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Kolmogorov's theory of turbulence and its rigorous 1d model.

Sergei Kuksin*

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Abstract

This paper is a synopsis of the recent book [9]. The latter is dedicated to the stochastic Burgers equation as a model for 1d turbulence, and the paper discusses its content in relation to the Kolmogorov theory of turbulence.

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1 Introduction

The goal of this paper is to discuss the content of the book [9], dedicated to a rigorous theory of 1d turbulence, in its relation to Kolmogorov's understanding of hydrodynamical turbulence, known as the K41 theory. At the origin of the book lie the results, obtained in PhD theses of two students of the author of this paper, A. Biryuk [4, 5] and A. Boritchev [7, 8] (the latter is another author of the book [9]). The theses, in their turn, were based on the previous work [17, 19, 18] on turbulence in the complex Ginzburg–Landau equation (see [6, Section 5] for this concept). The results of the two theses were developed further in subsequent publications of their authors, gave a material for an M2 lecture course which the author of this paper taught in Paris 7 and in some other universities, and were improved and edited in the lecture notes for that course [13]. Finally the results were significantly developed while working on the book [9]. More detailed references may be found in Chapters 5 and 9 of [9].

The paper is based on a number of zoom-seminars which we gave in the year 2020.

1.1 K41 theory

The K41 theory of turbulence was created by A. N. Kolmogorov in three articles [14, 15, 16], published in 1941 (partially based on the previous work of Taylor and von Karman–Howard); see in [20, §33−34], [11] and [9, Chapter 6]. This heuristic theory describes statistical properties of turbulent flows of fluids and gazes and is now the most popular theory of turbulence. We will discuss its basic concepts for the case of a fluid flow with velocity $u(t,x)$ of order 1, space–periodic of period one and with zero space-meanvalue. The Reynolds number of such a flow is

$$Rey = \nu^{-1},$$

where ν is the viscosity. If $Rey = \nu^{-1}$ is large, then the velocity field $u(t,x)$ becomes very irregular, and the flow becomes turbulent. The viscosity is the most important parameter for what follows; dependence on it is clearly indicated, and all constants below are independent from ν.

Kolmogorov postulated that the short scale in x features of a turbulent flow $u(t,x)$ display a universal behaviour which depends on particularities of the system only through a few parameters (in our setting – only through ν), and the K41 theory presents and discusses these universal features – the laws of the Kolmogorov theory.

The K41 theory is statistical. That is, it assumes that the velocity $u(t,x)$ is a random field over some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Moreover, u is assumed to be stationary in time and homogeneous in space. The K41 theory studies its short space-increments $u(t,x+r) - u(t,x)$, $|r| \ll 1$, and examines their moments as functions of r. Besides, for the decomposition of $u(t,x)$ in Fourier series

$$u(t,x) = \sum_{s\in\mathbb{Z}^3} \hat{u}_s(t) e^{2\pi i s\cdot x}, \quad \hat{u}_0(t) \equiv 0,$$

the theory examines the second moments of Fourier coefficients $\hat{u}_s(t)$ as functions of $|s|$ and ν.

Below we present the one-dimensional version of the Kolmogorov theory for a model, given by the stochastic Burgers equation and advocated by Burgers, Frisch, Sinai and some other mathematicians and physicists. Then we will discuss the basic statements of the K41 theory, their 1d versions and the proofs of the latter, suggested in [9].

1.2 Stochastic Burgers equation

The model for 1d turbulence we will talk about is given by the stochastic Burgers equation

$$u_t + uu_x - \nu u_{xx} = \partial_t \xi(t,x), \qquad x \in S^1 = \mathbb{R}/\mathbb{Z}, \quad \int u\,dx = \int \xi\,dx = 0, \qquad u(0,x) = u_0(x), \tag{1.1}$$

where ξ is a Wiener process in the space of functions of x,

$$\xi^\omega(t,x) = \sum_{s=\pm1,\pm2,\ldots} b_s \beta_s^\omega(t) e_s(x), \qquad 0 < B_0 = \sum_s b_s^2 < \infty. \tag{1.2}$$

Here $\{e_s, s = \pm1, \pm2, \ldots\}$ is the trigonometric basis in the space of 1-periodic function with zero mean:

$$\begin{cases} e_k = \sqrt{2}\cos(2\pi kx), \\ e_{-k} = \sqrt{2}\sin(2\pi kx), \end{cases} \quad k \in \mathbb{N};$$

$\{\beta_s^\omega(t)\}$ are standard independent Brownian processes, and $\{b_s\}$ are real numbers. For the purposes on this paper we assume that they fast converge to zero. Then ξ is a Wiener process in the space of functions of x, and for a.e. ω its realisation $\xi^\omega(t,x)$ is continuous in t and smooth in x.

