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To cite this version:

Arthur Bottois, Jérôme Lemoine, Arnaud Münch. Constructive exact controls for semi-linear wave equations. Annals of Mathematical Sciences and Applications, 2023, 8 (3), pp.629-675. 10.4310/AMSA.2023.v8.n3.a7 . hal-03112309v2

HAL Id: hal-03112309

https://hal.science/hal-03112309v2

Submitted on 14 May 2021

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Constructive exact controls for semi-linear wave equations

Arthur Bottois Jérôme Lemoine Arnaud Münch*

May 14, 2021

Abstract

The exact distributed controllability of the semilinear wave equation $\partial_{tt}y - \Delta y + g(y) = f\,1_\omega$ posed over multi-dimensional and bounded domains, assuming that $g \in C^1(\mathbb{R})$ satisfies the growth condition $\limsup_{|r|\to\infty} g(r)/(|r|\ln^{1/2}|r|) = 0$ has been obtained by Fu, Yong and Zhang in 2007. The proof based on a non constructive Leray-Schauder fixed point theorem makes use of precise estimates of the observability constant for a linearized wave equation. Assuming that g' does not grow faster than $\beta \ln^{1/2}|r|$ at infinity for $\beta > 0$ small enough and that g' is uniformly Hölder continuous on $\mathbb{R}$ with exponent $s \in (0,1]$, we design a constructive proof yielding an explicit sequence converging to a controlled solution for the semilinear equation, at least with order $1+s$ after a finite number of iterations. Numerical experiments in the two dimensional case illustrate the results.

AMS Classifications: 35L71, 49M15, 93E24.

Keywords: Semilinear wave equation, exact controllability, least-squares approach.

1 Introduction

Let Ω be a bounded domain of $\mathbb{R}^d$, $d \in \{2,3\}$ with $C^{1,1}$ boundary and $\omega \subset\subset \Omega$ be a non empty open set. Let $T > 0$ and denote $Q_T := \Omega \times (0,T)$, $q_T := \omega \times (0,T)$ and $\Sigma_T := \partial\Omega \times (0,T)$. We consider the semilinear wave equation

$$\begin{cases} \partial_{tt}y - \Delta y + g(y) = f1_\omega, & \text{in } Q_T, \\ y = 0, & \text{on } \Sigma_T, \\ (y(\cdot,0), \partial_t y(\cdot,0)) = (u_0,u_1), & \text{in } \Omega, \end{cases} \tag{1}$$

where $(u_0,u_1) \in \boldsymbol{V} := H_0^1(\Omega)\times L^2(\Omega)$ is the initial state of y and $f \in L^2(q_T)$ is a *control* function. Here and throughout the paper, $g : \mathbb{R} \to \mathbb{R}$ is a function of class C^1 such that $|g(r)| \le C(1+|r|)\ln(2+|r|)$ for every $r \in \mathbb{R}$ and some $C > 0$. Then, (1) has a unique global weak solution in $C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ (see [4, 6]).

The exact controllability for (1) in time T is formulated as follows: for any $(u_0,u_1),(z_0,z_1) \in \boldsymbol{V}$, find a control function $f \in L^2(q_T)$ such that the weak solution of (1) satisfies $(y(\cdot,T), \partial_t y(\cdot,T)) = (z_0,z_1)$. Assuming a growth condition on the nonlinearity g at infinity, this problem has been solved in [17].

Theorem 1. *[17] For any $x_0 \in \mathbb{R}^d\backslash\overline{\Omega}$, let $\Gamma_0 = \{x \in \partial\Omega, (x-x_0)\cdot\nu(x) > 0\}$ and, for any $\epsilon > 0$, $\mathcal{O}_\epsilon(\Gamma_0) = \{y \in \mathbb{R}^d \mid |y-x| \le \epsilon \, for \, x \in \Gamma_0\}$. Assume*

$(\mathbf{H_0})$ $T > 2\max_{x\in\overline{\Omega}}|x-x_0|$ *and* $\omega \subseteq \mathcal{O}_\epsilon(\Gamma_0)\cap\Omega$ *for some* $\epsilon > 0$.

If g satisfies

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$(\mathbf{H_1})$ $\limsup_{|r|\to\infty} \frac{|g(r)|}{|r|\ln^{1/2}|r|} = 0$

then (1) *is exactly controllable in time* T.

This result improves [28] where a stronger condition of the support ω is made, namely that ω is a neighborhood of $\partial\Omega$ and that $T > \mathrm{diam}(\Omega\backslash\omega)$. In Theorem 1, Γ_0 is the usual star-shaped part of the whole boundary of Ω introduced in [29].

A special case of Theorem 1 is when g is globally Lipschitz continuous, which gives the main result of [38], later generalized to an abstract setting in [22] using a global version of the inverse function theorem and improved in [36] for control domains ω satisfying the classical multiplier method of Lions [29].

Theorem 1 extends to the multi-dimensional case the result of [39] devoted to the one dimensional case under the condition $\limsup_{|r|\to\infty} \frac{|g(r)|}{|r|\ln^2|r|} = 0$, relaxed later on in [4], following [13], and in [30]. The exact controllability for subcritical nonlinearities is obtained in [11] assuming the sign condition $rg(r) \geq 0$ for every $r \in \mathbb{R}$. This latter assumption has been weakened in [20] to an asymptotic sign condition leading to a semi-global controllability result in the sense that the final data (z_0, z_1) is prescribed in a precise subset of $\boldsymbol{V}$. In this respect, we also mention in the one dimensional case [10] where a positive boundary controllability result is proved for a steady-state initial and final data specific class of initial and final data and for T large enough by a quasi-static deformation approach.

The proof given in [17, 28] is based on a fixed point argument introduced in [37, 39] that reduces the exact controllability problem to the obtention of suitable *a priori* estimates for the linearized wave equation with a potential (see Proposition 6 in appendix A). More precisely, it is shown that the operator $K : L^\infty(0,T;L^d(\Omega)) \to L^\infty(0,T;L^d(\Omega))$ where $y_\xi := K(\xi)$ is a controlled solution through the control function f_ξ of the linear boundary value problem

$$\begin{cases} \partial_{tt} y_\xi - \Delta y_\xi + y_\xi\, \widehat{g}(\xi) = -g(0) + f_\xi 1_\omega, & \text{in } Q_T, \\ y_\xi = 0, & \text{on } \Sigma_T, \\ (y_\xi(\cdot,0), \partial_t y_\xi(\cdot,0)) = (u_0, u_1), & \text{in } \Omega, \end{cases} \qquad \widehat{g}(r) := \begin{cases} \dfrac{g(r)-g(0)}{r} & r \neq 0, \\ g'(0) & r = 0 \end{cases}, \tag{2}$$

satisfying $(y_\xi(\cdot,T), y_{\xi,t}(\cdot,T)) = (z_0, z_1)$ has a fixed point. The control f_ξ is chosen in [28] as the one of minimal $L^2(q_T)$-norm. The existence of a fixed point for the compact operator K is obtained by using the Leray-Schauder's degree theorem. Precisely, it is shown under the growth assumption $(\mathbf{H_1})$ that there exists a constant $M = M(\|u_0, u_1\|_{\boldsymbol{V}}, \|z_0, z_1\|_{\boldsymbol{V}})$ such that K maps the ball $B_{L^\infty(0,T;L^d(\Omega))}(0, M)$ into itself.

The main goal of this article is to design an algorithm providing an explicit sequence $(f_k)_{k\in\mathbb{N}}$ that converges strongly to an exact control for (1). A first idea that comes to mind is to consider the Picard iterations $(y_k)_{k\in\mathbb{N}}$ associated with the operator K defined by $y_{k+1} = K(y_k)$, $k \geq 0$ initialized with any element $y_0 \in L^\infty(0,T;L^d(\Omega))$. The resulting sequence of controls $(f_k)_{k\in\mathbb{N}}$ is then so that $f_{k+1} \in L^2(q_T)$ is the control of minimal $L^2(q_T)$ norm for y_{k+1} solution of

$$\begin{cases} \partial_{tt} y_{k+1} - \Delta y_{k+1} + y_{k+1}\, \widehat{g}(y_k) = -g(0) + f_{k+1} 1_\omega, & \text{in } \quad Q_T, \\ y_{k+1} = 0, & \text{on } \Sigma_T, \\ (y_{k+1}(\cdot,0), \partial_t y_{k+1}(\cdot,0)) = (y_0, y_1), & \text{in } \Omega. \end{cases} \tag{3}$$

Such a strategy usually fails since the operator K is in general not contracting, even if g is globally Lipschitz. We refer to [16] providing numerical evidence of the lack of convergence in parabolic cases (see also Remark 6 in appendix A). A second idea is to use a Newton type method in order to find a zero of the C^1 mapping $\widetilde{F} : Y \mapsto W$ defined by

$$\widetilde{F}(y,f) := \Big(\partial_{tt} y - \Delta y + g(y) - f 1_\omega, y(\cdot\,,0) - u_0, \partial_t y(\cdot\,,0) - u_1, y(\cdot\,,T) - z_0, \partial_t y(\cdot\,,T) - z_1\Big) \tag{4}$$

for some appropriates Hilbert spaces Y and W (see further): given (y_0, f_0) in Y, the sequence $(y_k, f_k)_{k\in\mathbb{N}}$ is defined iteratively by $(y_{k+1}, f_{k+1}) = (y_k, f_k) - (Y_k, F_k)$ where F_k is a control for Y_k solution of

$$\begin{cases} \partial_{tt} Y_k - \Delta Y_k + g'(y_k)\, Y_k = F_k\, 1_\omega + \partial_{tt} y_k - \Delta y_k + g(y_k) - f_k 1_\omega, & \text{in} \quad Q_T, \\ Y_k = 0, & \text{on} \quad \Sigma_T, \\ Y_k(\cdot, 0) = u_0 - y_k(\cdot, 0), \partial_t Y_k(\cdot, 0) = u_1 - \partial_t y_k(\cdot, 0) & \text{in} \quad \Omega \end{cases} \tag{5}$$

such that $Y_k(\cdot, T) = -y_k(\cdot, T)$ and $\partial_t Y_k(\cdot, T) = -\partial_t y_k(\cdot, T)$ in Ω. This linearization makes appear an operator K_N, so that $y_{k+1} = K_N(y_k)$ involving the first derivative of g. However, as it is well known, such a sequence may fail to converge if the initial guess (y_0, f_0) is not close enough to a zero of F (see [16] where divergence is observed numerically for large data).

The controllability of nonlinear partial differential equations has attracted a large number of works in the last decades (see the monography [9] and references therein). However, as far as we know, few are concerned with the approximation of exact controls for nonlinear partial differential equations, and the construction of convergent control approximations for controllable nonlinear equations remains a challenge.

In this article, given any initial data $(u_0, u_1) \in \boldsymbol{V}$, we design an algorithm providing a sequence $(y_k, f_k)_{k\in\mathbb{N}}$ converging to a controlled pair for (1), under assumptions on g that are slightly stronger than the one done in Theorem 1. Moreover, after a finite number of iterations, the convergence is super-linear. This is done by introducing a quadratic functional measuring how much a pair $(y, f) \in Y$ is close to a controlled solution for (1) and then by determining a particular minimizing sequence enjoying the announced property. A natural example of an error (or least-squares) functional is given by $\widetilde{E}(y, f) := \frac{1}{2}\|\widetilde{F}(y, f)\|_W^2$ to be minimized over Y. Exact controllability for (1) is reflected by the fact that the global minimum of the nonnegative functional $\widetilde{E}$ is zero, over all pairs $(y, f) \in Y$ solutions of (1). In the line of recent works on the Navier-Stokes system (see [25, 26]), we determine, using an appropriate descent direction, a minimizing sequence $(y_k, f_k)_{k\in\mathbb{N}}$ converging to a zero of the quadratic functional.

The paper is organized as follows. In Section 2, we define the (nonconvex) least-squares functional E and the corresponding (nonconvex) optimization problem (6). We show that E is Gateaux-differentiable and that any critical point (y, f) for E such that $g'(y) \in L^\infty(0, T; L^d(\Omega))$ is also a zero of E. This is done by introducing an adequate descent direction (Y^1, F^1) for E at any (y, f) for which $E'(y, f) \cdot (Y^1, F^1)$ is proportional to $E(y, f)$. This instrumental fact compensates the failure of convexity of E and is at the base of the global convergence properties of the least-squares algorithm. The design of this algorithm is done by determining a minimizing sequence based on (Y^1, F^1), which is proved to converge to a controlled pair for the semilinear wave equation (1), in our main result (Theorem 2), under appropriate assumptions on g. Moreover, we prove that, after a finite number of iterations, the convergence is super-linear. Theorem 2 is proved in Section 3. We give in Section 4 several comments and mention notably that our least-squares approach coincides with the *damped Newton method* applied to a mapping similar to $\widetilde{F}$. Section 5 then illustrates the result with some numerical experiments in the two dimensional case and Section 6 provides some conclusion. In Appendix A, we recall some *a priori* estimates for the linearized wave equation with potential in $L^\infty(0, T; L^d(\Omega))$ and source term in $L^2(Q_T)$ and we show that the operator K is contracting if $\|\hat{g}'\|_{L^\infty(\mathbb{R})}$ is small enough.

As far as we know, the method introduced and analyzed in this work is the first one providing an explicit, algorithmic construction of exact controls for semilinear wave equations with non Lipschitz nonlinearity and defined over multi-dimensional bounded domains. It extends the one-dimensional study addressed in [34]. For parabolic equations with Lipschitz nonlinearity, we mention [23]. These works devoted to controllability problems takes their roots in earlier works, namely [25, 26], concerned with the approximation of solution of Navier-Stokes type problem, through least-square methods: they refine the analysis performed in [27, 32] inspired from the seminal contribution [2].

Notations. Throughout, we denote by $\|\cdot\|_\infty$ the usual norm in $L^\infty(\mathbb{R})$, by $(\cdot,\cdot)_X$ the scalar product of X (if X is a Hilbert space) and by $\langle\cdot,\cdot\rangle_{X,Y}$ the duality product between X and Y. The notation $\|\cdot\|_{2,q_T}$ stands for $\|\cdot\|_{L^2(q_T)}$ and $\|\cdot\|_p$ for $\|\cdot\|_{L^p(Q_T)}$, $p\in\mathbb{N}^\star$.

Given any $s\in[0,1]$, we introduce for any $g\in C^1(\mathbb{R})$ the following hypothesis :

$$(\overline{\mathbf{H}}_\mathbf{s})\quad [g']_s := \sup_{\substack{a,b\in\mathbb{R}\\ a\neq b}} \frac{|g'(a)-g'(b)|}{|a-b|^s} < +\infty$$

meaning that g' is uniformly Hölder continuous with exponent s. For $s=0$, by extension, we set $[g']_0 := 2\|g'\|_\infty$. In particular, g satisfies $(\overline{\mathbf{H}}_\mathbf{0})$ if and only if $g\in\mathcal{C}^1(\mathbb{R})$ and $g'\in L^\infty(\mathbb{R})$, and g satisfies $(\overline{\mathbf{H}}_\mathbf{1})$ if and only if g' is Lipschitz continuous (in this case, g' is almost everywhere differentiable and $g''\in L^\infty(\mathbb{R})$, and we have $[g']_s\le\|g''\|_\infty$).

We also denote by C a positive constant depending only on Ω and T that may vary from lines to lines.

In the rest of the paper, we assume that the open set ω and the time T satisfy $(\mathbf{H}_\mathbf{0})$.

2 The least-squares functional and its properties

2.1 The least-squares problem

We define the Hilbert space $\mathcal{H}$

$$\mathcal{H} = \Big\{(y,f)\in L^2(Q_T)\times L^2(q_T),\ y\in\mathcal{C}([0,T];H_0^1(\Omega))\cap\mathcal{C}^1([0,T];L^2(\Omega))\mid \partial_{tt}y-\Delta y\in L^2(Q_T)\Big\}$$

endowed with the scalar product

$$\begin{aligned}((y,f),(\overline{y},\overline{f}))_\mathcal{H} = (y,\overline{y})_2 + \big((y(\cdot,0),\partial_t y(\cdot,0)),&(\overline{y}(\cdot,0),\partial_t\overline{y}(\cdot,0))\big)_{\boldsymbol{V}}\\ &+(\partial_{tt}y-\Delta y,\partial_{tt}\overline{y}-\Delta\overline{y})_2+(f,\overline{f})_{2,q_T}\end{aligned}$$

and the norm $\|(y,f)\|_\mathcal{H} := \sqrt{((y,f),(y,f))_\mathcal{H}}$.

Remark 1. *We highlight that, endowed with the norm $\|(y,\partial_t y)\|_{L^\infty(0,T;\boldsymbol{V})} := \|y\|_{L^\infty(0,T;H_0^1(\Omega))}+\|\partial_t y\|_{L^\infty(0,T;L^2(\Omega))}$, $\mathcal{C}([0,T];H_0^1(\Omega))\cap\mathcal{C}^1([0,T];L^2(\Omega))$ is a Banach space and $\mathcal{H}\hookrightarrow\big(\mathcal{C}([0,T];H_0^1(\Omega))\cap\mathcal{C}^1([0,T];L^2(\Omega))\big)\times L^2(q_T)$ continuously. Indeed, if $(y,f)\in\mathcal{H}$, we easily deduce from [29, Lemme 3.6 p. 39] that*

$$\|(y,\partial_t y)\|_{L^\infty(0,T;\boldsymbol{V})}\le C(\|\partial_{tt}y-\Delta y\|_{L^2(Q_T)}+\|(y(\cdot,0),\partial_t y(\cdot,0))\|_{\boldsymbol{V}})$$

from which we deduce that $\|(y,\partial_t y)\|_{L^\infty(0,T;\boldsymbol{V})}+\|f\|_{L^2(q_T)}\le C\|(y,f)\|_\mathcal{H}$.

Let $(u_0,u_1),(z_1,z_1)\in\boldsymbol{V}$. We define the nonempty subspaces of $\mathcal{H}$

$$\mathcal{A} = \Big\{(y,f)\in\mathcal{H}\mid (y(\cdot,0),\partial_t y(\cdot,0))=(u_0,u_1),\ (y(\cdot,T),\partial_t y(\cdot,T))=(z_0,z_1)\Big\},$$
$$\mathcal{A}_0 = \Big\{(y,f)\in\mathcal{H}\mid (y(\cdot,0),\partial_t y(\cdot,0))=(0,0),\ (y(\cdot,T),\partial_t y(\cdot,T))=(0,0)\Big\}.$$

Remark that $(0,0)\in\mathcal{A}_0$ while $\mathcal{A}$ contains the controlled pairs for the linear wave equation.

