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PROCESS CONCEPTION OF LINEAR TRANSFORMATION FROM A FUNCTIONAL PERSPECTIVE

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In this proposal we discuss from an APOS (Action-Process-Object-Schema) viewpoint student conceptions involved in the construction of conceptions about domain, image and inverse image of a linear transformation from \mathbb{R}^2 to \mathbb{R}^2 as well as the relations between these notions. We present the design of a set of tasks which allows exploring different facets of the above concepts, evidenced by the analysis of the production of a student. Thanks to the design of the instrument it was possible to highlight some conceptions that may not be evident in typical teaching situations. Some suggestions related to teaching strategies are included.

Keywords: teaching and learning of linear and abstract algebra, teaching and learning of specific topics in university mathematics, linear transformation, task design, APOS theory

INTRODUCTION AND RESEARCH OBJECTIVES

The process of learning new concepts builds on previously constructed concepts, particularly in advanced mathematics. Domain, image and inverse image are among such previous concepts for the understanding of linear transformations. These concepts play an important role within linear algebra as well as in connection with other subjects such as calculus and analysis. In the teaching of linear transformations, very often the algebraic and algorithmic nature related to the linearity properties is favored, as opposed to more functional aspects. Zandieh et al. (2017) explore the relationship between a high school function conception and a university linear transformation conception. These authors advocate a unified function-transformation concept; according to APOS Theory this happens when the vector space Object gets assimilated into the function Schema as a possible domain (Roa-Fuentes & Oktaç, 2010). The relationship between these two concepts was also studied by Bagley et al. (2015) in the context of inverse, composition and identity.

Reports about students' difficulties are abundant in linear algebra education research (Dorier & Sierpinska, 2001). Some of these difficulties might be related to the lack of previous concepts that need to be constructed adequately and coordinated. We have not found any literature about the role that the concepts of domain, image and inverse

image play in the construction of the linear transformation concept, although the latter has been widely studied (Sierpinska, 2000; Andrews-Larson et al., 2017; Oktaç, 2018).

Given the context explained in the previous paragraphs, the aim of this study is to evidence student conceptions about domain, image, inverse image and linear transformation in \mathbb{R}^2 including the relations between them. Also of interest is to investigate the role that the former notions play in the construction of the linear transformation concept.

Given the importance of linear algebra as a support for developing advanced topics in mathematics, as well as their applications in different university programs and in real life, we consider it necessary to investigate these relations in order to contribute to existing strategies for an adequate understanding of these concepts.

THEORETICAL AND METHODOLOGICAL CONSIDERATIONS

In our research group we have studied the linear transformation concept from various angles including algebraic, geometrical and matrix representations (Ramírez-Sandoval et al., 2014); dynamical exploration (Camacho-Espinoza & Oktaç, 2018) and connections between research and teaching (Oktaç et al., 2019). The present study is a continuation of this general project that has as its objective to study the understanding of linear transformations.

In this study APOS Theory is adopted as a framework, as it provides a cognitive approach applied in the context of the understanding of advanced mathematical topics. The basic elements of this theory are known as *mental structures*, *stages* or *conceptions*, that are constructed by means of mental mechanisms. *Actions* are driven externally, where the individual can transform previously constructed Objects. *Processes* are developed when an internal stimulus replaces the external algorithms or rules via the mechanism *interiorization*. When Processes are *encapsulated* they become *Objects* to which Actions or Processes related to other concepts can be applied. Finally all these structures and their relationships can come together to form part of a *Schema*.

The research cycle related to APOS Theory has three components: Theoretical analysis (called a *genetic decomposition*); design and application of instruction; and collection of empirical data and their analysis. The construction of the linear transformation concept was studied theoretically and empirically from an APOS viewpoint (see Arnon et al., 2014). In previous research studies that we conducted, some student difficulties that were detected in the construction of this concept gave the impression that they might be related to the lack of construction of previous notions. In order to delve into the role that these concepts play, we designed a research study about student conceptions on these topics, some of whose results we report here.

We designed an instrument comprising of four questions that was applied to a total of 31 university students from three different institutions in Mexico who were enrolled in an introductory linear algebra course. In the design of the problems special care was

taken to involve different facets of the concepts in question. Usually in textbooks and in classroom activities only some of these aspects are offered to students, who get used to working on them from certain angles. When this angle is changed, and this is new to the student, it becomes easier to identify where the difficulties lie. In the case of linear transformations, the favored aspect can be the linearity properties over the functional characteristics.

