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Evaluation of Shewhart Time-Between-Events-and-Amplitude Control Charts for Correlated Data

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Short running head: “Shewhart TBEA Control Charts for Correlated Data”

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Keywords: Amplitude. Copulas. Machine breakdowns. Statistical Process
Monitoring. Time between events.

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Abstract

In recent years, several techniques based on control charts have been developed for the *simultaneous* monitoring of the time interval T and the amplitude X of events, known as TBEA (Time Between Events and Amplitude) charts. However, the vast majority of the existing works have some limitations. Firstly, they usually focus on statistics based on the ratio $\frac{X}{T}$ and, secondly, they only investigate a reduced number of potential distributions, i.e. the exponential distribution for T and the Normal distribution for X . Moreover, until now, very few research papers have considered the potential dependence between T and X . In this paper, we investigate three different statistics, denoted as Z_1 , Z_2 and Z_3 , for monitoring TBEA data in the case of three potential distributions (Gamma, Normal and Weibull), for both T and X , using Copulas as a mechanism to model the dependence. An illustrative example considering times between machine breakdowns and associated maintenance illustrates the use of TBEA control charts.

Keywords: Amplitude. Copulas. Machine breakdowns. Statistical Process Monitoring. Time between events.

1 Introduction

Control charts are one of the most effective techniques in Statistical Process Monitoring (SPM) to monitor manufacturing processes or critical events E such as the ones occurring in healthcare or in geological (earthquakes or volcanic eruptions) applications. The definition of an event E clearly depends on the situation considered. For instance, in the manufacturing industry, it means the occurrence of a nonconforming product; in reliability engineering, it means the failure of a specific system, etc. (see Wu et al.¹). In general, a critical event is defined by two important characteristics: i) the time interval T measured by the duration between two successive occurrences of the event E and, ii) its amplitude (cost, number of casualties, ...) X which reflects the corresponding effect, see Cheng et al.².

The problem of monitoring Time Between Events was introduced for the first time by Calvin³. Lucas⁴ and Vardeman and Ray⁵ extended this work by studying new control charts for the monitoring of Time Between Events (TBE). From that moment, many TBE control charts have been developed. The TBE exponential chart has been studied by Chan et al.⁶ and Xie et al.⁷. Bourke⁸ developed a geometric CUSUM chart for monitoring TBE data. Gan⁹, Borror et al.¹⁰ and Shafae et al.¹¹ investigated an exponential TBE cumulative sum (CUSUM) control chart. An exponentially weighted moving average (EWMA) control chart for monitoring the rate of occurrences of rare events has been developed by Gan¹². Liu et al.¹³ discussed the performance of continuous TBE charts among the CQC, CQC-r, exponential EWMA and exponential CUSUM charts. Zhang et al.¹⁴ investigated a Gamma chart to monitor the TBE and they developed a new method based on a random-shift model for calculating the out-of-control ATS. Xie et al.¹⁵ discussed the application of TBE control charts for health management. Recently Fang et al.¹⁶ proposed a generalized group runs TBE chart for a homogenous Poisson failure process.

As the effect of an event E is not only quantified by its frequency T but also by its amplitude X , a combined scheme for monitoring the time interval using a T chart and the amplitude using a X chart have been introduced by Wu et al.¹⁷. The corresponding control chart is called a Time Between Events and Amplitude (TBEA) chart. After the paper of Wu et al.¹⁷, several single TBEA charts have been developed (see Wu and Qu¹⁸, Qu et al.¹⁹, Qu et al.²⁰). Very recently Ali and Pievatolo²¹ introduced a new control chart based on the assumption of a renewal process with rewards, where the reward represents the magnitude, and a magnitude-over-threshold condition represents the occurrence of an event.

All the TBEA control charts presented above assume that the time between events T and their amplitudes X are *independent* variables. However, in practice, they may not necessarily be independent. More specifically, there are many situations where it seems expectable (and logical) that when the time between events becomes shorter, then the corresponding amplitude becomes larger. For instance, small amplitude earthquakes may occur with a low frequency (large time between events) and, suddenly, the frequency of the occurrence of these earthquakes may increase (shorter time between events) with a correlated increase in their amplitudes. As another example, some forest fires may occur with a low frequency and small amplitude (surface burned) during the “humid season” and, when the “dry season” comes, the time between the occurrence of forest fires tends to become shorter and their amplitude tends to increase (see the 2019 forest fires in Amazonia or Siberia). As the illustrative example will show, the contrary may also happen, i.e. time between events becomes shorter and amplitude becomes smaller. Whatever the situation, it seems important to investigate the dependency between time between events T and their amplitudes X . Until now, very few research papers have investigated TBEA control charts by considering the potential dependence between the two variables. Cheng and Mukherjee²² were the first to investigate a TBEA control chart by taking into account the dependence between the frequency and the amplitude using a T^2 chart based on transformed data. They have used a Smith-Adelfang-Tubbs (SAT) bivariate Gamma distribution to model the joint probability of T and X and they have shown that the proposed chart is more efficient when the shift size is moderate to large. Then, Cheng et al.² developed a multivariate exponentially weighted moving average (MEWMA) scheme also based on a bivariate Gamma distribution, to jointly monitor the frequency and the amplitude of an event. A comparison is conducted between three kinds of charts under eight shift domains and the results show that the proposed MEWMA procedure outperforms other charts in most scenarios. Very recently, Sanusi et al.²³ proposed a Max-EWMA type chart using the maximum of the absolute values of two EWMA statistics, one for controlling the magnitude and the other for the frequency of an event. The limitation of those studies is that they only considered the bivariate Gamma distribution to model the joint probability of T and X , while in this paper, we propose a general mechanism based on Copulas that allows a very flexible choice for the distribution of T and X .

In order to model the dependence between the time T and the amplitude X , in this paper we suggest the use of Copulas (popularized by Sklar²⁴) which allow the degree of association between these variables to be quantified without having to actually assume anything concerning their distributions. For instance,

Copulas have been used in the financial domain. Several researchers also used Copulas for defining and implementing control charts. Fatahi et al.²⁵ proposed a bivariate control chart based on Copulas. A Copula Markov CUSUM chart for monitoring bivariate auto-correlated data has been developed by Dokouhaki and Noorossana²⁶. Busababodhin and Amphanthong²⁷ studied various types of Copulas modeling for several multivariate control charts. Very recently, Sukparungsee et al.²⁸ proposed five different types of Copulas for the Hotelling's T^2 control chart and they evaluated the statistical measures of performance using Monte Carlo simulations.

In this paper we propose to extend the preliminary work in Rahali et al.²⁹ and use three kinds of Copulas in order to investigate the properties of one-sided Shewhart TBEA charts for three different statistics Z_1 , Z_2 and Z_3 which all depend on T (time) and X (amplitude). The main limitations of this contribution are i) the fact that a parametric distribution has to be chosen for X and T (without being exhaustive, this paper nevertheless investigates three distributions: Gamma, Normal and Weibull), ii) the parameters of these distributions, as well as the Copulas dependence parameter have to be known or, at least, accurately estimated.

Finally, it is important to note that the kind of events E we are interested in are different from the so-called "recurrence data" that can be found in survival analysis for which some right-censoring is likely to occur (i.e. at time t , the interarrival time between the $N(t)$ th event and the $(N(t) + 1)$ th event is *unobserved* and we only know that it is longer than $t - S_{N(t)}$, where S_k is the occurrence time of the k th event), see for instance, Wang et al.³⁰, Huang and Wang³¹ and Rondeau³². For example, in the case of cancer patients who have already been treated, the time-between-events is the time between two cancer recurrences. In this situation, right-censoring may (unfortunately) happen as, for some patients of the sample, after several recurrences of the disease, the last time-between-events is missing because they passed away (they never recovered from the cancer or they died because of some other reasons). Moreover, recurrence data in survival analysis, usually involve a sample (group of patients with the same disease, for instance) while, in our case, we only focus on a single phenomena. In conclusion, in our case, right-censoring is not supposed to happen because the kind of event E we are interested in (earthquakes, tsunamis, fires and, even, machine breakdown occurrences in our illustrative example) assumes that i) no matter what the situation is, this event E will necessarily re-happen "naturally" in the future, ii) it costs nothing to wait for the next occurrence of this event, iii) there is no sample to deal with.

The paper is structured as follows: in section 2 we introduce the dependence structure between T and X using Copulas, we define the statistics Z_1 , Z_2 and Z_3 and derive their cumulative distribution functions, we define the upper control limits of the corresponding TBEA charts and derive their time to signal properties. Then, in Section 3, we investigate the EATS properties of the three TBEA charts for 8 distributional scenarios, 3 different Copulas and several levels of correlation. Finally, in Section 4, we propose an illustrative example involving the time between consecutive machine breakdowns and their associate maintenance costs. Conclusions with comments and recommendations for future research are

given in Section 5.

2 TBEA charts

Let T be the time interval between two consecutive occurrences of a specific event E and let X be the corresponding magnitude of this event. In this paper, we assume that $(X, T) \in \mathbb{R}_+^2$ and their joint continuous c.d.f. (cumulative distribution function) is

$$F_{(T,X)}(t, x) = C(F_T(t), F_X(x)|\theta), \quad (1)$$

where $F_T(t)$ and $F_X(x)$ are the marginal c.d.f. of T and X , respectively, $C(u, v|\theta)$ is a Copula containing all information on the dependence structure between T and X while θ is a dependence parameter that measures the dependence between the marginals. Let

$$f_{(T,X)}(t, x) = c(F_T(t), F_X(x)|\theta)f_T(t)f_X(x) \quad (2)$$

be the joint p.d.f. (probability distribution function) of (X, T) where $f_T(t)$ and $f_X(x)$ are the marginal p.d.f.'s of T and X , respectively, and $c(u, v|\theta) = \frac{\partial C(u, v|\theta)}{\partial u \partial v}$ is the Copula density.

Let $\mu_T > 0$ and $\mu_X > 0$ be the (marginal) means of T and X , respectively. By definition, when the process is in-control, we have $\mu_T = \mu_{T_0}$, $\mu_X = \mu_{X_0}$ and, when the process is out-of-control, we have $\mu_T = \mu_{T_1}$, $\mu_X = \mu_{X_1}$. In order to not favor one random variable over the other one (their scale can be very different), the new random variables T' and X' are introduced as the in-control standardized counterparts of T and X , i.e.

$$T' = \frac{T}{\mu_{T_0}}, \quad (3)$$

$$X' = \frac{X}{\mu_{X_0}}. \quad (4)$$

By definition, when the process is in-control, $E(T') = E(X') = 1$. The marginal c.d.f. of T' and X' are $F_{T'}(t) = F_T(t\mu_{T_0})$ and $F_{X'}(x) = F_X(x\mu_{X_0})$. Let $F_{(T',X')}(t, x)$ and $f_{(T',X')}(t, x)$ be the joint c.d.f. and p.d.f. of $(X', T') \in \mathbb{R}_+^2$, respectively. It is easy to prove that

$$F_{(T',X')}(t, x) = C(F_T(t\mu_{T_0}), F_X(x\mu_{X_0})|\theta), \quad (5)$$

$$f_{(T',X')}(t, x) = \mu_{T_0}\mu_{X_0}c(F_T(t\mu_{T_0}), F_X(x\mu_{X_0})|\theta)f_T(t\mu_{T_0})f_X(x\mu_{X_0}). \quad (6)$$

As in Rahali et al.²⁹, in order to simultaneously monitor the time between an event E and its amplitude, we investigate *three* different statistics $Z = Z(T', X')$, denoted as Z_1 , Z_2 and Z_3 , all functions of T' and X' , having the following properties: i) $Z \uparrow$ if either $T' \downarrow$ or $X' \uparrow$, ii) $Z \downarrow$ if either $T' \uparrow$ or $X' \downarrow$.