We consider the following non convex extremal problem :

$$\inf_{(y,f)\in\mathcal{A}} E(y,f),\qquad E(y,f) := \frac{1}{2}\big\|\partial_{tt}y-\Delta y+g(y)-f\,1_\omega\big\|_2^2 \tag{6}$$

justifying the least-squares terminology we have used. Remark that we can write $\mathcal{A}=(\overline{y},\overline{f})+\mathcal{A}_0$ for any element $(\overline{y},\overline{f})\in\mathcal{A}$. The problem is therefore equivalent to the minimization of $E(\overline{y}+y,f+\overline{f})$ over $\mathcal{A}_0$ for any $(\overline{y},\overline{f})\in\mathcal{A}$.

The functional E is well-defined in $\mathcal{A}$. Precisely,

Lemma 1. *There exists a positive constant $C > 0$ such that $E(y, f) \leq C(1+\|(y, f)\|_{\mathcal{H}}^3)$ for any $(y, f) \in \mathcal{A}$.*

Proof. *A priori* estimate for the linear wave equation reads as

$$\|(y, \partial_t y)\|^2_{L^\infty(0,T;\boldsymbol{V})} \leq C\Big(\|\partial_{tt} y - \Delta y\|_2^2 + \|(u_0, u_1)\|^2_{\boldsymbol{V}}\Big)$$

for any y such that $(y, f) \in \mathcal{A}$. Using that $|g(r)| \leq C(1 + |r|)\log(2 + |r|)$ for every $r \in \mathbb{R}$ and some $C > 0$, we infer that

$$\begin{aligned}\|g(y)\|_2^2 &\leq C^2 \int_{Q_T} \Big((1 + |y|)\log(2 + |y|)\Big)^2 \\ &\leq C^2 \int_{Q_T} (1 + |y|)^3 \leq C^2(|Q_T|^3 + \|y\|^3_{L^3(Q_T)}) \\ &\leq C^2\big(|Q_T|^3 + \|y\|^3_{L^\infty(0,T;H_0^1(\Omega))}\big)\end{aligned}$$

for which we get $E(y, f) \leq C\big(\|\partial_{tt} y - \Delta y\|_2^2 + \|f\|^2_{2,q_T} + |Q_T|^3 + \|y\|^3_{L^\infty(0,T;H_0^1(\Omega))}\big)$ leading to the result. □

Within the hypotheses of Theorem 1, the infimum of the functional of E is zero and is reached by at least one pair $(y, f) \in \mathcal{A}$, solution of (1) and satisfying $(y(\cdot, T), \partial_t y(\cdot, T)) = (z_0, z_1)$. Conversely, any pair $(y, f) \in \mathcal{A}$ for which $E(y, f)$ vanishes is solution of (1). In this sense, the functional E is an *error functional* which measures the deviation of (y, f) from being a solution of the underlying nonlinear equation. A practical way of taking a functional to its minimum is through the use of gradient descent directions. In doing so, the presence of local minima is always something that may dramatically spoil the whole scheme. The unique structural property that discards this possibility is the convexity of the functional E. However, for nonlinear equation like (1), one cannot expect this property to hold for the functional E. Nevertheless, we are going to construct a minimizing sequence which always converges to a zero of E.

In order to construct such minimizing sequence, we formally look, for any $(y, f) \in \mathcal{A}$, for a pair $(Y^1, F^1) \in \mathcal{A}_0$ solution of the following formulation

$$\begin{cases} \partial_{tt} Y^1 - \Delta Y^1 + g'(y) \cdot Y^1 = F^1 1_\omega + \big(\partial_{tt} y - \Delta y + g(y) - f\, 1_\omega\big), & \text{in } Q_T, \\ Y^1 = 0, & \text{on } \Sigma_T, \\ (Y^1(\cdot, 0), \partial_t Y^1(\cdot, 0)) = (0, 0), & \text{in } \Omega. \end{cases} \tag{7}$$

Since (Y^1, F^1) belongs to $\mathcal{A}_0$, F^1 is a null control for Y^1. Among the controls of this linear equation, we select the control of minimal $L^2(q_T)$ norm. In the sequel, we shall call the corresponding solution $(Y^1, F^1) \in \mathcal{A}_0$ the solution of *minimal control norm*. We have the following property.

Proposition 1. *For any $(y, f) \in \mathcal{A}$, there exists a pair $(Y^1, F^1) \in \mathcal{A}_0$ solution of (7). Moreover, the pair (Y^1, F^1) of minimal control norm satisfies the following estimates :*

$$\|(Y^1, \partial_t Y^1)\|_{L^\infty(0,T;\boldsymbol{V})} + \|F^1\|_{2,q_T} \leq C e^{C\|g'(y)\|^2_{L^\infty(0,T;L^d(\Omega))}} \sqrt{E(y, f)}, \tag{8}$$

and

$$\|(Y^1, F^1)\|_{\mathcal{H}} \leq C\big(1 + \|g'(y)\|_{L^\infty(0,T;L^3(\Omega))}\big) e^{C\|g'(y)\|^2_{L^\infty(0,T;L^d(\Omega))}} \sqrt{E(y, f)} \tag{9}$$

for some positive constant $C > 0$.

Proof. The first estimate is a consequence of Proposition 7 using the equality $\|\partial_{tt} y - \Delta y + g(y) - f\, 1_\omega\|_2 = \sqrt{2E(y, f)}$. The second one follows from

$$\begin{aligned}\|(Y^1, F^1)\|_{\mathcal{H}} &\leq \|\partial_{tt} Y^1 - \Delta Y^1\|_2 + \|Y^1\|_2 + \|F^1\|_{2,q_T} + \|Y^1(\cdot, 0), \partial_t Y^1(\cdot, 0)\|_{\boldsymbol{V}} \\ &\leq \|Y^1\|_2 + \|g'(y) Y^1\|_2 + 2\|F^1\|_{2,q_T} + \sqrt{2}\sqrt{E(y, f)} \\ &\leq C\big(1 + \|g'(y)\|_{L^\infty(0,T;L^3(\Omega))}\big) e^{C\|g'(y)\|^2_{L^\infty(0,T;L^d(\Omega))}} \sqrt{E(y, f)}\end{aligned}$$

using that

$$\begin{aligned}\|g'(y)Y^1\|_2^2 &\leq \int_0^T \|g'(y)\|_{L^3(\Omega)}^2 \|Y^1\|_{L^6(\Omega)}^2 \\ &\leq T\|g'(y)\|_{L^\infty(0,T;L^3(\Omega))}^2 \|Y^1\|_{L^\infty(0,T;L^6(\Omega))}^2 \\ &\leq C\|g'(y)\|_{L^\infty(0,T;L^3(\Omega))}^2 \|Y^1\|_{L^\infty(0,T;H_0^1(\Omega))}^2.\end{aligned}$$

□

2.2 Main properties of the functional E

The interest of the pair $(Y^1, F^1) \in \mathcal{A}_0$ lies in the following result.

Proposition 2. *Assume that g satisfies $(\overline{\mathbf{H}}_\mathbf{s})$ for some $s \in [0,1]$. Let $(y,f) \in \mathcal{A}$ and let $(Y^1, F^1) \in \mathcal{A}_0$ be a solution of (7). Then the derivative of E at the point $(y,f) \in \mathcal{A}$ along the direction (Y^1, F^1) satisfies*

$$E'(y,f)\cdot(Y^1,F^1) = 2E(y,f). \tag{10}$$

Proof. We preliminary check that for all $(Y,F) \in \mathcal{A}_0$ the functional E is differentiable at the point $(y,f) \in \mathcal{A}$ along the direction $(Y,F) \in \mathcal{A}_0$. For any $\lambda \in \mathbb{R}$, simple computations lead to the equality

$$E(y+\lambda Y, f+\lambda F) = E(y,f) + \lambda E'(y,f)\cdot(Y,F) + h((y,f),\lambda(Y,F))$$

with

$$E'(y,f)\cdot(Y,F) := \big(\partial_{tt}y - \Delta y + g(y) - f\,1_\omega, \partial_{tt}Y - \Delta y + g'(y)Y - F\,1_\omega\big)_2 \tag{11}$$

and

$$\begin{aligned}h((y,f),\lambda(Y,F)) :=& \frac{\lambda^2}{2}\big(\partial_{tt}Y - \Delta Y + g'(y)Y - F\,1_\omega, \partial_{tt}Y - \Delta Y + g'(y)Y - F\,1_\omega\big)_2 \\ &+ \lambda\big(\partial_{tt}Y - \Delta Y + g'(y)Y - F\,1_\omega, l(y,\lambda Y)\big)_2 \\ &+ \big(\partial_{tt}y - \Delta y + g(y) - f\,1_\omega, l(y,\lambda Y)\big) + \frac{1}{2}(l(y,\lambda Y), l(y,\lambda Y))\end{aligned}$$

where $l(y,\lambda Y) := g(y+\lambda Y) - g(y) - \lambda g'(y)Y$. The application $(Y,F) \to E'(y,f)\cdot(Y,F)$ is linear and continuous from $\mathcal{A}_0$ to $\mathbb{R}$ as it satisfies

$$\begin{aligned}|E'(y,f)\cdot(Y,F)| &\leq \|\partial_{tt}y - \Delta y + g(y) - f\,1_\omega\|_2 \|\partial_{tt}Y - \Delta Y + g'(y)Y - F\,1_\omega\|_2 \\ &\leq \sqrt{2E(y,f)}\Big(\|(\partial_{tt}Y - \Delta Y)\|_2 + \|g'(y)\|_{L^\infty(0,T;L^3(\Omega))}\|Y^1\|_{L^\infty(0,T;H_0^1(\Omega))} + \|F\|_{2,q_T}\Big) \\ &\leq \sqrt{2E(y,f)}\max\big(1, \|g'(y)\|_{L^\infty(0,T;L^3(\Omega))}\big)\|(Y,F)\|_{\mathcal{H}}.\end{aligned} \tag{12}$$

Similarly, for all $\lambda \in \mathbb{R}^\star$,

$$\begin{aligned}\left|\frac{1}{\lambda}h((y,f),\lambda(Y,F))\right| \leq& \frac{|\lambda|}{2}\|\partial_{tt}Y - \Delta Y + g'(y)Y - F\,1_\omega\|_2^2 \\ &+ \left(|\lambda|\|\partial_{tt}Y - \Delta Y + g'(y)Y - F\,1_\omega\|_2 + \sqrt{2E(y,f)} + \frac{1}{2}\|l(y,\lambda Y)\|_2\right)\frac{1}{|\lambda|}\|l(y,\lambda Y)\|_2.\end{aligned}$$

For any $(x,y) \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$, we then write $g(x+\lambda y) - g(x) = \int_0^\lambda y g'(x+\xi y)d\xi$ leading to

$$\begin{aligned}|g(x+\lambda y) - g(x) - \lambda g'(x)y| &\leq |\int_0^\lambda |y||g'(x+\xi y) - g'(x)|d\xi| \\ &\leq |\int_0^\lambda |y|^{1+s}|\xi|^s \frac{|g'(x+\xi y) - g'(x)|}{|\xi y|^s}d\xi| \leq [g']_s|y|^{1+s}\frac{|\lambda|^{1+s}}{1+s}.\end{aligned}$$

It follows that

$$|l(y,\lambda Y)| = |g(y+\lambda Y) - g(y) - \lambda g'(y)Y| \le [g']_s \frac{|\lambda|^{1+s}}{1+s}|Y|^{1+s}$$

and

$$\frac{1}{|\lambda|}\big\|l(y,\lambda Y)\big\|_2 \le [g']_s \frac{|\lambda|^s}{1+s}\big\||Y|^{1+s}\big\|_2. \tag{13}$$

But $\big\||Y|^{1+s}\big\|_2^2 = \|Y\|_{2(s+1)}^{2(s+1)} \le C\|Y\|_{L^\infty(0,T;L^4(\Omega))}^{2(s+1)}$. Consequently, for $s>0$, $|\frac{1}{\lambda}|\|l(y,\lambda Y)\|_2 \to 0$ as $\lambda \to 0$ and $|h((y,f),\lambda(Y,F))| = o(\lambda)$. In the case $s=0$ leading to $g' \in L^\infty(\mathbb{R})$, the result follows from the Lebesgue dominated convergence theorem: we have

$$\Big|\frac{1}{\lambda}l(y,\lambda Y)\Big| = \Big|\frac{g(y+\lambda Y)-g(y)}{\lambda} - g'(y)Y\Big| \le 2\|g'\|_\infty |Y|, \quad a.e. \text{ in } Q_T$$

and $\big|\frac{1}{\lambda}l(y,\lambda Y)\big| = \big|\frac{g(y+\lambda Y)-g(y)}{\lambda} - g'(y)Y\big| \to 0$ as $\lambda \to 0$ a.e. in Q_T. It follows that $|\frac{1}{\lambda}|\|l(y,\lambda Y)\|_2 \to 0$ as $\lambda \to 0$ as well. We deduce that the functional E is differentiable at the point $(y,f) \in \mathcal{A}$ along the direction $(Y,F) \in \mathcal{A}_0$.

Eventually, the equality (10) follows from the definition of the pair (Y^1,F^1) given in (7). □

Remark that from the equality (11), the derivative $E'(y,f)$ is independent of (Y,F). We can then define the norm $\|E'(y,f)\|_{\mathcal{A}_0'} := \sup_{(Y,F)\in\mathcal{A}_0\setminus\{0\}} \frac{E'(y,f)\cdot(Y,F)}{\|(Y,F)\|_{\mathcal{H}}}$ associated with $\mathcal{A}_0'$, the topological dual of $\mathcal{A}_0$.

Combining the equality (10) and the inequality (8), we deduce the following estimate of $E(y,f)$ in term of the norm of $E'(y,f)$.

Proposition 3. *For any $(y,f) \in \mathcal{A}$, the following inequalities hold true:*

$$\begin{aligned}\frac{1}{\sqrt{2}\max\big(1,\|g'(y)\|_{L^\infty(0,T;L^3(\Omega))}\big)}\|E'(y,f)\|_{\mathcal{A}_0'} &\le \sqrt{E(y,f)} \\ &\le \frac{1}{\sqrt{2}}C\Big(1+\|g'(y)\|_{L^\infty(0,T;L^3(\Omega))}\Big)e^{C\|g'(y)\|^2_{L^\infty(0,T;L^d(\Omega))}}\|E'(y,f)\|_{\mathcal{A}_0'}\end{aligned} \tag{14}$$

where C is the positive constant from Proposition 1.

Proof. (10) rewrites $E(y,f) = \frac{1}{2}E'(y,f)\cdot(Y^1,F^1)$ where $(Y^1,F^1) \in \mathcal{A}_0$ is solution of (7) and therefore, with (9)

$$\begin{aligned}E(y,f) &\le \frac{1}{2}\|E'(y,f)\|_{\mathcal{A}_0'}\|(Y^1,F^1)\|_{\mathcal{A}_0} \\ &\le \frac{1}{2}C\big(1+\|g'(y)\|_{L^\infty(0,T;L^3(\Omega))}\big)e^{C\|g'(y)\|^2_{L^\infty(0,T;L^d(\Omega))}}\|E'(y,f)\|_{\mathcal{A}_0'}\sqrt{E(y,f)}.\end{aligned}$$

On the other hand, for all $(Y,F) \in \mathcal{A}_0$, the inequality (12), i.e.

$$|E'(y,f)\cdot(Y,F)| \le \sqrt{2E(y,f)}\max\big(1,\|g'(y)\|_{L^\infty(0,T;L^3(\Omega))}\big)\|(Y,F)\|_{\mathcal{A}_0}$$

leads to the left inequality. □

Consequently, any *critical* point $(y,f) \in \mathcal{A}$ of E (i.e., $E'(y,f)$ vanishes) such that $\|g'(y)\|_{L^\infty(0,T;L^3(\Omega))}$ is finite is a zero for E, a pair solution of the controllability problem. In other words, any sequence $(y_k,f_k)_{k>0}$ satisfying $\|E'(y_k,f_k)\|_{\mathcal{A}_0'} \to 0$ as $k \to \infty$ and for which $\|g'(y_k)\|_{L^\infty(0,T;L^3(\Omega))}$ is uniformly bounded is such that $E(y_k,f_k) \to 0$ as $k \to \infty$. We insist that this property does not imply the convexity of the functional E (and *a fortiori* the strict convexity of E, which actually does not hold here in view of the multiple zeros for E) but show that a minimizing sequence for E can not be stuck in a local minimum.

On the other hand, the left inequality indicates the functional E is flat around its zero set. As a consequence, gradient-based minimizing sequences may achieve a low speed of convergence (we refer to [33] and also [27] devoted to the Navier-Stokes equation where this phenomenon is observed).

We end this section with the following estimate.

Lemma 2. *Assume that g satisfies $(\overline{\mathbf{H}}_{\mathbf{s}})$ for some $s \in [0,1]$. For any $(y,f) \in \mathcal{A}$, let $(Y^1, F^1) \in \mathcal{A}_0$ be defined by (7). For any $\lambda \in \mathbb{R}$ the following estimate holds*

$$E\big((y,f) - \lambda(Y^1,F^1)\big) \leq E(y,f)\bigg(|1-\lambda| + |\lambda|^{1+s}\, c(y) E(y,f)^{s/2}\bigg)^2 \tag{15}$$

with

$$c(y) := \frac{C}{(1+s)\sqrt{2}}[g']_s d(y)^{1+s}, \quad d(y) := Ce^{C\|g'(y)\|^2_{L^\infty(0,T;L^d(\Omega))}}.$$

Proof. Estimate (13) applied with $Y = Y^1$ reads

$$\big\|l(y,\lambda Y^1)\big\|_2 \leq [g']_s \frac{|\lambda|^{1+s}}{1+s}\big\||Y^1|^{1+s}\big\|_2. \tag{16}$$

But $\big\||Y^1|^{1+s}\big\|_2^2 = \|Y^1\|^{2(s+1)}_{2(s+1)} \leq C\|Y^1\|^{2(s+1)}_{L^\infty(0,T;H^1_0(\Omega))}$ which together with (8) lead to

$$\big\||Y^1|^{1+s}\big\|_2 \leq C\bigg(Ce^{C\|g'(y)\|_{L^\infty(0,T;L^d(\Omega)}}\bigg)^{1+s} E(y,f)^{\frac{1+s}{2}}. \tag{17}$$

Eventually, we write

$$\begin{aligned}
&2E\big((y,f) - \lambda(Y^1,F^1)\big)\\
&= \Big\|\big(\partial_{tt}y - \Delta y + g(y) - f\,1_\omega\big) - \lambda\big(\partial_{tt}Y^1 - \Delta Y^1 + g'(y)Y^1 - F\,1_\omega\big) + l(y,-\lambda Y^1)\Big\|_2^2\\
&= \Big\|(1-\lambda)\big(\partial_{tt}y - \Delta y + g(y) - f\,1_\omega\big) + l(y,-\lambda Y^1)\Big\|_2^2\\
&\leq \Big(\big\|(1-\lambda)\big(\partial_{tt}y - \Delta y + g(y) - f\,1_\omega\big)\big\|_2 + \big\|l(y,-\lambda Y^1)\big\|_2\Big)^2\\
&\leq 2\bigg(|1-\lambda|\sqrt{E(y,f)} + [g']_s\frac{|\lambda|^{1+s}}{1+s}\big\||Y^1|^{1+s}\big\|_2\bigg)^2\\
&\leq 2\bigg(|1-\lambda|\sqrt{E(y,f)} + [g']_s\frac{|\lambda|^{1+s}}{1+s}C\bigg(Ce^{C\|g'(y)\|_{L^\infty(0,T;L^d(\Omega)}}\bigg)^{1+s} E(y,f)^{\frac{1+s}{2}}\bigg)^2
\end{aligned} \tag{18}$$

and we get the result. □

3 Convergence of a minimizing sequence for E

We now examine the convergence of an appropriate sequence $(y_k, f_k) \in \mathcal{A}$. In this respect, we observe that equality (10) shows that $-(Y^1, F^1)$ given by the solution of (7) is a descent direction for E. Therefore, we can define, for any fixed $m \geq 1$, a minimizing sequence $(y_k, f_k)_{k>0} \in \mathcal{A}$ as follows:

$$\begin{cases}
(y_0, f_0) \in \mathcal{A},\\
(y_{k+1}, f_{k+1}) = (y_k, f_k) - \lambda_k(Y^1_k, F^1_k), \quad k \in \mathbb{N},\\
\lambda_k = \operatorname{argmin}_{\lambda\in[0,m]} E\big((y_k,f_k) - \lambda(Y^1_k, F^1_k)\big),
\end{cases} \tag{19}$$

where $(Y_k^1, F_k^1) \in \mathcal{A}_0$ is the solution of minimal control norm of

$$\begin{cases} \partial_{tt} Y_k^1 - \Delta Y_k^1 + g'(y_k) \cdot Y_k^1 = F_k^1 1_\omega + (\partial_{tt} y_k - \Delta y_k + g(y_k) - f_k 1_\omega), & \text{in } Q_T, \\ Y_k^1 = 0, & \text{on } \Sigma_T, \\ (Y_k^1(\cdot,0), \partial_t Y_k^1(\cdot,0)) = (0,0), & \text{in } \Omega. \end{cases} \tag{20}$$

The real number $m \geq 1$ is arbitrarily fixed and is introduced in order to keep the sequence $(\lambda_k)_{k\in\mathbb{N}}$ bounded.

