Analogy between concepts (extended abstract)
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Analogical proportions are statements of the form "x is to y as z is to t", where x, y, z, t are items which can be represented in terms of the same set of features. They are items which can be represented in terms of the same set of features. A formal definition has been proposed for analogical proportions, and they are items which can be represented in terms of the same set of features.

1 Introduction

Analogical reasoning has raised the interest of psychologists and computer scientists for a long time; see, e.g., [Dedek and Kokinov, 2001; Prade and Richard, 2014]. We focus here on analogical proportions which are statements of the form "x is to y as z is to t", expressing an analogical parallel between pairs (x, y) and (z, t). A statement such as "Carlsen is to chess as Mozart is to music" introduces Carlsen as an exceptional virtuoso of chess, a quality that Mozart is particularly well known to have concerning music. It relates two types of items, here people and activities. It is an example of what we call relational proportions, which are statements of the form "object A has the same relationship with attribute a as object B with attribute b". This can be viewed as a special case of analogical proportions. In case where x, y, z, t are items which can be represented in terms of the same set of features, a formal definition has been proposed for analogical proportions in the setting of Boolean logic and then extended using multiple-valued logic for handling numerical features [Miclet and Prade, 2009; Prade and Richard, 2013; Dubois et al., 2016], by stating that "x differs from y as z differs from t and y differs from x as t differs from z". The nature of Relational Proportions (RP for short) suggests to handle them in the setting of formal concept analysis. This leads us to the question of defining analogical proportions between formal concepts. For more details and proofs, the reader is referred to [Barbot et al., 2019].

2 Analogical Proportions: Basics

Analogical proportions [Dorolle, 1949; Hesse, 1959] are usually characterized by three axioms. They acknowledge the interchangeability of pairs (x, y) and (z, t) in the proportion "x is to y as z is to t", and enforce the idea that y and z can be exchanged if the proportion is valid, just as in the equality of two numerical ratios where means can be exchanged.

Definition 1 (Axioms of Analogical Proportion). An analogical proportion (AP) on a set X is a quaternary relation on X, i.e., a subset of X^4. An element of this subset, written (x : y :: z : t), which reads "x is to y as z is to t", must obey the following axioms (studied in [Lepage, 2003]):

1. Reflexivity of 'as': (x : y :: x : y)
2. Symmetry of 'as': (x : y :: z : t) ⇔ (z : t :: x : y)
3. Exchange of means: (x : y :: z : t) ⇔ (x : z :: y : t)

Then, thanks to (2) and (3), it can be easily seen that (x : y :: z : t) ⇔ (t : y :: z : x) should also hold (exchange of the extremes). According to the first two axioms, four other formulations are equivalent to the canonical form (x : y :: z : t).

Definition 2 (A 4-tuple (x, y, z, t) of a lattice (L, ∨, ∧, ≤)^4 is a Factorial Analogue Proportion (FAP) (x : y :: z : t) if:

\[
\begin{align*}
x &= (x \land y) \lor (x \land z) \\
y &= (x \land y) \lor (y \land t) \\
z &= (z \land t) \lor (x \land z) \\
t &= (z \land t) \lor (y \land t)
\end{align*}
\]

Definition 3 (A 4-tuple (x, y, z, t) of (L, ∨, ∧, ≤)^4 is a Weak Analogue Proportion (WAP) when x ∧ t = y ∧ z and x ∨ t = y ∨ z. It is denoted x : y :: z : t.

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*This paper is an extended abstract of [Barbot et al., 2019] published in *Artificial Intelligence*.

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In the case of a distributive lattice (e.g., a Boolean lattice), this alternative definition is equivalent to the FAP. But, in general, a FAP is a WAP and the converse is false, which explains the use of adjective “weak” [Miclet et al., 2014].

**Example 1.** Let Σ be a finite set associated with the Boolean lattice \((2^Σ, ∪, ∩, ¬, ⊆)\). When saying that “\(x\) is to \(y\) as \(z\) is to \(t\)” where \(x, y, z, t \in Σ\), we express that \(x\) differs from \(y\) in the same way as \(z\) differs from \(t\). For example, if \(x = \{a, b, e\}\) and \(y = \{b, c, e\}\), we see that to transform \(x\) into \(y\), we have to remove \(a\) and add \(c\). Now, if \(z = \{a, d, e\}\), we can construct \(t\) with the same operations, to get \(t = \{c, d, e\}\). More formally, with this view, we should have \(x \setminus y = z \setminus t\) and \(y \setminus x = t \setminus z\). This is equivalent to \(x \cap t = y \cap z\) and \(x \cup t = y \cup z\). This relation linking \(x, y, z, t\) is clearly symmetrical, and satisfies the exchange of the means. Hence it is a correct definition of the AP in the Boolean setting [Miclet and Prade, 2009].

