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# ON THE ENERGY TRANSFER TO HIGH FREQUENCIES IN THE DAMPED/DRIVEN NONLINEAR SCHRÖDINGER EQUATION

GUAN HUANG AND SERGEI KUKSIN

ABSTRACT. We consider a damped/driven nonlinear Schrödinger equation in  $\mathbb{R}^n$ , where  $n$  is arbitrary,

$$\mathbb{E}u_t - \nu\Delta u + i|u|^2u = \sqrt{\nu}\eta(t, x), \quad \nu > 0,$$

under odd periodic boundary conditions. Here  $\eta(t, x)$  is a random force which is white in time and smooth in space. It is known that the Sobolev norms of solutions satisfy  $\|u(t)\|_m^2 \leq C\nu^{-m}$ , uniformly in  $t \geq 0$  and  $\nu > 0$ . In this work we prove that for small  $\nu > 0$  and any initial data, with large probability the Sobolev norms  $\|u(t, \cdot)\|_m$  with  $m > 2$  become large at least to the order of  $\nu^{-\kappa_{n,m}}$  with  $\kappa_{n,m} > 0$ , on time intervals of order  $\mathcal{O}(\frac{1}{\nu})$ . It proves that solutions of the equation develop short space-scale of order  $\nu$  to a positive degree, and rigorously establishes the (direct) cascade of energy for the equation.

## 1. INTRODUCTION

In this work we study a damped/driven nonlinear Schrödinger equation

$$u_t - \nu\Delta u + i|u|^2u = \sqrt{\nu}\eta(t, x), \quad x \in \mathbb{R}^n, \quad (1.1)$$

i.e. a CGL equation without linear dispersion, with cubic Hamiltonian nonlinearity and a random forcing. The dimension  $n$  is any,  $0 < \nu \leq 1$  is the viscosity constant and the random force  $\eta$  is white in time  $t$  and regular in  $x$ . The equation is considered under the odd periodic boundary conditions,

$$u(t, \dots, x_j, \dots) = u(t, \dots, x_j + 2\pi, \dots) = -u(t, \dots, x_j + \pi, \dots), \quad j = 1, \dots, n.$$

The latter implies that  $u$  vanishes on the boundary of the cube of half-periods  $K^n = [0, \pi]^n$ ,

$$u|_{\partial K^n} = 0.$$

We denote by  $\{\varphi_d(\cdot), d = (d_1, \dots, d_n) \in \mathbb{N}^n\}$  the trigonometric basis in the space of odd periodic functions,

$$\varphi_d(x) = \left(\frac{2}{\pi}\right)^{\frac{n}{2}} \sin(d_1 x_1) \cdots \sin(d_n x_n).$$

The basis is orthonormal with respect to the normalised scalar product  $\langle\langle \cdot, \cdot \rangle\rangle$  in  $L_2(K^n, \pi^{-n} dx)$ ,

$$\langle\langle u, v \rangle\rangle = \int_{K^n} \langle u(x), v(x) \rangle \pi^{-n} dx, \quad (1.2)$$

where  $\langle \cdot, \cdot \rangle$  is the real scalar product in  $\mathbb{C}$ ,  $\langle u, v \rangle = \Re u \bar{v}$ . It is formed by eigenfunctions of the Laplacian:

$$(-\Delta)\varphi_d = |d|^2\varphi_d.$$

The force  $\eta(t, x)$  is a random field of the form

$$\eta(t, x) = \frac{\partial}{\partial t} \xi(t, x), \quad \xi(t, x) = \sum_{d \in \mathbb{N}^n} b_d \beta_d(t) \varphi_d(x). \quad (1.3)$$

Here  $\beta_d(t) = \beta_d^R(t) + i\beta_d^I(t)$ , where  $\beta_d^R(t)$ ,  $\beta_d^I(t)$  are independent real-valued standard Brownian motions, defined on a complete probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  with a

filtration  $\{\mathcal{F}_t; t \geq 0\}$ . The set of real numbers  $\{b_d, d \in \mathbb{N}^n\}$  is assumed to form a non-zero sequence, satisfying

$$0 < B_{m_*} < \infty, \quad m_* = \min\{m \in \mathbb{Z} : m > n/2\}, \quad (1.4)$$

where for a real number  $k$  we set

$$B_k := \sum_{d \in \mathbb{N}^n} |d|^{2k} |b_d|^2 \leq \infty.$$

For  $m \geq 0$  we denote by  $H^m$  the Sobolev space of order  $m$ , formed by complex odd periodic functions, equipped with the homogeneous norm,

$$\|u\|_m = \|(-\Delta)^{\frac{m}{2}} u\|_0,$$

where  $\|\cdot\|_0$  is the  $L^2$ -norm on  $K^n$ ,  $\|u\|_0^2 = \langle u, u \rangle$  (see (1.2)). If we write  $u \in H^m$  as Fourier series,  $u(x) = \sum_{d \in \mathbb{N}^n} u_d \varphi_d(x)$ , then  $\|u\|_m^2 = \sum_{d \in \mathbb{N}^n} |d|^{2m} |u_d|^2$ .

Eq. (1.1) with small  $\nu$  belongs to a group of equations, describing turbulence in the CGL equations. These equations have got quite a lot of attention in physical literature as models for turbulence in various media, e.g. see [3, Chapter 5]. In particular – as a natural model for hydrodynamical turbulence since eq. (1.1) is obtained from the Navier-Stokes system by replacing the Euler term  $(u \cdot \nabla)u$ , which is a quadratic Hamiltonian nonlinearity, by  $i|u|^2 u$ , which is a cubic Hamiltonian nonlinearity, see [13].

The global solvability of eq. (1.1) for any space dimension  $n$  is established in [8, 10]. It is proved there that if

$$u(0, x) = u_0(x), \quad (1.5)$$

where  $u_0 \in H^m \cap C(K^n)$ ,  $m \in \mathbb{N}$ , and if  $B_m < \infty$ , then the problem (1.1), (1.5) has a unique strong solution  $u(t, x)$  in  $H^m$  which we write as  $u(t, x; u_0)$ , or  $u(t; u_0)$ , or  $u_\nu(t; u_0)$ . Its norm satisfies

$$\mathbb{E} \|u(t; u_0)\|_m^2 \leq C_m \nu^{-m}, \quad t \geq 0,$$

where  $C_m$  depends on  $\|u_0\|_m, |u_0|_\infty$  and  $B_m, B_{m_*}$ . Furthermore, denoting by  $C_0(K^n)$  the space of continuous complex functions on  $K^n$ , vanishing at  $\partial K^n$ , we have that the solutions  $u(t, x)$  define a Markov process in  $C_0(K^n)$ . Moreover, if the noise  $\eta(t, \cdot)$  is non-degenerate in the sense that in (1.3) all coefficients  $b_d$  are non-zero, then this process is mixing.<sup>1</sup>

Our goal is to study the growth of higher Sobolev norms for solutions of eq. (1.1) as  $\nu \rightarrow 0$  on time intervals of order  $\mathcal{O}(\frac{1}{\nu})$ . The main result of this work is the following.

**Theorem 1.** *For any real number  $m > 2$ , in addition to (1.4), assume that  $B_m < \infty$ . Then there exists  $\kappa_{n,m} > 0$  such that for every fixed quadruple  $(\delta, \kappa, \mathcal{X}, T_0)$ , where*

$$\kappa \in (0, \kappa_{n,m}), \quad \delta \in (0, \frac{1}{8}), \quad \mathcal{X}, T_0 > 0,$$

*there exists a  $\nu_0 > 0$  with the property that if  $0 < \nu \leq \nu_0$ , then for every  $u_0 \in H^m \cap C_0(K^n)$ , satisfying*

$$|u_0|_\infty \leq \mathcal{X}, \quad \|u_0\|_m \leq \nu^{-\kappa m}, \quad (1.6)$$

*the solution  $u(t, x; u_0)$  is such that*

(1)

$$\mathbb{P}\left\{ \sup_{t \in [t_0, t_0 + T_0 \nu^{-1}]} \|u_\nu^\omega(t)\|_m > \nu^{-m\kappa} \right\} \geq 1 - \delta, \quad \forall t_0 \geq 0.$$

<sup>1</sup>We note that solutions of equations (1.1) with complex  $\nu$  behave differently, and solubility of those equations with large  $n$  is unknown.

(2) If  $m$  is an integer,  $m \geq 3$ , then a possible choice of  $\kappa_{n,m}$  is  $\kappa_{n,m} = \frac{1}{35}$ , and there exists  $C \geq 1$ , depending on  $\kappa < \frac{1}{35}$ ,  $\mathcal{K}$ ,  $m$ ,  $B_{m_*}$  and  $B_m$ , such that

$$C^{-1}\nu^{-2m\kappa+1} \leq \mathbb{E}\left(\nu \int_{t_0}^{t_0+\nu^{-1}} \|u_\nu(s)\|_m^2 ds\right) \leq C\nu^{-m}, \quad \forall t_0 \geq 0. \quad (1.7)$$

A similar result holds for the classical  $C^k$ -norms of solutions:

**Proposition 2.** For any integer  $m \geq 2$  in addition to (1.4) assume that  $B_m < \infty$ . Then for every fixed triplet  $K, \mathcal{K}, T_0 > 0$  and any  $0 < \kappa < 1/16$  we have

$$\mathbb{P}\left\{\sup_{t \in [t_0, t_0+T_0\nu^{-1}]} |u_\nu^\omega(t; u_0)|_{C^m} > K\nu^{-m\kappa}\right\} \rightarrow 1 \quad \text{as } \nu \rightarrow 0, \quad (1.8)$$

for each  $t_0 \geq 0$ , if  $u_0$  satisfies  $|u_0|_\infty \leq \mathcal{K}$ ,  $|u_0|_{C^m} \leq \nu^{-\kappa m}$ . The rate of convergence depends only on the triplet and  $\kappa$ .