Given any $s \in [0,1]$, we set

$$\beta^\star(s) := \sqrt{\frac{s}{2C(2s+1)}} \tag{21}$$

where $C > 0$, only depending on Ω and T, is the constant appearing in Proposition 7. In this section, we prove our main result.

Theorem 2. *Assume that g' satisfies $(\overline{\mathbf{H}}_\mathbf{s})$ for some $s \in [0,1]$ and*

$(\mathbf{H_2})$ *There exists $\alpha \geq 0$ and $\beta \in [0, \beta^\star(s))$ such that $|g'(r)| \leq \alpha + \beta \ln^{1/2}(1+|r|)$ for every r in $\mathbb{R}$*

if $s \in (0,1]$ and

$(\mathbf{H_3})$ $\sqrt{2}C\|g'\|_\infty e^{C\|g'\|_\infty^2 |\Omega|^{2/d}} < 1$

if $s = 0$.

Then, for any $(y_0, f_0) \in \mathcal{A}$, the sequence $(y_k, f_k)_{k\in\mathbb{N}}$ defined by (19) strongly converges to a pair $(\overline{y}, \overline{f}) \in \mathcal{A}$ satisfying (1) and the condition $(y(\cdot,T), y_t(\cdot,T)) = (z_0, z_1)$, for all $(u_0,u_1),(z_0,z_1) \in \boldsymbol{V}$. Moreover, the convergence is at least linear and is at least of order $1+s$ after a finite number of iterations[1].

Consequently, the algorithm (19) provides a constructive way to approximate a control for the nonlinear wave equation (19).

The proof consists in showing that the decreasing sequence $(E(y_k, f_k))_{k\in\mathbb{N}}$ converges to zero. In view of (14), this property is related to the uniform property of the observability constant $e^{C\|g'(y_k)\|^2_{L^\infty(0,T;L^d(\Omega))}}$ with respect to k. In order to fix some notations and the main ideas of the proof of Theorem 2, we first prove in Section 3.1 the convergence of the sequence $(y_k, f_k)_{k\in\mathbb{N}}$ under the stronger condition that $g' \in L^\infty(\mathbb{R})$, sufficient to ensure the boundedness of the sequence $\left(e^{C\|g'(y_k)\|^2_{L^\infty(0,T;L^d(\Omega))}}\right)_{k\in\mathbb{N}}$. Then, in Section 3.2, we prove Theorem 2 by showing that under the assumption $(\mathbf{H_2})$, the sequence $(y_k, f_k)_{k\in\mathbb{N}}$ is actually bounded in $\mathcal{A}$. This implies the same property for the real sequence $e^{C\|g'(y_k)\|^2_{L^\infty(0,T;L^d(\Omega))}}$, and then the announced convergence.

3.1 Proof of the convergence under the additionnal assumption $g' \in L^\infty(\mathbb{R})$

We establish in this section the following preliminary result, which coincides with Theorem 2 in the simpler case $\beta = 0$.

Proposition 4. *Assume that g' satisfies $(\overline{\mathbf{H}}_\mathbf{s})$ for some $s \in [0,1]$ and that $g' \in L^\infty(\mathbb{R})$. If $s = 0$, assume moreover $(\mathbf{H_3})$. For any $(y_0, f_0) \in \mathcal{A}$, the sequence $(y_k, f_k)_{k\in\mathbb{N}}$ defined by (19) strongly converges to a pair $(\overline{y}, \overline{f}) \in \mathcal{A}$ satisfying (1) and the condition $(y(\cdot,T), y_t(\cdot,T)) = (z_0, z_1)$. Moreover, the convergence is at least linear and is at least of order $1+s$ after a finite number of iterations.*

Proceeding as in [26, 34], Proposition 4 follows from the following lemma.

[1] We recall that a sequence $(u_k)_{k\in\mathbb{N}}$ of real numbers converges to 0 with order $\alpha \geq 1$ if there exists $M > 0$ such that $|u_{k+1}| \leq M|u_k|^\alpha$ for every $k \in \mathbb{N}$. A sequence $(v_k)_{k\in\mathbb{N}}$ of real numbers converges to 0 at least with order $\alpha \geq 1$ if there exists a sequence $(u_k)_{k\in\mathbb{N}}$ of nonnegative real numbers converging to 0 with order $\alpha \geq 1$ such that $|v_k| \leq u_k$ for every $k \in \mathbb{N}$.

Lemma 3. *Under the hypotheses of Proposition 4, for any $(y_0, f_0) \in \mathcal{A}$, there exists a $k_0 \in \mathbb{N}$ such that the sequence $(E(y_k, f_k))_{k \geq k_0}$ tends to 0 as $k \to \infty$ with at least a rate $s+1$.*

Proof. Since $g' \in L^\infty(\mathbb{R})$, the nonnegative constant $c(y_k)$ in (15) is uniformly bounded w.r.t. k: we introduce the real $c > 0$ as follows

$$c(y_k) \leq c := \frac{C}{(1+s)\sqrt{2}} [g']_s \left(C e^{C\|g'\|_\infty^2 |\Omega|^{2/d}} \right)^{1+s}, \quad \forall k \in \mathbb{N}. \tag{22}$$

$|\Omega|$ denotes the measure of the domain Ω. For any $(y_k, f_k) \in \mathcal{A}$, let us then denote the real function p_k by

$$p_k(\lambda) := |1-\lambda| + \lambda^{1+s} c\, E(y_k, f_k)^{s/2}, \quad \lambda \in [0, m].$$

Lemma 2 with $(y, f) = (y_k, f_k)$ then allows to write that

$$\sqrt{E(y_{k+1}, f_{k+1})} = \min_{\lambda \in [0,m]} \sqrt{E((y_k, f_k) - \lambda(Y_k^1, F_k^1))} \leq p_k(\widetilde{\lambda_k}) \sqrt{E(y_k, f_k)} \tag{23}$$

with $p_k(\widetilde{\lambda_k}) := \min_{\lambda \in [0,m]} p_k(\lambda)$. Assume first that $s > 0$. We then easily check that the optimal $\widetilde{\lambda_k}$ is given by

$$\widetilde{\lambda_k} := \begin{cases} \dfrac{1}{(1+s)^{1/s} c^{1/s} \sqrt{E(y_k, f_k)}}, & \text{if} \quad (1+s)^{1/s} c^{1/s} \sqrt{E(y_k, f_k)} \geq 1, \\ 1, & \text{if} \quad (1+s)^{1/s} c^{1/s} \sqrt{E(y_k, f_k)} < 1 \end{cases}$$

leading to

$$p_k(\widetilde{\lambda_k}) := \begin{cases} 1 - \dfrac{s}{(1+s)^{\frac{1}{s}+1}} \dfrac{1}{c^{1/s} \sqrt{E(y_k, f_k)}}, & \text{if} \quad (1+s)^{1/s} c^{1/s} \sqrt{E(y_k, f_k)} \geq 1, \\ c\, E(y_k, f_k)^{s/2}, & \text{if} \quad (1+s)^{1/s} c^{1/s} \sqrt{E(y_k, f_k)} < 1. \end{cases} \tag{24}$$

Accordingly, we may distinguish two cases :

• If $(1+s)^{1/s} c^{1/s} \sqrt{E(y_0, f_0)} < 1$, then $c^{1/s} \sqrt{E(y_0, f_0)} < 1$, and thus $c^{1/s} \sqrt{E(y_k, f_k)} < 1$ for all $k \in \mathbb{N}$ since the sequence $(E(y_k, f_k))_{k \in \mathbb{N}}$ is decreasing. Hence (23) implies that

$$c^{1/s} \sqrt{E(y_{k+1}, f_{k+1})} \leq \left(c^{1/s} \sqrt{E(y_k, f_k)} \right)^{1+s} \quad \forall k \in \mathbb{N}.$$

It follows that $c^{1/s} \sqrt{E(y_k, f_k)} \to 0$ as $k \to \infty$ with a rate equal to $1+s$.

• If $(1+s)^{1/s} c^{1/s} \sqrt{E(y_0, f_0)} \geq 1$ then we check that the set $I := \{k \in \mathbb{N},\ (1+s)^{1/s} c^{1/s} \sqrt{E(y_k, f_k)} \geq 1\}$ is a finite subset of $\mathbb{N}$; indeed, for all $k \in I$, (23) implies that

$$c^{1/s} \sqrt{E(y_{k+1}, f_{k+1})} \leq \left(1 - \frac{s}{(1+s)^{\frac{1}{s}+1}} \frac{1}{c^{1/s} \sqrt{E(y_k, f_k)}} \right) c^{1/s} \sqrt{E(y_k, f_k)} = c^{1/s} \sqrt{E(y_k, f_k)} - \frac{s}{(1+s)^{\frac{1}{s}+1}} \tag{25}$$

and the strict decrease of the sequence $(c^{1/s} \sqrt{E(y_k, f_k)})_{k \in I}$. Thus there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, $(1+s)^{1/s} c^{1/s} \sqrt{E(y_k, f_k)} < 1$, that is I is a finite subset of $\mathbb{N}$. Arguing as in the first case, it follows that $\sqrt{E(y_k, f_k)} \to 0$ as $k \to \infty$.

It follows in particular from (24) that the sequence $(p_k(\widetilde{\lambda_k}))_{k \in \mathbb{N}}$ decreases as well.

If now $s = 0$, then $p_k(\lambda) = |1-\lambda| + \lambda c$ with $c = [g']_0 C e^{C\|g'\|_\infty^2 |\Omega|^{2/d}}$ and (23) with $\widetilde{\lambda_k} = 1$ leads to $\sqrt{E(y_{k+1}, f_{k+1})} \leq c \sqrt{E(y_k, f_k)}$. The convergence of $(E(y_k, f_k))_{k \in \mathbb{N}}$ to 0 holds if $c < 1$, i.e. $(\mathbf{H_3})$.

□

Proof. (of Proposition 4) In view of (9), we write

$$\left(1 + \|g'(y)\|_{L^\infty(0,T;L^3(\Omega))}\right) e^{C\|g'(y)\|^2_{L^\infty(0,T;L^d(\Omega))}} \leq (1 + \|g'\|_\infty |\Omega|^{1/3}) e^{C\|g'\|_\infty^2 |\Omega|^{2/d}} \leq e^{2C\|g'\|_\infty^2 |\Omega|^{2/d}}$$

using that $(1+u)e^{u^2} \leq e^{2u^2}$ for all $u \in \mathbb{R}^+$. It follows that

$$\sum_{n=0}^{k} |\lambda_n| \|(Y_n^1, F_n^1)\|_{\mathcal{H}} \leq m\, C e^{C\|g'\|_\infty^2 |\Omega|^{2/d}} \sum_{n=0}^{k} \sqrt{E(y_n, f_n)}. \tag{26}$$

Using that $p_n(\widetilde{\lambda}_n) \leq p_0(\widetilde{\lambda}_0)$ for all $n \geq 0$, we can write for $n > 0$,

$$\sqrt{E(y_n, f_n)} \leq p_{n-1}(\widetilde{\lambda}_{n-1})\sqrt{E(y_{n-1}, f_{n-1})} \leq p_0(\widetilde{\lambda}_0)\sqrt{E(y_{n-1}, f_{n-1})} \leq (p_0(\widetilde{\lambda}_0))^n \sqrt{E(y_0, f_0)}. \tag{27}$$

Then, using that $p_0(\widetilde{\lambda}_0) = \min_{\lambda \in [0,m]} p_0(\lambda) < 1$ (since $p_0(0) = 1$ and $p_0'(0) < 0$), we finally obtain the uniform estimate

$$\sum_{n=0}^{k} |\lambda_n| \|(Y_n^1, F_n^1)\|_{\mathcal{H}} \leq m\, C e^{C\|g'\|_\infty^2 |\Omega|^{2/d}} \frac{\sqrt{E(y_0, f_0)}}{1 - p_0(\widetilde{\lambda}_0)}$$

for which we deduce (since $\mathcal{H}$ is a complete space) that the serie $\sum_{n \geq 0} \lambda_n (Y_n^1, F_n^1)$ converges in $\mathcal{A}_0$. Writing from (19) that $(y_{k+1}, f_{k+1}) = (y_0, f_0) - \sum_{n=0}^{k} \lambda_n (Y_n^1, F_n^1)$, we conclude that (y_k, f_k) strongly converges in $\mathcal{A}$ to $(\overline{y}, \overline{f}) := (y_0, f_0) + \sum_{n \geq 0} \lambda_n (Y_n^1, F_n^1)$.

Let us now pass to the limit in (20). We write that $\|g(y_k) - g(\overline{y})\|_{L^2(Q_T)} \leq \|g'\|_\infty \|y_k - \overline{y}\|_{L^2(Q_T)}$ and thus $g(y_k) \to g(\overline{y})$ in $L^2(Q_T)$. Moreover, $(g'(y_k))_{k \in \mathbb{N}}$ is a bounded sequence of $L^2(Q_T)$ since $g' \in L^\infty$. Then, using that (Y_k^1, F_k^1) goes to zero as $k \to \infty$ in $\mathcal{A}_0$, we pass to the limit in (20) and get that $(\overline{y}, \overline{f}) \in \mathcal{A}$ solves

$$\begin{cases} \partial_{tt}\overline{y} - \Delta \overline{y} + g(\overline{y}) = \overline{f} 1_\omega, & \text{in} \quad Q_T, \\ \overline{y} = 0, & \text{on } \Sigma_T, \\ (\overline{y}(\cdot, 0), \partial_t \overline{y}(\cdot, 0)) = (y_0, y_1), & \text{in } \Omega. \end{cases} \tag{28}$$

Since the limit $(\overline{y}, \overline{f})$ belongs to $\mathcal{A}$, we have that $(\overline{y}(\cdot, T), \overline{y}_t(\cdot, T)) = (z_0, z_1)$ in Ω. Moreover, for all $k > 0$

$$\begin{aligned} \|(\overline{y}, \overline{f}) - (y_k, f_k)\|_{\mathcal{H}} &= \left\| \sum_{p=k+1}^{\infty} \lambda_p (Y_p^1, F_p^1) \right\|_{\mathcal{H}} \leq m \sum_{p=k+1}^{\infty} \|(Y_p^1, F_p^1)\|_{\mathcal{H}} \\ &\leq m\, C \sum_{p=k+1}^{\infty} \sqrt{E(y_p, f_p)} \leq m\, C \sum_{p=k+1}^{\infty} p_0(\widetilde{\lambda}_0)^{p-k} \sqrt{E(y_k, f_k)} \\ &\leq m\, C \frac{p_0(\widetilde{\lambda}_0)}{1 - p_0(\widetilde{\lambda}_0)} \sqrt{E(y_k, f_k)} \end{aligned} \tag{29}$$

and conclude from Lemma 3 the convergence of order at least $1 + s$ after a finite number of iterates. □

Remark 2. *In particular, along the sequence $(y_k, f_k)_{k \in \mathbb{N}}$ defined by (19),* (29) *is a kind of coercivity property for the functional E. We emphasize, in view of the non uniqueness of the zeros of E, that an estimate (similar to* (29)*) of the form $\|(\overline{y}, \overline{f}) - (y, f)\|_{\mathcal{H}} \leq C\sqrt{E(y, f)}$ does not hold for all $(y, f) \in \mathcal{A}$. We also insist in the fact the sequence $(y_k, f_k)_{k \in \mathbb{N}}$ and its limits $(\overline{y}, \overline{f})$ are uniquely determined from the initialization $(y_0, f_0) \in \mathcal{A}$ and from our selection criterion for the control F^1.*

Remark 3. *Estimate* (26) *implies the uniform estimate on the sequence $(\|(y_k, f_k)\|_{\mathcal{H}})_{k \in \mathbb{N}}$:*

$$\begin{aligned} \|(y_k, f_k)\|_{\mathcal{H}} &\leq \|(y_0, f_0)\|_{\mathcal{H}} + m\, C e^{C\|g'\|_\infty^2 |\Omega|^{2/d}} \sum_{n=0}^{k-1} \sqrt{E(y_n, f_n)} \\ &\leq \|(y_0, f_0)\|_{\mathcal{H}} + m\, C e^{C\|g'\|_\infty^2 |\Omega|^{2/d}} \frac{\sqrt{E(y_0, f_0)}}{1 - p_0(\widetilde{\lambda}_0)}. \end{aligned}$$

In particular, for $s > 0$ and the less favorable case for which $(1+s)^{1/s}c^{1/s}\sqrt{E(y_0,f_0)} \geq 1$, we get $\frac{\sqrt{E(y_0,f_0)}}{1-p_0(\widetilde{\lambda}_0)} = \frac{(1+s)^{\frac{1}{s}+1}}{s}c^{1/s}E(y_0,f_0)$, (see (24)) leading to

$$\|(y_k,f_k)\|_{\mathcal{H}} \leq \|(y_0,f_0)\|_{\mathcal{H}} + m\, C e^{C\|g'\|_\infty^2|\Omega|^{2/d}} \frac{(1+s)^{\frac{1}{s}+1}}{s} c^{1/s}E(y_0,f_0),$$

and then, in view of (22) to the explicit estimate in term of the data

$$\|(y_k,f_k)\|_{\mathcal{H}} \leq \|(y_0,f_0)\|_{\mathcal{H}} + m\,\frac{(1+s)}{s}\left(\frac{C[g']_s}{\sqrt{2}}\right)^{1/s}\left(Ce^{C\|g'\|_\infty^2|\Omega|^{2/d}}\right)^{\frac{2s+1}{s}} E(y_0,f_0).$$

The case $s=0$ under the hypothesis $c<1$ leads to $\|(y_k,f_k)\|_{\mathcal{H}} \leq \|(y_0,f_0)\|_{\mathcal{H}} + m\,\frac{c\sqrt{E(y_0,f_0)}}{1-c}$.