A first possible choice for the statistic Z (denoted as the Z_1 statistic) is

$$Z_1 = X' - T'. \quad (7)$$

If we integrate over all the couples $(X', T') \in \mathbb{R}_+^2$ satisfying $Z_1 = X' - T' \leq z$, then the c.d.f. $F_{Z_1}(z)$ of Z_1 is equal to

- if $z < 0$ (see Figure 1 (a), where the grey area represents the region satisfying $Z_1 = X' - T' \leq z < 0$)

$$F_{Z_1}(z) = \int_0^\infty \int_{x-z}^\infty f_{(T', X')}(t, x) dt dx. \quad (8)$$

- if $z > 0$ (see Figure 1 (b), where the grey area represents the region satisfying $Z_1 = X' - T' \leq z > 0$)

$$\begin{aligned} F_{Z_1}(z) &= \int_0^z \int_0^\infty f_{(T', X')}(t, x) dt dx + \int_z^\infty \int_{x-z}^\infty f_{(T', X')}(t, x) dt dx, \\ &= F_{X'}(z) + \int_z^\infty \int_{x-z}^\infty f_{(T', X')}(t, x) dt dx. \end{aligned} \quad (9)$$

Since $F_{X'}(z) = 0$ when $z < 0$ and $f_{(T', X')}(t, x) = 0$ when either $t < 0$ or $x < 0$, the last result can be generalized to $z \in \mathbb{R}$.

A second possible choice for the statistic Z (denoted as the Z_2 statistic) is

$$Z_2 = \frac{X'}{T'}. \quad (10)$$

The random variable Z_2 is defined on $[0, +\infty)$ and its c.d.f. $F_{Z_2}(z)$ can be obtained by integrating over all the couples $(X', T') \in \mathbb{R}_+^2$ satisfying $Z_2 = \frac{X'}{T'} \leq z$ (see Figure 1 (c), where the grey area represents the region satisfying $Z_2 = \frac{X'}{T'} \leq z$), i.e.

$$F_{Z_2}(z) = \int_0^\infty \int_{\frac{x}{z}}^\infty f_{(T', X')}(t, x) dt dx. \quad (11)$$

Finally, a possible third choice for the statistic Z (denoted as the Z_3 statistic) is

$$Z_3 = X' + \frac{1}{T'}. \quad (12)$$

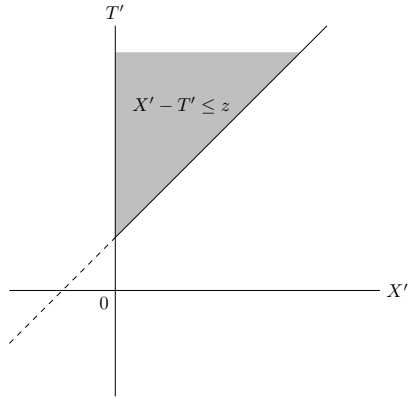
This statistic can be considered as a “hybrid” of the two previous ones. The random variable Z_3 is also defined on $[0, +\infty)$ and its c.d.f. $F_{Z_3}(z)$ can be obtained by integrating over all the couples $(X', T') \in \mathbb{R}_+^2$ satisfying $Z_3 = X' + \frac{1}{T'} \leq z$ (see Figure 1 (d), where the grey area represents the region satisfying $Z_3 = X' + \frac{1}{T'} \leq z$), i.e.

$$F_{Z_3}(z) = \int_0^z \int_{\frac{1}{z-x}}^\infty f_{(T', X')}(t, x) dt dx. \quad (13)$$

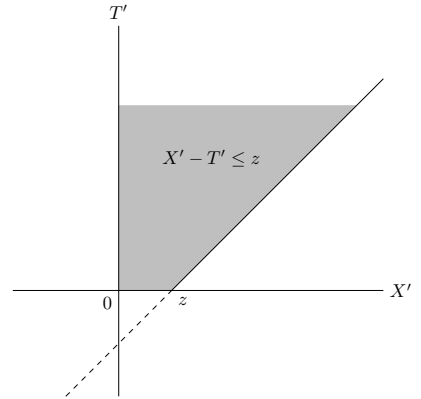
The control limits LCL_Z and UCL_Z of the TBFA (Time Between Event and Amplitude) charts based on the statistic $Z \in \{Z_1, Z_2, Z_3\}$ defined in equations (7), (10) and (12) are equal to

$$LCL_Z = F_Z^{-1}(\alpha_L | \mu_{T_0}, \mu_{X_0}) \quad (14)$$

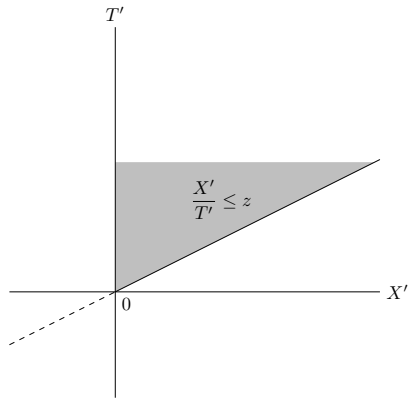
$$UCL_Z = F_Z^{-1}(1 - \alpha_U | \mu_{T_0}, \mu_{X_0}), \quad (15)$$



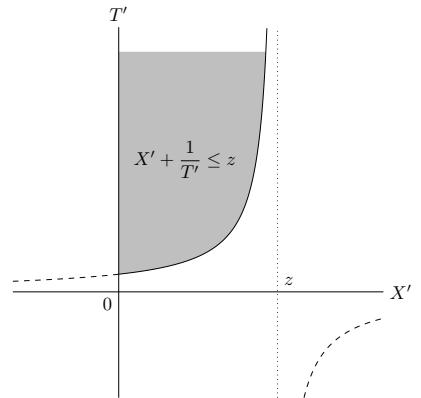
(a) Region satisfying $Z_1 = X' - T' \leq z < 0$



(b) Region satisfying $Z_1 = X' - T' \leq z > 0$



(c) Region satisfying $Z_2 = \frac{X'}{T'} \leq z$



(d) Region satisfying $Z_3 = X' + \frac{1}{T'} \leq z$

Figure 1: Integration areas used for statistics $Z_1 = X' - T' \leq z$ ((a) and (b)), $Z_2 = \frac{X'}{T'} \leq z$ (c) and $Z_3 = X' + \frac{1}{T'} \leq z$ (d).

where $F_Z^{-1}(\dots|\mu_{T_0}, \mu_{X_0})$ is the inverse c.d.f. of Z , $\alpha_L = \xi\alpha$ and $\alpha_U = (1 - \xi)\alpha$ are the lower-sided and upper-sided components of the Type I error α , respectively, with ξ being the detection power allocation factor defined as $\xi = \frac{\alpha_L}{\alpha_L + \alpha_U}$. Since it is important to consider the case of upward shift on Z , we will assume that $\xi = 0$ and, consequently, we only have a single upper-sided control limit UCL_Z .

A performance characteristic for the TBEA charts is the the average time to signal (ATS). The out-of-control ATS indicates the average time required to signal an out-of control case, whereas the in-control ATS_0 is used as a measure of the false alarm rate. The Type II error β of the TBEA charts based on the statistic $Z \in \{Z_1, Z_2, Z_3\}$ is equal to

$$\beta = F_Z(UCL_Z|\mu_T, \mu_X) - F_Z(LCL_Z|\mu_T, \mu_X). \quad (16)$$

As the TBEA chart is a Shewhart type chart, the average run length, $ARL = E(RL) = \frac{1}{1-\beta}$. Let $\ell \in \{1, 2, 3, \dots\}$ and T_1, T_2, T_3, \dots be the time between two consecutive events. The Time to Signal, TS is equal to

$$TS = \sum_{\ell=1}^{RL} T_\ell. \quad (17)$$

The Average Time to Signal and the Standard-Deviation of the Time to Signal are calculated by

$$ATS = E(TS) = E(T)E(RL) = \mu_T \times ARL = \frac{\mu_T}{1 - \beta}, \quad (18)$$

$$\begin{aligned} SDTS &= \sigma(TS) = \sqrt{V(T)E(RL) + E^2(T)V(RL)}, \\ &= \sqrt{\sigma_T^2 \times ARL + \mu_T^2 \times SDRL^2}, \\ &= \sqrt{\frac{\sigma_T^2}{1 - \beta} + \frac{\mu_T^2 \beta}{(1 - \beta)^2}}. \end{aligned} \quad (19)$$

When the process is in-control, we have $1 - \beta = \alpha$ and, consequently

$$ATS_0 = \frac{\mu_{T_0}}{\alpha} \Leftrightarrow \alpha = \frac{\mu_{T_0}}{ATS_0}.$$

3 Numerical analysis

In this section, we consider three types of distribution for T and X that depend on two parameters, a and b . We denote a_0 and b_0 as the in-control values of these parameters, respectively.

- The Gamma distribution has parameters $a > 0$, $b > 0$ and p.d.f.

$$f_G(x|a, b) = \frac{\exp(-\frac{x}{b})x^{a-1}}{b^a \Gamma(a)}. \quad (20)$$

- The Normal distribution has parameters $a, b > 0$ as in Rahali et al.²⁹, instead of the conventional notation (μ, σ) and p.d.f. $f_N(x|a, b)$.
- The Weibull distribution has parameters $a > 0, b > 0$ and p.d.f.

$$f_W(x|a, b) = \frac{a}{b} \left(\frac{x}{b}\right)^{a-1} \exp\left(-\left(\frac{x}{b}\right)^a\right). \quad (21)$$

To be more specific, in this paper, we have chosen to investigate

- the Gamma, Normal and Weibull distributions for the random variable X (amplitude).
- only the Gamma and the Weibull distributions for the random variable T (time). The Normal distribution is actually not the most suitable distribution for time data and we have decided to not investigate it.

In order to fairly compare the three TBEA charts defined in Section 2, we considered 8 in-control situations. Table 1 shows the in-control parameters a_0 and b_0 and the skewness coefficient γ_0 corresponding to an in-control mean $\mu_0 = 10$ and an in-control standard-deviation $\sigma_0 \in \{1, 2, 5\}$ (for the Normal distribution we only consider $\sigma_0 \in \{1, 2\}$). The formulae from which we obtain the values of a_0 and b_0 from μ_0 and σ_0 are given below:

- For the Gamma distribution:

$$a_0 = \left(\frac{\mu_0}{\sigma_0}\right)^2, \quad b_0 = \frac{\mu_0}{a_0}. \quad (22)$$

- For the Normal distribution:

$$a_0 = \mu_0, \quad b_0 = \sigma_0. \quad (23)$$

- For the Weibull distribution: a_0 is the solution of the equation (obtained using a univariate root finder)

$$\frac{\Gamma\left(\frac{2}{a_0} + 1\right)}{\Gamma^2\left(\frac{1}{a_0} + 1\right)} = \left(\frac{\sigma_0}{\mu_0}\right)^2 + 1, \quad (24)$$

and

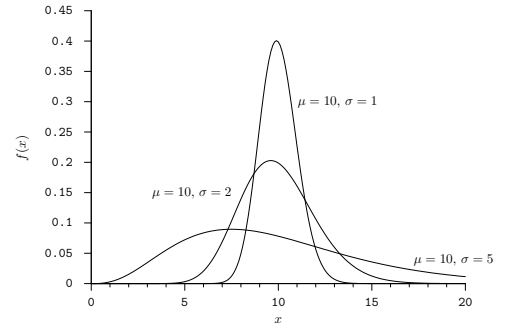
$$b_0 = \frac{\mu_0}{\Gamma\left(\frac{1}{a_0} + 1\right)}. \quad (25)$$

For the sake of clarity, the distributions corresponding to these 8 cases (i.e. Gamma with $\mu_0 = 10$ and $\sigma_0 \in \{1, 2, 5\}$, Normal with $\mu_0 = 10$ and $\sigma_0 \in \{1, 2\}$ and Weibull with $\mu_0 = 10$ and $\sigma_0 \in \{1, 2, 5\}$) are displayed in Figure 2.

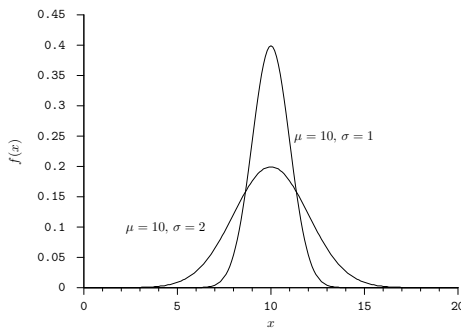
In this paper, we have chosen to investigate 3 types of bivariate Archimedean Copula:

Table 1: Distributions for T and X considered in this paper.