For a proof of this result see the extended version of our work [6]. Due to (1.8), for any  $m > 2 + n/2$  we have

$$\mathbb{P}\left\{\sup_{T_0 \leq t \leq t_0+T_0\nu^{-1}} \|u(t)\|_m \geq K\nu^{-\lfloor m - \frac{n}{2} \rfloor \kappa}\right\} \rightarrow 1 \quad \text{as } \nu \rightarrow 0,$$

for every  $K > 0$  and  $0 < \kappa < 1/16$ , where for  $a \in \mathbb{R}$  we denote  $\lfloor a \rfloor = \max\{n \in \mathbb{Z} : n < a\}$ . This improves the first assertion of Theorem 1 for large  $m$ .

We have the following two corollaries from Theorem 1, valid if the Markov process defined by the equation (1.1) is mixing:

**Corollary 3.** Assume that  $B_m < \infty$  for all  $m$  and  $b_d \neq 0$  for all  $d$ . Then eq. (1.1) is mixing and for any  $\kappa < 1/35$  and  $0 < \nu \leq \nu_0$  its unique stationary measure  $\mu_\nu$  satisfies

$$C^{-1}\nu^{-2m\kappa+1} \leq \int \|u\|_m^2 \mu_\nu(du) \leq C\nu^{-m}, \quad 3 \leq m \in \mathbb{N}. \quad (1.9)$$

Here  $C$  and  $\nu_0$  are as in Theorem 1.

**Corollary 4.** Under the assumptions of Corollary 3, for any  $u_0 \in C^\infty$  we have

$$\frac{1}{2}C^{-1}\nu^{-2m\kappa+1} \leq \mathbb{E}\|u(s; u_0)\|_m^2 \leq 2C\nu^{-m}, \quad 3 \leq m \in \mathbb{N},$$

if  $s \geq T(\nu, u_0, \kappa, B_m, B_{m_*})$ , where  $C$  is the same as in (1.9).

Theorem 1 rigorously establishes the energy cascade to high frequencies for solutions of eq. (1.1) with small  $\nu$ . Indeed, if  $u_0(x)$  and  $\eta(t, x)$  are smooth functions of  $x$  (or even trigonometric polynomials of  $x$ ), then in view of (1.7) for  $0 < \nu \ll 1$  and  $t \gtrsim \nu^{-1}$  a substantial part of the energy  $\frac{1}{2} \sum |u_d(t)|^2$  of a solution  $u(t, x; u_0)$  is carried by high modes  $u_d$  with  $|d| \gg 1$ . Relation (1.7) (valid for all integer  $m \geq 3$ ) also means that the averaged in time space-scale  $l_x$  of solutions for (1.1) satisfies  $l_x \in [\nu^{1/2}, \nu^{1/35}]$ , and goes to zero with  $\nu$  (see [1, 9]). We recall that the energy cascade to high frequencies and formation of short space-scale is the driving force of the Kolmogorov theory of turbulence, see [5].

We mention that in the work [12] the stochastic CGL equation

$$u_t - (\nu + i)\Delta u + i|u|^2 u = \sqrt{\nu}\eta(t, x), \quad 0 < \nu \leq 1, \quad (1.10)$$

with linear dispersion and white in time random force  $\eta$  as in (1.3) was considered under the odd periodic boundary conditions, and the inviscid limiting dynamics as  $\nu \rightarrow 0$  was examined. However, since the limiting equation (1.10) $_{\nu=0}$  is a regular PDE in difference with the equation (1.1) $_{\nu=0}$ , the results on the inviscid limit in [12] differ in spirit from those in our work, and we do not discuss them now.

Deterministic versions of the result of Theorem 1 for eq. (1.1) with  $\eta = 0$ , where  $\nu$  is a small non-zero complex number such that  $\Re \nu \geq 0$  and  $\Im \nu \leq 0$  are known,

see [9]. In particular, if  $\nu$  is a positive real number and  $u_0$  is a smooth function of order one, then for any integer  $m \geq 4$  a solution  $u_\nu(t, x; u_0)$  satisfies estimates (1.7) with the averaging  $\nu \mathbb{E} \int_t^{t+\nu^{-1}} \dots ds$  replaced by  $\nu^{1/3} \int_0^{\nu^{-1/3}} \dots ds$ , with the same upper bound and with the lower bound  $C_m \nu^{-\kappa_m m}$ , where  $\kappa_m \rightarrow 1/3$  as  $m \rightarrow \infty$ . Moreover, it was then shown in [2] that the lower bounds remain true with  $\kappa = 1/3$ , and that the estimates  $\sup_{t \in [0, |\nu|^{-1/3}]} \|u(t)\|_{C^m} \geq C_m |\nu|^{-m/3}$ ,  $m \geq 2$ , hold for smooth solutions of equation (1.1) with  $\eta = 0$  and any non-zero complex ‘‘viscosity’’  $\nu$ .

The better quality of the lower bounds for solutions of the deterministic equations is due to an extra difficulty which occurs in the stochastic case: when time grows, simultaneously with increasing of high Sobolev norms of a solution, its  $L_2$ -norm may decrease, which accordingly would weaken the mechanism, responding for the energy transfer to high modes. Significant part of the proof of Theorem 1 is devoted to demonstration that the  $L_2$ -norm of a solution cannot go down without sending up the second Sobolev norm.

If  $\eta = 0$  and  $\nu = i\delta \in i\mathbb{R}$ , then (1.1) is a Hamiltonian PDE (the defocusing Schrödinger equation), and the  $L_2$ -norm is its integral of motion. If this integral is of order one, then the results of [9] (see there Appendix 3) imply that at some point of each time-interval of order  $\delta^{-1/3}$  the  $C^m$ -norm of a corresponding solution will become  $\gtrsim \delta^{-m\kappa}$  if  $m \geq 2$ , for any  $\kappa < 1/3$ . Furthermore, if  $n = 2$  and  $\delta = 1$ , then due to [4] for  $m > 1$  and any  $M > 1$  there exists a  $T = T(m, M)$  and a smooth  $u_0(x)$  such that  $\|u_0\|_m < M^{-1}$  and  $\|u(T; u_0)\|_m > M$ .

The paper is organized as follows. In Section 2, we recall the results from [8, 10] on solutions of the equation (1.1). Next we show in Section 3 that if the noise  $\eta$  is non-degenerate, the  $L^2$ -norm of a solution of eq. (1.1) cannot stay too small on time intervals of order  $\mathcal{O}(\frac{1}{\nu})$  with high probability, unless its  $H^2$ -norm gets very large (see Lemma 12). Then in Section 4 we derive from this fact the assertion (1) of Theorem 1. We prove assertion (2) and both corollaries in Section 5.

Constants in estimates never depend on  $\nu$ , unless otherwise stated. For a metric space  $M$  we denote by  $\mathcal{B}(M)$  the Borel  $\sigma$ -algebra on  $M$ , and by  $\mathcal{P}(M)$  – the space of probability Borel measures on  $M$ . By  $\mathcal{D}(\xi)$  we denote the law of a r.v.  $\xi$ , and by  $|\cdot|_p$  – the norm in  $L_p(K^n)$ .

## 2. SOLUTIONS AND ESTIMATES

Strong solutions for the equation (1.1) are defined in the usual way:

**Definition 5.** Let  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$  be the filtered probability space as in the introduction. Let  $u_0$  in (1.5) be a r.v., measurable in  $\mathcal{F}_0$  and independent from the Wiener process  $\xi$  (e.g.,  $u_0(x)$  may be a non-random function). Then a random process  $u(t) = u(t; u_0) \in C_0(K^n)$ ,  $t \in [0, T]$ , adapted to the filtration, is called a strong solution of (1.1), (1.5), if

- (1) a.s. its trajectories  $u(t)$  belong to the space

$$\mathcal{H}([0, T]) := C([0, T], C_0(K^n)) \cap L^2([0, T], H^1);$$

- (2) we have

$$u(t) = u_0 + \int_0^t (\nu \Delta u - i|u|^2 u) ds + \sqrt{\nu} \xi(t), \quad \forall t \in [0, T], \quad a.s.,$$

where both sides are regarded as elements of  $H^{-1}$ .

If (1)-(2) hold for every  $T < \infty$ , then  $u(t)$  is a strong solution for  $t \in [0, \infty)$ . In this case a.s.  $u \in C([0, \infty), C_0(K^n)) \cap L^2_{loc}([0, \infty), H^1)$ .

Everywhere below when we talk about solutions for the problem (1.1), (1.5) we assume that the r.v.  $u_0$  is as in the definition above.

The global well-posedness of eq. (1.1) was established in [8, 10]:

**Theorem 6.** *For any  $u_0 \in C_0(K^n)$  the problem (1.1), (1.5) has a unique strong solution  $u^\omega(t, x; u_0)$ ,  $t \geq 0$ . The family of solutions  $\{u^\omega(t; u_0)\}$  defines in the space  $C_0(K^n)$  a Fellerian Markov process.*

In [8, 10] the theorem above is proved when (1.4) is replaced by the weaker assumption  $B_* < \infty$ , where  $B_* = \sum |b_d|$  (note that  $B_* \leq C_n B_{m_*}^{1/2}$ ).

The transition probability for the obtained Markov process in  $C_0(K^n)$  is

$$P_t(u, \Gamma) = \mathbb{P}\{u(t; u) \in \Gamma\}, \quad u \in C_0(K^n), \Gamma \in \mathcal{B}(C_0(K^n)),$$

and the corresponding Markov semigroup in the space  $\mathcal{P}(C_0(K^n))$  of Borel measures on  $C_0(K^n)$  is formed by the operators  $\{\mathcal{B}_t^*, t \geq 0\}$ ,

$$\mathcal{B}_t^* \mu(\Gamma) = \int_{C_0(K^n)} P_t(u, \Gamma) \mu(du), \quad t \in \mathbb{R}.$$

Then  $\mathcal{B}_t^* \mu = \mathcal{D}u(t; u_0)$  if  $u_0$  is a r.v., independent from  $\xi$  and such that  $\mathcal{D}(u_0) = \mu$ .