In the design of the research instrument, algebraic, geometrical as well as functional aspects of a linear transformation were used, integrating in this way different facets of the concept. By a functional aspect we mean considering the linear transformation as a correspondence that associates the vectors of a domain space to the vectors of an image space; this is related to a Process conception because of its generality. In what follows we go into more depth about this issue.

A Process conception of domain, coordinated with a Process conception of image, is required for the Process conception of linear transformation. Conceptions that students have about domain and image, and even about systems of linear equations and their solutions, can intervene in their conceptions about linear transformation and inverse image. All these considerations were taken into account in the design of the questions that made up the instrument, with the aim of investigating the relationships between the different elements related to the linear transformation Process, and hence offering suggestions for its construction and enrichment.

ANALYSIS OF DATA

We now present two of the questions from the instrument, their analysis as well as the results obtained from a student. With the application of the instrument we have realized that even if a student gives the appearance of having constructed a conception that allows the solution of a problem with success, when the problem is changed to one that explores different facets of the same concept, the conceptions that are evidenced by the new situation can be quite different from the ones detected initially. We now analyze the production of a student (E) who handles very efficiently algorithmic procedures, when working on questions 3 and 4 of the instrument.

Question 3 was stated as follows:

3.- Consider the linear transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ associated to the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

a) Determine its domain.

b) Determine its image.

c) ¿Does the vector $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ belong to the image of the transformation? If the answer is YES find the corresponding inverse image and graph it. If the answer is NO, justify.

d) Graph the domain of T .

e) Graph the image of T .

f) ¿Does the vector $\begin{pmatrix} 5 \\ 5 \end{pmatrix}$ belong to the image of the transformation? If the answer is YES find the corresponding inverse image and graph it. If the answer is NO, justify.

Different parts of Question 3 of the instrument allow an investigation into student conceptions on domain, image and inverse image of a linear transformation via its matrix. Action and Process conceptions about these concepts can be detected in the context of algebraic and geometric representations, as well as the way students relate them to the linear transformation concept.

Before we start analyzing E's responses and arguments, we would like to comment on the cognitive structures that are involved here.

Construction of a Process conception of domain allows the individual to accept that the transformation acts on all the vectors of the domain. This conception might be revealed with the use of generic vectors that represent *any vector* in the domain. However applying the linear transformation on general vectors does not necessarily imply a Process conception of the domain and hence, does not always lead to the construction of a Process conception of image.

A Process conception of image permits the individual determine how the domain and image vectors are related to each other, what conditions a vector should satisfy in order to belong to the image, how that set is represented geometrically and in general terms, how the transformation acts on the domain vectors.

Constructing a Process conception of inverse image requires the individual to identify the set of vectors in the domain that are related to a specific vector in the image through the linear transformation. It also requires determining the characteristics that vectors in the domain should satisfy in order to form part of the inverse image set. Included in this Process conception is the identification of its geometric representation.

In what follows we present empirical evidence that shows that even if an individual can work quite efficiently with algorithms and procedures in solving a problem, this does not imply that solid conceptions about the concepts in questions have been developed.

Items a) and d) of Question 3 aim at exploring the conceptions that a student might have about the domain of a linear transformation. Student E gives the impression that he has constructed a Process conception of domain, since he identifies the domain of the given transformation as \mathbb{R}^2 and graphs it shading the whole plane as shown in Figure 1.

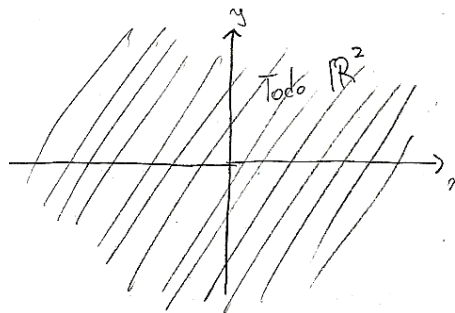


Figure 1. Domain of the linear transformation according to E

In order to determine the image of the transformation, E uses algebraic algorithms applied to generic vectors as shown in Figure 2.