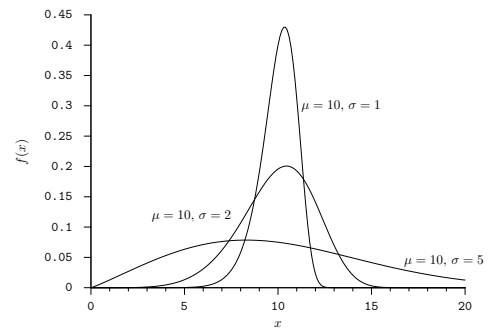
Distribution	a_0	b_0	μ_0	σ_0	γ_0
Gamma	100	0.1	10	1	0.2
	25	0.4	10	2	0.4
	4	2.5	10	5	1
Normal	10	1	10	1	0
	10	2	10	2	0
Weibull	12.1534	10.4304	10	1	-0.7155
	5.7974	10.7998	10	2	-0.3519
	2.1013	11.2906	10	5	0.5664



(a) Gamma with $\mu_0 = 10$ and $\sigma_0 \in \{1, 2, 5\}$



(b) Normal with $\mu_0 = 10$ and $\sigma_0 \in \{1, 2\}$



(c) Weibull with $\mu_0 = 10$ and $\sigma_0 \in \{1, 2, 5\}$

Figure 2: Distributions considered in this paper (a) Gamma with $\mu_0 = 10$ and $\sigma_0 \in \{1, 2, 5\}$, (b) Normal with $\mu_0 = 10$ and $\sigma_0 \in \{1, 2\}$ and (c) Weibull with $\mu_0 = 10$ and $\sigma_0 \in \{1, 2, 5\}$.

- The Frank³³ Copula defined as

$$C(u, v|\theta) = -\frac{1}{\theta} \ln \left(1 + \frac{(e^{-\theta u} - 1)(e^{-\theta v} - 1)}{e^{-\theta} - 1} \right), \quad (26)$$

where $\theta \in \mathbb{R} \setminus \{0\}$. This Copula is symmetric and can be used to model dependence structures with either positive or negative correlation. For the Frank Copula, it can be proven that the Kendall's rank correlation coefficient τ is related to the dependence parameter θ through the following equation

$$\tau = 1 + \frac{4(D_1(\theta) - 1)}{\theta}, \quad (27)$$

where $D_1(\theta)$ is the Debye function of the first kind defined as

$$D_1(\theta) = \frac{1}{\theta} \int_0^\theta \frac{t}{e^t - 1} dt. \quad (28)$$

By solving (27) for θ , we can obtain the value of θ in the function of τ .

- The Clayton³⁴ Copula defined as

$$C(u, v|\theta) = \max(0, u^{-\theta} + v^{-\theta} - 1)^{-\frac{1}{\theta}}, \quad (29)$$

where $\theta \in [-1, \infty) \setminus \{0\}$. This is an asymmetric Copula exhibiting a larger dependence in the negative tail than in the positive one. For the Clayton Copula, it can be proved that the Kendall's rank correlation coefficient τ is related to the dependence parameter θ through the following equation

$$\tau = \frac{\theta}{\theta + 2} \Leftrightarrow \theta = \frac{2\tau}{1 - \tau}. \quad (30)$$

- The Gumbel³⁵ (a.k.a. Gumbel-Hougaard) Copula is defined as

$$C(u, v|\theta) = \exp \left(- \left((-\ln(u))^\theta + (-\ln(v))^\theta \right)^{\frac{1}{\theta}} \right), \quad (31)$$

where $\theta \in [1, \infty)$. This is also an asymmetric Copula exhibiting a larger dependence in the positive tail than in the negative one. For the Gumbel Copula, it can be proved that the Kendall's rank correlation coefficient τ is related to the dependence parameter θ through the following equation

$$\tau = 1 - \frac{1}{\theta} \Leftrightarrow \theta = \frac{1}{1 - \tau}. \quad (32)$$

In order to facilitate the use of the Frank, Clayton and Gumbel Copulas, Table 2 simply provides pre-computed values of θ for several selected values of the Kendall's rank correlation coefficient $\tau \in \{0, 0.1, 0.2, \dots, 0.9\}$.

The upper control limits UCL_{Z_1} , UCL_{Z_2} and UCL_{Z_3} of the 3 TBEA charts based on the statistic $Z \in \{Z_1, Z_2, Z_3\}$ have been computed (and can be found in Tables 3, 4 and 5 for the Frank, Clayton and Gumbel Copulas, respectively) for

Table 2: Pre-computed values of θ for several selected values of $\tau \in \{0, 0.1, 0.2, \dots, 0.9\}$

τ	θ		
	Frank	Clayton	Gumbel
0.0	0.00	0.00	1.00
0.1	0.91	0.22	1.11
0.2	1.86	0.50	1.25
0.3	2.92	0.86	1.43
0.4	4.16	1.33	1.67
0.5	5.74	2.00	2.00
0.6	7.93	3.00	2.50
0.7	11.41	4.67	3.33
0.8	18.19	8.00	5.00
0.9	38.28	18.00	10.00

the scenarios described in Table 1 for T and X , and for 3 levels of the Kendall's coefficient $\tau \in \{0.2, 0.5, 0.8\}$ (corresponding to the dependence parameter θ in Table 2). For comparison purpose, the in-control value of ATS is the same as in Rahali et al.²⁹, i.e. $ATS_0 = 370$. For instance, in Table 3, if T follows a Gamma distribution with $\sigma_0 = 5$ and X follows a Normal distribution with $\sigma_0 = 2$ (remember that $\mu_0 = 10$ for all the scenarios in Table 1) then $UCL_{Z_1} = 0.748$, 0.592 and 0.438 when $\tau = 0.2, 0.5$ and 0.8 , respectively. From Tables 3–5, we can draw the following conclusions:

- When the statistic $Z \in \{Z_1, Z_2, Z_3\}$, the parameters in Table 1 and the type of Copula are fixed, the larger τ (i.e. the dependence between T and X), the smaller the control limit UCL_Z . This remark remains valid when a comparison is performed with Table 2 in Rahali et al.²⁹ where the random variables T and X were assumed independent (which corresponds to the case $\tau = 0$).
- With the parameters in Table 1, the value of τ and the type of Copula are fixed, the upper control limits of the statistic $Z \in \{Z_1, Z_2, Z_3\}$ always satisfy $UCL_{Z_1} < UCL_{Z_2} < UCL_{Z_3}$.
- With the parameters in Table 1, the value of τ and statistic $Z \in \{Z_1, Z_2, Z_3\}$ are fixed, the upper control limits are more or less the same no matter the type of Copula considered.

Since the upper control limit in (15) depends on the choice of the distribution for T and X and also on the values for μ_{T_0} and μ_{X_0} , it is important to evaluate the impact of the number m of Phase I data used for estimating them and to provide some guidelines concerning the minimum acceptable value for m . Table 6 gives the 95% confidence intervals $[UCL_{Z,m}^{\text{inf}}, UCL_{Z,m}^{\text{sup}}]$ for UCL_Z (obtained using Monte Carlo simulations) and their relative differences

$$\Delta = \frac{UCL_{Z,m}^{\text{sup}} - UCL_{Z,m}^{\text{inf}}}{UCL_{Z,\infty}},$$

as a function of $m \in \{20, 50, 100, 200, 500, 1000, 2000, \infty\}$, for the statistic $Z \in \{Z_1, Z_2, Z_3\}$, for some combinations of the Gamma, Normal and Weibull distributions and for the dependence parameter $\tau \in \{0.2, 0.8\}$ (Frank's Copula only). The smaller the value of Δ , the more "accurate" is the estimation of the upper control limit UCL_Z . The values of $\text{UCL}_{Z,\infty}$ have already been computed in Table 3. As for the previous tables, we assume that $\mu_{T_0} = \mu_{X_0} = 10$. The following conclusions can be drawn from Table 6:

- when m is small, the relative difference Δ can be very large. For instance, for the statistic Z_2 , for the case of Weibull ($\sigma_{T_0} = 5$) and Normal ($\sigma_{X_0} = 1$), we have $\Delta = 4.138$ when $m = 20$ and $\tau = 0.8$. As expected, when m becomes large, the relative difference Δ converge to 0 (i.e. $\text{UCL}_{Z,m}^{\text{inf}}$ and $\text{UCL}_{Z,m}^{\text{sup}}$ both converge to $\text{UCL}_{Z,\infty}$).
- The smaller the value of σ_{T_0} for T , the better. Taking the same example (i.e. statistic Z_2 , $m = 20$ and $\tau = 0.8$) but for the case of Gamma ($\sigma_{T_0} = 1$) and Normal ($\sigma_{X_0} = 2$), we have $\Delta = 0.251$ (to be compared with $\Delta = 4.138$).
- When σ_{T_0} is small, a better compromise is to choose the statistic Z_3 which yields smaller values for Δ , while when σ_{T_0} is large a better compromise is to choose the statistic Z_1 .
- The value of the dependence parameter τ seems to have a medium impact on Δ (the difference between the cases $\tau = 0.2$ and $\tau = 0.8$ is not that large). In most cases, if we want to estimate the upper control limit UCL_Z accurately (say $\Delta \leq 0.05$), it seems that m should be very large and, at least larger than 1000.

When an upward shift occurs for Z , three possible situations are likely to occur:

- an upward shift in the amplitude X from μ_{X_0} to $\mu_{X_1} = \delta_X \mu_{X_0}$ where $\delta_X \geq 1$ is the parameter quantifying the change in the amplitude,
- a downward shift in the time T from μ_{T_0} to $\mu_{T_1} = \delta_T \mu_{T_0}$ where $\delta_T \leq 1$ is the parameter quantifying the change in the time,
- or also a change in both the amplitude X from μ_{X_0} to $\mu_{X_1} = \delta_X \mu_{X_0}$ and in the time T from μ_{T_0} to $\mu_{T_1} = \delta_T \mu_{T_0}$.

As in Rahali et al.²⁹, we suggest to evaluate the performance of the proposed TBEA charts using

- the Expected Average Time to Signal, EATS_X for X (assuming $\delta_T = 1$) defined as

$$\text{EATS}_X = \sum_{\delta_X \in \Omega_X} f_{\delta_X}(\delta_X) \text{ATS}(\delta_X, 1), \quad (33)$$

- the Expected Average Time to Signal, EATS_T for T (assuming $\delta_X = 1$) defined as

$$\text{EATS}_T = \sum_{\delta_T \in \Omega_T} f_{\delta_T}(\delta_T) \text{ATS}(1, \delta_T), \quad (34)$$

Table 3: Upper control limits UCL_Z of the 3 TBFA charts based on the statistic $Z \in \{Z_1, Z_2, Z_3\}$ for the scenarios described in Table 1, for T and X , and for $\tau \in \{0.2, 0.5, 0.8\}$ assuming a Frank Copula