Introducing the slow time  $\tau = \nu t$  and denoting  $v(\tau, x) = u(\frac{\tau}{\nu}, x)$ , we rewrite eq. (1.1) in the following form, more convenient for some calculations:

$$\frac{\partial v}{\partial \tau} - \Delta v + i\nu^{-1}|v|^2 v = \tilde{\eta}(\tau, x), \quad (2.1)$$

where

$$\tilde{\eta}(\tau, x) = \frac{\partial}{\partial \tau} \tilde{\xi}(\tau, x), \quad \tilde{\xi}(\tau, x) = \sum_{d \in \mathbb{N}^n} b_d \tilde{\beta}_d(\tau) \varphi_d(x),$$

and  $\tilde{\beta}_d(\tau) := \nu^{1/2} \beta_d(\tau \nu^{-1})$ ,  $d \in \mathbb{N}^d$ , is another set of independent standard complex Brownian motions.

Let  $\Upsilon \in C^\infty(\mathbb{R})$  be any smooth function such

$$\Upsilon(r) = \begin{cases} 0, & \text{for } r \leq \frac{1}{4}; \\ r, & \text{for } r \geq \frac{1}{2}. \end{cases}$$

Writing  $v \in \mathbb{C}$  in the polar form  $v = r e^{i\Phi}$ , where  $r = |v|$ , and recalling that  $\langle \cdot, \cdot \rangle$  stands for the real scalar product in  $\mathbb{C}$ , we apply Itô's formula to  $\Upsilon(|v|)$  and obtain that the process  $\Upsilon(\tau) := \Upsilon(|v(\tau)|)$  satisfies

$$\begin{aligned} \Upsilon(\tau) &= \Upsilon_0 + \int_0^\tau \left[ \Upsilon'(r) (\nabla r - r |\nabla \Phi|^2) \right. \\ &\quad \left. + \frac{1}{2} \sum_{d \in \mathbb{N}^n} b_d^2 \left( \Upsilon''(r) \langle e^{i\Phi}, \varphi_d \rangle^2 + \Upsilon'(r) \frac{1}{r} (|\varphi_d|^2 - \langle e^{i\Phi}, \varphi_d \rangle^2) \right) \right] ds + \mathbb{W}(\tau), \end{aligned} \quad (2.2)$$

where  $\Upsilon_0 = \Upsilon(|v(0)|)$  and  $\mathbb{W}(\tau)$  is the stochastic integral

$$\mathbb{W}(\tau) = \sum_{d \in \mathbb{N}^n} \int_0^\tau \Upsilon'(r) b_d \varphi_d \langle e^{i\Phi}, d\tilde{\beta}_d(s) \rangle.$$

In [10] eq. (2.1) is considered with  $\nu = 1$  and, following [8], the norm  $|v(t)|_\infty$  of a solution  $v$  is estimated via  $\Upsilon(t)$  (since  $|v| \leq \Upsilon + 1/2$ ). But the nonlinear term  $i\nu^{-1}|v|^2 v$  does not contribute to eq. (2.2), which is the same as the  $\Upsilon$ -equation (2.3) in [10] (and as the corresponding equation in [8, Section 3.1]). So the estimates on  $|\Upsilon(t)|_\infty$  and the resulting estimates on  $|v(t)|_\infty$ , obtained in [10], remain true for solutions of (2.1) with any  $\nu$ . Thus we get the following upper bound for quadratic exponential moments of the  $L_\infty$ -norms of solutions:<sup>2</sup>

<sup>2</sup>In [8] polynomial moments of the random variables  $\sup_{\tau \leq s \leq \tau+T} |v(s)|_\infty^2$  are estimated, and in [10] these results are strengthened to the exponential bounds (2.3).

**Theorem 7.** For any  $T > 0$  there are constants  $c_* > 0$  and  $C > 0$ , depending only on  $B_*$  and  $T$ , such that for any r.v.  $v_0^\omega \in C_0(K^n)$  as in Definition 5, any  $\tau \geq 0$  and any  $c \in (0, c_*]$ , a solution  $v(\tau; v_0)$  of eq. (2.1) satisfies

$$\mathbb{E} \exp(c \sup_{\tau \leq s \leq \tau+T} |v(s)|_\infty^2) \leq C \mathbb{E} \exp(5c |v_0|_\infty^2) \leq \infty. \quad (2.3)$$

In [10] the result above is proved for a deterministic initial data  $v_0$ . The theorem's assertion follows by averaging the result of [10] in  $v_0^\omega$ .

The estimate (2.3) is crucial for derivation of further properties of solutions, including the given below upper bounds for their Sobolev norms, obtained in the work [8]. Since the scaling of the equation in [8] differs from that in (2.1) and the result there is a bit less general than in the theorem below, a sketch of the proof is given in Appendix B.

**Theorem 8.** Assume that  $B_m < \infty$  for some  $m \in \mathbb{N}$ , and  $v_0 = v_0^\nu \in H^m \cap C_0(K^n)$  satisfies

$$|v_0|_\infty \leq M, \quad \|v_0\|_m \leq M_m \nu^{-m}, \quad 0 < \nu \leq 1.$$

Then

$$\mathbb{E} \|v(\tau; v_0)\|_m^2 \leq C_m \nu^{-m}, \quad \forall \tau \in [0, \infty), \quad (2.4)$$

where  $C_{M,m}$  also depends on  $M$ ,  $M_m$  and  $B_m, B_{m_*}$ .

Neglecting the dependence on  $\nu$ , we have that if  $B_m < \infty$ ,  $m \in \mathbb{N}$ , and a r.v.  $v_0^\omega \in H^m \cap C_0(K^n)$  satisfies  $\mathbb{E} \|v_0\|_m^2 < \infty$  and  $\mathbb{E} \exp(c |v_0|_\infty^2) < \infty$  for some  $c > 0$ , then eq. (2.1) has a solution, equal  $v_0$  at  $t = 0$ , such that

$$\mathbb{E} \|v(\tau; v_0)\|_m^2 \leq e^{-\tau} \mathbb{E} \|v_0\|_m^2 + C, \quad \tau \geq 0, \quad (2.5)$$

$$\mathbb{E} \sup_{0 \leq \tau \leq T} \|v(\tau; v_0)\|_m^2 \leq C', \quad (2.6)$$

where  $C > 0$  depend on  $c, \nu, \mathbb{E} \exp(c |v_0|_\infty^2), B_{m_*}$  and  $B_m$ , while  $C'$  also depends on  $\mathbb{E} \|v_0\|_m^2 < \infty$  and  $T$ . See Appendix B.

As it is shown in [10], the estimate (2.3) jointly with an abstract theorem from [11], imply that under a mild nondegeneracy assumption on the random force the Markov process in the space  $C_0(K^n)$ , constructed in Theorem 6, is mixing:

**Theorem 9.** For each  $\nu > 0$ , there is an integer  $N = N(B_*, \nu) > 0$  such that if  $b_d \neq 0$  for  $|d| \leq N$ , then the equation (1.1) is mixing. I.e. it has a unique stationary measure  $\mu_\nu \in \mathcal{P}(C_0(K^n))$ , and for any probability measure  $\lambda \in \mathcal{P}(C_0(K^n))$  we have  $\mathcal{B}_t^* \lambda \rightarrow \mu_\nu$  as  $t \rightarrow \infty$ .

Under the assumption of Theorem 8, for any  $u_0 \in H^m$  the law  $\mathcal{D}u(t; u_0)$  of a solution  $u(t; u_0)$  is a measure in  $H^m$ . The mixing property in Theorem 9 and (2.4) easily imply

**Corollary 10.** If under the assumptions of Theorem 9  $B_m < \infty$  for some  $m \in \mathbb{N}$  and  $u_0 \in H^m$ , then  $\mathcal{D}(u(t; u_0)) \rightarrow \mu_\nu$  in  $\mathcal{P}(H^m)$ .

In view of Theorems 7, 8 with  $v_0 = 0$  and the established mixing, we have:

**Corollary 11.** Under the assumptions of Theorem 9, if  $v^{st}(\tau)$  is the stationary solution of the equation, then

$$\mathbb{E} \exp(c_* \sup_{\tau \leq s \leq \tau+T} |v^{st}(s)|_\infty^2) \leq C,$$

where the constant  $C > 0$  depends only on  $T$  and  $B_*$ . If in addition  $B_m < \infty$  for some  $m \in \mathbb{N} \cup \{0\}$ , then  $\mathbb{E} \|v^{st}(\tau)\|_m^2 \leq C_m \nu^{-m}$ , where  $C_m$  depends on  $B_*$  and  $B_m$ .

Finally we note that applying Itô's formula to  $\|v^{st}(\tau)\|_0^2$ , where  $v^{st}$  is a stationary solution of (2.1), and taking the expectation we get the balance relation

$$\mathbb{E}\|v^{st}(\tau)\|_1^2 = B_0. \quad (2.7)$$

We cannot prove that  $\mathbb{E}\|v^{st}(\tau)\|_0^2 \geq B' > 0$  for some  $\nu$ -independent constant  $B'$ , and cannot bound from below the energy  $\frac{1}{2}\mathbb{E}\|v(\tau; v_0)\|_0^2$  of a solution  $v$  by a positive  $\nu$ -independent quantity. Instead in next section we get a weaker conditional lower bound on the energies of solutions.

### 3. CONDITIONAL LOWER BOUND FOR THE $L^2$ -NORM OF SOLUTIONS

In this section we prove the following result:

**Lemma 12.** *Let  $B_2 < \infty$  and  $u(\tau; u_0)$ , where  $u_0 \in H^2 \cap C_0(K^n)$  is non-random, be a solution of eq. (2.1). Take any constants  $\chi > 0, \Gamma \geq 1, \tau_0 \geq 0$ , and define the stopping time*

$$\tau_\Gamma := \inf\{\tau \geq \tau_0 : \|u(\tau)\|_2 \geq \Gamma\}$$

(as usual,  $\tau_\Gamma = \infty$  if the set under the inf-sign is empty). Then

$$\mathbb{E} \int_{\tau_0}^{\tau \wedge \tau_\Gamma} \mathbb{I}_{[0, \chi]}(\|u(s)\|_0) ds \leq 2(1 + \tau - \tau_0) B_0^{-1} \chi \Gamma, \quad (3.1)$$

for any  $\tau > \tau_0$ .