We give here a simple example of FAP in a lattice.

**Proposition 1.** Let \(y\) and \(z\) be two elements of a lattice, the proportion \(y : y \lor z : y \land z : z\) is a FAP. We call it a Canonical Analogical Proportion (CAP).

### 3 Analogical Proportions in FCA

In order to derive more specifically the AP notion in a Formal Concept Analysis (FCA) framework, we first recall some basic elements of FCA, before studying the relations between several kinds of AP and their characterization in FCA.

FCA starts with a binary relation \(R\) defined between a set \(O\) of objects and a set \(A\) of attributes (or properties). The tuple \((O, A, R)\) is called a formal context. The notation \((o, a) \in R\) or \(oRa\) means that object \(o\) has attribute \(a\). We denote \(\alpha^\uparrow = \{a \in A \mid (o, a) \in R\}\) the attribute set of object \(o\) and \(\alpha^\downarrow = \{o \in O \mid (o, a) \in R\}\) the object set having attribute \(a\). Similarly, for any subset \(\alpha\) of \(O\), \(\alpha^\uparrow\) is defined as \(\{a \in A \mid \alpha \supseteq \alpha\}\). Then a formal concept is defined as a pair \((o, a)\), such that \(\alpha^\uparrow = o\) and \(\alpha^\downarrow = a\). One calls \(o\) the extension of the concept and \(a\) its intension.

The set of all formal concepts is equipped with a partial order (denoted \(\leq\)) defined as: \((o_1, a_1) \leq (o_2, a_2)\) iff \(o_1 \subseteq o_2\) (or, equivalently, \(a_2 \subseteq a_1\)). Then it is structured as a lattice, called the concept lattice of \(R\).

**Example 2.** The concept lattice of context \(R\) displayed in Figure 1 is shown in Figure 2.

Since a concept \(x\) is associated to a set of objects \(o_x\) and a set of attributes \(a_x\), the objective of this section is to relate the AP definitions with these sets, and to study the links between APs in a concept lattice and APs on object or attribute sets.

**Proposition 2.** Let \(x, y, z, t\) be 4 concepts, one has: \(x : y \lor z : z\) if \(a_x \cap a_t = a_y \cap a_z \) and \(o_x \cap o_t = o_y \cap o_z\).

![Figure 1: The formal context \(R\).](image1.png)

![Figure 2: The formal concept lattice of \(R\).](image2.png)

**Proposition 3.** Let \(x, y, z\) and \(t\) be 4 concepts, if \((a_x : a_y : a_z : a_t)\) then \(x : y \lor z : z\).

**Comments.** The converse is false. Let us consider the formal context \(R\) in Fig. 2. We have \(x : y \lor z : z\) due to Proposition 2. However, the Boolean APs \(a_x \lor a_y \lor a_z \) and \(o_x \lor o_y \lor o_z\) are both false. The WAP between sets of attributes: in a WAP, objects are allowed to possess attributes which are not shared by any other object concerned in the WAP. In this lattice, \(x, y, z\) and \(t\) are in WAP, but \(a_x \lor a_y \lor a_z \) and \(o_y \lor o_z\) are both false. Besides, \(x', y', z'\) and \(t'\) are in WAP and the FAP \(x' : y' \lor z' : z'\) is true, but \(a_x' \lor a_y' \lor a_z' \) and the FAP \(x' : y' \lor z' : t'\) are both false.

We give now a proposition which leads us to a corollary in which is defined yet another analogical proportion between formal concepts, the strongest of all.

**Proposition 4.** Let \(x, y, z, t\) be 4 concepts, if \((a_x : a_y : a_z : a_t)\) then \(x : y \lor z : t\).

**Corollary 1.** Let \(x, y, z, t\) be 4 concepts, the following 2 conjunctions are equivalent:

\[
\begin{align*}
& a_x \cup a_t = a_y \cup a_z \\
& o_x \cup o_t = o_y \cup o_z
\end{align*}
\]

This characterizes a particular case of FAP between concepts that we call a Strong Analogical Proportion (SAP). It is denoted \(x : y \lor z : t\).