Remark 4. *For $s>0$, recalling that the constant c is defined in (22), if $(1+s)^{1/s}c^{1/s}\sqrt{E(y_0,f_0)} \geq 1$, inequality (25) implies that*

$$c^{1/s}\sqrt{E(y_k,f_k)} \leq c^{1/s}\sqrt{E(y_0,f_0)} - k\frac{s}{(1+s)^{\frac{1}{s}+1}}, \quad \forall k \in I.$$

Hence, the number of iteration k_0 to achieve a rate $1+s$ is estimated as follows :

$$k_0 = \left\lfloor (1+s)\left(c^{1/s}(1+s)^{1/s}\sqrt{E(y_0,f_0)}\right) - \frac{1}{s}\right\rfloor + 1$$

where $\lfloor\cdot\rfloor$ denotes the integer part. As expected, this number increases with $\sqrt{E(y_0,f_0)}$ and $\|g'\|_\infty$. If $(1+s)^{1/s}c^{1/s}\sqrt{E(y_0,f_0)} < 1$, then $k_0 = 0$. In particular, as $s \to 0^+$, $k_0 \to \infty$ if $c>1$, i.e. if $(\mathbf{H_3})$ does not hold.

For $s=0$, the inequality $\sqrt{E(y_{k+1},f_{k+1})} \leq c\sqrt{E(y_k,f_k)}$ with $c<1$ leads to $k_0 = 0$.

We also have the following convergence result for the optimal sequence $(\lambda_k)_{k>0}$.

Lemma 4. *Assume that g' satisfies $(\overline{\mathbf{H}}_\mathbf{s})$ for some $s \in [0,1]$ and that $g' \in L^\infty(\mathbb{R})$. The sequence $(\lambda_k)_{k>k_0}$ defined in (19) converges to 1 as $k \to \infty$ at least with order $1+s$.*

Proof. In view of (18), we have, as long as $E(y_k,f_k) > 0$, since $\lambda_k \in [0,m]$

$$\begin{aligned}
(1-\lambda_k)^2 &= \frac{E(y_{k+1},f_{k+1})}{E(y_k,f_k)} - 2(1-\lambda_k)\frac{\big((\partial_{tt}y_k - \Delta y_k + g(y_k) - f_k\,1_\omega), l(y_k,\lambda_k Y_k^1)\big)_2}{E(y_k,f_k)} \\
&\qquad - \frac{\left\|l(y_k,\lambda_k Y_k^1)\right\|_2^2}{2E(y_k)} \\
&\leq \frac{E(y_{k+1},f_{k+1})}{E(y_k,f_k)} - 2(1-\lambda_k)\frac{\big((\partial_{tt}y_k - \Delta y_k + g(y_k) - f_k\,1_\omega), l(y_k,\lambda_k Y_k^1)\big)_2}{E(y_k,f_k)} \\
&\leq \frac{E(y_{k+1},f_{k+1})}{E(y_k,f_k)} + 2\sqrt{2}m\frac{\sqrt{E(y_k,f_k)}\|l(y_k,\lambda_k Y_k^1)\|_{L^2(Q_T)}}{E(y_k,f_k)} \\
&\leq \frac{E(y_{k+1},f_{k+1})}{E(y_k,f_k)} + 2\sqrt{2}m\frac{\|l(y_k,\lambda_k Y_k^1)\|_2}{\sqrt{E(y_k,f_k)}}.
\end{aligned}$$

But, from (16) and (17), we have $\|l(y_k,\lambda_k Y_k^1)\|_{L^2(Q_T)} \leq c\lambda_k^{1+s}E(y_k,f_k)^{\frac{1+s}{2}} \leq cm^{1+s}E(y_k,f_k)^{\frac{1+s}{2}}$ and thus

$$(1-\lambda_k)^2 \leq \frac{E(y_{k+1},f_{k+1})}{E(y_k,f_k)} + 2\sqrt{2}m^{2+s}cE(y_k,f_k)^{s/2}.$$

Consequently, since $E(y_k,f_k) \to 0$ and $\frac{E(y_{k+1},f_{k+1})}{E(y_k,f_k)} \to 0$ at least with order $1+s$, we deduce the result. □

3.2 Proof of Theorem 2

In this section, we relax the condition $g' \in L^\infty(\mathbb{R})$ and prove Theorem 3.2, for $s > 0$ under the assumption $(\mathbf{H_2})$. This assumption implies notably that $|g(r)| \leq C(1+|r|)\ln(2+|r|)$ for every $r \in \mathbb{R}$, mentioned in the introduction to state the well-posedness of (1). The case $\beta = 0$ corresponds to the case developed in the previous section, i.e. $g' \in L^\infty(\mathbb{R})$.

Within this more general framework, the difficulty is to have a uniform control with respect to k of the observability constant $Ce^{C\|g'(y_k)\|^2_{L^\infty(0,T;L^d(\Omega))}}$ appearing in the estimates for (Y^1_k, F^1_k), see Proposition 1. In other terms, we have to show that the sequence $(y_k, f_k)_{k\in\mathbb{N}}$ uniquely defined in (19) is uniformly bounded in $\mathcal{A}$, for any $(y_0, f_0) \in \mathcal{A}$.

We need the following intermediate result.

Lemma 5. *Let $C > 0$, only depending on Ω and T be the constant appearing in Proposition 7. Assume that g satisfies the growth condition $(\mathbf{H_2})$ and $2C\beta^2 \leq 1$. Then for any $(y, f) \in \mathcal{A}$,*

$$e^{C\|g'(y)\|^2_{L^\infty(0,T;L^d(\Omega))}} \leq 2\,C\max(1, e^{2C\alpha^2}|\Omega|^2)\left(1 + \frac{\|y\|_{L^\infty(0,T;L^{p^\star}(\Omega))}}{|\Omega|^{1/p^\star}}\right)^{2C\beta^2}$$

for any $p^\star \in \mathbb{N}^\star$ with $p^\star < \infty$ if $d = 2$ and $p^\star \leq 6$ if $d = 3$.

Proof. We use the following inequality (direct consequence of the inequality (3.8) in [28]):

$$e^{C\|g'(y)\|^2_{L^\infty(0,T;L^d(\Omega))}} \leq C\left(1 + \sup_{t\in(0,T)}\int_\Omega e^{C|g'(y)|^2}\right), \quad \forall (y, f) \in \mathcal{A}. \tag{30}$$

Writing that $|g'(y)|^2 \leq 2\big(\alpha^2 + \beta^2\ln(1+|y|)\big)$, we get that $\int_\Omega e^{C|g'(y)|^2} \leq e^{2C\alpha^2}\int_\Omega (1+|y|)^{2C\beta^2}$. Assuming $2C\beta^2 \leq p^\star$, Holder inequality leads to

$$\begin{aligned}\int_\Omega e^{C|g'(y)|^2} &\leq e^{2C\alpha^2}\left(\int_\Omega (1+|y|)^{p^\star}\right)^{\frac{2C\beta^2}{p^\star}}|\Omega|^{1-\frac{2C\beta^2}{p^\star}}\\ &\leq e^{2C\alpha^2}|\Omega|\left(1 + \frac{\|y\|_{L^{p^\star}(\Omega)}}{|\Omega|^{1/p^\star}}\right)^{2C\beta^2}.\end{aligned}$$

It follows, by (30), that for every $(y, f) \in \mathcal{A}$,

$$\begin{aligned}e^{C\|g'(y)\|^2_{L^\infty(0,T;L^d(\Omega))}} &\leq C\left(1 + e^{2C\alpha^2}|\Omega|\left(1 + \frac{\|y\|_{L^\infty(0,T;L^{p^\star}(\Omega))}}{|\Omega|^{1/p^\star}}\right)^{2C\beta^2}\right)\\ &\leq C\max(1, e^{2C\alpha^2}|\Omega|)\left(1 + \left(1 + \frac{\|y\|_{L^\infty(0,T;L^{p^\star}(\Omega))}}{|\Omega|^{1/p^\star}}\right)^{2C\beta^2}\right)\\ &\leq 2^{2C\beta^2}\,C\max(1, e^{2C\alpha^2}|\Omega|)\left(1 + \frac{\|y\|_{L^\infty(0,T;L^{p^\star}(\Omega))}}{|\Omega|^{1/p^\star}}\right)^{2C\beta^2}\end{aligned}$$

and the result. □

Lemma 6. *Assume that g satisfies the growth condition $(\mathbf{H_2})$ and $2C\beta^2 \leq 1$. For any $(y, f) \in \mathcal{A}$, the unique solution $(Y^1, F^1) \in \mathcal{A}_0$ of (7) satisfies*

$$\|(Y^1, \partial_t Y^1)\|_{L^\infty(0,T;\boldsymbol{V})} + \|F^1\|_{2,q_T} \leq d(y)\sqrt{E(y, f)}$$

with

$$d(y) := C_3(\alpha)\left(1 + \frac{\|y\|_{L^\infty(0,T;L^1(\Omega))}}{|\Omega|}\right)^{2C\beta^2}, \quad C_3(\alpha) := 2\,C\max(1, e^{2C\alpha^2}|\Omega|).$$

Proof. Lemma 5 with $p^\star = 1$ and estimate (8) lead to the result. □

With these notations, the term $c(y)$ in (15) rewrites as

$$c(y) = \frac{C}{(1+s)\sqrt{2}}[g']_s\, d(y)^{1+s}, \quad \forall (y,f) \in \mathcal{A},\, \forall s \in (0,1]. \tag{31}$$

Proof. (of Theorem 2) If the initialization $(y_0, f_0) \in \mathcal{A}$ is such that $E(y_0, f_0) = 0$, then the sequence $(y_k, f_k)_{k\in\mathbb{N}}$ is constant equal to (y_0, f_0) and therefore converges. We assume in the sequel that $E(y_0, f_0) > 0$.

We are going to prove that, for any $\beta < \beta^\star(s)$, there exists a constant $M > 0$ such that the sequence $(y_k)_{k\in\mathbb{N}}$ defined by (19) enjoys the uniform property

$$\|y_k\|_{L^\infty(0,T;L^1(\Omega))} \le M, \quad \forall k \in \mathbb{N}. \tag{32}$$

The convergence of the sequence $(y_k, f_k)_{k\in\mathbb{N}}$ in $\mathcal{A}$ will then follow by proceeding as in Section 3.1. Remark preliminary that the assumption $\beta < \beta^\star(s)$ implies $2C\beta^2 < \frac{s}{2s+1} \le 1$ since $s \in (0,1]$.

Proof of the uniform property (32) *for some M large enough*- As for $n = 0$, from any initialization (y_0, f_0) chosen in $\mathcal{A}$, it suffices to take M larger than $M_1 := \|y_0\|_{L^\infty(0,T;L^1(\Omega))}$. We then proceed by induction and assume that, for some $n \in \mathbb{N}$, $\|y_k\|_{L^\infty(0,T;L^1(\Omega))} \le M$ for all $k \le n$. This implies in particular that,

$$d(y_k) \le d_M(\beta) := C_3(\alpha)\left(1 + \frac{M}{|\Omega|}\right)^{2C\beta^2}, \quad \forall k \le n$$

and then

$$c(y_k) \le c_M(\beta) := \frac{C}{(1+s)\sqrt{2}}[g']_s\, d_M^{1+s}(\beta), \quad \forall k \le n. \tag{33}$$

Then, we write that $\|y_{n+1}\|_{L^\infty(0,T;L^1(\Omega))} \le \|y_0\|_{L^\infty(0,T;L^1(\Omega))} + \sum_{k=0}^n \lambda_k \|Y_k^1\|_{L^\infty(0,T;L^1(\Omega))}$. But, Lemma 6 implies that $\|Y_k^1\|_{L^\infty(0,T;L^1(\Omega))} \le d_M(\beta)\sqrt{E(y_k, f_k)}$ for all $k \le n$ leading to

$$\|y_{n+1}\|_{L^\infty(0,T;L^1(\Omega))} \le \|y_0\|_{L^\infty(0,T;L^1(\Omega))} + m\, d_M(\beta) \sum_{k=0}^n \sqrt{E(y_k, f_k)}. \tag{34}$$

Moreover, inequality (27) implies that $\sum_{k=0}^n \sqrt{E(y_k, f_k)} \le \frac{1}{1-p_0(\widetilde{\lambda_0})}\sqrt{E(y_0, f_0)}$ where $p_0(\widetilde{\lambda_0})$ is given by (24) with $c = c_M(\beta)$.

Now, we take M large enough so that $(1+s)^{1/s} c_M^{1/s}(\beta)\sqrt{E(y_0, f_0)} \ge 1$ i.e.

$$\left(\frac{C}{\sqrt{2}}\,[g']_s\right)^{1/s} C_3(\alpha)^{2/s}\left(1 + \frac{M}{|\Omega|}\right)^{\frac{4C\beta^2}{s}} \sqrt{E(y_0, f_0)} \ge 1. \tag{35}$$

Such M exists since $\sqrt{E(y_0, f_0)} > 0$ is independent of M and since the left hand side is of order $\mathcal{O}(M^{\frac{4C\beta^2}{s}})$ with $\frac{4C\beta^2}{s} > 0$. We denote by M_2 the smallest value of M such that (35) hold true.
Then, from (24), we get that $p_0(\widetilde{\lambda_0}) = 1 - \frac{s}{(1+s)^{\frac{1}{s}+1}} \frac{1}{c_M^{1/s}(\beta)\sqrt{E(y_0,f_0)}}$ and therefore

$$\frac{1}{1 - p_0(\widetilde{\lambda_0})} = \frac{(1+s)^{\frac{1}{s}+1}}{s} c_M^{1/s}(\beta)\sqrt{E(y_0, f_0)}$$

so that $\sum_{k=0}^n \sqrt{E(y_k, f_k)} \le \frac{(1+s)^{\frac{1}{s}+1}}{s} c_M^{1/s}(\beta) E(y_0, f_0)$. It follows from (34) that

$$\|y_{n+1}\|_{L^\infty(0,T;L^1(\Omega))} \le \|y_0\|_{L^\infty(0,T;L^1(\Omega))} + m\, d_M(\beta)\frac{(1+s)^{\frac{1}{s}+1}}{s} c_M^{1/s}(\beta) E(y_0, f_0).$$

The definition of $c_M(\beta)$ (see (33)) then gives

$$\|y_{n+1}\|_{L^\infty(0,T;L^1(\Omega))} \leq \|y_0\|_{L^\infty(0,T;L^1(\Omega))} + \frac{m(1+s)}{s}\left(\frac{C[g']_s}{\sqrt{2}}\right)^{1/s}\Big(C_3(\alpha)\Big)^{1+\frac{2}{s}} E(y_0,f_0)\left(1+\frac{M}{|\Omega|}\right)^{\frac{(2C\beta^2)(2s+1)}{s}}.$$

Now, we take $M>0$ large enough so that the right hand side is bounded by M, i.e.

$$\|y_0\|_{L^\infty(0,T;L^1(\Omega))} + \frac{m(1+s)}{s}\left(\frac{C[g']_s}{\sqrt{2}}\right)^{1/s}\Big(C_3(\alpha)\Big)^{1+\frac{2}{s}} E(y_0,f_0)\left(1+\frac{M}{|\Omega|}\right)^{\frac{(2C\beta^2)(2s+1)}{s}} \leq M. \tag{36}$$

Such M exists under the assumption $\beta<\beta^\star(s)$ equivalent to $\frac{(2C\beta^2)(2s+1)}{s}<1$. We denote by M_3 the smallest value of M such that (36) holds true. Eventually, taking $M:=\max(M_1,M_2,M_3)$, we get that $\|y_{n+1}\|_{L^\infty(0,T;L^1(\Omega))}\leq M$ as well. We have then proved by induction the uniform property (32) for some M large enough.

Proof of the convergence of the sequence $(y_k,f_k)_{k\in\mathbb{N}}$- In view of Lemma 5 with $p^\star=1$, the uniform property (32) implies that the observability constant $Ce^{C\|g'(y_k)\|^2_{L^\infty(0,T;L^d(\Omega))}}$ appearing in the estimates for (Y_k^1,F_k^1) (see Proposition 1) is uniformly bounded with respect to the parameter k. As a consequence, the constant $c(y_k)$ appearing in the instrumental estimate (15) is bounded by $c_M(\beta)$ given by (33). Consequently, the developments of Section 3.1 apply with $c=c_M(\beta)$. Theorem 2 then follows from the proof of Proposition 4 except for the limit in (20) with respect to k (since g' is not anymore in $L^\infty(Q_T)$). Since $g\in\mathcal{C}^1(\mathbb{R})$, a.e in Q_T there exists $0\leq\theta(x,t)\leq 1$ such that

$$\begin{aligned}|g(y_k(x,t))-g(\overline{y}(x,t))| &= |g'(\overline{y}(x,t)+\theta(x,t)y_k(x,t))||y_k(x,t)-\overline{y}(x,t)|\\ &\leq (\alpha+\beta\ln^{1/2}\big(1+|\overline{y}(x,t)+\theta(x,t)y_k(x,t)|\big))|y_k(x,t)-\overline{y}(x,t)|\\ &\leq (\alpha+\beta(|\overline{y}(x,t)|^{1/2}+|y_k(x,t)|^{1/2}))|y_k(x,t)-\overline{y}(x,t)|\end{aligned}$$

and thus

$$\|g(y_k)-g(\overline{y})\|_2\leq\big(\alpha|Q_T|^{1/4}+\beta(\|\overline{y}\|_2^{1/2}+\|y_k\|_2^{1/2})\big)\|y_k-\overline{y}\|_4.$$

Since $y_k\to\overline{y}$ in $L^4(Q_T)$, it follows that $g(y_k)\to g(\overline{y})$ in $L^2(Q_T)$. Moreover, since $(y_k)_{k\in\mathbb{N}}$ is a bounded sequence of $L^4(Q_T)$, the estimate

$$\|g'(y_k)\|_2\leq C(\alpha+\beta\|y_k\|_2^{1/2})\|y_k\|_4$$

implies that $(g'(y_k))_{k\in\mathbb{N}}$ is a bounded sequence of $L^2(Q_T)$. Then, using that (Y_k^1,F_k^1) goes to zero as $k\to\infty$ in $\mathcal{A}_0$, we pass to the limit in (20) and get that $(\overline{y},\overline{f})\in\mathcal{A}$ solves (28). □

Remark 5. *Remark that $M:=\max(M_2,M_3)$ since $M_3\geq M_1$. The constant M_2 can be made explicit since the constraint (35) implies that*

$$\left(\frac{C[g']_s}{\sqrt{2}}\right)^{1/s} C_3(\alpha)^{2/s}\left(1+\frac{M}{|\Omega|}\right)^{\frac{4C\beta^2}{s}}\sqrt{E(y_0,f_0)}\geq 1.$$

equivalent to

$$\left(1+\frac{M}{|\Omega|}\right)^{2C\beta^2}\geq C_3(\alpha)^{-1}\sqrt{E(y_0,f_0)}^{-s/2}\left(\frac{C}{\sqrt{2}}[g']_s\right)^{-1/2}.$$

In particular, M_2 is large for small values of $\sqrt{E(y_0,f_0)}$, for any $s>0$. On the other hand, the constant M_3 is no explicit, hence whether $M_2>M_3$ or $M_3>M_2$ depend on the values of $\sqrt{E(y_0,f_0)}$ and $\|y_0\|_{L^\infty(0,T;L^1(\Omega))}$. Remark that $\sqrt{E(y_0,f_0)}$ can be large and $\|y_0\|_{L^\infty(0,T;L^1(\Omega))}$ small, and vice versa.