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x+y \end{pmatrix}$$

Figure 2. Production of E to determine the image of the transformation

This might give the impression that the student has constructed a Process conception of image starting from a Process conception of domain, evidenced by the use of a general vector $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ on which the transformation acts through its matrix. However, this does not necessarily imply that the student has understood how the transformation acts on all the vectors in the domain, what the nature of this process is, and what properties it fulfills. One evidence that shows that E has not yet quite understood the nature of this image Process is that even though he determines the image set algebraically, he cannot associate it to the identity line and cannot graph it (see Figure 3).

e)

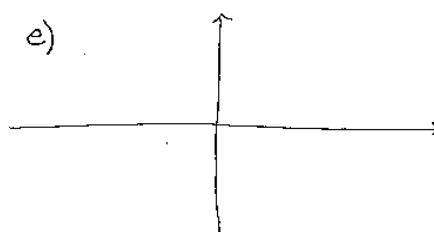
$$I_m(\tau) = \left\{ \begin{pmatrix} x+y \\ x+y \end{pmatrix} \in \mathbb{R}^2 \mid x, y \in \mathbb{R} \right\}$$


Figure 3. Image of the transformation and its graphical representation according to E

In this case the condition that the vectors of the image satisfy is that the x and y components should be equal. However, the algorithm used by E in order to find the image does not provide enough tools for him to realize this property; in parts c) and f) he does not make use of it to determine if the vectors (3,2) and (5,5) belong to the image set, as we shall see below.

In order to determine whether a vector belongs to the image set or not, E resorts to the Gaussian elimination technique applied to the augmented matrix of a linear system of equations. In the case of the vector (3,2) this system is inconsistent and for (5,5) it has infinitely many solutions (Figure 4).

$$\begin{pmatrix} 1 & 1 & | & 3 \\ 1 & 1 & | & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & | & 3 \\ 0 & 0 & | & -1 \end{pmatrix}, \text{ que es un sistema inconsistente}$$

$\therefore \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ no est\u00e1 en la imagen

$$\begin{pmatrix} 1 & 1 & | & 5 \\ 1 & 1 & | & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & | & 5 \\ 0 & 0 & | & 0 \end{pmatrix} \therefore \begin{matrix} x = 5 - t \\ y = t \end{matrix}, \text{ con } t \in \mathbb{R}$$

Figure 4. Gaussian reduction technique for vectors (3,2) and (5,5)

Setting up a system of equations related to a transformation in order to determine whether a vector belongs to the image set or not, might require a coordination between a linear transformation Process and a system of linear equations Process. This allows the individual to recognize that $y = T(x)$ if and only if the augmented system $(A: y)$ has a solution, where A is the matrix associated to T . Despite the skill in algorithmic manipulation of systems of linear equations, lack of Process conception might lead to the need to repeat this procedure every time regardless of the number of vectors to be determined as belonging to the image set or not. This is related to an Action conception.

In summary, somebody who can perform Actions on a generic vector of the form (x, y) may or may not have constructed a Process conception of image; this is related to understanding the way a linear transformation acts on its domain.

Finding the inverse image of a given vector can be conceivably more complex than establishing the image of a linear transformation. However a solid Process conception of system of linear equations concept helps E to determine the inverse image set of the vector $(5,5)$, using a parametric form and graphing the corresponding line as seen in Figure 5.

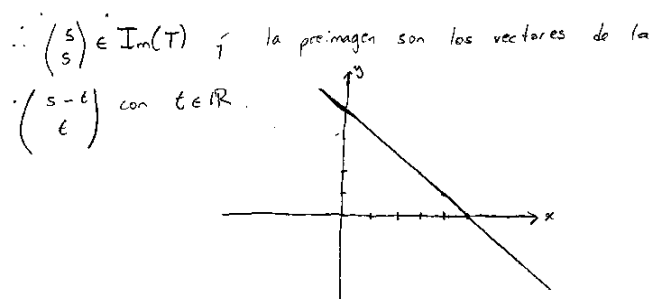


Figure 5. Parametric form of the inverse image of $(5,5)$ and its graph

On the other hand, although apparently E might have constructed a Process conception of the domain concept, there are three pieces of evidence that manifest that he has not constructed a Process conception of image concept: (i) he cannot graph the image, (ii) he cannot determine whether a vector belongs to the image without performing a specific action and (iii) he does not have it clear how the transformation acts on the vectors of the domain. Furthermore, although he was able to determine the inverse image of a particular vector, this is thanks to his solid conceptions about systems of linear equations.