T	$X \rightarrow$	Statistic Z_1															
		Gamma					Normal					Weibull					
\downarrow	σ_0	1	2	5	1	2	5	1	2	5	1	2	5	1	2	5	
Gamma	1	(0.233, 0.165, 0.083)	(0.413, 0.331, 0.261)	(1.124, 1.047, 0.996)	(0.229, 0.160, 0.080)	(0.385, 0.306, 0.230)	(1.052, 0.975, 0.914)	(0.218, 0.153, 0.072)	(0.353, 0.276, 0.201)	(1.045, 0.890, 0.744)	(0.354, 0.281, 0.195)	(0.440, 0.304, 0.142)	(1.045, 0.890, 0.744)	(0.706, 0.635, 0.553)	(0.749, 0.597, 0.431)	(1.176, 0.828, 0.401)	
	2	(0.356, 0.276, 0.204)	(0.468, 0.329, 0.164)	(1.122, 0.980, 0.819)	(0.354, 0.277, 0.201)	(0.458, 0.319, 0.152)	(1.052, 0.975, 0.914)	(0.218, 0.153, 0.072)	(0.353, 0.276, 0.201)	(1.045, 0.890, 0.744)	(0.354, 0.281, 0.195)	(0.440, 0.304, 0.142)	(1.045, 0.890, 0.744)	(0.706, 0.635, 0.553)	(0.749, 0.597, 0.431)	(1.176, 0.828, 0.401)	
	5	(0.702, 0.631, 0.566)	(0.754, 0.587, 0.446)	(1.216, 0.873, 0.431)	(0.705, 0.630, 0.563)	(0.748, 0.592, 0.438)	(1.176, 0.828, 0.401)	(0.706, 0.635, 0.553)	(0.749, 0.597, 0.431)	(1.176, 0.828, 0.401)	(0.706, 0.635, 0.553)	(0.749, 0.597, 0.431)	(1.176, 0.828, 0.401)	(0.706, 0.635, 0.553)	(0.749, 0.597, 0.431)	(1.176, 0.828, 0.401)	
Weibull	1	(0.250, 0.179, 0.111)	(0.412, 0.338, 0.283)	(1.128, 1.065, 1.023)	(0.249, 0.176, 0.102)	(0.387, 0.307, 0.248)	(1.053, 0.983, 0.939)	(0.238, 0.168, 0.084)	(0.359, 0.278, 0.214)	(1.044, 0.895, 0.788)	(0.411, 0.335, 0.265)	(0.483, 0.338, 0.184)	(1.044, 0.895, 0.788)	(0.408, 0.335, 0.254)	(0.466, 0.328, 0.163)	(1.044, 0.895, 0.788)	
	2	(0.412, 0.337, 0.270)	(0.501, 0.352, 0.208)	(1.128, 0.969, 0.868)	(0.411, 0.335, 0.265)	(0.483, 0.338, 0.184)	(1.044, 0.895, 0.788)	(0.408, 0.335, 0.254)	(0.466, 0.328, 0.163)	(1.044, 0.895, 0.788)	(0.411, 0.335, 0.265)	(0.483, 0.338, 0.184)	(1.044, 0.895, 0.788)	(0.408, 0.335, 0.254)	(0.466, 0.328, 0.163)	(1.044, 0.895, 0.788)	
	5	(0.772, 0.702, 0.643)	(0.803, 0.646, 0.515)	(1.226, 0.853, 0.427)	(0.775, 0.702, 0.641)	(0.806, 0.647, 0.503)	(1.185, 0.827, 0.397)	(0.777, 0.700, 0.626)	(0.806, 0.650, 0.490)	(1.185, 0.827, 0.397)	(0.777, 0.700, 0.626)	(0.806, 0.650, 0.490)	(1.185, 0.827, 0.397)	(0.777, 0.700, 0.626)	(0.806, 0.650, 0.490)	(1.185, 0.827, 0.397)	

T	$X \rightarrow$	Statistic Z_2															
		Gamma					Normal					Weibull					
\downarrow	σ_0	1	2	5	1	2	5	1	2	5	1	2	5	1	2	5	
Gamma	1	(1.264, 1.176, 1.086)	(1.429, 1.323, 1.230)	(2.123, 1.976, 1.855)	(1.263, 1.176, 1.083)	(1.406, 1.302, 1.206)	(2.045, 1.905, 1.794)	(1.254, 1.172, 1.078)	(1.382, 1.274, 1.179)	(2.045, 1.905, 1.794)	(1.522, 1.404, 1.303)	(1.606, 1.393, 1.183)	(2.142, 1.830, 1.583)	(1.517, 1.409, 1.280)	(1.587, 1.388, 1.161)	(2.142, 1.830, 1.583)	
	2	(1.522, 1.404, 1.303)	(1.606, 1.393, 1.183)	(2.191, 1.867, 1.645)	(1.518, 1.405, 1.301)	(1.595, 1.384, 1.168)	(2.142, 1.830, 1.583)	(1.517, 1.409, 1.280)	(1.587, 1.388, 1.161)	(2.142, 1.830, 1.583)	(1.522, 1.404, 1.303)	(1.606, 1.393, 1.183)	(2.142, 1.830, 1.583)	(1.517, 1.409, 1.280)	(1.587, 1.388, 1.161)	(2.142, 1.830, 1.583)	
	5	(3.403, 3.200, 3.006)	(3.403, 2.883, 2.539)	(3.503, 2.427, 1.556)	(3.451, 3.241, 2.990)	(3.354, 2.870, 2.448)	(3.538, 2.409, 1.493)	(3.538, 2.409, 1.493)	(3.454, 3.197, 2.925)	(3.369, 2.892, 2.389)	(3.538, 2.409, 1.493)	(3.403, 3.200, 3.006)	(3.403, 2.883, 2.539)	(3.451, 3.241, 2.990)	(3.354, 2.870, 2.448)	(3.538, 2.409, 1.493)	
Weibull	1	(1.299, 1.207, 1.123)	(1.446, 1.334, 1.254)	(2.121, 1.989, 1.908)	(1.299, 1.205, 1.115)	(1.422, 1.309, 1.222)	(2.051, 1.913, 1.828)	(1.293, 1.201, 1.098)	(1.395, 1.279, 1.194)	(2.051, 1.913, 1.828)	(1.673, 1.566, 1.463)	(1.714, 2.148, 1.266)	(2.185, 1.826, 1.618)	(1.678, 1.558, 1.429)	(1.703, 1.472, 1.217)	(2.185, 1.826, 1.618)	
	2	(1.673, 1.566, 1.463)	(1.714, 2.148, 1.266)	(2.219, 1.885, 1.683)	(1.676, 1.565, 1.453)	(1.716, 1.472, 1.243)	(2.185, 1.826, 1.618)	(1.678, 1.558, 1.429)	(1.703, 1.472, 1.217)	(2.185, 1.826, 1.618)	(1.673, 1.566, 1.463)	(1.714, 2.148, 1.266)	(2.185, 1.826, 1.618)	(1.678, 1.558, 1.429)	(1.703, 1.472, 1.217)	(2.185, 1.826, 1.618)	
	5	(4.722, 4.427, 4.099)	(4.539, 4.539, 3.490)	(4.348, 2.980, 1.900)	(4.695, 4.407, 4.111)	(4.545, 3.922, 3.319)	(4.443, 2.878, 1.721)	(4.443, 2.878, 1.721)	(4.660, 4.333, 4.024)	(4.512, 3.821, 3.218)	(4.443, 2.878, 1.721)	(4.722, 4.427, 4.099)	(4.539, 4.539, 3.490)	(4.695, 4.407, 4.111)	(4.545, 3.922, 3.319)	(4.443, 2.878, 1.721)	

T	$X \rightarrow$	Statistic Z_3															
		Gamma					Normal					Weibull					
\downarrow	σ_0	1	2	5	1	2	5	1	2	5	1	2	5	1	2	5	
Gamma	1	(2.260, 2.185, 2.110)	(2.426, 2.342, 2.283)	(3.145, 3.065, 3.026)	(2.254, 2.182, 2.105)	(2.400, 2.316, 2.247)	(3.058, 2.989, 2.938)	(2.244, 2.175, 2.091)	(2.370, 2.286, 2.216)	(3.058, 2.989, 2.938)	(2.514, 2.442, 2.378)	(2.577, 2.436, 2.298)	(3.098, 2.930, 2.831)	(2.520, 2.438, 2.364)	(2.558, 2.409, 2.245)	(3.098, 2.930, 2.831)	
	2	(2.514, 2.442, 2.378)	(2.577, 2.436, 2.298)	(3.173, 3.022, 2.921)	(2.520, 2.447, 2.377)	(2.566, 2.417, 2.269)	(3.098, 2.930, 2.831)	(2.520, 2.438, 2.364)	(2.558, 2.409, 2.245)	(3.098, 2.930, 2.831)	(2.514, 2.442, 2.378)	(2.577, 2.436, 2.298)	(3.098, 2.930, 2.831)	(2.520, 2.438, 2.364)	(2.558, 2.409, 2.245)	(3.098, 2.930, 2.831)	
	5	(4.538, 4.472, 4.412)	(4.489, 4.361, 4.292)	(4.504, 4.191, 3.940)	(4.553, 4.464, 4.414)	(4.494, 4.367, 4.285)	(4.478, 4.141, 3.929)	(4.478, 4.141, 3.929)	(4.589, 4.442, 4.389)	(4.559, 4.370, 4.225)	(4.478, 4.141, 3.929)	(4.538, 4.472, 4.412)	(4.489, 4.361, 4.292)	(4.553, 4.464, 4.414)	(4.494, 4.367, 4.285)	(4.478, 4.141, 3.929)	
Weibull	1	(2.296, 2.226, 2.162)	(2.442, 2.363, 2.304)	(3.129, 3.073, 3.044)	(2.295, 2.218, 2.151)	(2.412, 2.333, 2.273)	(3.053, 2.996, 2.950)	(2.287, 2.213, 2.131)	(2.388, 2.297, 2.235)	(3.053, 2.996, 2.950)	(2.688, 2.621, 2.564)	(2.715, 2.565, 2.447)	(3.154, 2.976, 2.883)	(2.690, 2.629, 2.548)	(2.707, 2.549, 2.396)	(3.154, 2.976, 2.883)	
	2	(2.688, 2.621, 2.564)	(2.715, 2.565, 2.447)	(3.233, 3.061, 2.964)	(2.693, 2.626, 2.566)	(2.708, 2.563, 2.422)	(3.154, 2.976, 2.883)	(2.690, 2.629, 2.548)	(2.707, 2.549, 2.396)	(3.154, 2.976, 2.883)	(2.688, 2.621, 2.564)	(2.715, 2.565, 2.447)	(3.154, 2.976, 2.883)	(2.690, 2.629, 2.548)	(2.707, 2.549, 2.396)	(3.154, 2.976, 2.883)	
	5	(5.873, 5.762, 5.703)	(5.825, 5.664, 5.608)	(5.748, 5.477, 5.221)	(5.861, 5.873, 5.683)	(5.780, 5.661, 5.588)	(5.661, 5.466, 5.192)	(5.661, 5.466, 5.192)	(5.909, 5.736, 5.744)	(5.860, 5.721, 5.623)	(5.873, 5.762, 5.703)	(5.825, 5.664, 5.608)	(5.748, 5.477, 5.221)	(5.861, 5.873, 5.683)	(5.780, 5.661, 5.588)	(5.661, 5.466, 5.192)	

Table 4: Upper control limits UCL_Z of the 3 TBFA charts based on the statistic $Z \in \{Z_1, Z_2, Z_3\}$ for the scenarios described in Table 1, for T and X , and for $\tau \in \{0.2, 0.5, 0.8\}$ assuming a Clayton Copula

T	$X \rightarrow$	Statistic Z_1								
		Gamma		Normal		Weibull				
Gamma	σ_0	1	(0.230, 0.166, 0.092)	(0.421, 0.361, 0.291)	(1.150, 1.099, 1.029)	(0.225, 0.163, 0.087)	(0.392, 0.332, 0.259)	(0.210, 0.146, 0.072)	(0.359, 0.300, 0.227)	(1.060, 1.003, 0.939)
	1	(0.334, 0.229, 0.168)	(0.469, 0.349, 0.201)	(1.142, 1.030, 0.893)	(0.332, 0.222, 0.159)	(0.452, 0.329, 0.180)	(0.328, 0.220, 0.134)	(0.429, 0.310, 0.162)	(1.064, 0.940, 0.807)	
	5	(0.669, 0.566, 0.539)	(0.729, 0.506, 0.379)	(1.244, 0.983, 0.583)	(0.666, 0.560, 0.530)	(0.720, 0.507, 0.345)	(0.667, 0.537, 0.497)	(0.716, 0.508, 0.329)	(1.192, 0.927, 0.553)	
Weibull	1	(0.237, 0.161, 0.100)	(0.414, 0.356, 0.303)	(1.143, 1.093, 1.038)	(0.232, 0.157, 0.090)	(0.388, 0.322, 0.272)	(0.218, 0.138, 0.066)	(0.355, 0.292, 0.233)	(1.054, 0.998, 0.958)	
	2	(0.377, 0.278, 0.237)	(0.479, 0.336, 0.198)	(1.129, 1.025, 0.913)	(0.378, 0.271, 0.229)	(0.460, 0.313, 0.174)	(0.371, 0.253, 0.198)	(0.446, 0.296, 0.149)	(1.058, 0.944, 0.834)	
	5	(0.735, 0.637, 0.614)	(0.770, 0.544, 0.450)	(1.257, 0.962, 0.569)	(0.731, 0.630, 0.603)	(0.767, 0.539, 0.411)	(0.729, 0.603, 0.569)	(0.765, 0.538, 0.386)	(1.197, 0.901, 0.529)	

