Proof Teaching at the University Level: the case of a lecturer who is mathematician and mathematics educator
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In this paper, we study the teaching of proof in an introductory mathematical analysis course taught by a lecturer who brought experience from not only the fields of teaching at the university and research in mathematics but also from research in mathematics education. The analysis showed that his experiences appeared to affect his teaching as he became aware of students’ needs and difficulties, they probably face, making a lesson potentially meaningful for them. In particular, regarding the proving process the lecturer developed a lesson where he filled the gap between informal and formal proving attempting to expand the students’ proof image around the theorem, using the semantic approach in proof teaching.

Keywords: Teaching and learning of analysis and calculus; teaching and learning of logic, reasoning and proof; lecture; semantic approach; proof image.

INTRODUCTION
The central role of proof in mathematics is widely accepted and lecturers’ attempts to teach proofs to students who study mathematics is an issue that concerns the research (e.g., Pinto & Karsenty, 2018; Weber, 2012). This paper explores proof teaching in an introductory mathematical analysis course taught in a mathematics department. Introductory courses are of significant importance for students’ learning as it is the first time the typical proving processes are introduced (Alcock, 2010).

Many studies that tried to shed light in proof teaching were based mainly on interviews with the lecturer(s) of the course (e.g., Alcock, 2010; Lai & Weber, 2014; Weber, 2012). Very few studies combined the lecturers’ underlying thoughts on proof and their teaching actions in a lecture with data from both observations and interviews (e.g., Pinto, 2019; Pinto & Karsenty, 2018; Weber, 2004). This study characterizes the teaching of proof in a lecture but goes in depth presenting a pattern that came up from lecture observations and systematic engagement with the data, and connects this pattern with the lecturer’s experiences. This pattern is an expression of the semantic style (Weber, 2004) and gives details of the way the lecturer understands the semantic style and develops it in his courses. Also, it connects semantic teaching style with proof image (Kidron and Dreyfus, 2014) in a way potentially meaningful for the students.

At this level the advanced university mathematical courses are usually taught by researcher mathematicians with main experience “on writing proofs for disciplinary, rather than pedagogical purposes” (Lai & Weber, 2014, p. 93). This fact leads them to give more emphasis on the formal aspects of proof (Alcock, 2010) even though they know that the ideas behind the proof are often provided in an informal way (Lai &
Weber, 2014). In their own research, lecturers themselves consider the use of informal ways for proving but pay less attention to these ways during the teaching in university courses. The study we report here is a part of a wider study on university teaching and examines the teaching of proof in an introductory course of mathematical analysis by a lecturer who brings experience from research in mathematics, research in mathematics education but also has many years of teaching experience. With this specific case, we try to understand how these different types of experience blend while he is teaching proofs. In particular, here we ask:

1. Which are the characteristics of proof teaching in an introductory mathematical analysis lecture?

2. What is the lecturer’s rationale underlying this teaching in relation to the different sources of experiences?

THEORETICAL BACKGROUND

In this paper, adopting a sociocultural perspective, teaching is considered an activity where the constructive actions and goals of the lecturer are socially and culturally framed and developed. The lecture in this context is an instructional activity, while the activity of teaching concerns the thoughts, decisions and judgments of a lecturer in planning, teaching and reflecting on the lesson (e.g., Petropoulou, et al., 2011).