4 Comments

Asymptotic condition. The asymptotic condition $(\mathbf{H_2})$ on g' is slightly stronger than the asymptotic condition $(\mathbf{H_1})$ made in [17]: this is due to our linearization of (1) which involves $r \to g'(r)$ while the linearization (2) in [17] involves $r \to (g(r) - g(0))/r$. There exist cases covered by Theorem 1 in which exact controllability for (1) is true but that are not covered by Theorem 2. Note however that the example $g(r) = a + br + cr \ln^{1/2}(1 + |r|)$, for any $a, b \in \mathbb{R}$ and for any $c > 0$ small enough (which is somehow the limit case in Theorem 1) satisfies $(\mathbf{H_2})$ as well as $(\overline{\mathbf{H}}_\mathbf{s})$ for any $s \in [0, 1]$.

While Theorem 1 was established in [17] by a nonconstructive Leray-Schauder fixed point argument, we obtain here, in turn, a new proof of the exact controllability of semilinear multi-dimensional wave equations, which is moreover constructive, with an algorithm that converges unconditionally, at least with order $1 + s$.

Minimization functional. Among all possible admissible controlled pair $(y, v) \in \mathcal{A}_0$, we have selected the solution (Y_1, F_1) of (7) that minimizes the functional $J(v) = \|v\|^2_{2,q_T}$. This choice has led to the estimate (8) which is one of the key points of the convergence analysis. The analysis remains true when one considers the quadratic functional $J(y, v) = \|w_1 v\|^2_{2,q_T} + \|w_2 y\|^2_2$ for some positive weight functions w_1 and w_2 (see for instance [7]).

Link with Newton method. Defining $F : \mathcal{A} \to L^2(Q_T)$ by $F(y, f) := (\partial_{tt} y - \Delta y + g(y) - f\, 1_\omega)$, we have $E(y, f) = \frac{1}{2}\|F(y, f)\|^2_2$ and we observe that, for $\lambda_k = 1$, the algorithm (19) coincides with the Newton algorithm associated to the mapping F (see (5)). This explains the super-linear convergence property in Theorem 2, in particular the quadratic convergence when $s = 1$. The optimization of the parameter λ_k gives to a global convergence property of the algorithm and leads to the so-called damped Newton method applied to F. For this method, global convergence is usually achieved with linear order under general assumptions (see for instance [12, Theorem 8.7]). As far as we know, the analysis of damped type Newton methods for partial differential equations has deserved very few attention in the literature. We mention [25, 35] in the context of fluids mechanics.

A variant. To simplify, let us take $\lambda_k = 1$, as in the standard Newton method. Then, for each $k \in \mathbb{N}$, the optimal pair $(Y^1_k, F^1_k) \in \mathcal{A}_0$ is such that the element (y_{k+1}, f_{k+1}) minimizes over $\mathcal{A}$ the functional $(z, v) \to J(z - y_k, v - f_k)$ with $J(z, v) := \|v\|_{2,q_T}$ (control of minimal $L^2(q_T)$ norm). Alternatively, we may select the pair (Y^1_k, F^1_k) so that the element (y_{k+1}, f_{k+1}) minimizes the functional $(z, v) \to J(z, v)$. This leads to the sequence $(y_k, f_k)_{k\in\mathbb{N}}$ defined by

$$\begin{cases} \partial_{tt} y_{k+1} - \Delta y_{k+1} + g'(y_k) y_{k+1} = f_{k+1} 1_\omega + g'(y_k) y_k - g(y_k) & \text{in } Q_T, \\ y_k = 0, & \text{on } \Sigma_T, \\ (y_{k+1}(\cdot, 0), \partial_t y_{k+1}(\cdot, 0)) = (u_0, u_1) & \text{in } \Omega. \end{cases} \tag{37}$$

In this case, for every $k \in \mathbb{N}$, (y_k, f_k) is a controlled pair for a linearized wave equation, while, in the case of the algorithm (19), (y_k, f_k) is a sum of controlled pairs (Y^1_j, F^1_j) for $0 \le j \le k$. This formulation used in [16] is different and the convergence analysis (at least in the least-squares setting) does not seem to be straightforward because the term $g'(y_k) y_k - g(y_k)$ is not easily bounded with respect to $\sqrt{E(y_k, f_k)}$.

Initialization with the controlled pair of the linear equation. The number of iterates to achieve convergence (notably to enter in a super-linear regime) depends on the size of the value $E(y_0, f_0)$. A natural example of an initialization $(y_0, f_0) \in \mathcal{A}$ is the unique solution of minimal control norm of (1) with $g = 0$ (i.e., in the linear case). Under the assumption $(\mathbf{H_2})$, this leads to the estimate

$$E(y_0, f_0) = \frac{1}{2}\|g(y_0)\|^2_2 \le |g(0)|^2 |Q_T| + 2 \int_{Q_T} |y_0|^2 \big(\alpha^2 + \beta^2 \ln(1 + |y_0|)\big).$$

Local controllability when removing the growth condition ($\mathbf{H_2}$). If the real $E(y_0, f_0)$ is small enough, we may remove the growth condition ($\mathbf{H_2}$) on g'.

Proposition 5. *Assume g' satisfies ($\overline{\mathbf{H}}_\mathbf{s}$) for some $s \in [0,1]$. Let $(y_k, f_k)_{k>0}$ be the sequence of $\mathcal{A}$ defined in (19). There exists a constant $C([g']_s)$ such that if $E(y_0, f_0) \leq C([g']_s)$, then $(y_k, f_k)_{k\in\mathbb{N}} \to (\overline{y}, \overline{f})$ in $\mathcal{A}$ where $\overline{f}$ is a null control for $\overline{y}$ solution of (1). Moreover, the convergence is at least linear and is at least of order $1+s$ after a finite number of iterations.*

Proof. In this proof, the notation $\|\cdot\|_{\infty,d}$ stands for $\|\cdot\|_{L^\infty(0,T;L^d(\Omega))}$. We note $D := \frac{C}{(1+s)\sqrt{2}}[g']_s$ and $e_k := c(y_k)E(y_k, f_k)^{s/2}$ with $c(y) := Dd(y)^{1+s}$ and $d(y) := Ce^{C\|g'(y)\|^2_{\infty,d}}$. (23) then reads

$$\sqrt{E(y_{k+1}, f_{k+1})} \leq \min_{\lambda\in[0,m]} \left(|1-\lambda| + \lambda^{1+s} e_k\right)\sqrt{E(y_k, f_k)}. \tag{38}$$

We write $|g'(y_k) - g'(y_k - \lambda_k Y_k^1)| \leq [g']_s |\lambda_k Y_k^1|^s$ so that

$$\|g'(y_{k+1})\|^2_{L^\infty(0,T;L^d(\Omega))} \leq \|g'(y_k)\|^2_{\infty,d} + \left([g']_s \lambda_k^s \|(Y_k^1)^s\|_{\infty,d}\right)^2 + 2\|g'(y_k)\|_{\infty,d}[g']_s\lambda_k^s\|(Y_k^1)^s\|_{\infty,d}$$

and

$$e^{C\|g'(y_{k+1})\|^2_{\infty,d}} \leq e^{C\|g'(y_k)\|^2_{\infty,d}} e^{C\left([g']_s\lambda_k^s\|(Y_k^1)^s\|_{\infty,d}\right)^2} e^{2C\|g'(y_k)\|_{\infty,d}\left([g']_s\lambda_k^s\|(Y_k^1)^s\|_{\infty,d}\right)}$$

leading to

$$\frac{c(y_{k+1})}{c(y_k)} \leq \left(e^{C\left([g']_s\lambda_k^s\|(Y_k^1)^s\|_{\infty,d}\right)^2} e^{2C\|g'(y_k)\|_{\infty,d}\left([g']_s\lambda_k^s\|(Y_k^1)^s\|_{\infty,d}\right)}\right)^{1+s}.$$

We infer that $\|(Y_k^1)^s\|_{\infty,d} = \|Y_k^1\|^s_{\infty,sd}$. Moreover, estimate (8) leads to

$$\|Y_k^1\|^s_{\infty,sd} \leq d^s(y_k)E(y_k, f_k)^{s/2} = \frac{c(y_k)^{\frac{s}{1+s}}}{D^{\frac{s}{1+s}}}E(y_k, f_k)^{s/2} \leq D^{-\frac{s}{1+s}}c(y_k)E(y_k, f_k)^{s/2}$$

using that $c(y_k) \geq 1$ (by increasing the constant C is necessary). Consequently,

$$e^{C\left([g']_s\lambda^s\|(Y_k^1)^s\|_{\infty,d}\right)^2} \leq e^{C\left([g']_s\lambda^s D^{-\frac{s}{1+s}}e_k\right)^2} := e^{C_1 e_k^2}.$$

Similarly,

$$\begin{aligned}
\|g'(y_k)\|_{\infty,d}\|(Y_k^1)^s\|_{\infty,d} &\leq \|g'(y_k)\|_{\infty,d} d^s(y_k)E(y_k, f_k)^{s/2} \\
&\leq \|g'(y_k)\|_{\infty,d}\left(Ce^{C\|g'(y)\|^2_{L^\infty(0,T;L^d(\Omega))}}\right)^s E(y_k, f_k)^{s/2} \\
&\leq \left(Ce^{C\|g'(y)\|^2_{L^\infty(0,T;L^d(\Omega))}}\right)^{s+1} E(y_k, f_k)^{s/2} \leq \frac{c(y_k)}{D}E(y_k, f_k)^{s/2} = \frac{e_k}{D}
\end{aligned}$$

using that $a \leq Ce^{Ca^2}$ for all $a \geq 0$ and $C > 0$ large enough. It follows that

$$e^{2C\|g'(y_k)\|_{\infty,d}\left([g']_s\lambda_k^s\|(Y_k^1)^s\|_{\infty,d}\right)} \leq e^{2C[g']_s\lambda_k^s\frac{e_k}{D}} := e^{C_2 e_k}$$

and then $\frac{c(y_{k+1})}{c(y_k)} \leq (e^{C_1 e_k^2 + C_2 e_k})^{1+s}$. By multiplying (38) by $c(y_{k+1})$, we obtain the inequality

$$e_{k+1} \leq \min_{\lambda\in[0,m]} \left(|1-\lambda| + e_k\lambda^{1+s}\right) \quad (e^{C_1 e_k^2 + C_2 e_k})^{1+s}\, e_k.$$

If $2e_k < 1$, the minimum is reached for $\lambda = 1$ leading $\frac{e_{k+1}}{e_k} \leq e_k(e^{C_1 e_k^2 + C_2 e_k})^{1+s}$. Consequently, if the initial guess (y_0, f_0) belongs to the set $\{(y_0, f_0) \in \mathcal{A}, e_0 < 1/2, e_0(e^{C_1 e_0^2 + C_2 e_0})^{1+s} < 1\}$, the sequence $(e_k)_{k>0}$ goes to zero as $k \to \infty$. Since $c(y_k) \geq 1$ for all $k \in \mathbb{N}$, this implies that the sequence $(E(y_k, f_k))_{k>0}$ goes to zero as well. Moreover, from (8), we get $D\|(Y_k^1, F_k^1)\|_{\mathcal{H}} \leq e_k\sqrt{E(y_k, f_k)}$ and repeating the arguments of the proof of Proposition 4, we conclude that the sequence $(y_k, f_k)_{k>0}$ converges to a controlled pair for (1). $\square$

These computations does not use the assumption $(\mathbf{H_2})$ on the nonlinearity g. However, the smallness assumption on e_0 requires a smallness assumption on $E(y_0, f_0)$ (since $c(y_0) > 1$). This is equivalent to assume the controllability of (1). Alternatively, in the case $g(0) = 0$, the smallness assumption on $E(y_0, f_0)$ is achieved as soon as $\|(u_0, u_1)\|_{\boldsymbol{V}}$ is small enough. Therefore, the convergence result stated in Proposition 5 is equivalent to the local controllability property for (1). Proposition 5 can also be seen as a consequence of the usual convergence of the Newton method: when $E(y_0, f_0)$ is small enough, i.e., when the initialization is close enough to the solution, then $\lambda_k = 1$ for every $k \in \mathbb{N}$ and we recover the standard Newton method.

Weakening of the condition $(\overline{\mathbf{H}}_\mathbf{s})$. Given any $s \in [0, 1]$, we introduce for any $g \in C^1(\mathbb{R})$ the following hypothesis :

$(\overline{\mathbf{H}}'_\mathbf{s})$ There exist $\overline{\alpha}, \overline{\beta}, \gamma \in \mathbb{R}^+$ such that $|g'(a) - g'(b)| \leq |a - b|^s \big(\overline{\alpha} + \overline{\beta}(|a|^\gamma + |b|^\gamma)\big), \quad \forall a, b \in \mathbb{R}$

which coincides with $(\overline{\mathbf{H}}_\mathbf{s})$ if $\gamma = 0$ for $\overline{\alpha} + \overline{\beta} = [g']_s$. If $\gamma \in (0, 1)$ is small enough and related to the constant β appearing in the growth condition $(\mathbf{H_2})$, Theorem 2 still holds if $(\overline{\mathbf{H}}_\mathbf{s})$ is replaced by the weaker hypothesis $(\overline{\mathbf{H}}'_\mathbf{s})$. Precisely, if g satisfies $(\mathbf{H_2})$ and $(\overline{\mathbf{H}}'_\mathbf{s})$ for some $s \in (0, 1]$, then the sequence $(y_k, f_k)_{k\in\mathbb{N}}$ defined by (19) fulfills the estimate

$$E(y_{k+1}, f_{k+1}) \leq E(y_k, f_k) \min_{\lambda\in[0,m]} \left(|1 - \lambda| + \lambda^{1+s}\, c(y_k) E(y_k, f_k)^{s/2} \right)^2$$

with $c(y) := \frac{1}{(1+s)\sqrt{2}} \left(\left(\overline{\alpha} + 2\overline{\beta}\|y_k\|^\gamma_{\infty,6\gamma}\right) + \overline{\beta} m^\gamma d(y)^\gamma E(y_0, f_0)^{\frac{\gamma}{2}} \right) d(y)^{1+s}$ and $d(y) := Ce^{C\|g'(y)\|^2_{L^\infty(0,T;L^d(\Omega))}}$. Using Lemma 5 with $p^\star = 6\gamma \leq 6$ and proceeding as in the proof of Theorem 2, one may prove by induction that the sequence $(\|y_k\|_{L^\infty(0,T;L^6(\Omega))})_{k\in\mathbb{N}}$ is uniformly bounded under the condition $\frac{\gamma + (2C\beta^2)(1+2s)}{s} < 1$ and then deduce the convergence of the sequence $(y_k, f_k)_{k\in\mathbb{N}}$.

5 Numerical illustrations in the two dimensional case

We illustrate in this section our results of convergence. We provide some practical details of the algorithm (19) then discuss some experiments in the two dimensional case performed with the software Freefem++ [19].

5.1 Algorithm and Approximation

We introduce a cut off χ of the form $\chi(t, x) = \chi_0(t)\chi_1(x)$ where $\chi_0 \in C_0^\infty(0, T)$ and $\chi_1 \in C_0^\infty(\omega)$ take values in $[0, 1]$ and then consider controls of minimal $L^2(\chi^{-1/2}, q_T)$ norm with $L^2(\chi^{-1/2}, q_T) := \{f : \int_{q_T} \chi^{-1} f^2 < \infty\}$. In order to determine linear controls, we employ the method introduced by Glowinski-Li-Lions in the seminal work [18] based on the unconstrained minimization of the conjugate functional. The algorithm (19) can be expended as follows :

1. *Initialization* - we define $(y_0, f_0) \in \mathcal{A}$ as the solution of the linear problem

$$\begin{cases} \partial_{tt} y_0 - \Delta y_0 = f_0 1_\omega, & \text{in } Q_T, \\ y_0 = 0, & \text{on } \Sigma_T, \\ (y_0(\cdot, 0), \partial_t y_0(\cdot, 0)) = (u_0, u_1), & \text{in } \Omega, \end{cases} \tag{39}$$

where f_0 is the control of minimal $L^2(\chi^{-1/2}, Q_T)$ norm. The control f_0 is given by $f_0 = \chi\varphi_0$ where φ_0 solves the adjoint problem

$$\begin{cases} \partial_{tt} \varphi - \Delta \varphi = 0, & \text{in } Q_T, \\ \varphi = 0, & \text{on } \Sigma_T, \\ (\varphi(\cdot, T), \partial_t \varphi(\cdot, T)) = (w_0, w_1), & \text{in } \Omega, \end{cases} \tag{40}$$

with $(w_0, w_1) \in \boldsymbol{H} := L^2(\Omega) \times H^{-1}(\Omega)$ the solution of the unconstrained extremal problem

$$\inf_{(w_0,w_1)\in\boldsymbol{H}} J_0^\star(w_0,w_1) := \frac{1}{2}\int_{q_T} \chi|\varphi|^2 - < u_0, \partial_t\varphi(\cdot,0) >_{H_0^1(\Omega),H^{-1}(\Omega)} + (u_1, \varphi(\cdot,0))_{L^2(\Omega)}$$
$$+ < z_0, w_1 >_{H_0^1(\Omega),H^{-1}(\Omega)} - (z_1, w_0)_{L^2(\Omega)}.$$

The resolution of this extremal problem is done using the Fletcher-Reeves conjugate gradient algorithm initialized with $(w_{0,0}, w_{1,0}) = (0,0)$. The stopping criterion used is $\|g_p\|_{\boldsymbol{H}} \leq 10^{-10}\|g_0\|_{\boldsymbol{H}}$ where $g_p := (z_p(T,\cdot) - z_0, z_1 - \partial_t z_p(T,\cdot))$ denotes the gradient of $J_0^\star$ at the iteration p and z_p the solution of (39) with control function χw_p where w_p solves (40) with initial data $(w_{0,p}, w_{1,p})$.