T	$X \rightarrow$	Statistic Z_2								
		Gamma		Normal		Weibull				
Gamma	σ_0	1	(1.257, 1.171, 1.088)	(1.435, 1.355, 1.264)	(2.140, 2.026, 1.909)	(1.250, 1.167, 1.082)	(1.411, 1.325, 1.239)	(1.236, 1.153, 1.071)	(1.383, 1.294, 1.208)	(2.055, 1.950, 1.836)
	1	(1.471, 1.320, 1.256)	(1.569, 1.370, 1.177)	(2.217, 1.959, 1.726)	(1.465, 1.308, 1.243)	(1.557, 1.351, 1.166)	(1.459, 1.287, 1.197)	(1.548, 1.338, 1.153)	(2.162, 1.900, 1.665)	
	5	(3.269, 2.980, 2.935)	(3.123, 2.507, 2.345)	(3.280, 2.147, 1.460)	(3.306, 2.942, 2.894)	(3.083, 2.416, 2.214)	(3.245, 2.883, 2.799)	(3.057, 2.318, 2.112)	(3.194, 2.154, 1.457)	
Weibull	1	(1.274, 1.170, 1.103)	(1.439, 1.344, 1.275)	(2.135, 2.022, 1.940)	(1.271, 1.162, 1.091)	(1.411, 1.316, 1.246)	(1.260, 1.145, 1.062)	(1.385, 1.285, 1.214)	(2.057, 1.957, 1.854)	
	2	(1.609, 1.454, 1.411)	(1.638, 1.380, 1.211)	(2.225, 1.942, 1.737)	(1.608, 1.441, 1.391)	(1.629, 1.347, 1.156)	(1.594, 1.411, 1.339)	(1.613, 1.331, 1.136)	(2.159, 1.880, 1.687)	
	5	(4.444, 4.134, 4.004)	(4.223, 3.370, 3.237)	(3.804, 2.298, 1.582)	(4.494, 4.097, 3.900)	(4.101, 3.249, 3.033)	(4.442, 3.965, 3.767)	(4.016, 3.116, 2.900)	(3.768, 2.212, 1.422)	

T	$X \rightarrow$	Statistic Z_3								
		Gamma		Normal		Weibull				
Gamma	σ_0	1	(2.247, 2.175, 2.102)	(2.428, 2.370, 2.308)	(3.143, 3.101, 3.052)	(2.241, 2.167, 2.094)	(2.398, 2.339, 2.274)	(2.227, 2.151, 2.076)	(2.369, 2.305, 2.238)	(3.068, 3.012, 2.958)
	1	(2.484, 2.382, 2.351)	(2.550, 2.392, 2.275)	(3.175, 3.058, 2.970)	(2.478, 2.373, 2.343)	(2.530, 2.364, 2.233)	(2.472, 2.357, 2.311)	(2.514, 2.342, 2.197)	(3.096, 2.973, 2.873)	
	5	(4.547, 4.414, 4.379)	(4.453, 4.253, 4.285)	(4.364, 3.975, 3.899)	(4.472, 4.386, 4.395)	(4.422, 4.241, 4.230)	(4.475, 4.371, 4.358)	(4.409, 4.196, 4.143)	(4.359, 3.910, 3.831)	
Weibull	1	(2.273, 2.191, 2.144)	(2.428, 2.365, 2.319)	(3.141, 3.103, 3.055)	(2.271, 2.184, 2.130)	(2.405, 2.331, 2.282)	(2.255, 2.158, 2.093)	(2.374, 2.298, 2.246)	(3.070, 2.998, 2.977)	
	2	(2.657, 2.558, 2.541)	(2.669, 2.490, 2.419)	(3.214, 3.082, 2.978)	(2.654, 2.555, 2.527)	(2.650, 2.456, 2.366)	(2.645, 2.536, 2.501)	(2.634, 2.426, 2.326)	(3.132, 2.985, 2.911)	
	5	(5.808, 5.682, 5.695)	(5.757, 5.522, 5.671)	(5.634, 5.223, 5.248)	(5.798, 5.670, 5.813)	(5.701, 5.528, 5.477)	(5.892, 5.721, 5.609)	(5.750, 5.527, 5.532)	(5.658, 5.243, 5.096)	

Table 5: Upper control limits UCL_Z of the 3 TBFA charts based on the statistic $Z \in \{Z_1, Z_2, Z_3\}$ for the scenarios described in Table 1, for T and X , and for $\tau \in \{0.2, 0.5, 0.8\}$ assuming a Gumbel Copula

		Statistic Z_1																							
T	$X \rightarrow$	Gamma					Normal					Weibull													
		1	2	5	1	2	1	2	5	1	2	5	1	2	5										
Gamma	σ_0	(0.228, 0.151, 0.063)	(0.384, 0.285, 0.225)	(1.092, 1.011, 0.977)	(0.225, 0.151, 0.064)	(0.361, 0.261, 0.193)	(0.218, 0.153, 0.072)	(0.339, 0.238, 0.159)	(1.009, 0.922, 0.875)	(0.358, 0.276, 0.196)	(0.449, 0.295, 0.124)	(1.054, 0.861, 0.754)	(0.356, 0.276, 0.191)	(0.444, 0.298, 0.128)	(0.357, 0.283, 0.184)	(0.435, 0.302, 0.136)	(0.982, 0.792, 0.674)	(0.704, 0.633, 0.560)	(0.749, 0.593, 0.431)	(1.122, 0.712, 0.283)	(0.707, 0.631, 0.552)	(0.756, 0.596, 0.422)	(0.710, 0.631, 0.534)	(0.757, 0.602, 0.419)	(1.098, 0.714, 0.303)
	1	(0.247, 0.167, 0.085)	(0.396, 0.305, 0.265)	(1.101, 1.030, 1.008)	(0.245, 0.165, 0.079)	(0.366, 0.282, 0.229)	(0.240, 0.168, 0.074)	(0.344, 0.251, 0.193)	(1.029, 0.944, 0.919)	(0.410, 0.336, 0.261)	(0.477, 0.317, 0.157)	(1.063, 0.892, 0.828)	(0.414, 0.335, 0.255)	(0.472, 0.316, 0.140)	(0.415, 0.334, 0.239)	(0.467, 0.324, 0.143)	(0.997, 0.821, 0.738)	(0.778, 0.699, 0.631)	(0.803, 0.649, 0.497)	(1.150, 0.719, 0.301)	(0.774, 0.699, 0.624)	(0.804, 0.651, 0.482)	(0.783, 0.703, 0.604)	(0.811, 0.659, 0.476)	(1.127, 0.729, 0.300)
	5	(1.257, 1.169, 1.069)	(1.402, 1.273, 1.191)	(2.049, 1.893, 1.805)	(1.257, 1.171, 1.071)	(1.386, 1.253, 1.166)	(1.253, 1.174, 1.078)	(1.369, 1.245, 1.139)	(1.986, 1.816, 1.738)	(1.523, 1.405, 1.294)	(1.590, 1.375, 1.147)	(2.101, 1.727, 1.545)	(1.524, 1.403, 1.283)	(1.588, 1.377, 1.153)	(1.516, 1.406, 1.266)	(1.589, 1.391, 1.161)	(2.056, 1.684, 1.491)	(3.455, 3.194, 3.020)	(3.402, 2.894, 2.470)	(3.539, 2.411, 1.479)	(3.441, 3.190, 2.968)	(3.369, 2.863, 2.363)	(3.488, 3.169, 2.881)	(3.435, 2.876, 2.292)	(3.507, 2.434, 1.452)
Weibull	1	(1.298, 1.198, 1.099)	(1.420, 1.295, 1.230)	(2.074, 1.923, 1.871)	(1.297, 1.201, 1.093)	(1.402, 1.269, 1.201)	(1.293, 1.201, 1.088)	(1.390, 1.249, 1.170)	(2.012, 1.861, 1.793)	(1.683, 1.554, 1.442)	(1.715, 1.470, 1.224)	(2.165, 1.776, 1.620)	(1.682, 1.556, 1.430)	(1.719, 1.467, 1.203)	(1.688, 1.549, 1.395)	(1.720, 1.481, 1.194)	(2.128, 1.720, 1.556)	(4.778, 4.353, 4.081)	(4.594, 3.984, 3.385)	(4.442, 2.979, 1.782)	(4.767, 4.399, 4.041)	(4.595, 3.851, 3.185)	(4.736, 4.364, 3.922)	(4.572, 3.855, 3.122)	(4.516, 2.915, 1.626)

		Statistic Z_2																							
T	$X \rightarrow$	Gamma					Normal					Weibull													
		1	2	5	1	2	1	2	5	1	2	5	1	2	5										
Gamma	σ_0	(2.253, 2.174, 2.086)	(2.403, 2.306, 2.255)	(3.103, 3.772, 3.720)	(2.250, 2.172, 2.085)	(2.378, 2.280, 2.225)	(2.245, 2.176, 2.085)	(2.354, 2.256, 2.188)	(3.032, 2.949, 2.914)	(2.516, 2.447, 2.372)	(2.567, 2.409, 2.252)	(3.125, 2.944, 2.870)	(2.520, 2.442, 2.364)	(2.562, 2.405, 2.232)	(2.522, 2.440, 2.348)	(2.563, 2.408, 2.220)	(3.052, 2.863, 2.790)	(4.550, 4.480, 4.396)	(4.484, 4.410, 4.262)	(4.508, 4.144, 3.946)	(4.581, 4.505, 4.418)	(4.515, 4.381, 4.253)	(4.592, 4.482, 4.374)	(4.471, 4.152, 3.849)	
	1	(2.297, 2.215, 2.139)	(2.425, 2.332, 2.286)	(3.116, 3.053, 3.034)	(2.290, 2.214, 2.130)	(2.400, 2.304, 2.252)	(2.292, 2.216, 2.119)	(2.371, 2.274, 2.214)	(3.038, 2.967, 2.948)	(2.696, 2.615, 2.562)	(2.707, 2.565, 2.424)	(3.192, 3.009, 2.929)	(2.696, 2.628, 2.543)	(2.701, 2.547, 2.394)	(2.696, 2.629, 2.536)	(2.704, 2.558, 2.381)	(3.125, 2.922, 2.844)	(5.787, 5.789, 5.752)	(5.817, 5.701, 5.644)	(5.719, 5.521, 5.171)	(5.877, 5.859, 5.760)	(5.820, 5.710, 5.570)	(5.878, 5.734, 5.706)	(5.798, 5.687, 5.468)	
	5	(1.257, 1.169, 1.069)	(1.402, 1.273, 1.191)	(2.049, 1.893, 1.805)	(1.257, 1.171, 1.071)	(1.386, 1.253, 1.166)	(1.253, 1.174, 1.078)	(1.369, 1.245, 1.139)	(1.986, 1.816, 1.738)	(1.523, 1.405, 1.294)	(1.590, 1.375, 1.147)	(2.101, 1.727, 1.545)	(1.524, 1.403, 1.283)	(1.588, 1.377, 1.153)	(1.516, 1.406, 1.266)	(1.589, 1.391, 1.161)	(2.056, 1.684, 1.491)	(3.455, 3.194, 3.020)	(3.402, 2.894, 2.470)	(3.539, 2.411, 1.479)	(3.441, 3.190, 2.968)	(3.369, 2.863, 2.363)	(3.488, 3.169, 2.881)	(3.435, 2.876, 2.292)	(3.507, 2.434, 1.452)
Weibull	1	(1.298, 1.198, 1.099)	(1.420, 1.295, 1.230)	(2.074, 1.923, 1.871)	(1.297, 1.201, 1.093)	(1.402, 1.269, 1.201)	(1.293, 1.201, 1.088)	(1.390, 1.249, 1.170)	(2.012, 1.861, 1.793)	(1.683, 1.554, 1.442)	(1.715, 1.470, 1.224)	(2.165, 1.776, 1.620)	(1.682, 1.556, 1.430)	(1.719, 1.467, 1.203)	(1.688, 1.549, 1.395)	(1.720, 1.481, 1.194)	(2.128, 1.720, 1.556)	(4.778, 4.353, 4.081)	(4.594, 3.984, 3.385)	(4.442, 2.979, 1.782)	(4.767, 4.399, 4.041)	(4.595, 3.851, 3.185)	(4.736, 4.364, 3.922)	(4.572, 3.855, 3.122)	(4.516, 2.915, 1.626)

Table 6: 95% confidence intervals for UCL_Z and their relative differences Δ as a function of m , for the statistic $Z \in \{Z_1, Z_2, Z_3\}$, for some combinations of the Gamma, Normal and Weibull distributions and for the dependence parameter $\tau \in \{0.2, 0.8\}$ (Frank's Copula only)