**Comments.** From Corollary 1, \(a_x \lor a_y \lor a_z \) and \(o_x \lor o_y \lor o_z\) imply the FAP \(x : y \lor z : t\). However, the reciprocal is false. Let us consider the concept lattice displayed in Figure 2: we have the FAP \(y : T \lor z : z\) (which is a CAP), but \(o_y \lor o_z \neq o_T \lor o_L\) and \(a_y \lor a_z \neq a_T \lor a_L\).

### 4 Formal Concepts and Relational Proportion

#### 4.1 From a RP to Concepts in AP

We study here if we can deduce from a relational proportion \(A\) is the \(B\) of \(a\), or \(A\) is to \(a\) as \(B\) is to \(b\), formal concepts in WAP and an analogical complex from this knowledge.

As an example, we have found in a web magazine the following proportion “Massimiliano Alajmo is the Mozart of Italian cooking”. The background knowledge allowing to understand this relational proportion is the following: music and Italian cooking are disciplines practiced by humans,
such disciplines can be practiced with different levels of ability. Mozart is a musician and Mozart is a genius in music
discipline. Since the quality “to be a genius” is not possessed by
everybody, there must exist many “ordinary gifted”
musicians. Then, the background knowledge can be expressed
by the formal context on the left side of Figure 3. where $o_1$
stands for Mozart, $o_2$ for one of “ordinary gifted” musicians,
a $a_1$ is the attribute “practices music”, $a_2$ “is a genius” and $a_3$
“has an ordinary ability”.

Now, when the new data “Alajmo is the Mozart of Italian
cooking” is introduced, the knowledge extends as follows:
Alajmo practices Italian cooking, and he has something in
common with Mozart that is not Italian cooking. The
relational proportion is a reduced form of “Alajmo is to Italian
cooking as Mozart is to music”. Since Mozart has only
the other attribute “Genius”, Alajmo must have it. Moreover,
since cooking is a discipline practiced by humans, there
must exist some ordinary gifted Italian cook. At last, we
must introduce the notion of non-genius in our universe.
If we do not, we implicitly suppose that everybody is a genius
for some activity. The knowledge is now expressed by
the formal context on the right side of Figure 3 where $o_3$
stands for Alajmo, $a_4$ an ordinary gifted Italian cook and $a_4$
Italian cooking. This context is called the analogical context.
Considering the associated concept lattice, the closest analogical
proportion to “Alajmo is the Mozart of Italian cooking” is
$(\{o_3\}, \{a_2, a_4\}) : (\{o_4\}, \{a_3, a_4\}) \stackrel{\text{WAP}}{\supseteq} (\{o_1\}, \{a_1, a_2\}) : (\{o_2\}, \{a_1, a_3\})$ which translates into “Mozart is to some ordinary
musician as Alajmo is to some ordinary cook”.

More formally, from the relational proportion “$o_1$ is the
$2$ of $a_2$”, we can derive an analogical context as above. It is
composed of objects $o_1$ and $o_2$, described by four attributes: $a$
is possessed by $o_1$ and not by $o_2$, $b$ is possessed by $o_2$ and not
by $o_1$, $b$ is possessed both by $o_1$ and $o_2$ and $b$ is some attribute
not possessed by $o_1$ nor $o_2$. Secondly we complete the context
with two objects $o_3$ and $o_4$ that are the complements of $o_2$
and $o_1$ with respect to the four attributes. The result is the
analogical context where $a_1 = b$, $a_2 = a$, $a_3 = \sim b$ and $a_4 = \sim b$.

4.2 Analogical Complex

In the previous subsection, it turns out that the analogical context
is an interesting pattern, from which we can construct
relational proportions. A more general definition of this pattern,
notated as analogical complex, has been given in [Miclet
and Nicolas, 2015]. We present it here through an example.

Let us consider the context in Fig. 4, called SmallZoo, ex-
ttracted from the Zoo data base [Lichman, 2013]. The context
of Fig. 5 is a subcontext of SmallZoo with special charac-
teristics described in Fig. 6. Any subcontext with such
characteristics is called ‘analogical complex’. In the left part of

$$
\begin{array}{c|ccc}
\hline
& a_1 & a_2 & a_3 \\
\hline
o_1 & \times & \times & \\
o_2 & \times & \times & \\
o_3 & \times & \times & \\
o_4 & \times & \times & \\
\hline
\end{array}
$$

Figure 3: Constructing a formal context from a RP

Fig. 6, a ‘$\times$’ between a subset of objects (say, $o_1$) and a sub-
set of attributes (say, $a_4$) means that every object of $o_1$
is in relation with every attribute of $a_4$, and a ‘blank’ between
a subset of objects (say, $o_1$) and a subset of attributes (say, $a_1$)
is in relation with no attribute of $a_1$. In practice, the right table is sufficient to define an
analogical complex, since the left one is the same for all of
them. Thus, an analogical complex is defined by the 8 sets
$o_1, …, o_4, a_1, …, a_4$, and it is said to be complete when none
of these sets are empty. From the analogical complex
structure, we derive a formal definition of a relational proportion.