Regarding mathematical proof teaching in university, Weber (2004) investigated the traditional instruction “definition – theorem – proof”, showing that there is not a single paradigm of teaching in the specific instruction. Weber described in detail the teaching styles of the lecturer and offered insights into the reasons he used these specific styles, through lesson observations and interviews with him. Three different styles of proof teaching were identified: the logico–structural, the procedural, and the semantic. The first had to do with the typical mathematical argument. The second was about the emphasis given on the techniques and the general structure of the proof. The last concerned the idea behind the proof while informal representations, such as diagrams, metaphors, generic examples, everyday language, took part during the teaching process. The researcher mentioned that the semantic approach was a style a lecturer followed in order to help students gain rich images of the mathematical concepts. Within this style, the lecturer usually worked with intuitive descriptions of the concepts and focused on the links among them. In contrast, in the first two styles (logico–structural and procedural) informal representations of the mathematical concepts were rarely used. Relevant to the semantic proof production was the concept of proof image described by Kidron and Dreyfus (2014). As they described “if an individual has attempted to understand why a given claim is true, this individual may have a proof image” (Kidron & Dreyfus, 2014, p. 309). During the process of creating a proof, a collection was made of previous constructs, ideas, knowledge and examples that seemed to be useful and fitted to a specific problem. This collection could lead to the proof image and was not necessarily communicable but was complete and provided explanation with certainty. Thus, the individual could use the proof image to move to
the formal proof with logical links. The difference between proof image and semantic reasoning was “the entity characteristic of a proof image” which “implies a complete image of the proof rather than specific instantiations of the mathematical object being explored” (Kidron & Dreyfus, 2014, p. 304).

Studies on the teaching of advanced mathematical concepts indicated that lecturers’ teaching was affected by multiple factors (Weber, 2004). A combination of knowledge about mathematics and pedagogical concepts, skills and experience, goals for the course and understandings about how students learn and what they have to learn affect teaching decisions and actions in a lecture (Weber, 2004). In order to understand why a lecturer chooses these specific actions we have to get an insight into all these aspects. The aim of this study is to investigate the teaching of proof at the university when the lecturer, a mathematician with research experience in mathematics and with long teaching experience at university mathematics teaching is also aware of pedagogical issues coming from his research on mathematics education. This paper attempts to shed light on an expression of the semantic style. In particular, the emphasis is on the pattern of proof teaching that is based on lecturer’s understanding of students’ needs and difficulties.

**METHODODOLOGY: DATA COLLECTION AND ANALYSIS**

The study was based on an introductory, proof-oriented, mathematical analysis course, taught in the mathematics department of a central Greek University, for a period of a semester. The content of the course included limits of sequences and functions, theorems about continuous functions, the definition of derivative, and applications of the previous concepts. The course was compulsory and taught in two parallel classes of approximately 100 students in each class. We conducted a case study, focusing on one of the lecturers of the course. The lecturer has a 20 – year experience in teaching this course and he is an exemplary case of a lecturer. Except of teaching experience, he is a researcher both in mathematics and mathematics education and his research concerns the area of mathematical analysis and its teaching. More specific, his research in mathematics is on functional analysis while in mathematics education is on students’ learning of advanced mathematical concepts, the role of counter examples in teaching and learning of mathematics, as well as on teaching in undergraduate level and teachers’ professional development.

In total, 17 lectures (34 academic hours) were observed during a semester. The lectures were audio – recorded and transcribed while field notes were kept. There were three meetings/ semi-structured interviews with the lecturer. The first meeting took place during the observations of the lectures, the second at the end of the semester, and the third after an initial analysis of the data. The meetings were also audio-recorded and transcribed.

The analysis of the data was done in three stages. In the first stage, we divided each lecture into episodes according to the accomplishment of teaching a theorem. We came up with 52 episodes. Grounded approaches were used for the analysis of the episodes.
Firstly, in each episode were mentioned codes for the informal representations that the lecturer presented and for his teaching actions. The codes were merged or refined after the continuous engagement with the data and comparison with the current literature (e.g., Petropoulou et al., 2011; Fukawa – Connelly et al., 2017). In the second stage of the analysis, the interaction between teaching actions and informal representations led to the identification of the process of proving the lecturer followed and of emerging patterns. In the last part of the analysis the data from the lectures were connected with the data from the interviews. After the transcription of the discussion meetings, we tried to gain deeper insight into one identified pattern. We explored the reasons why the lecturer followed this pattern while he was teaching proof and we investigated the underlying rationale of each phase of the pattern. We tried make relations with the academic fields he participated (research in mathematics, prior teaching experiences, research in mathematics education). In this stage of the analysis the lecturer played an important role in confirming the interpretations.

