous study (Kawazoe & Okamoto, 2016). According to our previous result (ibid.), Task 2 (8) was expected to be difficult for the participants. Task 2 (8) contains four vectors and any three of them do not lie on the same plane, hence it cannot be reduced to the case of less than or equal to three vectors. Task 2 (3) also contains four vectors, but it can be reduced to the case of three vectors because the vectors  $a, b, c$  lie on the same plane. The terms ‘dimension’, ‘span’, and ‘linearly independent’ were used in the texts in these tasks. Since the participants were in the second-year or higher, they had already learned them when they were in the first-year.

The aim of Task 3 is to investigate students’ understanding of the definition of basis. For any set of vectors listed in (1)-(6), one can determine their linear (in)dependence without computation. Only (2) and (6) are basis, and the others are not.

In Task 4, Q1 is a task in the embodied world, and Q2 is a task in the symbolic world. The two pictures in Q1 was taken from Task 2. According to the result of our previous study (ibid.), determining linear (in)dependence of four spatial vectors is problematic. Q1(1) and Q2(2) present essentially the same situation, and Q1(2) and Q2(1) present essentially the same situation. Q1(1) and Q1(2) can be answered by drawing vectors representing linear combinations, or by using the fact on vector subspaces spanned by two or three vectors. Q2(1) and Q2(2) can be answered by using numerical computation

(with or without the use of the Gaussian elimination), but they also can be answered with geometrical reasoning.

## METHODOLOGY AND DATA COLLECTION

We implemented four-weeks lessons whose design is described in the above. Task 1 and 2 were conducted at the beginning of the first lesson. Task 3 was conducted at the third week, and Task 4 was conducted at the beginning of the fifth week lesson. Participants' answers for Task 1 were analyzed whether they have an image of dimension less than or equal to three and whether they have an image of increment of dimension. Participants' reasoning for Task 4 Q1 were analyzed with APOS theory. Participants' reasoning for Task 4 Q2 were classified into two types: algebraical reasoning, and geometrical reasoning. For other tasks, participants' answers were evaluated depending on their correctness. Then, the relations between the results of these tasks were investigated.

The study was conducted in the fall semester in the academic year 2018. All data were collected during the first five weeks in the linear algebra class for engineering students who had failed in the previous year or before. The number of students in the class were 58. Among the 58 students, 38 of them worked out all the tasks from Task 1 to Task 4. In this study, the data of the 38 participants was statistically analyzed.

## RESULTS

### The result of each task and setting of groups

*Task 1.* For Task 1 (1), almost all participants described their images for dimension 1, 2, 3, as 'line', 'plane', 'space', respectively. For Task 1 (2), only 11 (28.9%) of them could draw their image of increment of dimension as extending outside the space. We set two groups according to the result of Task 1 (2):  $GI+$  is the group of 11 participants having a geometric image of increment of dimension,  $GI-$  is the group of the others.

*Task 2.* The percentages of correct answers for Task 2 were as follows: (1) 97.4%, (2) 94.7%, (3) 65.8%, (4) 97.4%, (5) 89.5%, (6) 94.7%, (7) 86.8%, (8) 52.6%, (9) 89.5%, (10) 86.8%. The percentages of correctness for (3) and (8) were much lower, compared with the others. The pictures of (3) and (8) contain four vectors. The number of vectors in the others is less than four. The result of Task 2 was almost the same as the one in our previous study (Kawazoe & Okamoto, 2016), except for the result of (3). In the previous study, the percentage of correct answers for (3) was 84.5%. The median of the number of correct answers per participant was 9. We set two groups according to the result of Task 2:  $GV+$  is the group of participants who answered correctly to more than 8 questions, and  $GV-$  is the group of the others.

*Task 3.* For Q1, the number of participants who could describe two conditions in the definition of basis correctly was 23 (60.5%). While 34 (89.5%) of the participants could describe linear independence of the vectors correctly as one of the conditions, 24 (63.2%) of them could described correctly that the vectors span  $V$  or that any vector in

$V$  can be represented as a linear combination of the vectors. 8 (21.1%) of the them described ‘ $\dim V=n$ ’ as one of the conditions, which is a wrong answer because ‘ $\dim V$ ’ is defined after the definition of basis is introduced.

For Q2, while the percentages of correct answers for (2), (3), (4) were high, those of (1), (5), (6) were relatively low: (1) 78.9%, (2) 97.4%, (3) 94.7%, (4) 94.7%, (5) 78.9%, (6) 65.8%. As for reasoning in (1), (3), (4), and (5), we evaluated whether a participant could answer correctly based on the necessary and sufficient conditions to be a basis. Hence, for a participant who described ‘ $\dim V=n$ ’ in Q1, we evaluated his/her answer for Q2 whether it was logically correct based on his/her answer in Q1. The percentages of correct answers for reasoning were as follows: (1) 65.8%, (3) 63.2%, (4) 36.8%, (5) 65.8%. The median of the number of errors in Q2 (including errors in reasoning in the case of non-basis) per participant was 2. We set two groups according to the number of incorrect answers for Task 3 Q2:  $B+$  is the group of participants whose incorrect answers were less than or equal to 2, and  $B-$  is the group of the others.