**Proof.** We establish the result by adapting the proof from [16] (also see [11, Theorem 5.2.12]) to non-stationary solutions. The argument relies on the concept of local time for semi-martingales (see e.g. [14, Chapter VI.1] for details of the concept). By  $[\cdot]_b$  we denote the quasinorm  $[u]_b^2 = \sum_d |u_d|^2 b_d^2$ .

Without loss of generality we assume  $\tau_0 = 0$ . Otherwise we just need to replace  $u(\tau, x)$  by the process  $\tilde{u}(\tau, x) := u(\tau + \tau_0, x)$ , apply the lemma with  $\tau_0 = 0$  and with  $u_0$  replaced by the initial data  $\tilde{u}_0^\omega = u^\omega(\tau_0; u_0)$ , and then average the estimate in the random  $\tilde{u}_0^\omega$ .

Let us write the solution  $u(\tau; u_0)$  as  $u(\tau) = \sum_{d \in \mathbb{N}^n} u_d(\tau) \varphi_d$ . For any fixed function  $g \in C^2(\mathbb{R})$ , consider the process

$$f(\tau) = g(\|u(\tau \wedge \tau_\Gamma)\|_0^2).$$

Since

$$\partial_u g(\|u\|_0^2) = 2g'(\|u\|_0^2) \langle u, \cdot \rangle, \quad \partial_{uu} g(\|u\|_0^2) = 4g''(\|u\|_0^2) \langle u, \cdot \rangle \langle u, \cdot \rangle + 2g'(\|u\|_0^2) \langle \cdot, \cdot \rangle,$$

then by Itô's formula we have

$$f(\tau) = f(0) + \int_0^{\tau \wedge \tau_\Gamma} A(s) ds + \sum_{d \in \mathbb{N}^n} b_d \int_0^{\tau \wedge \tau_\Gamma} 2g'(\|u(s)\|_0^2) \langle u_d(s), d\beta_d(s) \rangle, \quad (3.2)$$

where

$$\begin{aligned} A(s) &= 2g'(\|u\|_0^2) \langle u, \Delta u - \frac{1}{\nu} i|u|^2 u \rangle + 2 \sum_d b_d^2 (g''(\|u\|_0^2) |u_d|^2 + g'(\|u\|_0^2)) \\ &= -2g'(\|u\|_0^2) \|u\|_1^2 + 2g''(\|u\|_0^2) [u]_b^2 + 2g'(\|u\|_0^2) B_0, \quad u = u(s). \end{aligned} \quad (3.3)$$

**Step 1:** We firstly show that for any bounded measurable set  $G \subset \mathbb{R}$ , denoting by  $\mathbb{I}_G$  its indicator function, we have the following equality

$$\begin{aligned} 2\mathbb{E} \int_0^{\tau \wedge \tau_\Gamma} \mathbb{I}_G(f(s)) (g'(\|u(s)\|_0^2))^2 [u(s)]_b^2 ds &= \int_{-\infty}^{\infty} \mathbb{I}_G(a) \\ &\quad \left[ \mathbb{E}(f(\tau) - a)_+ - \mathbb{E}(f(0) - a)_+ - \mathbb{E} \int_0^{\tau \wedge \tau_\Gamma} \mathbb{I}_{(a+\infty)}(f(s)) A(s) ds \right] da. \end{aligned} \quad (3.4)$$

Let  $L(\tau, a)$ ,  $(\tau, a) \in [0, \infty) \times \mathbb{R}$ , be the local time for the semi martingale  $f(\tau)$  (see e.g. [14, Chapter VI.1]). Since in view of (3.2) the quadratic variation of the process  $f(\tau)$  is

$$d\langle f, f \rangle_s = \sum_d (2g'(\|u\|_0^2) |u_d| b_d)^2 = 4(g'(\|u\|_0^2))^2 [u]_b^2,$$

then for any bounded measurable set  $G \subset \mathbb{R}$ , we have the following equality (known as the occupation time formula, see [14, Corollary VI.1.6]),

$$\int_0^{\tau \wedge \tau_\Gamma} \mathbb{I}_G(f(s)) 4(g'(\|u(s)\|_0^2))^2 [u(s)]_b^2 ds = \int_{-\infty}^{\infty} \mathbb{I}_G(a) L(\tau, a) da. \quad (3.5)$$

For the local time  $L(\tau, a)$ , due to Tanaka's formula (see [14, Theorem VI.1.2]) we have

$$\begin{aligned} (f(\tau) - a)_+ &= (f(0) - a)_+ \\ &+ \sum_{d \in \mathbb{N}^n} b_d \int_0^{\tau \wedge \tau_\Gamma} \mathbb{I}_{(a, +\infty)}(f(s)) 2g'(\|u(s)\|_0^2) \langle u_d(s), d\beta_d(s) \rangle \\ &+ \int_0^{\tau \wedge \tau_\Gamma} \mathbb{I}_{(a, +\infty)}(f(s)) A(s) ds + \frac{1}{2} L(\tau, a). \end{aligned} \quad (3.6)$$

Taking expectation of both sides of (3.5) and (3.6) we obtain the required equality (3.4).

**Step 2:** Let us choose  $G = [\rho_0, \rho_1]$  with  $\rho_1 > \rho_0 > 0$ , and  $g(x) = g_{\rho_0}(x) \in C^2(\mathbb{R})$  such that  $g'(x) \geq 0$ ,  $g(x) = \sqrt{x}$  for  $x \geq \rho_0$  and  $g(x) = 0$  for  $x \leq 0$ . Then due to the factors  $\mathbb{I}_G(f)$  and  $\mathbb{I}_G(a)$  in (3.4), we may there replace  $g(x)$  by  $\sqrt{x}$ , and accordingly replace  $g(\|u\|_0^2)$ ,  $g'(\|u\|_0^2)$  and  $g''(\|u\|_0^2)$  by  $\|u\|_0$ ,  $\frac{1}{2}\|u\|_0^{-1}$  and  $-\frac{1}{4}\|u\|_0^{-3}$ . So the relation (3.4) takes the form

$$\begin{aligned} \mathbb{E} \int_0^{\tau \wedge \tau_\Gamma} \mathbb{I}_G(f(s)) \|u(s)\|_0^{-2} [u(s)]_b^2 &= 2 \int_{\rho_0}^{\rho_1} [\mathbb{E}(f(\tau) - a)_+ - \mathbb{E}(f(0) - a)_+] da \\ &- 2 \int_{\rho_0}^{\rho_1} \left\{ \mathbb{E} \int_0^{\tau \wedge \tau_\Gamma} \mathbb{I}_{(a, +\infty)}(f(s)) \left[ \frac{2}{2\|u(s)\|_0} (B_0 - \|u(s)\|_1^2) - \frac{2}{4\|u(s)\|_0^3} [u(s)]_b^2 \right] ds \right\} da. \end{aligned}$$

Since the l.h.s. of the above equality is non-negative, we have

$$\begin{aligned} &\int_{\rho_0}^{\rho_1} \left[ \mathbb{E} \int_0^{\tau \wedge \tau_\Gamma} \mathbb{I}_{(a, +\infty)}(f(s)) \frac{1}{\|u(s)\|_0^3} (B_0 \|u(s)\|_0^2 - \frac{1}{2} [u(s)]_b^2) ds \right] da \\ &\leq \int_{\rho_0}^{\rho_1} \mathbb{E} \left[ ((f(\tau) - a)_+ - (f(0) - a)_+) + \int_0^{\tau \wedge \tau_\Gamma} \mathbb{I}_{(a, +\infty)}(f(s)) \frac{\|u(s)\|_1^2}{\|u(s)\|_0} ds \right] da. \end{aligned} \quad (3.7)$$

Noting that

$$B_0 \|u\|_0^2 - \frac{1}{2} [u]_b^2 = \sum_{d \in \mathbb{N}^n} (B_0 - \frac{1}{2} b_d^2) |u_d|^2 \geq \frac{B_0}{2} \|u\|_0^2,$$

that by the definition of the stopping time  $\tau_\Gamma$

$$(f(\tau) - a)_+ - (f(0) - a)_+ \leq \Gamma,$$

and that by interpolation,

$$\int_0^{\tau \wedge \tau_\Gamma} \frac{\|u(s)\|_1^2}{\|u(s)\|_0} ds \leq \int_0^{\tau \wedge \tau_\Gamma} \|u(s)\|_2 ds \leq (\tau \wedge \tau_\Gamma) \Gamma,$$

we derive from (3.7) the relation

$$\frac{B_0}{2} \int_{\rho_0}^{\rho_1} \left( \mathbb{E} \int_0^{\tau \wedge \tau_\Gamma} \mathbb{I}_{(a, +\infty)}(f(s)) \|u(s)\|_0^{-1} ds \right) da \leq (\rho_1 - \rho_0) \Gamma (1 + \tau).$$

When  $\rho_0 \rightarrow 0$ , we have  $g(x) \rightarrow \sqrt{x}$  and  $f(\tau) \rightarrow \|u(\tau \wedge \tau_\Gamma)\|_0$ . So sending  $\rho_0$  to 0 and using Fatou's lemma we get from the last estimate that

$$\int_0^{\rho_1} \mathbb{E} \int_0^{\tau \wedge \tau_\Gamma} \mathbb{I}_{(a, \infty)}(\|u(s)\|_0) \|u(s)\|_0^{-1} ds da \leq 2\rho_1(1 + \tau)B_0^{-1}\Gamma.$$