RESULTS

Through the analysis of the data we identified a pattern that the lecturer followed during the teaching of a theorem. This pattern consists of four phases: *posing the problem*, *formulation of the conjecture and informal proving*, *formal proving* and *reflecting*. The phases will be illustrated using an episode referring to the theorem of the uniqueness of the limit of a sequence. In this episode, the identified teaching actions are typical of the way the lecturer dealt with proof teaching. In the end of the section we give a first insight into the lecturer’s rationale of his teaching actions through the analysis of the interviews, highlighting the sources of the different experiences.

This theorem was taught at the 9th lecture of the course. The definition of the limit of the sequence was introduced in the previous lectures and was explained through the use of different representations (e.g., verbal, graphical).

<table>
<thead>
<tr>
<th>Episode</th>
<th>Teaching actions</th>
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<tbody>
<tr>
<td>[1] Let’s see something. If we have a convergent sequence, can this sequence converge to more than one numbers?</td>
<td>Posing the problem/ rhetoric question</td>
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<tr>
<td>[2] Do you know a sequence that converges, for example, to both 1 and ½? What do you think?</td>
<td>Making the problem more specific/ posing question to the students</td>
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<tr>
<td>[a student responds that the limit is unique]</td>
<td></td>
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<tr>
<td>[3] We have already discussed the definition of the limit of a sequence. We have not proved that there is only one limit. I want you to think intuitively.</td>
<td>Pointing out the need for facing the problem/</td>
</tr>
<tr>
<td>Line</td>
<td>Inviting Students to Think Intuitively</td>
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<td>----------------------------------------------------------------------</td>
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<tr>
<td>[a student gives a general description of the idea of proof]</td>
<td>Conjecturing</td>
</tr>
<tr>
<td>[4] Exactly, so you conjecture that the limit is unique.</td>
<td>Explaining the key idea using a diagram</td>
</tr>
<tr>
<td>[5] Let’s see. We have a sequence that converges to both numbers, ( \alpha ) and ( \beta ). Because these numbers are different we can find an interval around ( \alpha ), an ( \varepsilon ) here [showing in the area around limit ( \alpha )] and an ( \varepsilon ) there [showing the area around the second limit ( \beta )], so these two intervals will be disjoined.</td>
<td>Pointing out the importance of the key idea</td>
</tr>
<tr>
<td>[6] This is an important property of real numbers. We have already used it to prove that ((-1)^n) is not convergent. For every two different numbers, we can find an ( \varepsilon ) so these two intervals will be disjoined.</td>
<td>Informal proving/using a diagram and informal language</td>
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<td>[7] If the sequence converges to both numbers then from a point onwards all the terms will be there [showing the area around the first limit ( \alpha )] and from this point onwards all the terms will be there [showing the area around the second limit ( \beta )], that is a contradiction.</td>
<td>Informal proving using an alternative definition of limit</td>
</tr>
<tr>
<td>[8] Or if you want, if we have these two intervals, outside these intervals we would have finite numbers, so here [showing the area around the second limit ( \beta )] we would have finite numbers. That is contradiction.</td>
<td></td>
</tr>
<tr>
<td>[9] Let’s prove it.</td>
<td>Writing the statement of the theorem</td>
</tr>
<tr>
<td>Theorem: if the sequence ( a_n ) converges to both ( \alpha ) and ( \beta ), then ( \alpha = \beta ).*</td>
<td></td>
</tr>
<tr>
<td>[10] The proof is what we described previously. Typical now. We will prove by contradiction. We will assume that a ( \alpha \neq \beta ).</td>
<td>Stating the method of proving</td>
</tr>
<tr>
<td>[11] Let’s assume that ( \alpha \neq \beta ). Let’s ( \alpha &lt; \beta ). If ( \beta &lt; \alpha ) we would do the same.</td>
<td>Starting the formal proving</td>
</tr>
</tbody>
</table>
| [12] [Making a new diagram – as in [5]] which \( \varepsilon \) we should take to have contradiction? We want an \( \varepsilon \) so these two intervals would be disjoined. | Posing rhetorical questions related to a specific proving step/
The distance between \( \alpha, \beta \) is \( \beta - \alpha \). Thus the distance of \( \alpha, \beta \) from their mid-point \( \frac{\beta+\alpha}{2} \) is \( \frac{\beta-\alpha}{2} \).