Assume now that $(y_k, f_k)_{0\leq k\leq n} \in \mathcal{A}$ is computed for some $n \geq 0$.

2. *Evaluation of the least-squares functional* - If $\sqrt{2E(y_n,f_n)} \leq 10^{-5}$, then the algorithm stops. The real $E(y_n,f_n)$ is defined as $E(y_n,f_n) = \frac{1}{2}\|c_n\|^2_{L^2(Q_T)}$ where c_n solves

$$\int_{Q_T} c_n\bar{c} + \int_{Q_T} \partial_{tt}y_n\bar{c} + \nabla y_n\cdot\nabla\bar{c} + g(y_n)\bar{c} - \int_{q_T} f_n\bar{c} = 0, \quad \forall \bar{c} \in L^2(0,T;H_0^1(\Omega)).$$

3. *Descent direction* - We compute the controlled pair $(Y_n^1, F_n^1) \in \mathcal{A}_0$

$$\begin{cases} \partial_{tt}Y_n^1 - \Delta Y_n^1 + A_nY_n^1 = F_n^1 1_\omega + B_n & \text{in } Q_T, \\ Y_n^1 = 0 & \text{on } \Sigma_T, \\ (Y_n^1(\cdot,0), \partial_t Y_n^1(\cdot,0)) = (0,0) & \text{in } \Omega, \end{cases} \tag{41}$$

with potential $A_n := g'(y_n)$ and source term $B_n := \partial_{tt}y_n - \Delta y_n + g(y_n) - f_n 1_\omega$. The control F_n^1 of minimal $L^2(\chi^{-1/2}, Q_T)$ norm is given by $F_n^1 = \chi\varphi_n$ where φ_n solves the adjoint problem

$$\begin{cases} \partial_{tt}\varphi - \Delta\varphi + A_n\varphi = 0, & \text{in } Q_T, \\ \varphi = 0, & \text{on } \Sigma_T, \\ (\varphi(\cdot,T), \partial_t\varphi(\cdot,T)) = (w_0,w_1), & \text{in } \Omega, \end{cases} \tag{42}$$

with $(w_0, w_1) \in \boldsymbol{H}$ the solution of the unconstrained extremal problem

$$\inf_{(w_0,w_1)\in\boldsymbol{H}} J_n^\star(w_0,w_1) := \frac{1}{2}\int_{Q_T} \chi|\varphi|^2 + \int_{Q_T} B_n\varphi.$$

The resolution of this extremal problem is done using the Fletcher-Reeves conjugate gradient algorithm initialized with the minimizer of $J_{n-1}^\star$. The stopping criterion used is $\|g_p\|_{\boldsymbol{H}} \leq 10^{-5}$ where $g_p := (z_p(T,\cdot), -\partial_t z_p(T,\cdot))$ denotes the gradient at the iteration p and z_p the solution of (41) with control χw_p where w_p solves (42) with initial data $(w_{0,p}, w_{1,p})$.

4. *Optimal descent step and update* Compute the optimal step λ_n minimizer in $[0,1]$ of $\lambda \to E((y_n,f_n) - \lambda(Y_n^1, F_n^1))$ defined by

$$2E\big((y_n,f_n) - \lambda(Y_n^1,F_n^1)\big) = \Big\|(1-\lambda)\big(\partial_{tt}y_n - \Delta y_n + g(y_n) - f_n\,1_\omega\big) + l(y_n, -\lambda Y_n^1)\Big\|_2^2$$

with $l(y,\lambda Y) := g(y+\lambda Y) - g(y) - \lambda g'(y)Y$. This is done with 12 iterations of the trichotomy method on the interval $[0,1]$. Then, set $(y_{n+1}, f_{n+1}) = (y_n,f_n) - \lambda_n(Y_n^1, F_n^1)$ and return to step 2.

In the sequel, we denote by $k^\star = \min\{k \mid \sqrt{2E(y_k,f_k)} \leq 10^{-5}\}$ and define the corresponding approximation of the solution

$$(y^\star, f^\star) = (y_0, f_0) - \sum_{k=0}^{k^\star} \lambda_k(Y_k^1, F_k^1) \in \mathcal{A}.$$

Then, in order to measure *a posteriori* the quality of this approximation, we shall compute the relative term

$$\mathcal{E}_T := \frac{\|(y, \partial_t y)(\cdot, T; f^\star)\|_{\boldsymbol{V}}}{\|(y, \partial_t y)(\cdot, T; 0)\|_{\boldsymbol{V}}},$$

where $y(\cdot, \tau, f^\star)$ (resp. $y(\cdot, \tau, 0)$) is the solution at time τ of (1) with control equal to $f = f^\star$ (resp. $f = 0$).

The introduction of the cut off together with regularity properties on the data (y_0, y_1) make the controlled pairs (y_0, f_0) and (Y_k^1, F_k^1) regular as well (we refer to [14]). This allows to give a meaning to $B_k = \partial_{tt} y_k - \Delta y_k + g(y_k) - f_k 1_\omega$ as an $L^2(Q_T)$ function. Moreover, this involves stability properties of the standard finite dimensional approximations with respect to the discretization parameters. Precisely, we use a time marching method combining the explicit scheme with centered finite differences in time and a finite element approximation for the space variable. We use a uniform discretization $(t_i)_{i=0,N}$ of the time interval $[0, T]$ and denote by $\delta t = T/N$ the time discretization parameter. Moreover, we consider a family $\mathcal{T} = \{\mathcal{T}_h, h > 0\}$ of regular triangulations of Ω such that $\overline{\Omega} = \cup_{K \in \mathcal{T}_h} K$. The family is indexed by $h = \max_{K \in \tau_h} |K|$. For every time t_i, the variable $Y_k^1(\cdot, t_i), F_k^1(\cdot, t_i)$ are approximated in the space $P_h = \{p_h \in C(\overline{Q_T}); (p_h)_{|K} \in \mathbb{P}_1(K), \forall K \in \mathcal{T}_h\}$ where $\mathbb{P}_k(K)$ denotes the space of polynomials of degree 1. We refer to [3] for convergent results in this setting. We also refer to [1, 5, 15, 31].

5.2 Experiments

We consider a two-dimensional case for which $\Omega = (0, 1)^2$. The controllability time is equal to $T = 3$ and the control domain ω is depicted on Figure 1. Moreover, for any real constant c_g, we consider the nonlinear function g defined as follows :

$$g(r) = -c_g\, r \ln^{1/2}(2 + |r|), \quad \forall r \in \mathbb{R}.$$

We check that g satisfies $(\overline{\mathbf{H}}_\mathbf{s})$ for $s = 1$ and $(\mathbf{H_2})$ for $|c_g|$ small enough. Remark that the unfavorable situation for which the norm of the uncontrolled corresponding solution grows corresponds to strictly positives values of c_g. As for the initial and final conditions, we consider $(u_0, u_1) = (100 \sin(\pi x_1) \sin(\pi x_2), 0)$ in Ω and $(z_0, z_1) = (0, 0)$ respectively. Moreover, we mainly use a regular triangulation with fineness $h = 1/64$. A time step equal to $\delta t = h/3$ is then selected to satisfy the CFL condition arising from the explicit scheme with respect to the time variable.

Table 1 collects some norms from the sequence $(y_k, f_k)_{k \in \mathbb{N}}$ associated with the value $c_g = 1$. The convergence of the algorithm is observed after 2 iterations. The optimal steps λ_k are equal to one so that the algorithm (19) coincides with the Newton algorithm (see previous section). Figure 2 depicts with respect to the time variable the $L^2(\Omega)$ norm of the controlled solution $y^\star = y_{k=2}$ (red line) to be compared with the $L^2(\Omega)$ norm of the controlled solution $y_{k=0}$ of the linear equation (blue line) (equivalently, the controlled solution corresponding to $c_g = 0$) used to initialize the algorithm. The effect of the nonlinearity is reduced as the dynamics of the two controlled solutions are similar. The figure also depicts the $L^2(\Omega)$ norm of the uncontrolled solution (blue dotted line) and displays a periodic behavior. Similarly, Figure 3 depicts the $L^2(\chi, q_T)$ norm of the null control $f^\star = f_{k=2}$ (red line) and $f_{k=0}$ (blue line). By construction, these controls vanish at the initial and final time. The corresponding value of the relative error $\mathcal{E}_{T,h} \approx 1.15 \times 10^{-3}$ indicates a notable reduction of the solution at time T thought the action of the control.

Table 2 and Figures 4 and 5 collect the results obtained for the value $c_g = 5$. The relative error takes the value $\mathcal{E}_T = 1.08 \times 10^{-3}$. The convergence is quadratic and is obtained after 4 iterations.

Table 3 and Figures 6 and 7 collects the results obtained for the value $c_g = 10$. We compute the relative error $\mathcal{E}_T = 5.83 \times 10^{-5}$. The convergence is observed after 4 iterations. As before, the optimal steps are very close to one. The main difference with the previous situations for which $c_g = 1$ and $c_g = 5$ is the behavior of the uncontrolled solution which grows exponentially with respect to the time variable, as shown in Figure 6. As expected, this larger value of c_g induces a larger gap between the

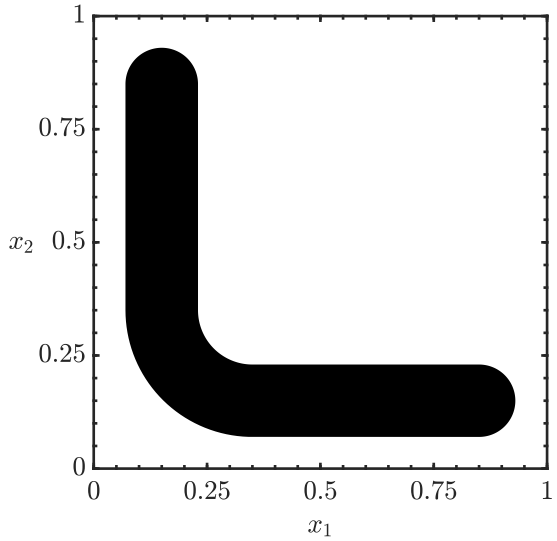

Figure 1: Control domain $\omega \subset \Omega = (0,1)^2$ (black part).

$\sharp$iterate k	$\sqrt{2E(y_k, f_k)}$	$\frac{\|y_k - y_{k-1}\|_{L^2(Q_T)}}{\|y_{k-1}\|_{L^2(Q_T)}}$	$\frac{\|f_k - f_{k-1}\|_{L^2_\chi(q_T)}}{\|f_{k-1}\|_{L^2_\chi(q_T)}}$	$\|y_k\|_{L^2(Q_T)}$	$\|f_k\|_{L^2_\chi(q_T)}$	λ_k
0	7.44×10^1	–	–	38.116	732.22	1
1	8.83×10^{-1}	1.65×10^{-1}	3.37×10^{-1}	37.2	697.423	1
2	7.14×10^{-6}	2.66×10^{-4}	9.90×10^{-4}	37.201	697.615	–

Table 1: $c_g = 1$; Norms of (y_k, f_k) w.r.t. k defined by the algorithm (19).

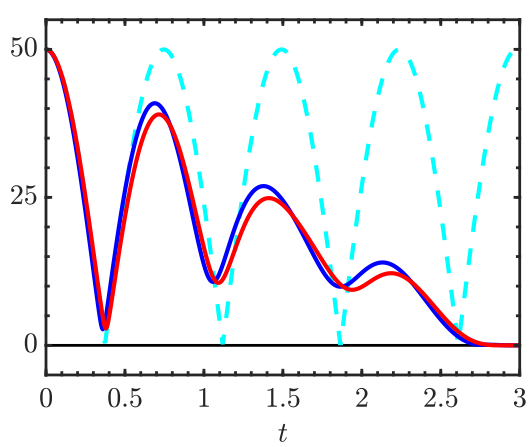

Figure 2: $c_g = 1$ – (▬) $\|y_2(\cdot,t)\|_{L^2(\Omega)}$; (▬) $\|y_0(\cdot,t)\|_{L^2(\Omega)}$; (- -) $\|y(\cdot,t;0)\|_{L^2(\Omega)}$.

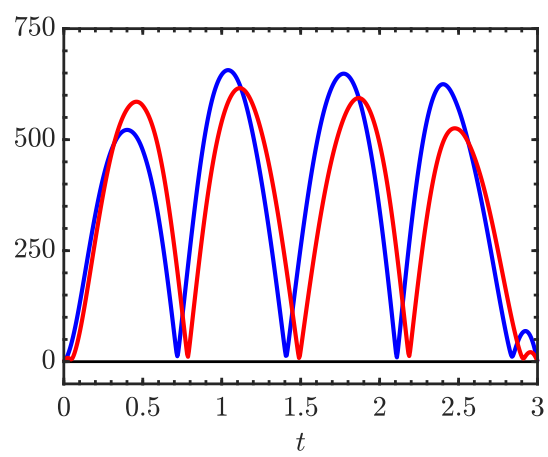

Figure 3: $c_g = 1$ – (▬) $\|f_2(\cdot,t)\|_{L^2_\chi(\omega)}$; (▬) $\|f_0(\cdot,t)\|_{L^2_\chi(\omega)}$.

nonlinear control and the linear one. We observe notably that the nonlinear control $f^\star$ acts stronger from the begining, precisely in order to balance the initial exponential growth of the solution outside the subset ω. We also observe that the control reduces the oscillations of the corresponding controlled solution (in comparison with the solution of the linear equation). For larger values of c_g, we suspect a different dynamic between the control yielding to the first values of the optimal step λ_k far from one (as

$\sharp$iterate k	$\sqrt{2E(y_k, f_k)}$	$\frac{\|y_k - y_{k-1}\|_{L^2(Q_T)}}{\|y_{k-1}\|_{L^2(Q_T)}}$	$\frac{\|f_k - f_{k-1}\|_{L^2_\chi(q_T)}}{\|f_{k-1}\|_{L^2_\chi(q_T)}}$	$\|y_k\|_{L^2(Q_T)}$	$\|f_k\|_{L^2_\chi(q_T)}$	λ_k
0	3.72×10^2	–	–	38.116	732.22	1
1	4.58×10^1	9.01×10^{-1}	1.07×10^0	30.219	665.222	1
2	9.12×10^{-1}	6.36×10^{-2}	1.57×10^{-1}	30.563	734.688	1
3	1.69×10^{-4}	6.34×10^{-4}	1.43×10^{-3}	30.567	734.56	1
4	9.31×10^{-11}	1.15×10^{-7}	1.78×10^{-7}	30.567	734.559	–

Table 2: $c_g = 5$; Norms of (y_k, f_k) w.r.t. k defined by the algorithm (19).

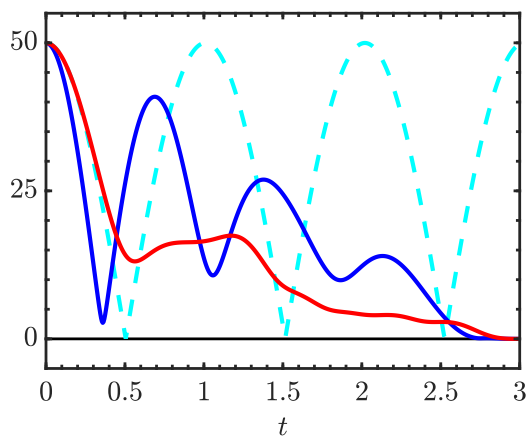

Figure 4: $c_g = 5$ – (▬) $\|y_4(\cdot, t)\|_{L^2(\Omega)}$; (▬) $\|y_0(\cdot, t)\|_{L^2(\Omega)}$; (- - -) $\|y(\cdot, t; 0)\|_{L^2(\Omega)}$.

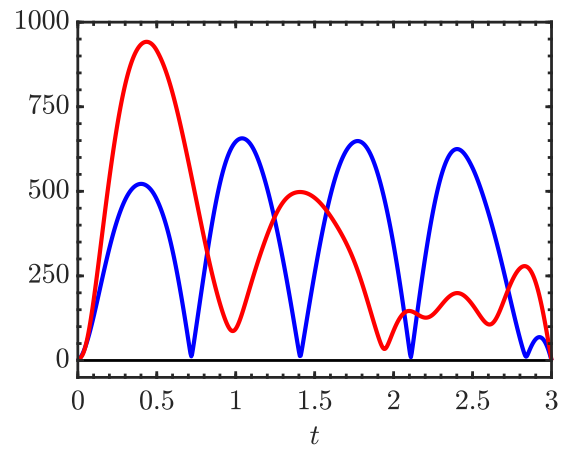

Figure 5: $c_g = 5$ – (▬) $\|f_4(\cdot, t)\|_{L^2_\chi(\omega)}$; (▬) $\|f_0(\cdot, t)\|_{L^2_\chi(\omega)}$.

observed in [26] for the resolution of the Navier-Stokes system with large values of the Reynolds number). However, for larger values of c_g (for instance $c_g = 20$), the exponential growth behavior leads to numerical instabilities and overflow in the computation of the controlled pair (Y_k^1, F_k^1) solution of (41), where the potential $g'(y_k)$ appears. This leads to the divergence of the conjugate gradient algorithm including for very fine discretizations and the non convergence of the least-squares algorithm.

$\sharp$iterate k	$\sqrt{2E(y_k, f_k)}$	$\frac{\|y_k - y_{k-1}\|_{L^2(Q_T)}}{\|y_{k-1}\|_{L^2(Q_T)}}$	$\frac{\|f_k - f_{k-1}\|_{L^2_\chi(q_T)}}{\|f_{k-1}\|_{L^2_\chi(q_T)}}$	$\|y_k\|_{L^2(Q_T)}$	$\|f_k\|_{L^2_\chi(q_T)}$	λ_k
0	7.44×10^2	–	–	38.116	732.22	1
1	1.63×10^2	1.79×10^0	9.30×10^{-1}	58.691	667.602	1
2	1.62×10^0	8.42×10^{-2}	1.41×10^{-1}	60.781	642.643	1
3	1.97×10^{-3}	1.21×10^{-3}	4.66×10^{-3}	60.745	643.784	1
4	5.11×10^{-10}	6.43×10^{-7}	2.63×10^{-6}	60.745	643.785	–

Table 3: $c_g = 10$; Norms of (y_k, f_k) w.r.t. k defined by the algorithm (19).

For negative values of c_g leading to $rg(r) \geq 0$ for every r, the situation is more favorable from a computational viewpoint. Table 4 and Figures 8 and 9 are concerned with the value $c_g = -20$. The convergence is observed after 4 iterations and leads to $\mathcal{E}_T = 1.11 \times 10^{-3}$. We observe that the uncontrolled solution oscillates faster as c_g decreases. This leads to an oscillatory dynamic of the optimal control pair $(y^\star, f^\star)$. We also observe that the norm of the control $f^\star$ is significantly greater than f_0, the initial control associated with the linear case.