*Task 4.* The percentages of correct answers for Task 4 were as follows: Q1(1) 89.5%, Q1(2) 55.3%, Q2(1) 86.8%, Q2(2) 89.5%. The pictures in Q1(1) and Q1(2) are same as in Task 2 (3) and Task 2 (8), respectively. While the percentage of correct answers for Q1(2) remained still low, the one for Q1(1) was much improved from the result of Task 2 (3). Though Q1(2) is essentially same as Q2(1) from a geometrical viewpoint, the results of them were different. According to the reasoning in Q1, we set the following groups: For  $j=1, 2$ ,  $O_j+$  is the group of participants showing *Object* conceptions in the reasoning for Q1(j),  $O_j-$  is the group of participants showing *Action/Process* conceptions or giving no reason in the reasoning for Q1(j). According to the reasoning in Q2(j), we set the following groups: For  $j=1, 2$ ,  $GR_j+$  is the group of participants using geometrical reasoning for Q2(j),  $GR_j-$  is the group of the others.

### The relations between the results of each task

In the following analysis, we used Fisher’s exact test instead of the Chi-square test because there were small numbers in cross-tabulation.

*Relation between understanding in the embodied world and understanding of basis:* Fisher’s exact test indicated that having a geometric image of increment of dimension (Task 1 (2)) and the result of Task 3 Q2 were positively associated ( $p<0.05$ , Table 1). Fisher’s exact test also indicated that showing Object conceptions in reasoning for Task 4 Q1(2) and the result of Task 3 Q2 were positively associated ( $p<0.05$ , Table 2). On the other hand, we could not find any significant relation between  $O_{1+/-}$  and  $B+/-$ .

	$B+$	$B-$
$GI+$	9	2
$GI-$	11	16

**Table 1: Relation between the results of Task 1(2) and Task 3 Q2**

	$B+$	$B-$
$O_{2+}$	8	1
$O_{2-}$	12	17

**Table 2: Relation between having Object conception and the result of Task 3 Q2**

*Relation between understandings in the embodied world and in the symbolic world:* Fisher's exact test indicated that showing Object conception in reasoning for Task 4 Q1(2) and the number of correct answers in determining linear (in)dependence in Task 4 were positively associated ( $p < 0.01$ , Table 3), where  $NC$  means the number of correct answers in determining linear (in)dependence in Task 4. On the other hand, we could not find any significant relation between  $O_{1+/-}$  and the result of Task 4. Fisher's exact test also indicated that the use of geometrical reasoning for Task 4 Q2 and the number of correct answers in determining linear (in)dependence in Task 4 were positively associated ( $p < 0.05$ , Table 4), where  $GR^+ = GR_1 \cup GR_2^+$ ,  $GR^- = GR_1 \cap GR_2^-$ , and  $NC$  is the same as in Table 3. Fisher's exact test also indicated significant correlations for  $GR_{1+/-}$  ( $p < 0.05$ ) and for  $GR_{2+/-}$  ( $p < 0.05$ ).

	$NC=4$	$NC < 4$
$O_{2+}$	8	1
$O_{2-}$	9	20

**Table 3: Relation between having Object conception and the result of Task 4**

	$NC=4$	$NC < 4$
$GR^+$	10	5
$GR^-$	7	16

**Table 4: Relation between the use of geometrical reasoning and the result of Task 4**

*Difference of understanding of linear (in)dependence between before and after of four-weeks lessons:* The picture in Task 4 Q1(1) and Q1(2) are same as the one in Task 2 (3) and (8), respectively. McNemar's test indicated that there was a significant difference between the results of Task 2 (3) and Task 4 Q1(1) ( $p < 0.05$ , Table 5), where the participants were divided into two groups depending on whether their answers for Task 2(3) were correct ( $T_{2(3)+}$ ) or not ( $T_{2(3)-}$ ), and they were divided into two groups depending on whether their answers for Task 4 Q1(1) were correct ( $T_{4Q1(1)+}$ ) or not ( $T_{4Q1(1)-}$ ). On the other hand, Fisher's exact test indicated that the result of Task 2 and the number of correct answers in determining linear (in)dependence in Task 4 Q1 were positively associated ( $p < 0.01$ , Table 6), where  $NC_{Q1}$  means the number of correct answers in determining linear (in)dependence in Task 4 Q1.