As the l.h.s. above is not smaller than

$$\frac{1}{\chi} \int_0^{\rho_1} \mathbb{E} \int_0^{\tau \wedge \tau_\Gamma} \mathbb{I}_{(a, \chi]}(\|u(s)\|_0) ds da,$$

then

$$\frac{1}{\rho_1} \int_0^{\rho_1} \mathbb{E} \int_0^{\tau \wedge \tau_\Gamma} \mathbb{I}_{(a, \chi]}(\|u(s)\|_0) ds da \leq 2(1 + \tau)B_0^{-1}\Gamma\chi. \quad (3.8)$$

By the monotone convergence theorem

$$\lim_{a \rightarrow 0} \mathbb{E} \int_0^{\tau \wedge \tau_\Gamma} \mathbb{I}_{(a, \chi]}(\|u(s)\|_0) ds = \mathbb{E} \int_0^{\tau \wedge \tau_\Gamma} \mathbb{I}_{(0, \chi]}(\|u(s)\|_0) ds,$$

so we get from (3.8) that

$$\mathbb{E} \int_0^{\tau \wedge \tau_\Gamma} \mathbb{I}_{(0, \chi]}(\|u(s)\|_0) ds \leq 2(1 + \tau)B_0^{-1}\Gamma\chi. \quad (3.9)$$

**Step 3:** We continue to verify that

$$\mathbb{E} \int_0^{\tau \wedge \tau_\Gamma} \mathbb{I}_{\{0\}}(\|u(s)\|_0) ds = 0. \quad (3.10)$$

To do this let us fix any index  $d \in \mathbb{N}^n$  such that  $b_d \neq 0$ . The process  $u_d(\tau)$  is a semimartingale,  $du_d = v_d ds + b_d d\beta_d$ , where  $v_d(s)$  is the  $d$ -th Fourier coefficient of  $\Delta u + \frac{1}{\nu}i|u|^2 u$  for the solution  $u(\tau) = \sum_d u_d(\tau)\varphi_d$  which we discuss. Consider the stopping time

$$\tau_R = \inf\{s \leq \tau \wedge \tau_\Gamma : |u(s)|_\infty \geq R\}.$$

Due to (2.3) and (2.6),  $\mathbb{P}(\tau_R = \tau \wedge \tau_\Gamma) \rightarrow 1$  as  $R \rightarrow \infty$ . Let us denote  $u_d^R(\tau) = u_d(\tau \wedge \tau_R)$ . To prove (3.10) it suffices to verify that

$$\pi(\delta) := \mathbb{E} \int_0^{\tau \wedge \tau_\Gamma} \mathbb{I}_{\{|u_d(s)| < \delta\}} ds \rightarrow 0 \quad \text{as } \delta \rightarrow 0.$$

If we replace above  $u_d$  by  $u_d^R$ , then the obtained new quantity  $\pi^R(\delta)$  differs from  $\pi(\delta)$  at most by  $\mathbb{P}(\tau_R < \tau \wedge \tau_\Gamma)$ . The process  $u_d^R$  is an Ito process with a bounded drift. So by [7, Theorem 2.2.2, p. 52],  $\pi^R(\delta)$  goes to zero with  $\delta$ . Thus, given any  $\varepsilon > 0$ , we firstly choose  $R$  sufficiently big and then  $\delta$  sufficiently small to achieve  $\pi(\delta) < \varepsilon$ , for a suitable  $\delta(\varepsilon) > 0$ . So (3.10) is verified. Jointly with (3.9) this proves (3.1).  $\square$

#### 4. LOWER BOUNDS FOR SOBOLEV NORMS OF SOLUTIONS

In this section we work with eq. (1.1) in the original time scale  $t$  and provide lower bounds for the  $H^m$ -norms of its solutions with  $m > 2$ . This will prove the assertion (1) of Theorem 1. As always, the constants do not depend on  $\nu$ , unless otherwise stated.

**Theorem 13.** *For any integer  $m \geq 3$ , if  $B_m < \infty$  and*

$$0 < \kappa < \frac{1}{35}, \quad T_0 \geq 0, \quad T_1 > 0,$$

*then for any r.v.  $u_0(x) \in H^m \cap C_0(K^n)$ , satisfying*

$$\mathbb{E}\|u_0\|_m^2 < \infty, \quad \mathbb{E}\exp(c|u_0|_\infty^2) \leq C < \infty \quad (4.1)$$

*for some  $c, C > 0$ , we have*

$$\mathbb{P}\left\{ \sup_{T_0 \leq t \leq T_0 + T_1 \nu^{-1}} \|u(t; u_0)\|_m \geq K\nu^{-m\kappa} \right\} \rightarrow 1 \quad \text{as } \nu \rightarrow 0, \quad (4.2)$$

for every  $K > 0$ .

**Proof.** Consider the complement to the event in (4.2):

$$Q^\nu = \left\{ \sup_{T_0 \leq t \leq T_0 + \frac{T_1}{\nu}} \|u(t)\|_m < K\nu^{-m\kappa} \right\}.$$

We will prove the assertion (4.2) by contradiction. Namely, we assume that there exists a  $\gamma > 0$  and a sequence  $\nu_j \rightarrow 0$  such that

$$\mathbb{P}(Q^{\nu_j}) \geq 5\gamma \quad \text{for } j = 1, 2, \dots, \quad (4.3)$$

and will derive a contradiction. Below we write  $Q^{\nu_j}$  as  $Q$  and always suppose that

$$\nu \in \{\nu_1, \nu_2, \dots\}.$$

The constants in the proof may depend on  $\mathcal{K}, K, \gamma, B_{m \vee m_*}$ , but not on  $\nu$ .

Without loss of generality we assume that  $T_1 = 1$ . For any  $T_0 > 0$ , due to (2.5) and (2.3) the r.v.  $\tilde{u}_0 := u(T_1)$  satisfies (4.1) with  $c$  replaced by  $c/5$ . So considering  $\tilde{u}(t, x) = u(t + T_0, x)$  we may assume that  $T_0 = 0$ .

Let us denote  $J_1 = [0, \frac{1}{\nu}]$ . Due to Theorem 7,

$$\mathbb{P}(Q_1) \geq 1 - \gamma, \quad Q_1 = \left\{ \sup_{t \in J_1} |u(t)|_\infty \leq C_1(\gamma) \right\},$$

uniformly in  $\nu$ , for a suitable  $C_1(\gamma)$ . Then, by the definition of  $Q$  and Sobolev's interpolation,

$$\|u^\omega(t)\|_l \leq C_{l, \gamma} \nu^{-l\kappa}, \quad \omega \in Q \cap Q_1, \quad t \in J_1, \quad (4.4)$$

for  $l \in [0, m]$  (and any  $\nu \in \{\nu_1, \nu_2, \dots\}$ ).

Denote  $J_2 = [0, \frac{1}{2\nu}]$  and consider the stopping time

$$\tau_1 = \inf \left\{ t \in J_2 : \|u(t)\|_2 \geq C_{2, \gamma} \nu^{-2\kappa} \right\} \leq \frac{1}{2\nu}.$$

Then  $\tau_1 = \frac{1}{2\nu}$  for  $\omega \in Q \cap Q_1$ . So due to (3.1) with  $\Gamma = C_{2, \gamma} \nu^{-2\kappa}$ , for any  $\chi > 0$ , we have

$$\begin{aligned} \mathbb{E} \left( \nu \int_{J_2} \mathbb{I}_{[0, \chi]}(\|u(s)\|_0) ds \mathbb{I}_{Q \cap Q_1}(\omega) \right) &= \mathbb{E} \left( \nu \int_0^{\frac{1}{2\nu} \wedge \tau_1} \mathbb{I}_{[0, \chi]}(\|u(s)\|_0) ds \mathbb{I}_{Q \cap Q_1}(\omega) \right) \\ &\leq \mathbb{E} \left( \nu \int_0^{\frac{1}{2\nu} \wedge \tau_1} \mathbb{I}_{[0, \chi]}(\|u(s)\|_0) ds \right) \leq C\nu^{-2\kappa} \chi. \end{aligned}$$

Consider the event

$$\Lambda = \left\{ \omega \in Q \cap Q_1 : \|u(s)\|_0 \leq \chi, \quad \forall s \in J_2 \right\}.$$

Due to the above, we have,

$$\mathbb{P}(\Lambda) \leq 2\mathbb{E} \left( \nu \int_{J_2} \mathbb{I}_{[0, \chi]}(\|u(s)\|_0) ds \mathbb{I}_{Q \cap Q_1}(\omega) \right) \leq 2C\nu^{-2\kappa} \chi.$$

So  $\mathbb{P}(\Lambda) \leq \gamma$  if we choose

$$\chi = c_3(\gamma) \nu^{2\kappa}, \quad c_3(\gamma) = \gamma(2C)^{-1}. \quad (4.5)$$

Let us set

$$Q_2 = (Q \cap Q_1) \setminus \Lambda, \quad \mathbb{P}(Q_2) \geq 3\gamma, \quad (4.6)$$

and for  $\chi$  as in (4.5), consider the stopping time

$$\tilde{\tau}_1 = \inf \left\{ t \in J_2 : \|u(t)\|_0 \geq \chi \right\}.$$

Then  $\tilde{\tau}_1 \leq \frac{1}{2\nu}$  for all  $\omega \in Q_2$ . Consider the function

$$v(t, x) := u(\tilde{\tau}_1 + t, x), \quad t \in [0, \frac{1}{2\nu}].$$

It solves eq. (1.1) with modified Wiener processes and with initial data  $v_0(x) = u^\omega(\tilde{\tau}_1, x)$ , satisfying

$$\|v_0^\omega\|_0 \geq \chi = c\nu^{2\kappa} \quad \text{if } \omega \in Q_2. \quad (4.7)$$

Now we introduce another stopping time, in terms of  $v(t, x)$ :

$$\tau_2 = \inf\{t \in [0, \frac{1}{2\nu}] : \|v(t)\|_m \geq K\nu^{-m\kappa}\} \leq \frac{1}{2\nu}.$$

For  $\omega \in Q_2$ ,  $\tau_2 = \frac{1}{2\nu}$  and in view of (4.4)

$$\|v^\omega(t)\|_l \leq C_3(\gamma)\nu^{-l\kappa}, \quad t \in [0, \frac{1}{2\nu}], \quad l \in [0, m], \quad \forall \omega \in Q_2. \quad (4.8)$$