Table 5 associated with the value $c_g = 5$ provides a numerical evidence of the convergence of the

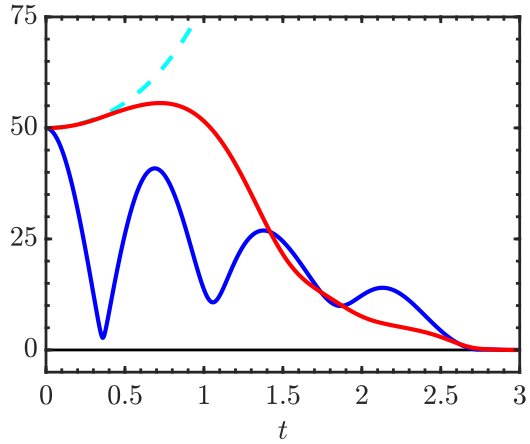

Figure 6: $c_g = 10$ – (▬) $\|y_4(\cdot,t)\|_{L^2(\Omega)}$; (▬) $\|y_0(\cdot,t)\|_{L^2(\Omega)}$; (▬) $\|y(\cdot,t;0)\|_{L^2(\Omega)}$.

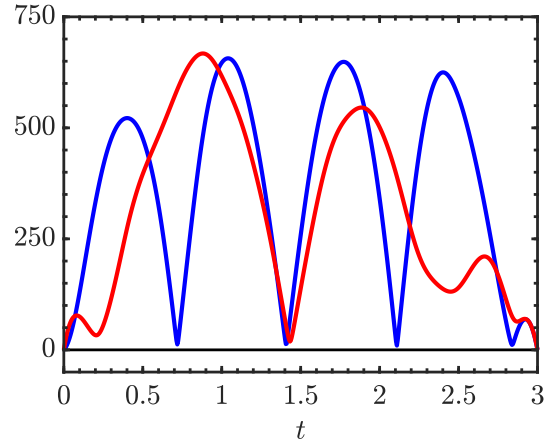

Figure 7: $c_g = 10$ – (▬) $\|f_4(\cdot,t)\|_{L^2_\chi(\omega)}$; (▬) $\|f_0(\cdot,t)\|_{L^2_\chi(\omega)}$.

♯iterate k	$\sqrt{2E(y_k,f_k)}$	$\frac{\|y_k-y_{k-1}\|_{L^2(Q_T)}}{\|y_{k-1}\|_{L^2(Q_T)}}$	$\frac{\|f_k-f_{k-1}\|_{L^2_\chi(q_T)}}{\|f_{k-1}\|_{L^2_\chi(q_T)}}$	$\|y_k\|_{L^2(Q_T)}$	$\|f_k\|_{L^2_\chi(q_T)}$	λ_k
0	1.49×10^3	–	–	38.116	732.22	1
1	2.70×10^2	1.36×10^0	1.78×10^0	41.413	1474.93	0.987
2	1.65×10^1	1.64×10^{-1}	1.65×10^{-1}	43.041	1537.65	1
3	1.39×10^{-2}	3.12×10^{-3}	5.60×10^{-3}	43.071	1539.62	1
4	2.68×10^{-9}	1.46×10^{-6}	4.09×10^{-6}	43.071	1539.62	–

Table 4: $c_g = -20$; Norms of (y_k, f_k) w.r.t. k defined by the algorithm (19).

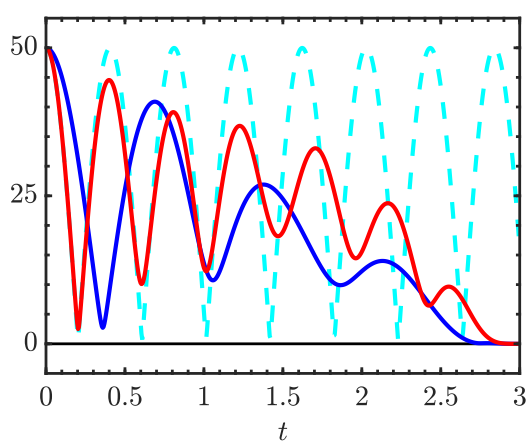

Figure 8: $c_g = -20$ – (▬) $\|y_4(\cdot,t)\|_{L^2(\Omega)}$; (▬) $\|y_0(\cdot,t)\|_{L^2(\Omega)}$; (▬) $\|y(\cdot,t;0)\|_{L^2(\Omega)}$.

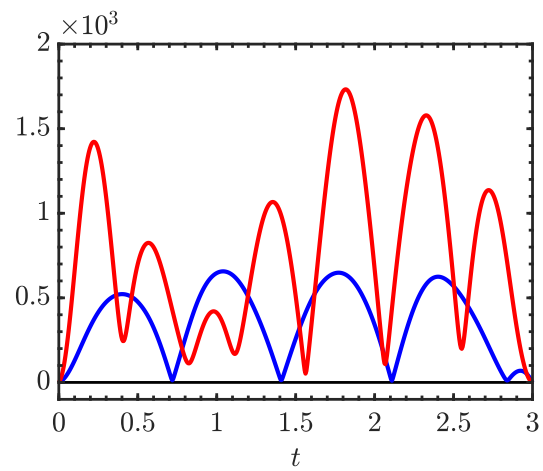

Figure 9: $c_g = -20$ – (▬) $\|f_4(\cdot,t)\|_{L^2_\chi(\omega)}$; (▬) $\|f_0(\cdot,t)\|_{L^2_\chi(\omega)}$.

approximation $(y_h^\star, f_h^\star)$ with respect to the value of h. Actually, in view of the inequality

$$\|f - f_k^h\| \leq \|f - f_k\| + \|f_k - f_k^h\|, \quad \forall k \in \mathbb{N}, \forall h > 0$$

the convergence result stated in Theorem 2 for the sequence $(f_k)_{k\in\mathbb{N}}$ and the convergence, for any k, of the approximation $(f_k^h)_{h>0}$ of the linear control f_k implies that f_k^h is a finite dimensional approximation

of f a control for (1). We observe that the level of the discretization has no influence on the speed of convergence of the least-squares algorithm.

h	$n_h^\star$	$\|y_h^\star\|_{L^2(Q_T)}$	$\|f_h^\star\|_{L^2_\chi(q_T)}$	$\mathcal{E}_{T,h}$
1/10	4	27.706	786.599	1.02×10^{-1}
1/20	4	30.155	713.949	3.14×10^{-2}
1/40	4	30.518	738.217	4.17×10^{-3}
1/80	4	30.581	735.318	4.88×10^{-4}
1/100	4	30.588	735.149	2.72×10^{-4}
1/120	4	30.592	735.052	2.15×10^{-4}

Table 5: $c_g = 5$ - Norm of $(y_h^\star, f_h^\star)$ w.r.t. h.

To end this numerical section, we compare our least-squares approach with two fixed point methods. We first consider the method associated with the algorithm (3) mentioned in the introduction. With the same data and initialization, Table 6 collects some norms with respect to k for $c_g = 5$. The $L^2(\chi^{-1/2}, q_T)$ norm of the control is smaller than the one from the least-squares algorithm (527.226 vs 734.559) but leads to a larger $L^2(Q_T)$ norm of the controlled solution (36.806 vs 30.567). The convergence is linear and reached after 9 iterations leading to $\mathcal{E}_T \approx 1.12 \times 10^{-3}$. Figure 10 displays the time evolution of the norms of y_k and f_k for the final iteration $k = 9$. We observe that the approximation obtained differs from those of Figures 4 and 5. For these data, the sequence converges for $|c_g| < 15$ approximately. For larger values, we observe the non-convergence of the method suggesting that the operator K is not contracting in general.

$\sharp$iterate k	$\sqrt{2E(y_k, f_k)}$	$\frac{\|y_{k+1}-y_k\|_{L^2(Q_T)}}{\|y_k\|_{L^2(Q_T)}}$	$\frac{\|f_{k+1}-f_k\|_{L^2_\chi(q_T)}}{\|f_k\|_{L^2_\chi(q_T)}}$	$\|y_k\|_{L^2(Q_T)}$	$\|f_k\|_{L^2_\chi(q_T)}$
0	3.72×10^2	1.02×10^0	1.33×10^0	38.116	732.22
1	4.79×10^1	5.85×10^{-2}	1.73×10^{-1}	37.945	562.213
2	2.65×10^0	3.35×10^{-3}	1.55×10^{-2}	36.798	530.787
3	1.54×10^{-1}	3.05×10^{-4}	9.84×10^{-4}	36.812	526.864
4	1.39×10^{-2}	4.70×10^{-5}	8.77×10^{-5}	36.807	527.209
5	2.13×10^{-3}	9.24×10^{-6}	1.81×10^{-5}	36.806	527.221
6	4.20×10^{-4}	1.88×10^{-6}	3.93×10^{-6}	36.806	527.225
7	8.55×10^{-5}	4.07×10^{-7}	8.81×10^{-7}	36.806	527.226
8	1.85×10^{-5}	8.97×10^{-8}	1.99×10^{-7}	36.806	527.226
9	4.08×10^{-6}	–	–	36.806	527.226

Table 6: $c_g = 5$; Norms for the sequence defined by the fixed point algorithm (3).

The second fixed point method is associated with the operator $\Lambda : L^2(Q_T) \to L^2(Q_T)$ defined as follows $y = \Lambda(z)$ where y is a controlled solution of

$$\begin{cases} \partial_{tt} y - \Delta y = f 1_\omega - g(z) & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ (y, \partial_t y)(\cdot, 0) = (u_0, u_1) & \text{in } \Omega \end{cases} \tag{43}$$

satisfying $(y(\cdot, T), \partial_t y(\cdot, T)) = (z_0, z_1)$. f is selected as the control of minimal $L^2(\chi^{-1/2}, Q_T)$ norm. Any fixed point of Λ is a controlled solution for (1). Theorem 1 implies the existence of at least one fixed point for Λ. The controllability of system (43) allows to define the sequence $(y_k)_{k\in\mathbb{N}}$ as follows:

$$\begin{cases} y_0 \in L^2(Q_T), \\ y_{k+1} = \Lambda(y_k), \quad k \geq 0. \end{cases} \tag{44}$$

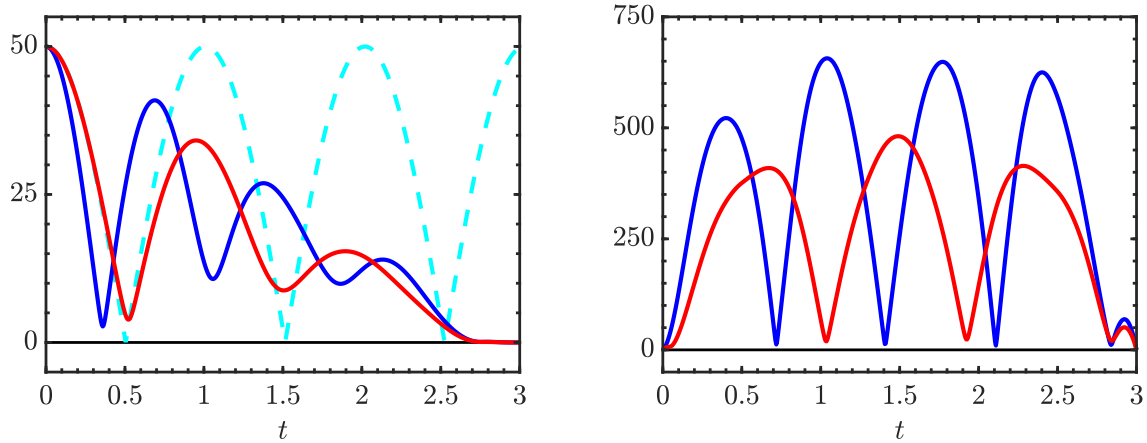

Figure 10: Fixed point algorithm (3) ; $c_g = 5$; **Left**: $\|y_9(\cdot,t)\|_{L^2_\chi(\omega)}$ (▬) and $\|y_0(\cdot,t)\|_{L^2_\chi(\omega)}$ (▬) vs t; **Right**: $\|f_9(\cdot,t)\|_{L^2_\chi(\omega)}$ (▬) and $\|f_0(\cdot,t)\|_{L^2_\chi(\omega)}$ (▬) vs t.

With the same data and initialization, Table 7 collects some norms with respect to k for $c_g = 5$. f_k is the control of minimal $L^2(\chi^{-1/2}, q_T)$ for y_k solution of (43). The L^2 norm of the controlled pair is greater than the one obtained from the least-squares algorithm. The convergence is significantly lower and reached after 39 iterations leading to $\mathcal{E}_T \approx 1.41 \times 10^{-3}$. The convergence is again linear. Figures 11 depicts the time evolution of the norms of y_k and f_k for the final iteration $k = 39$. We check that the approximation obtained differs from those of Figures 4 and 5. For these data, the sequence converges for $|c_g| < 7$ approximately. For larger values, we observe the non-convergence of the method suggesting that the operator Λ is not contracting in general.

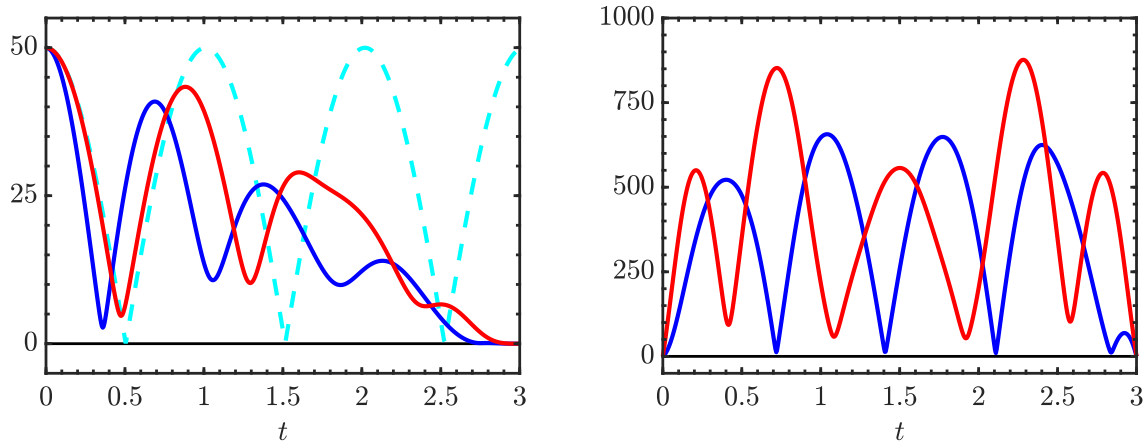

Figure 11: Fixed point algorithm (44) ; $c_g = 5$; **Left**: $\|y_{39}(\cdot,t)\|_{L^2_\chi(\omega)}$ (▬) and $\|y_0(\cdot,t)\|_{L^2_\chi(\omega)}$ (▬) vs t; **Right**: $\|f_{39}(\cdot,t)\|_{L^2_\chi(\omega)}$ (▬) and $\|f_0(\cdot,t)\|_{L^2_\chi(\omega)}$ (▬) vs t.

6 Conclusion

Exact controllability of (1) has been established in [17], under a growth condition on g, by means of a Leray-Schauder fixed point argument that is not constructive. In this paper, under a slightly

$\sharp$iterate k	$\sqrt{2E(y_k, f_k)}$	$\frac{\|y_{k+1}-y_k\|_{L^2(Q_T)}}{\|y_k\|_{L^2(Q_T)}}$	$\frac{\|f_{k+1}-f_k\|_{L^2_\chi(q_T)}}{\|f_k\|_{L^2_\chi(q_T)}}$	$\|y_k\|_{L^2(Q_T)}$	$\|f_k\|_{L^2_\chi(q_T)}$
0	3.72×10^2	8.26×10^{-1}	1.71×10^0	38.116	732.22
1	3.11×10^2	3.77×10^{-1}	7.11×10^{-1}	48.341	1330.18
2	1.80×10^2	1.49×10^{-1}	3.32×10^{-1}	46.01	1264.46
3	6.89×10^1	6.16×10^{-2}	1.31×10^{-1}	44.116	965.409
4	2.70×10^1	2.94×10^{-2}	4.71×10^{-2}	43.696	879.298
5	1.27×10^1	1.64×10^{-2}	2.22×10^{-2}	43.66	859.09
6	7.08×10^0	9.78×10^{-3}	1.24×10^{-2}	43.702	849.733
7	4.24×10^0	6.07×10^{-3}	7.53×10^{-3}	43.757	844.619
8	2.64×10^0	3.85×10^{-3}	4.71×10^{-3}	43.804	841.847
9	1.68×10^0	2.48×10^{-3}	3.02×10^{-3}	43.841	840.273
10	1.08×10^0	1.61×10^{-3}	1.95×10^{-3}	43.867	839.376
11	7.05×10^{-1}	1.05×10^{-3}	1.27×10^{-3}	43.886	838.853
12	4.62×10^{-1}	6.90×10^{-4}	8.36×10^{-4}	43.899	838.546
13	3.03×10^{-1}	4.54×10^{-4}	5.51×10^{-4}	43.909	838.364
14	2.00×10^{-1}	3.00×10^{-4}	3.64×10^{-4}	43.915	838.256
15	1.32×10^{-1}	1.98×10^{-4}	2.41×10^{-4}	43.919	838.192
16	8.75×10^{-2}	1.31×10^{-4}	1.60×10^{-4}	43.922	838.154
17	5.80×10^{-2}	8.72×10^{-5}	1.06×10^{-4}	43.924	838.131
18	3.85×10^{-2}	5.80×10^{-5}	7.09×10^{-5}	43.926	838.118
19	2.56×10^{-2}	3.86×10^{-5}	4.73×10^{-5}	43.927	838.11
20	1.71×10^{-2}	2.57×10^{-5}	3.16×10^{-5}	43.927	838.105
21	1.14×10^{-2}	1.72×10^{-5}	2.11×10^{-5}	43.928	838.103
22	7.61×10^{-3}	1.15×10^{-5}	1.41×10^{-5}	43.928	838.101
23	5.09×10^{-3}	7.67×10^{-6}	9.45×10^{-6}	43.928	838.1
24	3.40×10^{-3}	5.13×10^{-6}	6.33×10^{-6}	43.928	838.1
25	2.28×10^{-3}	3.43×10^{-6}	4.24×10^{-6}	43.928	838.099
39	8.71×10^{-6}	–	–	43.929	838.099

Table 7: $c_g = 5$; Norms for the sequence defined by the fixed point algorithm (44).

stronger growth condition and under the additional assumption that g' is uniformly Hölder continuous with exponent $s \in [0, 1]$, we have designed an explicit algorithm and proved its convergence of a controlled solution of (1). Moreover, the convergence is super-linear of order greater than or equal to $1 + s$ after a finite number of iterations. Our approach gives a new and constructive proof of the exact controllability of (1). We emphasize that the method is general and may be applied to any other equations or systems - not necessarily of hyperbolic nature - for which a precise observability estimate for the linearized problem is available: we refer to [23, 24] addressing the case of the heat equation. Among the open issues, we mention the extension of this constructive approach to the case of the boundary controllability (analyzed for instance [38]). Preliminaries numerical experiments reported are in agreement with the theoretical convergence: in particular the rate of convergence is quadratic to be compared with the linear rate observed with algorithm derived from simpler linearizations. Eventually, we have observed some numerical difficulties when linear controls are approximated using the standard dual method, due to the exponential growth of the uncontrolled solution. From this point of view, the method (developed in [7, 8]) based on the direct resolution of the optimality system (41)-(42) may be more robust with respect to the size of the potential. Preliminaries experiments seem promising and will be discussed in a future work.