As we mentioned before, a coordination between the Processes of domain and image concepts is necessary so that a Process conception of linear transformation concept can be constructed. If any of these Processes is absent, it is very probable that the conceptions about linear transformations would be limited.

In order to consolidate our interpretations about these conceptions, we now turn into the work that E has produced when working on Question 4 of the instrument, hence examining his conceptions in a different context.

Question 4 was stated as follows:

Find the rule of a linear transformation under which the image of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is $\begin{pmatrix} 0 \\ -4 \end{pmatrix}$ and where the transformation is the combination of a rotation followed by a dilation. Determine the matrices of rotation and dilatation for the transformations used. Justify your response.

This question was designed so that it would be accessible to students with Action, Process or Object conceptions who would employ different kinds of strategies. Figure 6 shows the work of E.

$$\textcircled{4} \quad T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot 0 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} (-4) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -4 \end{pmatrix}$$

$$\text{Definimos } T: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad \& \quad T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -4x \end{pmatrix} = y \begin{pmatrix} 1 \\ 0 \end{pmatrix} - 4x \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\text{Sea } A = \begin{pmatrix} 0 & 0 \\ -4 & 0 \end{pmatrix} \text{ la matriz de rotaci3n.}$$

Figure 6. E's solution to Question 4

Here E shows an Action conception about image, by choosing to work with a single vector and its image, where $T(1,0) = (0, -4)$. As can be seen from his strategy, he employs a generic vector (x, y) , which might indicate a Process conception, however he only tries to make sure that the conditions of the problem are satisfied by these two vectors and does not take into account the rest of the domain vectors. We should note that the image vector denoted by $(y, -4x)$ does not correspond to rotation and dilation transformations. It is possible that he is mixing up a rotation with a reflection about the line $y = x$. In order to find the image of (x, y) he interchanges the coordinates $(x, y) \rightarrow (y, x)$ and then multiplies the second one by -4 , that is $(x, y) \rightarrow (y, x) \rightarrow (y, -4x)$. The linear transformation that he finds complies with the condition $T(1,0) = (0, -4)$. However the rest of the domain vectors do not actually configure into the solution. This confirms the lack of a Process conception of linear transformation concept, because of a lack of a Process conception of image concept, and probably lack of a Process conception of domain concept as well.

CONCLUSIONS

The design of the instrument enabled us to look into the conceptions related to the concepts of domain, image and inverse image of a linear transformation from different angles and how these conceptions intervene in the construction of the linear transformation concept. Question 3 allows the exploration of conceptions about notions

that until now have not been prioritized in research as key in the construction of the concept of linear transformation.

Question 4 was designed to explore Action, Process and Object conceptions of the concepts mentioned before. Even though its complete solution requires an Object conception of linear transformation, students can tackle the problem partially evoking Action and Process conceptions. The analysis of the production of a student when confronted with these questions allowed us to investigate and demonstrate the structures involved and those that are to be constructed for the learning of the linear transformation concept.

When we talk about exploring different facets of concepts and conceptions we refer to the different roles that are played by them in different situations. For example in Question 3 the student is asked directly to find the domain, the image and the inverse image set. The student could not graph the image set, which indicated the absence of a Process conception. On the other hand, he was able to graph the domain, which could give the impression that he has constructed a Process conception for this concept. However Question 4 involves the same concept, namely domain, in a different way; the student has to use it in order to solve the problem, even though it is not even mentioned in the statement. Presenting situations that involve the concept in question from different angles allow us to identify the real conceptions that the student has constructed.

In teaching situations usually algorithmic and algebraic aspects of the linear transformation concept are favored, leading to Action-based strategies on the part of the students. The construction of a Process conception requires the interiorization of these Actions, leading to an internal control over them, including the ability to work with general situations. The functional aspect of a linear transformation is precisely related to a Process conception, in which the concepts of domain and image play an important role, which should be taken into consideration when establishing teaching strategies.

As for the limitations of the study, we mention that since it was only conducted in the context of \mathbb{R}^2 , care should be taken when generalizing into other contexts.

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