As usual, $u^\omega(t,x)$ is a solution of (1.1) if a.s.

$$u(t) - u_0 + \int_0^t (uu_x - \nu u_{xx})\,ds = \xi(t), \qquad \forall\, t \geq 0.$$

For an integer $m \geq 0$ we denote by H^m the L_2–Sobolev space of order m of functions on S^1 with zero mean-value, equipped with the homogeneous norm

$$\|u\|_m^2 = \int (u^{(m)}(x))^2 dx.$$

It is not hard to see that if $u_0 \in H^r$, $r \geq 1$, then there is a solution u of (1.1) such that $u^\omega \in C(\mathbb{R}_+, H^r)$ a.s., and any two solutions coincide a.s. We will denote a solution of (1.1), regarded as a random process in a space of functions of x, as $u(t;u_0)$ or $u^\nu(t;u_0)$. Regarding u as a random field of (t,x) we will write it as $u(t,x;u_0)$ or $u^\nu(t,x;u_0)$.

As we will soon explain, in average solutions of (1.1) are of order one, i.e. for any u_0, $\mathbb{E}|u^\nu(t;u_0)|^2_{L_2} \sim 1$ uniformly in $t \geq 1$ and $\nu \in (0,1]$. Since the order of magnitude of a solution u^ν is $\sqrt{\mathbb{E}|u^\nu(t)|^2_{L_2}}$ and its space-period is one, then the Reynolds number of u^ν is $\sim \nu^{-1}$. So eq. (1.1) with small ν describes 1d turbulence (called by Uriel Frisch *burgulence*).

The goals, related to eq. (1.1) as a 1d model of turbulence, are:
1) to study solutions $u^\nu(t,x)$ for small ν and for $1 \leq t \leq \infty$;
2) to relate the obtained results with the theory of turbulence, regarding the Burgers equation (1.1) as a 1d hydrodynamical equation.

Inspired by the heuristic work on the stochastic Burgers equation by U. Frisch with collaborators (e.g. see [1, 3]), Sinai and others in the influential paper [10] used the Lax-Oleinik formula to write down the limiting dynamics of (1.1) as $\nu \to 0$, and next studied the obtained limiting solutions $u^0(t,x)$ of the inviscid stochastic Burgers equation $(1.1)\,|_{\nu=0}$. The research was continued by Khanin and some other mathematicians, e.g. see [12] and references in [9]. It has led to a beautiful theory which is related to 1d turbulence and casts light on the problem 1) above, but so far this approach has not allowed to obtain for the limiting dynamics analogies of the K41 laws.

On the contrary, in [9] we study eq. (1.1) for small but positive ν, i.e. not when $\nu \to 0$, but when $0 < \nu \ll 1$ is fixed, using basic tools from PDEs and stochastic calculus. This approach allows to get relations, similar to those claimed by the K41 theory, and to rigorously justify the heuristic theory of burgulence, built in [1, 3].

2 Apriori estimates

We start with a-priori estimates for equation (1.1). The key starting point is the Oleinik inequality, which we apply to solutions of (1.1) with fixed ω. The inequality was proved by Oleinik for the free Burgers equation, but her argument applies to the stochastic equation (1.1) trajectory-wise and implies the following result:

Theorem 2.1. *For any initial data $u_0 \in H^1$, any $p \geq 1$ and any $\nu, \theta \in (0,1]$, uniformly in $t \geq \theta$ we have:*

$$\mathbb{E}\big(|u^\nu(t;u_0)|^p_{L_\infty} + |u^\nu_x(t;u_0)|^p_{L_1} + |u^+_x(t;u_0)|^p_{L_\infty}\big) \leq C_p \theta^{-p} \tag{2.1}$$

(here $v^+ = \max(v,0)$). The constant C_p depends only on the random force in (1.1).