		Statistic Z_1			
$T \rightarrow$		Gamma ($\sigma_{T_0} = 1$)		Weibull ($\sigma_{T_0} = 5$)	
$X \rightarrow$		Normal ($\sigma_{X_0} = 2$)		Normal ($\sigma_{X_0} = 1$)	
m		$\tau = 0.2$	$\tau = 0.8$	$\tau = 0.2$	$\tau = 0.8$
20		[0.170, 0.606], 1.132	[0.102, 0.438], 1.460	[0.468, 1.009], 0.698	[0.396, 0.818], 0.658
50		[0.280, 0.580], 0.779	[0.149, 0.417], 1.165	[0.622, 0.993], 0.478	[0.521, 0.784], 0.410
100		[0.300, 0.527], 0.589	[0.173, 0.351], 0.773	[0.658, 0.908], 0.322	[0.551, 0.752], 0.313
200		[0.320, 0.470], 0.389	[0.180, 0.306], 0.547	[0.695, 0.866], 0.220	[0.566, 0.719], 0.238
500		[0.337, 0.434], 0.251	[0.196, 0.275], 0.343	[0.718, 0.836], 0.152	[0.595, 0.685], 0.140
1000		[0.350, 0.418], 0.176	[0.206, 0.261], 0.239	[0.735, 0.814], 0.101	[0.606, 0.671], 0.101
2000		[0.357, 0.409], 0.135	[0.212, 0.252], 0.173	[0.743, 0.803], 0.077	[0.614, 0.664], 0.078
∞		[0.385, 0.385], 0.000	[0.230, 0.230], 0.000	[0.775, 0.775], 0.000	[0.641, 0.641], 0.000

		Statistic Z_2			
$T \rightarrow$		Gamma ($\sigma_{T_0} = 1$)		Weibull ($\sigma_{T_0} = 5$)	
$X \rightarrow$		Normal ($\sigma_{X_0} = 2$)		Normal ($\sigma_{X_0} = 1$)	
m		$\tau = 0.2$	$\tau = 0.8$	$\tau = 0.2$	$\tau = 0.8$
20		[1.170, 1.714], 0.386	[1.084, 1.387], 0.251	[2.067, 21.097], 4.053	[1.792, 18.804], 4.138
50		[1.287, 1.694], 0.289	[1.134, 1.379], 0.203	[2.748, 20.836], 3.852	[2.551, 18.738], 3.937
100		[1.313, 1.573], 0.184	[1.153, 1.305], 0.126	[3.094, 10.941], 1.671	[2.731, 8.690], 1.449
200		[1.332, 1.513], 0.128	[1.164, 1.267], 0.085	[3.494, 7.435], 0.839	[3.008, 6.659], 0.888
500		[1.358, 1.468], 0.078	[1.173, 1.243], 0.058	[3.757, 6.166], 0.513	[3.326, 5.504], 0.529
1000		[1.370, 1.452], 0.058	[1.183, 1.231], 0.039	[4.017, 5.737], 0.366	[3.526, 5.008], 0.360
2000		[1.380, 1.438], 0.041	[1.188, 1.222], 0.028	[4.196, 5.372], 0.250	[3.638, 4.637], 0.243
∞		[1.406, 1.406], 0.000	[1.206, 1.206], 0.000	[4.695, 4.695], 0.000	[4.111, 4.111], 0.000

		Statistic Z_3			
$T \rightarrow$		Gamma ($\sigma_{T_0} = 1$)		Weibull ($\sigma_{T_0} = 5$)	
$X \rightarrow$		Normal ($\sigma_{X_0} = 2$)		Normal ($\sigma_{X_0} = 1$)	
m		$\tau = 0.2$	$\tau = 0.8$	$\tau = 0.2$	$\tau = 0.8$
20		[2.199, 2.648], 0.187	[2.110, 2.453], 0.152	[2.927, 24.828], 3.736	[2.936, 24.298], 3.758
50		[2.284, 2.605], 0.133	[2.168, 2.444], 0.122	[4.027, 23.506], 3.323	[3.697, 24.143], 3.597
100		[2.312, 2.538], 0.094	[2.184, 2.360], 0.078	[4.162, 12.156], 1.363	[4.118, 12.259], 1.432
200		[2.331, 2.483], 0.063	[2.198, 2.327], 0.057	[4.542, 9.032], 0.766	[4.362, 8.795], 0.780
500		[2.354, 2.450], 0.040	[2.214, 2.293], 0.035	[4.876, 7.392], 0.429	[4.821, 7.415], 0.456
1000		[2.362, 2.434], 0.030	[2.224, 2.280], 0.024	[5.070, 6.912], 0.314	[4.969, 6.871], 0.334
2000		[2.374, 2.424], 0.020	[2.230, 2.271], 0.018	[5.289, 6.564], 0.217	[5.223, 6.400], 0.207
∞		[2.400, 2.400], 0.000	[2.247, 2.247], 0.000	[5.861, 5.861], 0.000	[5.683, 5.683], 0.000

- the Expected Average Time to Signal, EATS for both X and T defined as

$$\text{EATS}_{XT} = \sum_{\delta_X \in \Omega_X} \sum_{\delta_T \in \Omega_T} f_{\delta_X}(\delta_X) f_{\delta_T}(\delta_T) \text{ATS}(\delta_X, \delta_T), \quad (35)$$

where Ω_X and Ω_T are the “range of possible shifts” for δ_X and δ_T , respectively, and $f_{\delta_X}(\delta_X)$ and $f_{\delta_T}(\delta_T)$ are the p.m.f. (probability mass functions) of the shifts δ_X and δ_T over Ω_X and Ω_T , respectively. To study the efficiency of the upper-sided TBEA control charts for an increase in the amplitude X and / or a decrease in the time between events T , we will use the same situation discussed in Rahali et al.²⁹ with $\Omega_X = \{1.1, 1.2, \dots, 1.9, 2\}$ and $\Omega_T = \{0.5, 0.55, \dots, 0.9, 0.95\}$.

As the results obtained in Tables 3–5 are quite similar, from now on, we will only present the results for the Frank Copulas. Results for the Clayton and the Gumbel Copulas have also been obtained (and can be requested from the corresponding author) but due to a lack of space, they will not be presented here. Results in Rahali et al.²⁹ showed that, for *independent* T and X , the choice of the statistic (Z_1 , Z_2 or Z_3) to be monitored clearly depends on the kind of expected shift:

- If the shift is due to a change in the amplitude X , the statistic Z_1 is the better choice as it allows us to obtain smaller EATS_X values.
- If the shift is due to a change in the time between events T , the statistic Z_3 is more appropriate as it allows us to obtain smaller EATS_T values.
- If the shift is due to a change in the amplitude X and in the time between events T , the overall best option is the statistic Z_1 as it allows us to obtain smaller EATS_{XT} values.

Now, we would like to investigate if these findings remain valid when a dependence exists between the time between events T and the corresponding amplitude X using the Frank Copula. For the same scenarios as in Table 1 and for the control limits in Table 3, the EATS_{XT} values corresponding to $\tau \in \{0.2, 0.5, 0.8\}$ (i.e. small, medium and strong dependence) have been computed and listed in Tables 7, 8 and 9, respectively (results for EATS_X and EATS_T are not presented here but can also be requested from the corresponding author). From these tables, some interesting findings can be drawn:

- When the statistic $Z \in \{Z_1, Z_2, Z_3\}$ and the parameters in Table 1 are fixed, the larger the τ , the smaller the values of EATS_{XT} .
- For $\tau = 0.2$ (see bold values in Table 7) the statistic Z_1 seems to be the best option as it gives the smallest EATS_{XT} values (in 65% of the cases with an average EATS_{XT} value $\overline{\text{EATS}}_{XT} = 14.46$), followed by the statistic Z_2 (in 35% of the cases with an average EATS_{XT} value $\overline{\text{EATS}}_{XT} = 22.08$).
- For $\tau = 0.5$ (see bold values in Table 8), the statistic Z_1 produces the smallest EATS_{XT} values (in 67% of the cases with an average EATS_{XT} value $\overline{\text{EATS}}_{XT} = 12.35$), followed by the statistic Z_2 (in 33% of the cases with an average EATS_{XT} value $\overline{\text{EATS}}_{XT} = 16.34$).

- For $\tau = 0.8$ (see bold values in Table 9), the most efficient statistic is Z_1 (in 67% of the cases with an average $EATS_{XT}$ value $\overline{EATS}_{XT} = 9.74$), followed by Z_2 (in 33% of the cases with an average $EATS_{XT}$ value $\overline{EATS}_{XT} = 13.16$).

These results clearly show that, irrespective of the level of dependence, for a simultaneous change in the amplitude X and in the time between events T , the overall best option is the statistic Z_1 or, eventually, the statistic Z_2 but the statistic Z_3 cannot be considered as potential efficient monitoring statistic. Whereas, in Rahali et al.²⁹, the most efficient statistic is Z_1 with an average EATS value of $\overline{EATS} = 14.91$, followed by Z_2 with an average EATS value of $\overline{EATS} = 29.15$, and Z_3 with an average EATS value of $\overline{EATS} = 11.74$. This also allows us to conclude that the statistic Z_1 is the best option whether there is a dependence between T and X or not.

In order to clarify how much the dependence between T and X has a negative impact on the efficiency of the TBEA charts, we have also recomputed the \overline{EATS}_{XT} in the case of a Frank Copula with $\tau = 0.5$ using the control limits obtained in Rahali et al.²⁹ (instead of the ones provided in this paper), i.e. by assuming that T and X are actually independent random variables. The results are presented in Table 10 and, with a simple comparison with the results obtained in Table 8, it can be seen that neglecting the dependence between T and X clearly increases the \overline{EATS}_{XT} values (by about 60% for Z_1 and by about 200% for Z_2 , on the average). Similar results have also been obtained for $\tau = 0.2$ and $\tau = 0.8$ (these results are not presented in this paper but they can be requested from the corresponding author). As expected, the larger τ is, the stronger the negative impact on the TBEA charts. This clearly emphasizes the fact that using specific control limits by taking into account of the dependence between T and X , improves the efficiency of the TBEA charts.

4 Illustrative example

For one of its bottleneck machine, a company recorded (see Table 11) from 08/01/12 to 27/12/18 (format DD/MM/YY) all the breakdown dates (in days) as well as the estimated corresponding incurred costs (X_i , in euros) which include all the repair and restart costs (spare parts, manpower) and the cost of manufacturing disruption. From these dates, it is easy to obtain the times between two consecutive breakdowns (T_i in days). For instance, the machine started to operate on 08/01/12 for the first time, its first breakdown happened on 10/03/12, i.e. $T_1 = 62$ days later, and the costs incurred for repairing and restarting the machine have been estimated as $X_1 = 4890$ euros. The data in Table 11 are divided into

- 30 breakdowns recorded for 5 years, from 2012 to 2016, and used here as a Phase I data set.
- 14 breakdowns recorded for 2 years, from 2017 to 2018, and used here as a Phase II data set.