Definition 4. Let $(o_1, a_1)$ be a complete analogical com-
pex in a formal context, the following sets are said to be in
the formal relational proportion (RP) $(o_1$ is to $a_3$ as $o_2$ is to
$a_2$), and we write $(o_1 \uparrow a_3 \equiv o_2 \uparrow a_2)$.

Comments. The reduced form of the RP would be $(o_1$ is
the $o_2$ of $a_3$). Besides, from the same complex, we can ex-
tract the RPs $(o_1 \uparrow a_2 \equiv o_3 \uparrow a_1)$, $(o_2 \uparrow a_4 \equiv o_4 \uparrow a_1)$
and $(o_3 \uparrow a_3 \equiv o_4 \uparrow a_2)$. Since the operator $\equiv$ is com-
mutative, it gives a total of 8, but permuting the extremes and
the means in a RP may lead to awkward phrasings.

$$
\begin{array}{cccc|cccc}
& a_1 & a_2 & a_3 & a_4 \\
\hline
o_1 & \times & \times & \\
o_2 & \times & \times & \\
o_3 & \times & \times & \\
o_4 & \times & \times & \\
\hline
\end{array} \quad \begin{array}{c|c|c|c|c}
\hline
o_1, a_2 & a_3 & a_4 \\
\hline
o_1, \{o_2\} & \{a_3\} & \{a_4\} \\
o_2, \{o_5\} & \{a_6, a_3, a_7\} & \{a_4\} \\
o_3, \{o_8\} & \{a_1, a_2\} & \{a_4\} \\
o_4, \{o_7\} & \{a_1, a_2\} & \{a_4\} \\
\hline
\end{array}
$$

Figure 6: Notation of an analogical complex

<table>
<thead>
<tr>
<th>SmallZoo</th>
<th>hair</th>
<th>feathers</th>
<th>eggs</th>
<th>milk</th>
<th>airborne</th>
<th>aquatic</th>
<th>predator</th>
<th>toothed</th>
</tr>
</thead>
<tbody>
<tr>
<td>$o_0$ aardvark</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
</tr>
<tr>
<td>$o_1$ chicken</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
</tr>
<tr>
<td>$o_2$ crow</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
</tr>
<tr>
<td>$o_3$ dolphin</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
</tr>
<tr>
<td>$o_4$ duck</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
</tr>
<tr>
<td>$o_5$ fruitbat</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
</tr>
<tr>
<td>$o_6$ kiwi</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
</tr>
<tr>
<td>$o_7$ mink</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
</tr>
<tr>
<td>$o_8$ penguin</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
</tr>
<tr>
<td>$o_9$ platypus</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
<td>$x$</td>
</tr>
</tbody>
</table>

Figure 4: The context SmallZoo

Figure 5: An analogical complex extracted from SmallZoo
Example 3. The complex described in Fig. 6 implies all attributes but \(a_0\) (predator) and objects \(o_1\) and \(o_2\) (chicken and crow), \(o_5\) (fruitbat), \(o_8\) (penguin) and \(o_7\) (mink). From this context, the RP in reduced form “a fruitbat is the mink of airborne animals” can be derived for instance, meaning that fruitbat and mink have hair, are toothed and produce milk, but that the mink is aquatic at the contrary of the fruitbat. Of course, the interest of such phrases has to be taken in context: the SmallZoo data base is supposed to be the only knowledge.

4.3 WAP and Analogical Complex

We explore now the links between WAP between concepts and complete analogical complexity, and then the formal RPs.

First, we are interested in defining non degenerated WAPs, called complete and bi-complete, forbidding inclusion between two of its concepts. It is a key notion for building WAP between concepts with a sound cognitive interpretation.

Definition 5. Let us consider \((x: y \mathcal{WAP} z: t)\), this WAP is complete through attributes when \((a_x \cap a_y) \subset a_z\), \((a_x \cap a_z) \subset a_y\), \((a_y \cap a_z) \subset a_x\), \((a_x \cap a_y) \subset a_z\), \((a_y \cap a_z) \subset a_x\), and \((a_z \cap a_y) \subset a_x\). A WAP complete through objects is similarly defined using subsets of objects. A WAP is bi-complete when

- if it is complete through attributes and \((x \cup y) \mathcal{WAP} (z \cup t)\) is complete through objects.
- or if it is complete through objects and \((x \cup y) \mathcal{WAP} (z \cup t)\) is complete through attributes.