Decomposing a solution $u(t;u_0)$ in Fourier series,

$$u(t,x;u_0) = \sum \hat{u}^\nu_k(t;u_0) e^{2\pi i k x}, \tag{2.2}$$

and using that $|\hat{u}^\nu_k(t;u_0)| \leq |u_x(t;u_0)|_{L_1}/2\pi|k|$ we derive from (2.1) an important consequence:

$$\mathbb{E}|\hat{u}^\nu_k(t;u_0)|^p \leq C'_p |k|^{-p} \theta^{-p}, \quad p \geq 1, \;\; |k| \geq 1, \tag{2.3}$$

if $t \geq \theta$, for any $u_0 \in H^1$.

2.1 Upper bounds for moments of Sobolev norms of solutions

The very powerful estimate (2.1), jointly with some PDE tricks, allows to bound from above moments of all Sobolev norms of solutions. Namely, denoting

$$X_j(t) = \mathbb{E}\|u(t)\|_j^2, \quad B_m = \sum_{s\in\mathbb{Z}^*} |2\pi s|^{2m} b_s^2 < \infty\,, \qquad j, m \in \mathbb{N},$$

and applying to eq. (1.1) Ito's formula, estimate (2.1) and the Gagliardo-Nirenberg inequality we get that

$$\begin{aligned}\frac{d}{dt}X_m(t) &\leq B_m - 2\nu X_{m+1}(t) + C_m X_{m+1}(t)^{\frac{2m}{2m+1}} \\ &= B_m - X_{m+1}(t)^{\frac{2m}{2m+1}} \left(2\nu X_{m+1}(t)^{\frac{1}{2m+1}} - C_m\right), \quad t \geq \theta.\end{aligned}$$

Using once again (2.1) jointly with basic PDE inequalities we obtain

$$X_m(t) \leq C'_m X_{m+1}(t)^{\frac{2m-1}{2m+1}}, \quad t \geq \theta.$$

It can be derived from these two relations that second moments of L_2-Sobolev norms of solutions are bounded uniformly in the initial data:

Theorem 2.2. *For any $u_0 \in H^1$, every $m \in \mathbb{N}$, $0 < \nu \leq 1$ and every $\theta > 0$,*

$$\mathbb{E}\|u^\nu(t; u_0)\|_m^2 \leq C(m, \theta)\nu^{-(2m-1)} \quad \textit{if} \quad t \geq \theta. \tag{2.4}$$

Jointly with (2.1) and the Gagliardo-Nirenberg inequality this result implies upper bounds on moments of all L_p–Sobolev norms of solutions of (1.1). A remarkable feature of the Burgers equation is that these estimates [1] are asymptotically sharp when $\nu \to 0$. In the next section we prove this fact for the basic inequalities (2.4).

2.2 Lower bounds

The Ito formula, applied to $\frac{1}{2}\|u(t)\|_0^2$, where $u(t)$ satisfies (1.1), implies the balance of energy relation

$$\mathbb{E}\int \tfrac{1}{2}|u(T+\sigma, x)|^2 dx - \mathbb{E}\int \tfrac{1}{2}|u(T, x)|^2 dx + \nu\mathbb{E}\int_T^{T+\sigma}\int |u_x(s, x)|^2 dx ds = \sigma B_0,$$

where $T, \sigma > 0$. Let $T \geq 1$. By (2.1) the first two terms are bounded by a constant C_* which depends only on the random force. If $\sigma \geq \sigma_* = 4C_*/B_0$, then $C_* \leq \frac{1}{4}\sigma B_0$ and we get that

[1] Except those for moments of the L_1–Sobolev norms.

$$\nu\mathbb{E}\,\frac{1}{\sigma}\int_T^{T+\sigma}\!\!\int |u_x(s,x)|^2 dxds \geq \tfrac{1}{2}B_0.$$

For any random process $f^\omega(t)$ we denote by $\langle\!\langle f\rangle\!\rangle$ its averaging in ensemble and local averaging in time,

$$\langle\!\langle f\rangle\!\rangle = \langle\!\langle f(t)\rangle\!\rangle = \mathbb{E}\,\frac{1}{\sigma}\int_T^{T+\sigma} f(s)\,ds,$$

where $T \geq 1$ and $\sigma \geq \sigma_*$ are parameters of the averaging. In this notation the just proved result reeds $\langle\!\langle \|u^\nu\|_1^2\rangle\!\rangle \geq \nu^{-1}\,\frac{1}{2}B_0$. But by Theorem 2.2 $\langle\!\langle \|u^\nu\|_1^2\rangle\!\rangle \leq \nu^{-1}C$. So

$$\langle\!\langle \|u^\nu\|_1^2\rangle\!\rangle \sim \nu^{-1},$$

where $\sim$ means that the ratio of the two quantities is bounded from below and from above, uniformly in ν and in the parameters $T \geq 1$ and $\sigma \geq \sigma_*$, entering the definition of the brackets $\langle\!\langle\cdot\rangle\!\rangle$.