In the reliability terminology, where the life of a machine is modeled as a “bath-tub” curve, our Phase I must be considered as the *constant failure rate* period

Table 7: $EATS_{XT}$ values for the 3 TBEA charts based on statistics $Z \in \{Z_1, Z_2, Z_3\}$, Frank Copula, $\tau = 0.2$

		Statistic Z_1								
T	$X \rightarrow$	Gamma			Normal		Weibull			
\downarrow	σ_0	1	2	5	1	2	1	2	5	
Gamma	1	7.9	11.3	44.3	8.0	10.0	7.8	9.6	38.8	
	2	9.6	12.8	48.8	9.7	12.2	9.4	11.5	37.3	
	5	16.1	17.3	49.2	15.89	18.4	16.1	17.9	43.3	
Weibull	1	8.2	11.3	50.8	8.1	10.6	8.2	9.8	36.5	
	2	11.4	13.3	47.3	11.3	12.8	11.4	12.4	35.7	
	5	18.3	21.4	43.4	17.8	21.0	18.5	20.1	42.3	

		Statistic Z_2								
T	$X \rightarrow$	Gamma			Normal		Weibull			
\downarrow	σ_0	1	2	5	1	2	1	2	5	
Gamma	1	8.1	10.1	8.0	8.1	9.6	7.9	9.2	25.4	
	2	11.2	13.2	27.3	11.1	12.5	11.1	12.4	26.2	
	5	32.5	31.2	38.8	30.7	34.1	32.6	32.1	38.0	
Weibull	1	8.6	10.3	28.4	8.5	9.9	8.5	9.5	24.2	
	2	15.2	15.5	31.6	16.3	15.6	16.7	15.6	26.9	
	5	43.8	44.6	48.8	45.8	45.9	45.6	46.3	41.9	

		Statistic Z_3								
T	$X \rightarrow$	Gamma			Normal		Weibull			
\downarrow	σ_0	1	2	5	1	2	1	2	5	
Gamma	1	8.0	10.6	40.5	8.0	9.7	7.9	11.1	29.6	
	2	11.5	13.5	37.0	11.3	12.5	12.0	12.2	26.5	
	5	50.2	47.9	49.9	55.9	49.3	52.6	49.3	51.3	
Weibull	1	8.7	10.7	38.8	8.73	10.2	8.5	9.6	32.7	
	2	18.2	17.4	37.7	17.7	17.6	17.9	16.8	29.3	
	5	73.4	84.9	72.4	78.3	72.4	75.3	68.8	72.3	

Table 8: $EATS_{XT}$ values for the 3 TBEA charts based on statistics $Z \in \{Z_1, Z_2, Z_3\}$, Frank Copula, $\tau = 0.5$

		Statistic Z_1								
T	$X \rightarrow$	Gamma			Normal		Weibull			
\downarrow	σ_0	1	2	5	1	2	1	2	5	
Gamma	1	7.4	9.8	43.0	7.4	9.1	7.4	8.5	37.4	
	2	8.4	9.4	43.9	8.5	9.2	8.5	9.0	29.8	
	5	14.1	13.4	31.9	14.1	13.9	14.1	13.8	27.6	
Weibull	1	7.6	10.1	43.3	7.5	9.2	7.6	8.6	34.4	
	2	9.9	10.2	43.3	9.5	9.5	10.1	9.4	29.9	
	5	16.4	15.3	29.1	15.9	15.9	16.8	16.0	29.8	

		Statistic Z_2								
T	$X \rightarrow$	Gamma			Normal		Weibull			
\downarrow	σ_0	1	2	5	1	2	1	2	5	
Gamma	1	7.5	8.7	7.5	7.5	8.4	7.5	8.0	21.7	
	2	9.9	9.8	22.6	9.4	9.3	9.6	9.3	18.3	
	5	30.6	26.2	24.1	29.1	26.5	27.7	26.9	23.9	
Weibull	1	7.7	8.9	26.9	7.7	8.4	7.6	8.2	20.4	
	2	13.7	11.4	24.2	12.5	11.3	13.2	12.3	21.2	
	5	43.5	40.9	32.4	42.5	39.4	40.6	41.5	32.1	

		Statistic Z_3								
T	$X \rightarrow$	Gamma			Normal		Weibull			
\downarrow	σ_0	1	2	5	1	2	1	2	5	
Gamma	1	7.5	9.6	41.0	7.5	8.7	7.4	9.5	31.0	
	2	10.5	10.0	30.8	10.2	9.9	10.5	9.9	27.9	
	5	47.6	47.8	55.5	51.3	42.3	54.9	45.8	43.8	
Weibull	1	7.9	9.8	37.9	7.7	8.9	7.7	8.3	34.6	
	2	15.9	15.0	32.2	16.3	13.9	16.8	14.7	26.2	
	5	72.8	71.9	67.9	70.9	74.0	67.1	77.3	71.7	

Table 9: EATS_{XT} values for the 3 TBEA charts based on statistics $Z \in \{Z_1, Z_2, Z_3\}$, Frank Copula, $\tau = 0.8$

		Statistic Z_1								
T	$X \rightarrow$	Gamma			Normal		Weibull			
\downarrow	σ_0	1	2	5	1	2	1	2	5	
Gamma	1	7.3	8.6	40.7	7.3	8.1	7.3	7.7	31.9	
	2	7.7	7.4	31.3	7.7	7.4	7.6	7.4	26.9	
	5	12.5	10.4	13.5	12.3	10.2	11.9	13.8	11.8	
Weibull	1	7.3	9.2	40.6	7.3	8.3	7.3	7.8	33.5	
	2	8.9	7.9	37.5	8.6	7.6	8.2	7.4	30.1	
	5	14.5	12.4	13.3	14.6	11.4	13.5	11.4	11.8	

		Statistic Z_2								
T	$X \rightarrow$	Gamma			Normal		Weibull			
\downarrow	σ_0	1	2	5	1	2	1	2	5	
Gamma	1	7.3	7.9	7.3	7.3	7.6	7.3	7.5	19.9	
	2	8.4	7.5	17.7	8.2	7.4	8.2	7.4	14.5	
	5	28.9	22.8	11.7	26.6	24.0	25.3	20.1	10.1	
Weibull	1	7.3	8.3	24.5	7.2	7.8	7.3	7.5	21.3	
	2	11.3	8.4	20.6	10.7	7.9	10.2	7.8	16.3	
	5	39.0	34.8	18.0	42.9	32.8	37.1	32.9	14.8	

		Statistic Z_3								
T	$X \rightarrow$	Gamma			Normal		Weibull			
\downarrow	σ_0	1	2	5	1	2	1	2	5	
Gamma	1	7.3	8.4	41.9	7.3	8.0	7.3	9.1	32.6	
	2	9.7	8.6	30.5	9.7	8.0	9.5	7.8	24.5	
	5	56.6	48.0	49.9	53.9	46.7	45.0	47.8	48.7	
Weibull	1	7.4	9.2	41.5	7.3	8.3	7.3	7.8	31.4	
	2	16.1	11.8	35.9	14.6	12.0	14.9	10.9	24.7	
	5	70.8	73.6	75.1	72.7	65.3	66.8	71.6	70.9	

Table 10: $EATS_{XT}$ values for the 3 TBEA charts based on statistics $Z \in \{Z_1, Z_2, Z_3\}$ for a Frank Copula with $\tau = 0.5$ using the control limits obtained in Rahali et al.²⁹, i.e. by assuming that T and X are actually independent random variables

		Statistic Z_1								
T	$X \rightarrow$	Gamma			Normal		Weibull			
\downarrow	σ_0	1	2	5	1	2	1	2	5	
Gamma	1	9.5	16.4	62.3	9.4	14.7	8.7	13.4	51.0	
	2	13.8	29.6	81.7	13.4	29.3	12.6	26.5	74.0	
	5	22.4	60.7	188.2	22.7	54.8	22.4	52.2	197.3	
Weibull	1	10.2	16.9	58.9	10.2	15.4	9.7	14.2	50.8	
	2	16.2	36.6	87.5	16.4	34.7	15.7	30.5	78.7	
	5	26.6	72.2	209.6	26.2	65.1	26.7	59.6	229.8	

		Statistic Z_2								
T	$X \rightarrow$	Gamma			Normal		Weibull			
\downarrow	σ_0	1	2	5	1	2	1	2	5	
Gamma	1	9.8	16.3	51.1	9.4	15.4	8.9	14.3	44.1	
	2	14.5	28.9	98.9	14.4	27.2	14.4	24.8	99.3	
	5	37.9	45.9	108.8	37.7	47.6	38.3	46.3	110.1	
Weibull	1	10.9	18.0	49.7	10.8	17.1	10.4	16.4	45.9	
	2	20.1	33.1	108.5	21.2	32.3	20.7	31.1	124.7	
	5	53.4	61.6	108.5	52.9	61.9	54.4	64.1	112.5	

		Statistic Z_3								
T	$X \rightarrow$	Gamma			Normal		Weibull			
\downarrow	σ_0	1	2	5	1	2	1	2	5	
Gamma	1	9.4	15.3	54.4	9.3	14.1	8.9	13.0	44.9	
	2	15.1	22.6	66.9	14.8	22.8	14.5	21.9	63.5	
	5	53.2	58.2	75.6	54.7	58.2	54.9	59.9	75.0	
Weibull	1	10.7	16.2	51.8	10.7	15.3	10.5	14.6	45.4	
	2	22.5	29.9	70.8	22.5	29.7	22.3	28.6	69.9	
	5	74.6	82.1	88.1	75.0	79.8	78.7	78.4	88.8	

of the machine, while our Phase II period will be considered as the beginning of its potentially *increasing failure rate* period (this will be confirmed later on). The first part of the “bathtub” curve, corresponding to a *decreasing failure rate* period is not presented here, i.e. all the early failures have been fixed and there is no report about this relatively short period.

The data, T_i and X_i in Table 11, have also been plotted in Figure 3 with (○) and (●) corresponding to Phases I and II, respectively. From the bottom-most plot of Figure 3, a slight positive correlation between the time T between consecutive breakdowns and the corresponding incurred costs X can be seen. More precisely, it seems that when the time between consecutive breakdowns T is smaller (larger), the corresponding cost seems to be also smaller (larger). Investigations (during the period 2012 to 2016) about this phenomena have shown that, when a breakdown occurs and is fixed, then if the next breakdown occurs after a *short* period of time, it is often (but not always) due to similar causes and, consequently, i) the time for searching the breakdown causes are reduced and ii) the spare parts costs are also reduced as they have already been purchased for the previous breakdown. On the contrary, when the next breakdown occurs after a *long* period of time, the causes are usually different from the previous breakdown and need i) more time to be searched and ii) new spare parts to be purchased. In order to analytically confirm this fact, the Kendall’s and Spearman’s rank correlation coefficients have been computed, $\tau = 0.4657$ and $\rho = 0.6129$ along with their corresponding p -values 0.00035 and 0.00032, respectively, confirming a positive correlation between T and X . In the case of a Frank Copula, the relationship between θ and τ is given by equations (27) and (28). As a consequence, if τ is known, then θ can be obtained by numerically solving the following equation:

$$\tau - 1 - \frac{4}{\theta} \times \left(\frac{1}{\theta} \int_0^\theta \frac{t}{e^t - 1} dt - 1 \right) = 0.$$

In our case, for $\tau = 0.4657$, we have $\theta = 5.14$. In Table 12, we provide the estimated values for μ_{T_0} , σ_{T_0} , μ_{X_0} and σ_{X_0} , as well as the estimated values for a_0 and b_0 for the distributions considered in Table 1. Based on the Kolmogorov-Smirnov distance D_{KS} (the smaller, the better) the Gamma distribution will be chosen for the time T and the Weibull distribution for the cost X .

Assuming an in-control ATS value, $ATS_0 = 9125$ days (i.e. 25 years), the upper control limits of the 3 TBEA charts based on statistics Z_1 , Z_2 and Z_3 are equal to $UCL_{Z_1} = 0.57$, $UCL_{Z_2} = 2.06$ and $UCL_{Z_3} = 3.18$, respectively. The TBEA charts, corresponding to the statistics Z_1 , Z_2 and Z_3 , are plotted in Figure 4 along with the upper control limits $UCL_{Z_1} = 0.57$, $UCL_{Z_2} = 2.06$ and $UCL_{Z_3} = 3.18$. As it can be seen, the Phase I part of these charts seems to confirm the fact that during the period from 2012 to 2016, the time between consecutive breakdowns and their corresponding costs were a stable process. But, from 2017, things seems to have changed as several out-of-control situations (see values in bold in Table 11) have been detected by the 3 TBEA charts on 14/05/18, 24/11/18 and 27/12/18 due to more frequent breakdowns and an increasing maintenance cost (due to an aging machine). Every time an out-of-control situation is detected, the production is stopped, the sources of the

Table 11: Phase I and II data sets corresponding to the time (T_i in days) between two consecutive breakdowns, amplitudes (X_i as the repair and restart cost in euros) and the values of the statistics Z_1 , Z_2 and Z_3 .