A Appendix: controllability results for the linearized wave equation

We recall in this section some *a priori* estimates for the linearized wave equation with potential in $L^\infty(0,T;L^d(\Omega))$ and right hand side in $L^2(Q_T)$. We first recall the crucial observability type estimate proved in [17, Theorem 2.2] (see also [28, Theorem 2.1]).

Proposition 6. *[17] Assume that ω and T satisfy the assumptions of Theorem 1. For any $A \in L^\infty(0,T;L^d(\Omega))$, and $(\phi_0,\phi_1) \in \boldsymbol{H} := L^2(\Omega) \times H^{-1}(\Omega)$, the weak solution ϕ of*

$$\begin{cases} \partial_{tt}\phi - \Delta\phi + A\phi = 0, & \text{in } Q_T, \\ \phi = 0, & \text{on } \Sigma_T, \\ (\phi(\cdot,0), \partial_t\phi(\cdot,0)) = (\phi_0,\phi_1), & \text{in } \Omega, \end{cases} \tag{45}$$

satisfies the observability inequality $\|\phi_0,\phi_1\|_{\boldsymbol{H}} \le Ce^{C\|A\|^2_{L^\infty(0,T;L^d(\Omega))}}\|\phi\|_{2,q_T}$ for some $C>0$ only depending on Ω and T.

Classical arguments then lead to following controllability result.

Proposition 7. *[17] Let $A \in L^\infty(0,T;L^d(\Omega))$, $B \in L^2(Q_T)$ and $(z_0,z_1) \in \boldsymbol{V}$. Assume that ω and T satisfy the assumptions of Theorem 1. There exists a control function $u \in L^2(q_T)$ such that the weak solution of*

$$\begin{cases} \partial_{tt}z - \Delta z + Az = u1_\omega + B, & \text{in } Q_T, \\ z = 0, & \text{on } \Sigma_T, \\ (z(\cdot,0), \partial_t z(\cdot,0)) = (z_0,z_1), & \text{in } \Omega, \end{cases} \tag{46}$$

satisfies $(z(\cdot,T), z_t(\cdot,T)) = (0,0)$ in Ω. Moreover, the unique pair (u,z) of minimal control norm satisfies

$$\|u\|_{2,q_T} + \|(z,\partial_t z)\|_{L^\infty(0,T;\boldsymbol{V})} \le C\Big(\|B\|_2 + \|z_0,z_1\|_{\boldsymbol{V}}\Big)e^{C\|A\|^2_{L^\infty(0,T;L^d(\Omega))}} \tag{47}$$

for some constant $C>0$ only depending on Ω and T.

Let $p^\star \in \mathbb{N}^\star$ such that $p^\star < \infty$ if $d=2$ and $p^\star < 6$ if $d=3$. We next discuss some properties of the operator $K : L^\infty(0,T;L^{p^\star}(\Omega)) \to L^\infty(0,T;L^{p^\star}(\Omega))$ defined by $K(\xi) = y_\xi$, a null controlled solution of the linear boundary value problem (2) with the control f_ξ of minimal $L^2(q_T)$ norm. Proposition 7 with $B = -g(0)$ gives

$$\|(y_\xi,\partial_t y_\xi)\|_{L^\infty(0,T;\boldsymbol{V})} \le C\Big(\|u_0,u_1\|_{\boldsymbol{V}} + \|g(0)\|_2\Big)e^{C\|\widehat{g}(\xi)\|^2_{L^\infty(0,T;L^d(\Omega))}} \tag{48}$$

where the function $\hat{g}$ is defined in (2). We assume that $g \in C^1(\mathbb{R})$ satisfies the following asymptotic condition (slightly weaker than $(\mathbf{H_1})$): there exists a $\overline{\beta}$ small enough such that $\limsup_{|r|\to\infty} \frac{|g(r)|}{|r|\ln^{1/2}|r|} \le \overline{\beta}$, i.e.

- $(\mathbf{H_1'})$ There exist $\overline{\alpha} \ge 0$ and $\overline{\beta} \ge 0$ small enough such that $|g(r)| \le \overline{\alpha} + \overline{\beta}(1+|r|))\ln^{1/2}(1+|r|)$ for every r in $\mathbb{R}$.

This implies that $\hat{g}$ satisfies $|\widehat{g}(r)| \le \overline{\alpha} + \overline{\beta}\ln^{1/2}(1+|r|)$ for every $r \in \mathbb{R}$ and some constant $\overline{\alpha} > 0$. This also implies that $\widehat{g}(\xi) \in L^\infty(0,T;L^d(\Omega))$ for any $\xi \in L^\infty(0,T;L^{p^\star}(\Omega))$. Assuming $2C\overline{\beta}^2 \le 1$ and proceeding as in the proof of Lemma 5, we get

$$e^{C\|\widehat{g}(\xi)\|^2_{L^\infty(0,T;L^d(\Omega))}} \le C_1\left(1 + \frac{\|\xi\|_{L^\infty(0,T;L^{p^\star}(\Omega))}}{|\Omega|^{1/p^\star}}\right)^{2C\overline{\beta}^2}, \quad \forall \xi \in L^\infty(0,T;L^{p^\star}(\Omega))$$

for some $C_1 = C_1(\alpha)$. Using (48), we then infer that

$$\|y_\xi\|_{L^\infty(0,T;L^{p^\star}(\Omega))} \leq C\Big(\|u_0,u_1\|_{\boldsymbol{V}} + \|g(0)\|_2\Big)C_1\left(1+\frac{\|\xi\|_{L^\infty(0,T;L^{p^\star}(\Omega))}}{|\Omega|^{1/p^\star}}\right)^{2C\overline{\beta}^2}, \quad \forall \xi \in L^\infty(0,T;L^{p^\star}(\Omega)).$$

Taking $\overline{\beta}$ small enough so that $2C\overline{\beta}^2 < 1$, we conclude that there exists $M > 0$ such that $\|\xi\|_{L^\infty(0,T;L^{p^\star}(\Omega))} \leq M$ implies $\|K(\xi)\|_{L^\infty(0,T;L^{p^\star}(\Omega))} \leq M$. This is the argument (introduced in [39] for the one dimensional case and) implicitly used in [28] to prove the controllability of (1). Note that, in contrast to $\overline{\beta}$, M depends on $\|u_0,u_1\|_{\boldsymbol{V}}$ (and increases with $\|u_0,u_1\|_{\boldsymbol{V}}$).

The following result gives an estimate of the difference of two controlled solutions.

Lemma 7. *Let $A \in L^\infty(0,T;L^d(\Omega))$, $a \in L^\infty(0,T;L^{d+\epsilon}(\Omega))$ for any $\epsilon > 0$, $B \in L^2(Q_T)$ and $(u_0,u_1) \in \boldsymbol{V}$. Let u and v be the null controls of minimal $L^2(q_T)$ norm for y and z respectively weak solutions of*

$$\begin{cases} \partial_{tt}y - \Delta y + Ay = u1_\omega + B & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ (y(\cdot,0),\partial_t y(\cdot,0)) = (u_0,u_1) & \text{in } \Omega, \end{cases} \tag{49}$$

and

$$\begin{cases} \partial_{tt}z - \Delta z + (A+a)z = v1_\omega + B & \text{in } Q_T, \\ z = 0 & \text{on } \Sigma_T, \\ (z(\cdot,0),\partial_t z(\cdot,0)) = (u_0,u_1) & \text{in } \Omega. \end{cases} \tag{50}$$

Then,

$$\|y-z\|_{L^\infty(0,T;H_0^1(\Omega))} \leq C\|a\|_{L^\infty(0,T;L^{d+\epsilon}(\Omega))}\big(\|B\|_2 + \|u_0,u_1\|_{\boldsymbol{V}}\big)e^{C\|A+a\|^2_{L^\infty(0,T;L^d(\Omega))}}e^{C\|A\|^2_{L^\infty(0,T;L^d(\Omega))}}$$

for some constant $C > 0$ only depending on Ω and T.

Proof. The controls of minimal $L^2(q_T)$ norm for y and z are given by $u = \phi 1_\omega$ and $v = \phi_a 1_\omega$ where ϕ and ϕ_a respectively solve the adjoint equations

$$\begin{cases} \partial_{tt}\phi - \Delta\phi + A\phi = 0 & \text{in } Q_T, \\ \phi = 0 & \text{on } \Sigma_T, \\ (\phi(\cdot,0),\partial_t\phi(\cdot,0)) = (\phi_0,\phi_1) & \text{in } \Omega, \end{cases} \quad , \quad \begin{cases} \partial_{tt}\phi_a - \Delta\phi_a + (A+a)\phi_a = 0 & \text{in } Q_T, \\ \phi_a = 0 & \text{on } \Sigma_T, \\ (\phi_a(\cdot,0),\partial_t\phi_a(\cdot,0)) = (\phi_{a,0},\phi_{a,1}) & \text{in } \Omega, \end{cases}$$

for some appropriate $(\phi_0,\phi_1),(\phi_{a,0},\phi_{a,1}) \in \boldsymbol{H}$. In particular, $\phi,\phi_a \in C([0,T];L^2(\Omega)) \cap C^1([0,T];H^{-1}(\Omega))$. Hence $Z := z - y$ solves

$$\begin{cases} \partial_{tt}Z - \Delta Z + (A+a)Z = \Phi 1_\omega - ay & \text{in } Q_T, \\ Z = 0 & \text{on } \Sigma_T, \\ (Z(\cdot,0),\partial_t Z(\cdot,0)) = (0,0) & \text{in } \Omega, \end{cases} \tag{51}$$

and $\Phi := \phi_a - \phi$ solves

$$\begin{cases} \partial_{tt}\Phi - \Delta\Phi + (A+a)\Phi = -a\phi & \text{in } Q_T, \\ \Phi = 0 & \text{on } \Sigma_T, \\ (\Phi(\cdot,0),\partial_t\Phi(\cdot,0)) = (\phi_{a,0}-\phi_0,\phi_{a,1}-\phi_1) & \text{in } \Omega. \end{cases}$$

In particular (since $a \in L^\infty(0,T;L^{d+\epsilon}(\Omega))$ and $\phi \in L^\infty(0,T;L^2(\Omega))$), we get that $a\phi \in L^\infty(0,T;H^{-1}(\Omega))$ and therefore $(\Phi,\Phi_t) \in C([0,T];\boldsymbol{H})$, see [21, Theorem 2.3]. We decompose $\Phi := \Psi + \psi$ where Ψ and ψ solve respectively

$$\begin{cases} \partial_{tt}\Psi - \Delta\Psi + (A+a)\Psi = 0 & \text{in } Q_T, \\ \Psi = 0 & \text{on } \Sigma_T, \\ (\Psi(\cdot,0),\partial_t\Psi(\cdot,0)) = (\phi_{a,0}-\phi_0,\phi_{a,1}-\phi_1) & \text{in } \Omega, \end{cases} \quad , \quad \begin{cases} \partial_{tt}\psi - \Delta\psi + (A+a)\psi = -a\phi & \text{in } Q_T, \\ \psi = 0 & \text{on } \Sigma_T, \\ \psi(\cdot,0),\partial_t\psi(\cdot,0)) = (0,0) & \text{in } \Omega, \end{cases}$$

and we deduce that $\Psi 1_\omega$ is the control of minimal $L^2(q_T)$ norm for Z solution of

$$\begin{cases} \partial_{tt} Z - \Delta Z + (A+a)Z = \Psi 1_\omega + \Big(\psi 1_\omega - ay\Big) & \text{in } Q_T, \\ Z = 0 & \text{on } \Sigma_T, \\ (Z(\cdot,0), \partial_t Z(\cdot,0)) = (0,0) & \text{in } \Omega. \end{cases}$$

Proposition 7 implies that

$$\|\Psi\|_{2,q_T} + \|(Z,\partial_t Z)\|_{L^\infty(0,T;\boldsymbol{V})} \leq C\|\psi 1_\omega - ay\|_2 e^{C\|A+a\|^2_{L^\infty(0,T;L^d(\Omega))}}.$$

Moreover, [28, Lemma 2.4] applied to ψ leads to

$$\|(\psi,\partial_t\psi)\|_{L^\infty(0,T;\boldsymbol{H})} \leq \|a\varphi\|_{L^\infty(0,T;H^{-1}(\Omega))} e^{C\|A+a\|_{L^\infty(0,T;L^d(\Omega))}}$$

and $\|\psi\|_{L^2(q_T)} \leq C\|a\|_{L^\infty(0,T;L^{d+\epsilon}(\Omega))}\|\phi\|_2 e^{C\|A+a\|^2_{L^\infty(0,T;L^d(\Omega))}}$. But, using again [28, Lemma 2.4], we infer that $\|\phi\|_2 \leq C\|\phi_0,\phi_1\|_{\boldsymbol{H}} e^{C\|A\|_{L^\infty(0,T;L^d(\Omega))}}$ while [28, Theorem 2.1] gives $\|\phi_0,\phi_1\|_{\boldsymbol{H}} \leq Ce^{C\|A\|^2_{L^\infty(0,T;L^d(\Omega))}}\|\phi\|_{2,q_T}$. Since $u = \phi 1_\omega$, we obtain $\|\phi\|_2 \leq C^2 e^{C\|A\|^2_{L^\infty(0,T;L^d(\Omega))}}\|u\|_{2,q_T} e^{C\|A\|_{L^\infty(0,T;L^d(\Omega))}}$ and then

$$\|\psi\|_{L^2(q_T)} \leq C\|a\|_{L^\infty(0,T;L^{d+\epsilon}(\Omega))} e^{C\|A\|^2_{L^\infty(0,T;L^d(\Omega))}} e^{C\|A+a\|^2_{L^\infty(0,T;L^d(\Omega))}}\|u\|_{2,q_T}$$

from which we deduce that

$$\begin{aligned} \|Z\|_{L^\infty(0,T;H^1_0(\Omega))} &\leq C\Big(\|\psi\|_{2,q_T} + \|a\|_{L^\infty(0,T;L^{d+\epsilon}(\Omega))}\|y\|_2\Big) e^{C\|A+a\|^2_{L^\infty(0,T;L^d(\Omega))}} \\ &\leq C\|a\|_{L^\infty(0,T;L^{d+\epsilon}(\Omega))}\Big(\|B\|_2 + \|u_0,u_1\|_{\boldsymbol{V}}\Big) e^{C\|A+a\|^2_{L^\infty(0,T;L^d(\Omega))}} e^{C\|A\|^2_{L^\infty(0,T;L^d(\Omega))}} \end{aligned}$$

leading to the result. □

This result allows to establish the following property for the operator K.

Lemma 8. *Under the assumptions done in Theorem 1, let $M = M(\|u_0,u_1\|_{\boldsymbol{V}},\overline{\beta})$ be such that K maps $B_{L^\infty(0,T;L^{d+\epsilon}(\Omega))}(0,M)$ into itself and assume that $\hat{g}' \in L^\infty(\mathbb{R})$. For any $\xi^i \in B_{L^\infty(0,T;L^{d+\epsilon}(\Omega))}(0,M)$, $i=1,2$, there exists $c(M)>0$ such that*

$$\|K(\xi^2) - K(\xi^1)\|_{L^\infty(0,T;H^1_0(\Omega))} \leq c(M)\|\hat{g}'\|_\infty \|\xi^2 - \xi^1\|_{L^\infty(0,T;L^{d+\epsilon}(\Omega))}.$$

Proof. For any $\xi^i \in B_{L^\infty(0,T;L^{p^\star}(\Omega))}(0,M)$, $i=1,2$, let $y_{\xi^i} = K(\xi^i)$ be the null controlled solution of

$$\begin{cases} \partial_{tt} y_{\xi^i} - \Delta y_{\xi^i} + y_{\xi^i}\,\widehat{g}(\xi^i) = -g(0) + f_{\xi^i} 1_\omega & \text{in } Q_T, \\ y_{\xi^i} = 0 & \text{on } \Sigma_T, \\ (y_{\xi^i}(\cdot,0), \partial_t y_{\xi^i}(\cdot,0)) = (u_0,u_1) & \text{in } \Omega, \end{cases}$$

with the control $f_{\xi^i}1_\omega$ of minimal $L^2(q_T)$ norm. We observe that y_{ξ^2} is solution of

$$\begin{cases} \partial_{tt} y_{\xi^2} - \Delta y_{\xi^2} + y_{\xi^2}\,\widehat{g}(\xi^1) + y_{\xi^2}(\widehat{g}(\xi^2) - \widehat{g}(\xi^1)) = -g(0) + f_{\xi^2} 1_\omega & \text{in } Q_T, \\ y_{\xi^2} = 0 & \text{on } \Sigma_T, \\ (y_{\xi^2}(\cdot,0), \partial_t y_{\xi^2}(\cdot,0)) = (u_0,u_1) & \text{in } \Omega. \end{cases}$$

It follows from Lemma 7 applied with $B = -g(0)$, $A = \hat{g}(\xi^1)$, $a = \hat{g}(\xi^2) - \hat{g}(\xi^1)$, that

$$\|y_{\xi^2} - y_{\xi^1}\|_{L^\infty(0,T;H^1_0(\Omega))} \leq A(\xi^1,\xi^2)\|\widehat{g}(\xi^2) - \widehat{g}(\xi^1)\|_{L^\infty(0,T;L^{d+\epsilon}(\Omega))} \tag{52}$$

where the positive constant

$$A(\xi^1,\xi^2) := C\Big(\|g(0)\|_2 + \|u_0,u_1\|_{\boldsymbol{V}}\Big) e^{C\|\hat{g}(\xi^1)\|^2_{L^\infty(0,T;L^d(\Omega))}} e^{C\|\hat{g}(\xi^2)\|^2_{L^\infty(0,T;L^d(\Omega))}}$$

is bounded by some $c(M)>0$ for every $\xi^i \in B_{L^\infty(0,T;L^{d+\epsilon}(\Omega))}(0,M)$. The result follows from (52). □

Remark 6. *By Lemma 8, if* $\|\hat{g}'\|_\infty < 1/c(M)$ *then the operator* $K : L^\infty(0,T;L^{d+\epsilon}(\Omega)) \to L^\infty(0,T;L^{d+\epsilon}(\Omega))$ *is contracting. Note however that the bound depends on the norm* $\|u_0, u_1\|_{\boldsymbol{V}}$ *of the initial data to be controlled.*

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