Now the Gagliardo-Nirenberg inequality jointly with (2.1) imply:

$$\langle\!\langle |u_x^\nu|_{L_2}^2\rangle\!\rangle \leq C'_m \langle\!\langle \|u^\nu\|_m^2\rangle\!\rangle^{\frac{1}{2m-1}} \langle\!\langle |u_x^\nu|_{L_1}^2\rangle\!\rangle^{\frac{2m-2}{2m-1}} \leq C_m \langle\!\langle \|u^\nu\|_m^2\rangle\!\rangle^{\frac{1}{2m-1}}, \quad m \in \mathbb{N}.$$

Using the already obtained lower bound for the first Sobolev norm we get from here lower bounds for the second moments of all norms $\|u^\nu\|_m$:

$$\langle\!\langle \|u^\nu\|_m^2\rangle\!\rangle \geq C''_m \nu^{-(2m-1)} \qquad \forall m \in \mathbb{N}.$$

Combining this with the upper bound in Theorem 2.2 we get:

Theorem 2.3. *For any u_0, any $0 < \nu \leq 1$ and every $m \in \mathbb{N}$,*

$$\langle\!\langle \|u^\nu(t;u_0)\|_m^2\rangle\!\rangle \sim \nu^{-(2m-1)}. \tag{2.5}$$

This theorem and the Oleinik estimate turn out to be a powerful and efficient tool to study turbulence in the 1d Burgers equation (1.1) (the burgulence). In particular, they imply that

$$\langle\!\langle \|u^\nu(t;u_0)\|_m^2\rangle\!\rangle \sim 1 \qquad \forall u_0 \in H^1,\ \forall\, m \leq 0.$$

Here the upper bound $\langle\!\langle \|u^\nu(t;u_0)\|_m^2\rangle\!\rangle \leq C^{-1}$ for $m \leq 0$ immediately follows from (2.1), while derivation of the lower estimate $\langle\!\langle \|u^\nu(t;u_0)\|_m^2\rangle\!\rangle \geq C_m^{-1}$ for $m \leq 0$ requires some efforts.

We stress that we do not know if $\langle\!\langle \|u^\nu\|_m^2\rangle\!\rangle$ admits an asymptotic expansion as $\nu \to 0$, i.e. if it is true that

$$\langle\!\langle \|u^\nu\|_m^2\rangle\!\rangle = C_m \nu^{-(2m-1)} + o(\nu^{-(2m-1)}), \qquad m \in \mathbb{N},$$

for a suitable constants C_m.

3 Burgulence and K41

3.1 Dissipation scale

By a direct analogy with K41, the basic quantity, characterizing a solution $u^\nu(t,x)$ of (1.1) as a 1d turbulent flow is its *dissipation scale* l_d, a.k.a. *Kolmogorov's inner scale*. To define the *mathematical dissipation scale* $l_d(u)$ of any random field $u^\nu(t,x)$, continuous in (t,x), H^1-smooth in x and depending on $\nu \in (0,1]$, we write it as Fourier series (2.2) and set l_d to be the smallest number of the form $l_d = \nu^{-c_d}$, $c_d > 0$, such that for $|s| \gg l_d$ the averaged squared norm of the s-sth Fourier coefficient $\hat{u}_s(t)$ decays with s very fast. Namely, $c_d > 0$ is such that for every $\gamma > 0$ we have

$$\langle\langle |\hat{u}_s(t)|^2 \rangle\rangle \le C_{N,\gamma}|s|^{-N} \quad \text{if} \quad |s| \ge \nu^{-c_d-\gamma},$$

and c_d is the smallest number with this property. If such a $c_d > 0$ does not exist, then the inner scale $l_d(u)$ is not defined.