**Step 1:** Let us estimate from above the increment  $\mathcal{E}(t, x) = |v(t \wedge \tau_2, x)|^2 - |v_0(x)|^2$ . Due to Itô's formula, we have that

$$\begin{aligned} \mathcal{E}(t, x) &= 2\nu \int_0^{t \wedge \tau_2} \left( \langle v(s, x), \Delta v(s, x) \rangle + \sum_{d \in \mathbb{N}^n} b_d^2 \varphi_d^2(x) \right) ds + \sqrt{\nu} M(t, x), \\ M(t, x) &= \int_0^{t \wedge \tau_2} \sum_{d \in \mathbb{N}^n} b_d \varphi_d(x) \langle v(s, x), d\beta_d(s) \rangle. \end{aligned}$$

We treat  $M$  as a martingale  $M(t)$  in the space  $H^1$ . Since in view of (A.3) for  $0 \leq s < \tau_2$  we have

$$\|v(s)\varphi_d\|_1 \leq C(|v(s)|_\infty \|\varphi_d\|_1 + |v(s)|_1 |\varphi_d|_\infty) \leq C(\zeta d + \zeta^{(m-1)/m} \nu^{-\kappa}),$$

where  $\zeta = \sup_{0 \leq s \leq \frac{1}{2\nu}} |u(s)|_\infty$  (the assertion is empty if  $\tau_2 = 0$ ), then for any  $0 < T_* \leq \frac{1}{2\nu}$

$$\mathbb{E}\|M(T_*)\|_1^2 \leq \int_0^{T_*} \mathbb{E} \sum_d b_d^2 \|\varphi_d v(s)\|_1^2 ds \leq CT_* \nu^{-2\kappa}, \quad (4.9)$$

where we used that  $B_1 < \infty$ . So by Doob's inequality

$$\mathbb{P}\left(\sup_{0 \leq s \leq T_*} \|M(s)\|_1^2 \geq r^2\right) \leq CT_* r^{-2} \nu^{-2\kappa}, \quad \forall r > 0. \quad (4.10)$$

Let us choose

$$T_* = \nu^{-b}, \quad b \in (0, 1),$$

where  $b$  will be specified later. Then  $1 \leq T_* \leq \frac{1}{2\nu}$  if  $\nu$  is sufficiently small, so due to (4.10)

$$\mathbb{P}(Q_3) \geq 1 - \gamma, \quad Q_3 = \left\{ \sup_{0 \leq \tau \leq T_*} \|M(\tau)\|_1 \leq C_4(\gamma) \nu^{-\kappa} \sqrt{T_*} \right\},$$

for a suitable  $C_4(\gamma)$  (and for  $\nu \ll 1$ ); thus  $\mathbb{P}(Q_2 \cap Q_3) \geq 2\gamma$ . Since  $\|\langle v, \Delta v \rangle\|_1 \leq C|v|_\infty \|v\|_3$  by (A.2) and  $\|\sum_d b_d \varphi_d\|_1 \leq C$ , then in view of (4.8) and the definition of  $Q_3$ ,

$$\|\mathcal{E}^\omega(\tau)\|_1 \leq C(\gamma)(\nu^{1-3\kappa} T_* + \nu^{\frac{1}{2}-\kappa} T_*^{1/2}), \quad \forall \tau \in [0, T_*], \quad \forall \omega \in Q_2 \cap Q_3. \quad (4.11)$$

**Step 2:** For any  $x \in K^n$ , denoting  $R(t) = |v(t, x)|^2$ ,  $a(t) = \Delta v(t, x)$  and  $\xi(t) = \xi(t, x)$ , we write the equation for  $v(t) := v(t, x)$  as an Itô process:

$$dv(t) = (-iRv + \nu a)dt + \sqrt{\nu} d\xi(t). \quad (4.12)$$

Setting  $w(t) = e^{i \int_0^t R(s) ds} v(t)$ , we observe that  $w$  also is an Itô process,  $w(0) = v_0$  and  $dw = e^{-i \int_0^t R(s) ds} dw - iRv dt$ . From here and (4.12),

$$w(t) = v_0 + \nu \int_0^t e^{i \int_0^s R(s') ds'} a(s) ds + \sqrt{\nu} \int_0^t e^{i \int_0^s R(s') ds'} d\xi(s).$$

So  $v(t \wedge \tau_2) = v(t \wedge \tau_2, x)$  can be written as

$$v(t \wedge \tau_2, x) = I_1(t \wedge \tau_2, x) + I_2(t \wedge \tau_2, x) + I_3(t \wedge \tau_2, x), \quad (4.13)$$

where

$$\begin{aligned} I_1(t, x) &= e^{-i \int_0^t |v(s, x)|^2 ds} v_0, \quad I_2(t, x) = \nu \int_0^t e^{-i \int_s^t |v(s', x)|^2 ds'} \Delta v(s, x) ds, \\ I_3(t, x) &= \sqrt{\nu} e^{-i \int_0^t |v(s', x)|^2 ds'} \int_0^t e^{i \int_0^s |v(s', x)|^2 ds'} d\xi(s, x). \end{aligned}$$

Our next goal is to obtain a lower bound for  $\|v(T_*)\|_1$  when  $\omega \in Q_2 \cap Q_3$ , using the above decomposition (4.13).

**Step 3:** We first deal with the stochastic term  $I_3(t)$ . For  $0 \leq s \leq s_1 \leq T_* \wedge \tau_2$  we set

$$W(s, s_1, x) := \exp(i \int_s^{s_1} |v(s', x)|^2 ds'), \quad F(s, s_1, x) := \int_s^{s_1} |v(s', x)|^2 ds'; \quad (4.14)$$

then  $W(s, s_1, x) = \exp(iF(s, s_1, x))$ . The functions  $F$  and  $W$  are periodic in  $x$ , but not odd. Speaking about them we understand  $\|\cdot\|_m$  as the non-homogeneous Sobolev norm, so  $\|F\|_m^2 = \|F\|_0^2 + \|(-\Delta)^{m/2} F\|_0^2$ , etc. We write  $I_3$  as

$$I_3(t) = \sqrt{\nu} \overline{W}(0, t \wedge \tau_2, x) \int_0^{t \wedge \tau_2} W(0, s, x) d\xi(s, x). \quad (4.15)$$

In view of (A.1),

$$\|\exp(iF(s, s_1, \cdot))\|_k \leq C_k (1 + |F(s, s_1, \cdot)|_\infty)^{k-1} \|F(s, s_1, \cdot)\|_k, \quad k \in \mathbb{N}. \quad (4.16)$$

For any  $s \in J = [0, T_* \wedge \tau_2)$ , by (A.3) and the definition of  $\tau_2$ , we have that  $v := v(s)$  satisfies

$$\|v\|_1^2 \leq C |v|_\infty \|v\|_1 \leq C |v|_\infty \|v\|_0^{1-1/m} \|v\|_m^{1/m} \leq C' |v|_\infty^{2-1/m} \nu^{-\kappa} \quad (4.17)$$

(this assertion is empty if  $\tau_2 = 0$  since then  $J = \emptyset$ ). So for  $s, s_1 \in J$ ,

$$|F(s, s_1, \cdot)|_\infty \leq |s_1 - s| \sup_{s' \in J} |v(s')|^2, \quad \|F(s, s_1, \cdot)\|_k \leq C \nu^{-\kappa k} |s_1 - s| \left( \sup_{s' \in J} |v(s')|_\infty \right)^{2-k/m}$$

for  $k \leq m$ . Then, due to (4.16),

$$\|W(0, s \wedge \tau_2, \cdot)\|_1 \leq C' T_* \nu^{-\kappa} (1 + \sup_{s \in J} |v(s)|_\infty)^2. \quad (4.18)$$

Consider the stochastic integral in (4.15),

$$N(t, x) = \int_0^t W(0, s, x) d\xi(s, x).$$

The process  $t \mapsto W(0, t, x)$  is adapted to the filtration  $\{\mathcal{F}_t\}$ , and

$$dW(0, t, x) = i |v(t, x)|^2 W(0, t, x) dt.$$

So integrating by parts (see, e.g., [14, Proposition IV.3.1]) we re-write  $N$  as

$$N(t, x) = W(0, t, x) \xi(t, x) - i \int_0^t \xi(s, x) |v(s, x)|^2 W(0, s, x) ds,$$

and we see from (4.15) that

$$I_3(t) = \sqrt{\nu} \xi(t \wedge \tau_2, x) + i \sqrt{\nu} \int_0^{t \wedge \tau_2} \xi(s, x) |v(s, x)|^2 W(s, t \wedge \tau_2, x) ds. \quad (4.19)$$

Due to (1.4) and since  $B_m < \infty$ , the Wiener process  $\xi(t, x)$  satisfies

$$\mathbb{E} \|\xi(T_*, x)\|_1^2 \leq C B_1 T_*,$$

and

$$\mathbb{E} \sup_{0 \leq t \leq T_*} |\xi(t, \cdot)|_\infty \leq \sum_{d \in \mathbb{N}^n} b_d (\mathbb{E} \sup_{0 \leq t \leq T_*} |\beta_d(t) \varphi_d|_\infty) \leq C B_* \sqrt{T_*},$$