Phase I							Phase II						
Date	i	T_i	X_i	$Z_{1,i}$	$Z_{2,i}$	$Z_{3,i}$	Date	i	T_i	X_i	$Z_{1,i}$	$Z_{2,i}$	$Z_{3,i}$
10/03/12	1	62	4890	-0.064	0.939	1.939	11/01/17	1	63	5080	-0.043	0.960	1.962
28/05/12	2	79	6180	-0.092	0.932	1.995	21/03/17	2	69	5350	-0.090	0.923	1.935
25/07/12	3	58	3730	-0.231	0.766	1.770	07/05/17	3	47	3770	-0.036	0.955	2.015
27/08/12	4	33	2930	0.032	1.057	2.377	15/07/17	4	69	4590	-0.243	0.792	1.782
20/11/12	5	85	7600	0.093	1.065	2.230	14/10/17	5	91	5940	-0.344	0.777	1.848
20/02/13	6	92	5580	-0.434	0.722	1.768	18/12/17	6	65	5420	-0.008	0.993	2.002
30/04/13	7	69	4570	-0.247	0.789	1.778	26/02/18	7	70	4580	-0.262	0.779	1.767
06/07/13	8	67	5230	-0.080	0.930	1.937	21/04/18	8	54	5430	0.181	1.197	2.189
18/08/13	9	43	4470	0.174	1.238	2.274	14/05/18	9	23	5740	0.770	2.972	3.721
28/09/13	10	41	3420	-0.005	0.993	2.128	27/06/18	10	44	6110	0.488	1.654	2.574
22/11/13	11	55	3460	-0.234	0.749	1.770	22/08/18	11	56	7340	0.533	1.561	2.536
08/02/14	12	78	5360	-0.241	0.818	1.839	29/10/18	12	68	8160	0.495	1.429	2.516
05/04/14	13	56	4470	-0.047	0.951	1.956	24/11/18	13	26	4800	0.529	2.199	3.236
28/05/14	14	53	4470	0.004	1.004	2.015	27/12/18	14	33	6570	0.768	2.371	3.113
08/07/14	15	41	3320	-0.025	0.964	2.108							
27/09/14	16	81	4910	-0.382	0.722	1.720							
29/10/14	17	32	5010	0.470	1.864	2.854							
07/01/15	18	70	6630	0.152	1.128	2.182							
30/03/15	19	82	5710	-0.238	0.829	1.873							
20/05/15	20	51	5130	0.171	1.198	2.192							
16/07/15	21	57	5330	0.110	1.114	2.111							
01/09/15	22	47	5010	0.215	1.269	2.266							
22/10/15	23	51	3660	-0.126	0.855	1.895							
15/11/15	24	24	3340	0.268	1.657	3.129							
12/01/16	25	58	3600	-0.257	0.739	1.743							
14/03/16	26	62	5560	0.072	1.068	2.074							
28/04/16	27	45	5760	0.401	1.524	2.473							
25/06/16	28	58	6440	0.317	1.322	2.318							
16/08/16	29	52	6310	0.393	1.445	2.408							
09/11/16	30	85	6300	-0.169	0.883	1.967							

Table 12: Estimated values for μ_{T_0} , σ_{T_0} , μ_{X_0} and σ_{X_0} as well as the estimated values for a_0 and b_0 for the distributions in Table 1.

	T			X		
	$\mu_{T_0} = 58.9, \sigma_{T_0} = 17.28$			$\mu_{X_0} = 4946, \sigma_{X_0} = 1165.43$		
	a_0	b_0	D_{KS}	a_0	b_0	D_{KS}
Gamma	11.6488	5.0562	0.0818	18.0107	274.6135	0.1228
Normal	58.9000	17.2573	0.1207	4946	1165.4349	0.1182
Weibull	3.8123	65.1584	0.1264	4.8472	5396.4958	0.1129

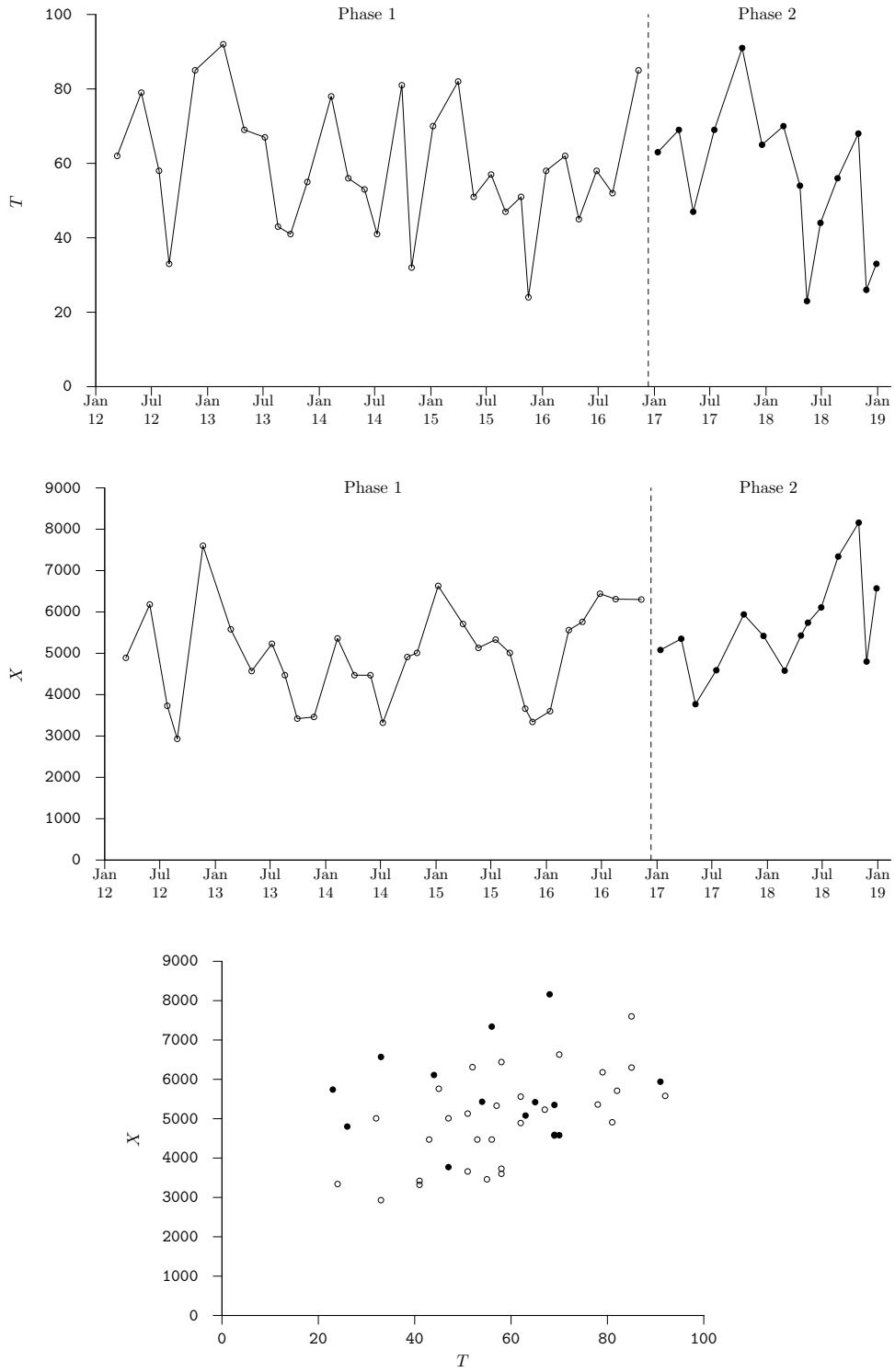


Figure 3: Phase I (\circ) and II (\bullet) data corresponding to the time (T in days) between breakdowns and amplitudes (X as the repair and restart cost in euros) corresponding to the data set in Table 11.

breakdown are searched for, analyzed and repaired. The machine is restarted as soon as possible in order to decrease the manufacturing disruption.

5 Conclusions

In this paper, we investigated simple Shewhart type control charts for several statistics and many distributional scenarios for simultaneously monitoring TBEA data. The Frank Copulas has been used as a mechanism to model the dependence between the time T and the amplitude X . The main result of this paper is that, for a simultaneous change in the amplitude X and in the time between events T , the overall best choice is the statistic Z_1 followed by Z_2 , regardless of the level of dependence. On the contrary, the statistic Z_3 should not be considered as a potentially efficient statistic for monitoring.

In terms of potential future research, TBEA type control charts could be extended, for instance, in order to be able to monitor *resilience* type data which are known to depend on three characteristics: the time between disruptions (say T_1), the performance loss X and the time needed for recovery (say T_2) (see³⁶ and³⁷). In this context, the idea would be to develop new TBEA type control charts for which the time is no longer a univariate random variable but a bivariate one $\mathbf{T} = (T_1, T_2)$ and the amplitude X remains the same as for traditional TBEA type data.

Similar statistics to the ones used in this paper can also be investigated in the case where time between events T is a univariate random variable and the amplitude is no longer a univariate random variable X but is a multivariate random vector \mathbf{X} (containing the different types of amplitude corresponding to a particular kind of negative event). In addition, other continuous distributions like the Beta or Rayleigh distributions (as in Ali and Riaz³⁸ for instance), as well as discrete distributions for the amplitude X like the Binomial or Poisson distribution could also be considered.

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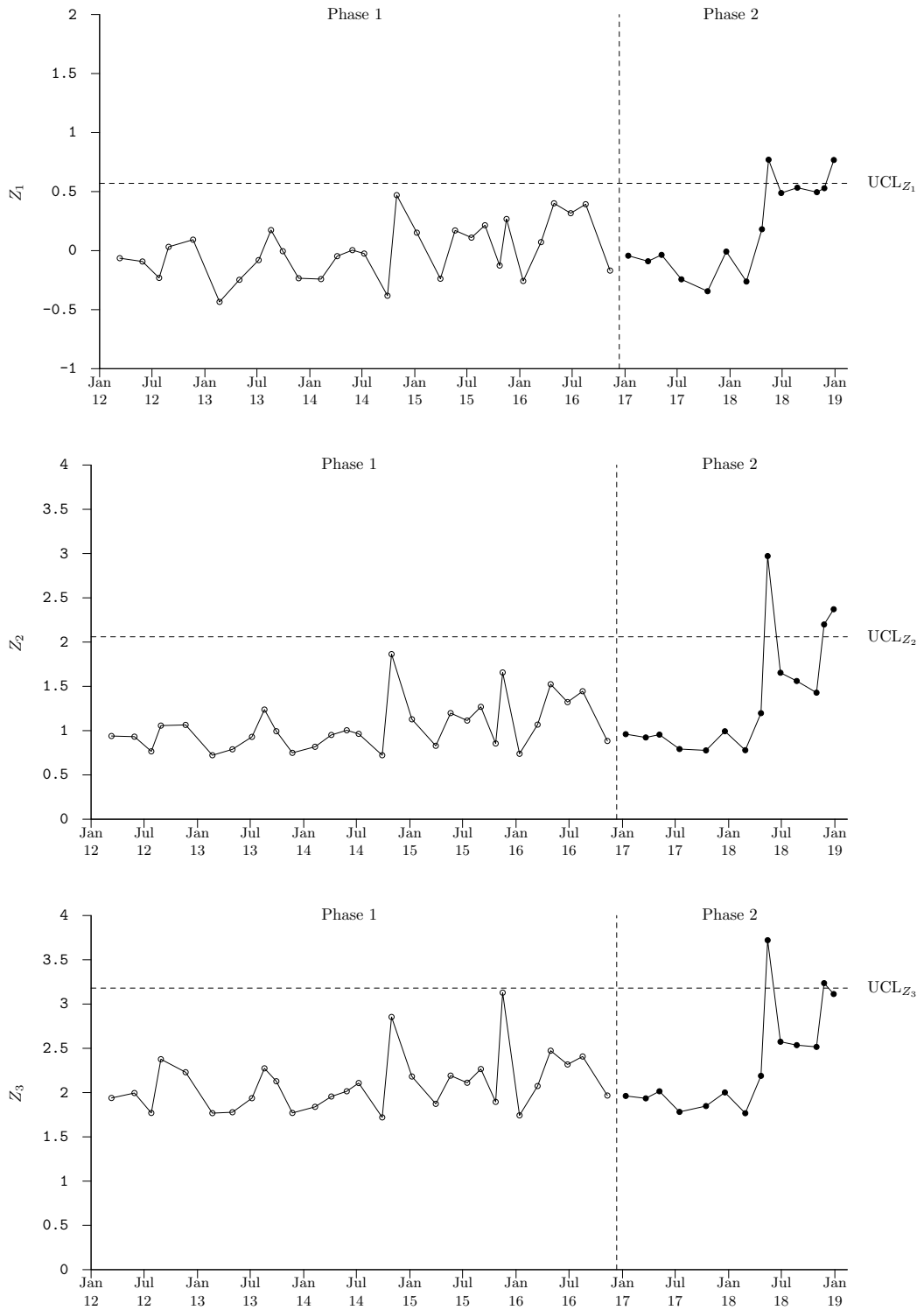


Figure 4: Statistics Z_1 , Z_2 and Z_3 corresponding to the data set in Table 11.

ing techniques.

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