Theorem 2.3 and estimates (2.3) with $p = 2$ imply:

Theorem 3.1. *The mathematical dissipation space-scale l_d of any solution u of* (1.1) *equals ν^{-1}.*

In physics, the dissipative scale l_d is defined modulo a constant factor, so for the Burgers equation the physical dissipative scale is $l_d = \text{Const}\,\nu^{-1}$. It was Burgers himself who first predicted its correct value.

Now let us consider the set of integers $[C_1, \infty)$, regarded as the set of indices s of Fourier coefficients $\hat{u}_s$, and the closed interval $[0, c_1]$, $c_1 \le 1/2$, regarded as the set of increments of x. Using the physical dissipative scale l_d we divide both of them to two sets, called the *dissipation and inertial ranges*:[2]
– in Fourier presentation the *dissipation range* is $I_{diss} = (l_d, \infty) = (C\nu^{-1}, \infty) \cap \mathbb{N}$, and the *inertial range* is $I_{inert} = [\,\text{const}, l_d] = [C_1, C\nu^{-1}] \cap \mathbb{N}$.
– in the x-presentation the dissipation range is $I^x_{diss} = [0, c\nu) \subset [0, 1/2]$, and the inertial range is $I^x_{inert} = [c\nu, c_1] \subset [0, 1/2]$.

The constants C, C_1 and c, c_1 do not depend on ν and may change from one group of results to another.

Dissipation scale in K41. In K41 the hydrodynamical dissipation scale is predicted to be $l^K_d = \text{Const}\,\nu^{-3/4}$. Accordingly, in the Fourier presentation the inertial range of the K41 theory is $I^K_{inert} = [C_1, C\nu^{-3/4}]$, while in the x-presentation it is $I^{xK}_{inert} = [c\nu^{3/4}, c_1]$.

3.2 Moments of small-scale increments

For a random field $u = u^\nu(t,x)$, $t \ge 0$, $x \in S^1$, we consider the moments of its space-increments, average them in (x,t) and organise the result in the *structure function of u*:

$$S_{p,l}(u) = \langle\langle |u(\cdot + l) - u(\cdot)|^p_{L_p} \rangle\rangle, \quad p > 0, \ \ |l| \le 1/2. \tag{3.1}$$

[2] In this paper we do not deal with the energy range, so we do not define it.

If $u = u^\nu(t,x;u_0)$ with some $u_0 \in H^1$, then a.s. for $t > 0$ u is a smooth function of x, so for very small x the function $S(u)$ behaves as $|l|^p$. It turns out that for l not that small it behaves differently:

Theorem 3.2. *For $u = u^\nu$ as above and for $|l|$ in the inertial range $[c\nu, c_1]$ we have*

$$S_{p,l}(u^\nu) \sim |l|^{\min(p,1)} \quad \forall\, p > 0. \tag{3.2}$$

While for $|l|$ in the dissipation range $[0, c)$,

$$S_{p,l}(u^\nu) \sim |l|^p \nu^{1-\min(p,1)} \quad \forall\, p > 0. \tag{3.3}$$

The constants c and c_1 depend only on the force (1.2).

In [1] U. Frisch with collaborators obtained the assertion (3.2) by a convincing heuristic argument. We rigorously derive (3.2) and (3.3) from Theorems 2.1 and 2.3, using some ideas from the paper above.

Moments of small-scale increments in K41. For water turbulence the structure function is defined as above with the difference that there the increment of the velocity field $u(x+r)-u(x)$ (usually) is replaced by its projection on the direction of the vector r. Since the K41 theory deals with stationary and homogeneous vector fields, then there the structure function of a velocity field $u(t,x)$, $x \in \mathbb{T}^3$, is defined as

$$S^{\|}_{p,r}(u) = \mathbb{E}\Big|\big(u(t,x+r) - u(t,x)\big) \cdot \frac{r}{|r|}\Big|^p \tag{3.4}$$

(the r.h.s. does not depend on t and x). The K41 theory predicts that if the viscosity of the fluid is $\nu \ll 1$ (so the Reynolds number is large), then

$$S^{\|}_{2,r}(u) \sim |r|^{2/3} \quad \text{for } |r| \in I^{xK}_{inert}\ . \tag{3.5}$$