(we recall that  $B_* = \sum_{d \in \mathbb{N}^n} |b_d| < \infty$ ). Therefore,

$$\mathbb{P}(Q_4) \geq 1 - \gamma, \quad Q_4 = \left\{ \sup_{0 \leq t \leq T_*} (\|\xi(t)\|_1 \vee |\xi(t)|_\infty) \leq C T_*^{1/2} \right\},$$

with a suitable  $C = C(\gamma)$ . Let

$$\tilde{Q} = \bigcap_{i=1}^4 Q_i,$$

then  $\mathbb{P}(\tilde{Q}) \geq \gamma$ . As  $\tau_2 = T_*$  for  $\omega \in \tilde{Q}$ , then due to (4.17), (4.18), (4.19) and (A.3), for  $\omega \in \tilde{Q}$  we have

$$\begin{aligned} \sup_{0 \leq t \leq T_*} \|I_3^\omega(t)\|_1 &\leq \sqrt{\nu} \sup_{0 \leq t \leq T_*} \left( \|\xi^\omega(t)\|_1 + \int_0^t \|\xi^\omega(s)|v^\omega(s)|^2 W^\omega(s,t)\|_1 ds \right) \\ &\leq CT_*^{5/2} \nu^{\frac{1}{2}-\kappa}. \end{aligned} \quad (4.20)$$

**Step 4:** We then consider the term  $I_2 = \nu \int_0^{t \wedge \tau_2} \bar{W}(s, t \wedge \tau_2, x) \Delta v(s, x) ds$ . To bound its  $H^1$ -norm we need to estimate  $\|W \Delta v\|_1$ . Since

$$\|\partial_x^a W \partial_x^b v\|_0 \leq C \|W\|_3^{1/3} \|v\|_3^{2/3} |v|_\infty^{1/3} \quad \text{if } |a| = 1, |b| = 2,$$

(see [17, Proposition 3.6]), we have

$$\|W \Delta v\|_1 \leq C (\|v\|_3 + \|W\|_3^{1/3} \|v\|_3^{2/3} |v|_\infty^{1/3}).$$

Then in view of (4.16) and (4.8), for  $\omega \in \tilde{Q}$

$$\|W \Delta v\|_1 \leq C (\nu^{-3\kappa} + (T_*^3 \nu^{-3\kappa})^{1/3} \nu^{-2\kappa}) \leq C \nu^{-3\kappa} T_*,$$

and accordingly

$$\sup_{0 \leq t \leq T_*} \|I_2^\omega(t)\|_1 \leq \nu \sup_{0 \leq t \leq T_*} \int_0^t \|W^\omega(s, T_*) \Delta v^\omega(s)\|_1 ds \leq C \nu^{1-3\kappa} T_*^2, \quad \forall \omega \in \tilde{Q}. \quad (4.21)$$

**Step 5:** Now we estimate from below the  $H^1$ -norm of the term  $I_1^\omega(T_*, x)$ ,  $\omega \in \tilde{Q}$ . Writing it as  $I_1^\omega(T_*, x) = e^{-iT_*|v_0(x)|^2} e^{-i \int_0^{T_*} \mathcal{E}(s, x) ds} v_0(x)$  we see that

$$\|I_1^\omega(T_*)\|_1 \geq \|\nabla(\exp(-iT_*|v_0|^2)v_0)\|_0 - \|\nabla(\exp(-i \int_0^{T_*} \mathcal{E}(s) ds)v_0)\|_0 - \|v_0\|_1.$$

This first term on the r.h.s is

$$T_* \|v_0 \nabla(|v_0|^2)\|_0 = T_* \frac{2}{3} \|\nabla|v_0|^3\|_0 \geq CT_* \| |v_0|^3 \|_0 \geq CT_* \|v_0\|_0^3 \geq CT_* \nu^{6\kappa}, \quad C > 0,$$

where we have used the fact that  $u|_{\partial K^n} = 0$ , Poincaré's inequality and (4.7).

For  $\omega \in \tilde{Q}$  and  $0 \leq s \leq T_*$ , in view of (4.11), the second term is bounded by

$$\left\| \left( \int_0^{T_*} \nabla \mathcal{E}(s) ds \right) v_0 \right\|_0 \leq CT_* |v_0|_\infty \sup_{0 \leq s \leq T_*} \|\mathcal{E}(s)\|_1 \leq CT_* (\nu^{1-3\kappa} T_* + \nu^{\frac{1}{2}-\kappa} T_*^{1/2}).$$

Therefore, using (4.11), we get for the term  $I_1^\omega(T_*)$  the following lower bound:

$$\|I_1^\omega(T_*)\|_1 \geq C \left( \nu^{6\kappa} T_* - T_* (\nu^{1-3\kappa} T_* + \nu^{\frac{1}{2}-\kappa} T_*^{1/2}) - \nu^{-\kappa} \right).$$

Recalling  $T_* = \nu^{-b}$  we see that if we assume that

$$\begin{cases} 6\kappa - b < -\kappa, \\ 6\kappa - b < 1 - 3\kappa - 2b, \\ 6\kappa - b < 1/2 - \kappa - \frac{3}{2}b, \end{cases} \quad (4.22)$$

then for  $\omega \in \tilde{Q}$ ,

$$\|I_1^\omega(T_*)\|_1 \geq C \nu^{6\kappa} T_*, \quad C > 0, \quad (4.23)$$

provided that  $\nu$  is sufficiently small.

**Step 6:** Finally, remembering that  $\tau_2 = T_*$  for  $\omega \in \tilde{Q}$  and combining the relations (4.20), (4.21) and (4.23) to estimate the terms of (4.13), we see that for  $\omega \in \tilde{Q}$  we have

$$\|v^\omega(T_*)\|_1 \geq \|I_1^\omega(T_*)\|_1 - \|I_2^\omega(T_*)\|_1 - \|I_3^\omega(T_*)\|_1 \geq \frac{1}{2} C_1 \nu^{6\kappa-b}, \quad C_1 > 0, \quad (4.24)$$

if we assume in addition to (4.22) that

$$6\kappa - b < \frac{1}{2} - \kappa - \frac{5}{2}b, \quad (4.25)$$

and  $\nu$  is small. Note that this relation implies the last two in (4.22).

Combining (4.8) and (4.24) we get that

$$\nu^{-b+7\kappa} \leq C_2^{-1}, \quad (4.26)$$

for all sufficiently small  $\nu$ . Thus we have obtained a contradiction with the existence of the sets  $Q^{\nu_j}$  as at the beginning of the proof if (for a chosen  $\kappa$ ) we can find a  $b \in (0, 1)$  which meets (4.22), (4.25) and

$$-b + 7\kappa < 0.$$

Noting that this is nothing but the first relation in (4.22), we see that we have obtained a contradiction if

$$\kappa < \frac{1}{7}b, \quad \kappa < \frac{1}{14} - \frac{3}{14}b,$$

for some  $b \in (0, 1)$ . We see immediately that such a  $b$  exists if and only if  $\kappa < \frac{1}{35}$ .  $\square$

**Amplification.** If we replace the condition  $m \geq 3$  with the weaker assumption

$$\mathbb{R} \ni m > 2,$$

then the statement (4.2) remains true for  $0 < \kappa < \kappa(n, m)$  with a suitable (less explicit) constant  $\kappa(n, m) > 0$ . In this case we obtain a contradiction with the assumption (4.3) by deriving a lower bound for  $\|v(T_*)\|_\alpha$ , where  $\alpha = \min\{1, m-2\} \in (0, 1]$ , using the decomposition (4.13). The proof remains almost identical except that now, firstly, we bound  $\|I_2\|_\alpha$  ( $\alpha < 1$ ) from above using the following estimate from [15, Theorem 5, p. 206] (also see there p. 14):

$$\|W\Delta u\|_\alpha \leq C\|u\|_{2+\alpha}(|W|_\infty + |W|_\infty^{1-\frac{2\alpha}{n}}\|W\|_2^{\frac{2\alpha}{n}});$$

and, secondly, estimate  $\|I_1^\omega(T_*)\|_\alpha$  ( $\alpha < 1$ ) from below as

$$\|I_1^\omega(T_*)\|_\alpha \geq \|I_1^\omega(T_*)\|_1^{2-\alpha} \|I_2^\omega(T_*)\|_2^{-1+\alpha},$$

which directly follows from Sobolev's interpolation. See [6] for more details.

## 5. LOWER BOUNDS FOR TIME-AVERAGED SOBOLEV NORMS

In this section we prove the assertion (2) of Theorem 1. We provide each space  $H^r$ ,  $r \geq 0$ , with the scalar product

$$\langle\langle u, v \rangle\rangle_r := \langle\langle (-\Delta)^{\frac{r}{2}} u, (-\Delta)^{\frac{r}{2}} v \rangle\rangle,$$

corresponding to the norm  $\|u\|_r$ . Let  $u(t) = \sum u_d(t)\varphi_d$  be a solution of eq. (1.1). Applying Itô's formula to the functional  $\|u\|_m^2$ , we have for any  $0 \leq t < t' < \infty$  the relation

$$\begin{aligned} \|u(t')\|_m^2 &= \|u(t)\|_m^2 + 2 \int_t^{t'} \langle\langle u(s), \nu\Delta u(s) - i|u(s)|^2 u(s) \rangle\rangle_m ds \\ &\quad + 2\nu B_m(t' - t) + 2\sqrt{\nu} M(t, t'), \end{aligned} \quad (5.1)$$

where  $M$  is the stochastic integral

$$M(t, t') := \int_t^{t'} \sum_{d \in \mathbb{N}^n} b_d |d|^{2m} \langle u_d(s), d\beta_d(s) \rangle.$$

Let us fix a  $\gamma \in (0, \frac{1}{8})$ . Due to Theorems 7 and 13, for small enough  $\nu$  there exists an event  $\Omega_1 \subset \Omega$ ,  $\mathbb{P}(\Omega_1) \geq 1 - \gamma/2$ , such that for all  $\omega \in \Omega_1$  we have:

- a)  $\sup_{0 \leq t \leq \frac{1}{\nu}} |u^\omega(t)|_\infty \leq C(\gamma)$ , for a suitable  $C(\gamma) > 0$ ;
- b) there exist  $t_\omega \in [0, \frac{1}{3\nu}]$  and  $t'_\omega \in [\frac{2}{3\nu}, \frac{1}{\nu}]$  satisfying

$$\|u^\omega(t_\omega)\|_m, \|u^\omega(t'_\omega)\|_m \geq \nu^{-m\kappa}. \quad (5.2)$$