	$T_{4Q1(1)+}$	$T_{4Q1(1)-}$
$T_{2(3)+}$	23	2
$T_{2(3)-}$	11	2

**Table 5: Relation between the results of Task 2 (3) and Task 4 Q1(1)**

	$NC_{Q1} = 2$	$NC_{Q1} < 2$
$GV^+$	16	7
$GV^-$	3	12

**Table 6: Relation between the result of Task 2 and the result of Task 4 Q1**

## DISCUSSIONS

As for the first research question, we observed some relations between understanding in the embodied world and understanding in the symbolic world. The analysis for Table

1 indicated that having a geometric image of increment of dimension and understanding of basis in the symbolic world were positively associated. The analysis for Table 4 indicated that the use of geometrical reasoning in the symbolic world and understanding of linear (in)dependence in both embodied and symbolic world were positively associated. The analysis for Table 2 and 3 indicated that having Object conception for linear (in)dependence in the embodied world, especially for the case of four spatial vectors such that any three of them do not lie on the same plane (as in the picture of Task 2(8) and Task 4 Q1(2)), was positively associated with understanding of basis in the symbolic world (Table 2), and also positively associated with understanding of linear independence in both embodied and symbolic world (Table 3).

As for the second research question, we observed that the effectiveness of the implemented instruction emphasizing geometric images was limited. The analysis for Table 5 indicated that understanding of linear dependence of four spatial vectors in the picture of Task 2 (3) had been improved during the four-weeks lessons. On the other hand, the result of Task 4 and the analysis for Table 6 indicated that understanding of linear dependence of four spatial vectors in the picture of Task 2 (8) had not been improved. Improving students' understanding of Task 2 (8) was more important because conceptual understanding of linear dependence in the case of Task 2 (8) was related to understanding of basis and linear independence in the symbolic world. How should we consider this result? There may be the following two possibilities: one is that the geometrical instruction implemented in this study was insufficient and it can be more improved; the other is that there is a limitation of students' perception even in the embodied world and it is cognitively hard to overcome such limitation. In the latter case, we should take into account of such limitation in teaching linear algebra, and it may lead us to reconsider how to design a linear algebra course under the framework with Tall's model of three worlds, especially to reconsider the balance and integration between geometric and algebraic presentation. However, the two possibilities need to be carefully examined in the future study.

Finally, we should mention the limitations of the study. First, the sample size was small. Second, the participants were not ordinary because they were students who had failed to pass the subject in the earlier years. Hence, further studies with a larger number of first-year students are needed.

## **ACKNOWLEDGMENTS**

This work was supported by the JSPS KAKENHI Grant Number JP15K12392.

## **REFERENCES**

- Arnon, I., Cottrill, J., Dubinsky, E., Oktaç, A., Fuentes, S. R., Trigueros, M., & Weller, K. (2014). *APOS Theory: A Framework for Research and Curriculum Development in Mathematics Education*. New York, NY: Springer.
- Donevska-Todorova, A. (2018). Fostering students' competencies in linear algebra with digital resources. In S. Stewart, C. Andrews-Larson, A. Berman, & M. Zandieh

- (Eds.), *Challenges and Strategies in Teaching Linear Algebra* (pp. 261-276). Cham, Switzerland: Springer International Publishing.
- Dorier, J.-L., & Sierpinska, A. (2001). Research into the teaching and learning of linear algebra. In D. Holton (Ed.), *The Teaching and Learning of Mathematics at University Level: An ICMI Study* (pp. 255-273). Dordrecht, The Netherlands: Kluwer Academic Publishers.
- Hannah, J., Stewart, S., & Thomas, M. O. J. (2014). Teaching linear algebra in the embodied, symbolic and formal worlds of mathematical thinking: Is there a preferred order? In P. Liljedahl, C. Nicol, S. Oesterle, & D. Allan (Eds.), *Proceedings of the Joint Meeting of PME 38 and PME-NA 36* (Vol. 3, pp. 241-248). Vancouver, Canada: PME.
- Kawazoe, M., Okamoto, M., & Takahashi, T. (2014). Students' mental representations of geometric vectors and sets related to linear algebra. In P. Liljedahl, C. Nicol, S. Oesterle, & D. Allan (Eds.), *Proceedings of the Joint Meeting of PME 38 and PME-NA 36* (Vol. 6, p. 126). Vancouver, Canada: PME.
- Kawazoe, M., & Okamoto, M. (2016). The influence of students' mental representation on understanding of linear independency of spatial vectors. *Journal of Japan Society of Mathematical Education*, 98, 17-24. [in Japanese]
- Kawazoe, M. (2018). Students' conception of spanned space and its relation to conception of linear independence. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.), *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 5, p. 82). Umeå, Sweden: PME.
- Lakoff, G., & Núñez, R. (2000). *Where Mathematics Comes From: The Embodied Mind Brings Mathematics into Being*. New York, NY: Basic Books.
- Sierpinska, A. (2000). On some aspects of students' thinking in linear algebra. In J.-L. Dorier (Ed.), *On the Teaching of Linear Algebra* (pp. 209-246). Dordrecht, The Netherlands: Kluwer Academic Publishers.
- Stewart, S., & Thomas, M. O. J. (2007). Embodied, symbolic and formal aspects of basic linear algebra concepts. In J. H. Woo, H. C. Lew, K. S. Park, & D. Y. Seo (Eds.), *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 201-208). Seoul, Korea: PME.
- Tall, D. (2013). *How Humans Learn to Think Mathematically: Exploring the Three Worlds of Mathematics*. New York, NY: Cambridge University Press.