This is the celebrated *2/3 law of the K41 theory.* The theory states that the third moment of the speed's increments without the modulus sign behaves similarly:

$$\Big\langle \big((u(x+r) - u(x)) \cdot r/|r|\big)^3 \Big\rangle \sim -r \quad \text{for } |r| \in I^{xK}_{inert}\ . \tag{3.6}$$

The dimension argument, used by Kolmogorov to derive (3.5), also implies that

$$S^{\|}_{p,r}(u) \sim |r|^{p/3} \quad \text{for} \quad p > 0 \text{ if } \ |r| \in I^{xK}_{inert}. \tag{3.7}$$

This relation, although not claimed in the K41 papers, was frequently suggested in later works, related to the Kolmogorov theory.

Burgulence compare to K41. In (3.5), (3.7) the structure function behaves as $|r|$, raised to a degree, proportional to p, while in (3.2) the degree is a nonlinear function of p. Based on that, the relation in (3.2) sometime is called the abnormal scaling. The linear in p behaviour of the exponent in (3.7) now is frequently put to doubt. Indeed, it implies that for any $p, q > 0$ the ratio

$$(S^{\|}_{p,r})^{1/p} \big/ (S^{\|}_{q,r})^{1/q} \sim C_{p,q} \quad \text{for } |r| \in I^{xK}_{inert}\,, \tag{3.8}$$

where $C_{p,q}$ is an r-independent quantity. If $u(x+r)-u(x) =: \zeta$ was a Gaussian r.v., then the relations (3.8) would hold as equalities with absolute constants $C_{p,q}$, independent from ζ. But it is well known from experiments that increments of the velocity field u of a fluid with small viscosity are not Gaussian, so the Gaussian-like behaviour, manifested by (3.8), looks suspicious. On the contrary, if $u = u^\nu(t; u_0)$ is a solution of (1.1), then in view of (3.2), for $p, q \geq 1$ we have

$$S_{p,r}^{1/p} / S_{q,r}^{1/q} \sim C_{p,q}\, |r|^{1/p-1/q} \quad \text{for } |r| \in I^x_{inert} = [c\nu, c_1],$$

which is big if $p > q$ and $|r| \in I^x_{inert}$ is small (the latter may be achieved if $\nu \ll 1$). This very non-Gaussian behaviour [3] of the increments of u shows that solutions of (1.1) with small ν are random fields, far from Gaussian.

3.3 Distribution of energy along the spectrum

The second celebrated law of the Kolmogorov theory deals with distribution of fluid's energy along the spectrum. For a solution $u = u^\nu(t, x; u_0)$ of (1.1), regarded as a 1d turbulent flow, consider $\frac{1}{2}\langle\langle|\hat{u}_s|^2\rangle\rangle$. By Parseval's identity,

$$\left\langle\left\langle \tfrac{1}{2}\int |u|^2 dx \right\rangle\right\rangle = \sum_s \tfrac{1}{2}\langle\langle|\hat{u}_s|^2\rangle\rangle.$$

The l.h.s. above is energy of the fluid. So the quantities $\frac{1}{2}\langle\langle|\hat{u}_s|^2\rangle\rangle$ characterize distribution of energy along the spectrum.

Next, for any $k \in \mathbb{N}$ we define $E_k(u)$ as the averaging of $\frac{1}{2}\langle\langle|\hat{u}_s|^2\rangle\rangle$ along the layer J_k around k, defined as $J_k = \{n \in \mathbb{Z}^* : M^{-1}k \leq |n| \leq Mk\}$, $M > 1$. I.e.,

$$E_k(u) = \langle\langle e_k(u)\rangle\rangle, \tag{3.9}$$

where

$$e_k(u) = \frac{1}{|J_k|}\sum_{n\in J_k} \tfrac{1}{2}|\hat{u}_n|^2$$

is the averaged energy of the k-th mode of u. The function $k \mapsto E_k(u)$ is called the *energy spectrum* of u. It follows immediately from the definition of $l_d(u)$ with $u = u^\nu(t; u_0)$ that for $k \gg l_d$ $E_k(u)$ decays faster than any negative degree of k, uniformly in ν. But for $k \leq l_d$ the behaviour of E_k is quite different. Namely, Theorem 3.2 and relations (2.3) imply the following spectral power law for "1d Burgers fluid":

Theorem 3.3. *Let u be a solution of eq.* (1.1) *with any $u_0 \in H^1$. Then for k in the inertial range, $1 \leq k \leq C\nu^{-1}$, we have:*

$$E_k(u^\nu) \sim k^{-2}, \tag{3.10}$$

with suitable $C > 0$ and $M > 1$.