Since for the martingale  $M(0, t)$  we have that

$$\mathbb{E}|M(0, \frac{1}{\nu})|^2 \leq B_m \mathbb{E} \int_0^{\frac{1}{\nu}} \|u(s)\|_m^2 ds =: X_m,$$

then by Doob's inequality

$$\mathbb{P}(\Omega_2) \geq 1 - \frac{\gamma}{2}, \quad \Omega_2 = \left\{ \sup_{0 \leq t \leq \frac{1}{\nu}} |M(0, t)| \leq c(\gamma) X_m^{1/2} \right\}.$$

Now let us set  $\hat{\Omega} = \Omega_1 \cap \Omega_2$ . Then  $\mathbb{P}(\hat{\Omega}) \geq 1 - \gamma$  for small enough  $\nu$ , and for any  $\omega \in \hat{\Omega}$  there are two alternatives:

i) there exists a  $t_\omega^0 \in [0, \frac{1}{3\nu}]$  such that  $\|u^\omega(t_\omega^0)\|_m = \frac{1}{3}\nu^{-\kappa m}$ . Then from (5.1) and (5.2) in view of (A.4) we get

$$\frac{8}{9}\nu^{-2m\kappa} + 2\nu \int_{t_\omega^0}^{t'_\omega} \|u^\omega(s)\|_{m+1}^2 ds \leq C(m, \gamma) \int_0^{\frac{1}{\nu}} \|u^\omega(s)\|_m^2 ds + 2B_m + 2\sqrt{\nu}c(\gamma)X_m^{1/2}.$$

ii) There exists no  $t \in [0, \frac{1}{3\nu}]$  with  $\|u^\omega(t)\|_m = \frac{1}{3}\nu^{-\kappa m}$ . In this case, since  $\|u^\omega(t)\|_m$  is continuous with respect to  $t$ , then due to (5.2)  $\|u^\omega(t)\|_m > \frac{1}{3}\nu^{-m\kappa}$  for all  $t \in [0, \frac{1}{3\nu}]$ . This leads to the relation

$$\frac{1}{27}\nu^{-2m\kappa-1} \leq \int_0^{\frac{1}{\nu}} \|u^\omega(s)\|_m^2 ds.$$

In both cases for  $\omega \in \hat{\Omega}$  we have:

$$\frac{1}{27}\nu^{-2m\kappa} \leq C'(m, \gamma) \int_0^{\frac{1}{\nu}} \|u(s)\|_m^2 ds + 2B_m + \nu c(\gamma)^2 + X_m.$$

It implies that

$$\mathbb{E}\nu \int_0^{\frac{1}{\nu}} \|u(\tau)\|_m^2 d\tau \geq C\nu^{-2m\kappa+1}$$

(for small enough  $\nu$ ), and gives the lower bound in (1.7).

The upper bound follows directly from Theorem 8.

*Proof of Corollaries 3 and 4:* Since  $B_k < \infty$  for each  $k$  and all coefficients  $b_d$  are non-zero, then eq. (1.1) is mixing in the spaces  $H^m$ ,  $m \in \mathbb{N}$ , see Corollary 10. As the stationary solution  $v^{st}$  satisfies Corollary 11 with any  $m$ , then for each  $\mu \in \mathbb{N}$  and  $M > 0$ , interpolating the norm  $\|u\|_\mu$  via  $\|u\|_0$  and  $\|u\|_m$  with  $m$  sufficiently large we get that the stationary measure  $\mu_\nu$  satisfies

$$\int \|u\|_\mu^M \mu_\nu(du) < \infty \quad \forall \mu \in \mathbb{N}, \forall M > 0. \quad (5.3)$$

Similar, in view of (2.5) and Theorem 7,

$$\mathbb{E}\|u(t; u_0)\|_\mu^M \leq C_\nu(u_0) \quad \forall t \geq 0, \quad (5.4)$$

for each  $u_0 \in C^\infty$  and every  $\mu$  and  $M$  as in (5.3). Now let us consider the integral in (1.7) and write it as

$$J_t := \nu \int_t^{t+\nu^{-1}} \mathbb{E}\|u(s)\|_m^2 ds.$$

Replacing the integrand in  $J_t$  with  $\mathbb{E}(\|u_\nu(s)\|_m \wedge N)^2$ ,  $N \geq 1$ , using the convergence

$$\mathbb{E}(\|u(s; v_0)\|_m \wedge N)^2 \rightarrow \int (\|u\|_m \wedge N)^2 \mu_\nu(du) \quad \text{as } s \rightarrow \infty \quad \forall N, \quad (5.5)$$

which follows from Corollary 10, and the estimates (5.3), (5.4) we get that

$$J_t \rightarrow \int \|u\|_m^2 \mu_\nu(du) \quad \text{as } t \rightarrow \infty. \quad (5.6)$$

This convergence and (1.7) imply the assertion of Corollary 3.

Now the convergence (5.5) jointly with estimates (5.3), (5.4) and (1.8) imply Corollary 4.

## APPENDIX A. SOME ESTIMATES

For any integer  $l \in \mathbb{N}$  and  $F \in H^l$  we have that

$$\|\exp(iF(x))\|_l \leq C_l(1 + |F|_\infty)^{l-1} \|F\|_l. \quad (\text{A.1})$$

Indeed, to verify (A.1) it suffices to check that for any non-zero multi-indices  $\beta_1, \dots, \beta_{l'}$ , where  $1 \leq l' \leq l$  and  $|\beta_1| + \dots + |\beta_{l'}| = l$ , we have

$$\|\partial_x^{\beta_1} F \dots \partial_x^{\beta_{l'}} F\|_0 \leq C |F|_\infty^{l'-1} \|F\|_l. \quad (\text{A.2})$$

But this is the assertion of Lemma 3.10 in [17]. Similarly,

$$\|FG\|_r \leq C_r(|F|_\infty \|G\|_r + |G|_\infty \|F\|_r), \quad F, G \in H^r, \quad r \in \mathbb{N}, \quad (\text{A.3})$$

see [17, Proposition 3.7] (this relation is known as Moser's estimate). Finally, since for  $|\beta| \leq m$  we have  $|\partial_x^\beta v|_{2m/|\beta|} \leq C |v|_\infty^{1-|\beta|/m} \|v\|_m^{|\beta|/m}$  (see relation (3.17) in [17]), then

$$\|\langle |v|^2 v, v \rangle_m\| \leq C_m \|v\|_m^2 |v|_\infty^2, \quad \|\langle |v|^2 v, v \rangle_m\| \leq C'_m \|v\|_{\frac{2m}{m+1}}^{\frac{2m}{m+1}} |v|_\infty^{\frac{2m+4}{m+1}}. \quad (\text{A.4})$$

## APPENDIX B. PROOF OF THEOREM 8

Applying Ito's formula to a solution  $v(\tau)$  of eq. (2.1) we get a slow time version of the relation (5.1):

$$\|v(\tau)\|_m^2 = \|v_0\|_m^2 + 2 \int_0^\tau (-\|v\|_{m+1}^2 - \nu^{-1} \langle i|v|^2 v, v \rangle_m) ds + 2B_m \tau + 2M(\tau), \quad (\text{B.1})$$

where  $M(\tau) = \int_0^\tau \sum_d b_d |d|^{2m} \langle v_d(s), d\beta_d(s) \rangle$ . Since in view of (A.4)

$$\mathbb{E} \|\langle |v|^2 v, v \rangle_m\| \leq C_m (\mathbb{E} \|v\|_{m+1}^2)^{\frac{m}{m+1}} \mathbb{E} (|v|_\infty^{2m+4})^{\frac{1}{m+1}},$$

then denoting  $\mathbb{E} \|v(\tau)\|_r^2 =: g_r(\tau)$ ,  $r \in \mathbb{N} \cup \{0\}$ , taking the expectation of (B.1), differentiating the result and using (2.3), we get that

$$\frac{d}{d\tau} g_m \leq -2g_{m+1} + C_m \nu^{-1} g_{m+1}^{\frac{m}{m+1}} + 2B_m \leq -2g_{m+1} (1 - C'_m \nu^{-1} g_m^{-\frac{1}{m}} + 2B_m), \quad (\text{B.2})$$

since  $g_m \leq g_0^{1/(m+1)} g_{m+1}^{m/(m+1)} \leq C_m g_{m+1}^{m/(m+1)}$ . We see that if  $g_m \geq (2\nu^{-1} C'_m)^m$ , then the r.h.s. of (B.2) is

$$\leq -g_{m+1} + 2B_m \leq -C_m^{-1} g_m^{(m+1)/m} + 2B_m \leq -\tilde{C}_m \nu^{-m-1} + 2B_m, \quad (\text{B.3})$$

which is negative if  $\nu \ll 1$ . So if

$$g_m(\tau) < (2\nu^{-1} C'_m)^m \quad (\text{B.4})$$

at  $\tau = 0$ , then (B.4) holds for all  $\tau \geq 0$  and (2.4) follows. If  $g_m(0)$  violates (B.4), then in view of (B.2) and (B.3), for  $\tau \geq 0$ , while (B.4) is false, we have that

$$\frac{d}{d\tau} g_m \leq -C_m g_m^{(m+1)/m} + 2B_m,$$

which again implies (2.4). Besides, in view of (B.2),

$$\frac{d}{d\tau} g_m \leq -g_m + C_m(\nu, |v_0|_\infty, B_{m_*}, B_m).$$

This relation immediately implies (2.5).

Now let us return to eq. (B.1). Using Doob's inequality and (2.4) we find that

$$\mathbb{E} \left( \sup_{0 \leq \tau \leq T} |M(\tau)|^2 \right) \leq C < \infty.$$

Next, applying (A.4) and Young's inequality we get

$$\int_0^\tau (-\|v\|_{m+1}^2 - \nu^{-1} \langle i|v|^2 v, v \rangle_m) ds \leq C_m \int_0^\tau |v(s)|_\infty^{2m+3} ds, \quad \forall 0 \leq \tau \leq T.$$

Finally, using in (B.1) the last two displayed formulas jointly with (2.3) we obtain (2.6).

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