We can take an \( \varepsilon \) less than or even than \( \frac{\beta-\alpha}{2} \).

Let's say \( \varepsilon = \frac{\beta-\alpha}{3} \). This is positive and \( \alpha + \varepsilon < \beta - \varepsilon \).

What does it mean that \( \alpha_n \) converges to \( \alpha \)?

There is \( n_1 > 0 \), I will write it this way, \( \alpha - \varepsilon < \alpha_n < \alpha + \varepsilon \ \forall \ n \geq n_1 \).

Similarly, because \( \alpha_n \) converge to \( \beta \), there is a \( n_2 > 0 \), not necessarily similar with the previous one, \( \beta - \varepsilon < \alpha_n < \beta + \varepsilon \ \forall \ n \geq n_2 \).

This means that if I find a natural number greater than \( n_1 \) and \( n_2 \) then for this index the corresponded term will be here [adding \( \alpha_n \) at the diagram second at 13].

So, for \( n > \max\{n_1, n_2\} \) we have \( \alpha_n < \alpha + \varepsilon < \beta - \varepsilon < \alpha_n \) which is a contradiction.

There is a property that is crucial in this proof. The key idea of this proof is that we can separate two different real numbers with disjoined intervals. I say this because it will be useful in future courses.
In the episode, the lecturer follows the pattern of *posing the problem, formulation of the conjecture and informal proving, formal proving and reflecting* phases. During the phase *posing the problem* ([1]-[3]) the lecturer sets the problem by asking the students whether a sequence can converge in more than one numbers [1]. He specifies the question [2] and points out the necessity of facing the problem [3]. The purpose of this phase is to show to the students that any mathematical result originates from a problem.

The *formulation of the conjecture and informal proving phase* follows ([4]-[8]). The lecturer in this phase is trying to develop an inquiry that will lead to the conjecture and to the informal proof of this conjecture. The lecturer uses informal representations like diagrams to support the proving process. Also, the language that he uses seems to be both formal and informal. In the episode, the lecturer uses the student’s answer for the formulation of the conjecture [4], draws a diagram and uses informal language and diagrams to explain the key idea [5], [6], [7], [8]. At the end of this phase a proof image has been presented to the students creating a certainty that the conjecture is true.

Then, in the phase of *formal proving* ([9] – [16]) the lecturer translates the informal arguments of the proof to formal ones by using mathematical language. The lecturer in that phase writes on the board the statement of the theorem and the typical proof. During the typical proof, he makes references to the informal proof, translating it step by step, combining different representations, so as to create links between the previous phases of the proving process. In the episode at the beginning, he writes the statement of the theorem [9]. The lecturer makes a new diagram to support his teaching actions. The difference is that this diagram is gradually updated through the proving process. The diagram keeps the structure of the process compact and links the previous phase of the informal proving with the phase of formal proving. There is an interplay between formal and informal proving as the lecturer explains the steps first informally using the diagram or questions [12, [13] and then translates them in a formal way [14], [15].

After the completion of the proof the lecturer takes a few minutes (*reflecting phase*) to reflect on the key idea that arises from the proving process and seems to be useful for the students [17]. Therefore, in the last phase, the lecturer sums up what happened in the previous phase. He focuses on the key ideas of the proof as well as he investigates the necessity of the theorem conditions and its reverse when is needed.

In general, in the phase of formulation of the conjecture and informal proving the teaching has characteristics from the semantic style. The lecturer tries to make students understand what the theorem is about and help them construct meaning of the proving process using several informal representations. At the end of this phase he has also developed the proof image of the theorem. The steps he makes are explicit and came from what he thinks will be useful for the students in order to be ready for the typical proof. The transition from the developed proof image to the formal proof happens with the identification of the logical links, which in our case is the translation of the previous phase step by step. In the last part of the proving process the lecturer seems to focus on the key ideas of the proof, a characteristic of the semantic teaching style, but also, he
separates constructs, ideas etc. that can be used as a proof image for the proof of another theorem.