In order to derive RPs from an AP between concepts, we consider a complete WAP through attributes (a similar reasoning can be done from a complete WAP through objects) and introduce a process to extract an analogical complex.

Due to the completeness, sets \(a_1 = (a_x \cap a_z) \setminus a_y\), \(a_2 = (a_y \cap a_z) \setminus a_x\), \(a_3 = (a_x \cap a_z) \setminus a_y\) and \(a_4 = (a_y \cap a_z) \setminus a_x\) are nonempty. We also define \(o_1 = o_2\), the set of objects proper to \(x\) (that appear in \(o_2\) but not in the objects of \(y, z\) and \(t\)) and similarly \(o_3 = o_y\), \(o_5 = o_z\) and \(o_4 = o_t\).

By construction, every \(o \in o_1\) is in relation with every \(a \in a_1 \cup a_2\). It is also the case between \(o_2\) and \(a_2 \cup a_3\), \(o_3\) and \(a_1 \cup a_3\), \(o_4\) and \(a_1 \cup a_2\). For all the other combinations, for instance \(o_1\) and \(a_1\), for any \(o \in o_1\), there exists \(a \in a_1\) such that \(o \notin a^2\). However, these properties do not guarantee that the subcontext \((o_{1,4}, a_{1,4})\) is an analagical schema, even if it is a closed schema: it can exist an object \(o \in o_1\) in relation with an attribute \(a \in a_1\), where \((i, j) \in \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 2), (3, 3), (4, 3), (4, 4)\}\). In such a case, either \(a = o\) is removed and this postprocessing permits to obtain an analagical schema. But this schema is not necessarily a complex, since the associated subcontext may be not maximal. Then a second postprocessing maximises the schema into complex, adding new attributes and/or objects chosen among those which do not appear in \(a_{1,2} \cup \ldots \cup a_{1,4}\) or \(o_{1,2} \cup \ldots \cup o_{1,4}\).

This method may lead to several complexes, according to the choices in both postprocessings. This set of complexes is a sub-lattice of the lattice of complexes and in case of a bi-complete WAP as input, one of the output lattices is entirely made of complete analogical complexes [Barbot et al., 2019].

Example 4. In SmallZoo, \(x = \{(o_1, o_2, o_4), (o_1, o_2, a_4)\}\), \(y = \{(o_5), (o_1, o_4, a_7)\}\), \(z = \{(o_4, o_8), (o_1, o_2, a_5)\}\), \(t = \{(o_7, o_3), (o_1, o_3, a_5, a_6)\}\) are concepts in complete WAP through attributes. At the beginning, \(o_1 = (o_1, o_2)\), \(o_2 = \{o_5\}\), \(o_3 = \{o_8\}\), \(o_4 = \{o_7, o_3\}\), \(o_5 = \{a_5\}\), \(o_6 = \{a_0, o_3\}\), \(a_3 = \{a_1, o_2\}\), \(a_4 = \{a_4\}\). Due to the relation between \(o_5\) and \(o_2\), the first postprocessing applied on the subcontext shown in Fig. 7 can remove (either \(o_5\) or \(o_2\). After removing \(o_5\), the table in Fig. 8 is an analogical schema and we can check that it is maximal in SmallZoo. Note that we had to remove \(o_5\), the postprocessings would have produced the analogical complex detailed in Fig. 5.

For example, from the complete analogical complex described above, we can derive the following RP: “the chicken and the crow are to the feathers as the fruitbat is to the hair, the milk and the teeth”. It makes sense when considering that all these animals share the attribute “airborne”.

5 Conclusion

We have shown how relational proportions can be identified in a formal context. Relational proportions offer a basis for concise forms of explanations. Indeed, if \(B\) has some well-known features, the proportion “object \(A\) is to attribute \(a\) as object \(B\) is to attribute \(b\)” provides an argument for stating that “object \(A\) is the \(B\) of \(a\)”, when \(A\) possesses these well-known features also, as in “Carlsen is the Mozart of chess”. It is worth pointing out that two cognitive capabilities, namely conceptual categorization and analogical reasoning can be handled together in the setting of formal concept analysis. This short presentation has left aside the algorithmic side (based on the identification of formal complexes), which is discussed in the long version of [Barbot et al., 2019].

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References