**Lecturer’s rationale underlying proof teaching**

During the first interview, the lecturer stated that the main goal of his teaching was to show to the students how mathematics are produced and how they should study and understand mathematics. In his words, “The biggest problem of the students, I realized it latter was that they don’t know how to study mathematics… They study mathematics in the same way as they did in school. This way doesn’t help them now”. He emphasized that he attempts to “gradually introduce the students to the new learning culture that is different from that of school and to the mathematical production that is based on proofs”.

Regarding the emerged pattern, in the second interview the lecturer said that “this is a lesson addressed to future mathematicians, so I try not only to show what the theorem says and solve exercises but to present how our thoughts develop and lead to a conclusion”. Then, when the problem leads to a conjecture, he said that “we don’t know if it is right or not, so we start to think informally to understand what is going on. In the end of this process we have a strong belief that the conjecture is true”. He added that “we sum up what we proved (…) because this is meaningful for the students in order to understand the theorem and get a holistic view of the proof”. The third phase, he stated that “Here I repeat the previous phase, step by step, translating every step in a formal way in order to make links”. In the last phase, he restated the key ideas because “the key ideas will probably help the students in similar situations or in other courses”.

Research in mathematics helped the lecturer adopt this particular teaching approach. Nevertheless, it was the research in mathematics education that gave shape to his teaching by following a specific pattern while he presents proofs. In the last interview, the lecturer made explicit the sources of his teaching decisions and actions:

The reason to teach a theorem following this pattern came both from my research in mathematics and research in mathematics education. In the research in mathematics the starting point is a problem. By the use of informal and formal tools, a conjecture about the answer of this problem is formulated and then usually informally the researcher develops a general process of the proof of the conjecture. The last phase is to write the formal proof, and then they are sure that the theorem has been proved. I try to follow a similar process in my teaching in order to make explicit to the students how we think when we do mathematics. If I taught only the theorem and the formal proof I would teach the mathematical product and not the thinking that led to this.

The research in mathematics education made him more conscious of the teaching goals he should try to achieve during the lecture:

My involvement with research in mathematics education helps me to see that common practices in mathematics research, as the use of different representations and the connections between them, are very important in learning and teaching of mathematics.
So, I try to adapt my teaching with the way that mathematicians work, using ideas and results from the research in mathematics education. I used to emphasize this process in the past, but now I am aware of the importance of this in my teaching. Also the research results from mathematics education on students’ difficulties helps me to focus on these difficulties.

**DISCUSSION**

The present study brings into account the teaching of the proof during a lecture. We studied a case of a lecturer who brings experience from both research in mathematics and mathematics education, and also, has many years of teaching experience at the university level. All the above seemed to affect the lecturer’s teaching as he became aware of students’ needs and difficulties they probably face. What makes this case particular is that the lecturer is not only a mathematician who teaches at the university level (e.g., Paterson et al., 2011). He also draws in research in mathematics education to make the proving process potential meaningful for the students. A previous research of Petropoulou, Potari and Zachariades (2011) had shown that these experiences of a lecturer affected the way he taught. For instance, in that study, in many incidents sensitivity to students’ needs seemed to balance with a challenging mathematical content. In this study, the lecturer focuses on the process of mathematical proof production taking into account students’ possible learning needs. In the course, the lecturer tried to develop students’ proof image as it is not expected that students would have their own during the first year. This case of a lecturer brings closer the distance between mathematical teaching and mathematical production. The inquiry the lecturer promotes during the proving process is an expression of semantic style and supports the development of students’ proof image.

This study highlights the importance of proof and proving at the university level and especially in the teaching of an introductory analysis course for the students of a mathematical department in a lecture format. Although there are studies at the university level focusing on teaching practices and specific teaching actions (Viirman, 2015), our research goes beyond the identification of teaching actions. It offers a global characterization of proof teaching indicating interrelations among different teaching actions. The phases of the pattern appear as a characterization of the semantic style of proof teaching and seems to have potential in developing students’ proof image around the proof. The pattern is not linear. For instance, as we described in the episode, in the phase of formal proof, the lecturer used also informal explanation. The process is nested and the pattern appears again in a way similar to the mathematical production.

Further research is needed to study the impact of this kind of proof teaching in students’ learning. This would enrich our understanding of the multifaceted process that is the act of proving at the university level.
REFERENCES