[3] The function above with $p = 4$, $q = 2$ is called the *flatness* of the random variable $u(x+l) - u(x)$. It equals three for any Gaussian r.v.

For solutions of (1.1), Jan Burgers already in 1940 predicted that $E_k \sim |k|^{-2}$ for $|k| \le \mathrm{Const}\,\nu^{-1}$, i.e. exactly the spectral power law above.

We do not know if the theorem's assertion remains true for any $M > 1$ (with a suitable $C(M)$).

Let us briefly explain how (3.10) follows from Theorem 3.2. For a solution $u = u^\nu(t, x; u_0)$ relation (2.3) implies the upper bound for energy spectrum, $E_k(u^\nu) \le Ck^{-2}$ for each k, as well as that

$$\sum_{|n|\le M^{-1}k} |n|^2 \langle\langle |\hat u_n|^2\rangle\rangle \le CM^{-1}k, \qquad \sum_{|n|\ge Mk} \langle\langle |\hat u_n|^2\rangle\rangle \le C'M^{-1}k^{-1}. \tag{3.11}$$

Now consider the sum $\Sigma_k = \sum\limits_{|n|\le Mk} |n|^2 \langle\langle |\hat u_n|^2\rangle\rangle$. Since $|\alpha| \ge |\sin\alpha|$, then

$$\Sigma_k \ge \frac{k^2}{\pi^2}\Big(\sum_{n=-\infty}^{\infty} \sin^2(n\pi k^{-1})\langle\langle |\hat u_n|^2\rangle\rangle - \sum_{|n|>Mk} \sin^2(n\pi k^{-1})\langle\langle |\hat u_n|^2\rangle\rangle\Big).$$

By Parseval's identity, $|u(t,\cdot+y) - u(t,\cdot)|^2_{L_2} = 4\sum_{n\in\mathbb{Z}^*} \sin^2(n\pi y)|\hat u_n(t)|^2$. Applying the averaging $\langle\!\langle\cdot\rangle\!\rangle$ to this equality we get that $S_{2,1/k}(u)$ is equal to $4\sum_n \sin^2(n\pi k^{-1})\langle\!\langle \hat u_n(t)\rangle\!\rangle^2$. So

$$\Sigma_k \ge \frac{k^2}{\pi^2}\Big(\frac14 S_{2,1/k}(u) - \sum_{|n|>Mk} \langle\langle |\hat u_n|^2\rangle\rangle\Big).$$

Using the second inequality in (3.11) and Theorem 3.2 we find that $\Sigma_k \ge k^2C_1k^{-1} - C_2M^{-1}k$. Since

$$E_k \ge \frac{1}{2k^3M^3}\Big(\Sigma_k - \sum_{|n|\le M^{-1}k} |n|^2\langle\langle |\hat u_n|^2\rangle\rangle\Big),$$

then using the just obtained lower bound for Σ_k and the first inequality in (3.11) we get that $E_k \ge C^{-1}k^{-2}$, if M is large enough.

Distribution of energy along the spectrum in K41. For the water turbulence the K41 theory predicts that E_k obeys the celebrated Kolmogorov–Obukhov law:

$$E_k \sim |k|^{-5/3} \quad \text{for } k \text{ in the inertial range.} \tag{3.12}$$

Experiments and numerical study of the corresponding equations convincingly show that this law is close to reality, see [11, Section 5.1].

3.4 Relation between the two laws of turbulence.

Let us first note that the definitions of the structure function S and of the energy spectrum E_k in Section 3 apply in the case when $u^\nu(t,x)$ does not depend on t and ω (and u has no relation with eq. (1.1)). Then the averaging $\langle\!\langle\cdot\rangle\!\rangle$ may be

dropped in the definitions of the objects. In this case the proof of Theorem 3.3, sketched in Section 3.3, shows that if $u^\nu \in H^1$ depends on $\nu \in (0,1]$ and if for all ν it satisfies

1) relation (3.2) with $p = 2$ for $|l| \in [c\nu, c_1]$,

2) relation (2.3), which for $u(x)$ as above reeds $|\hat{u}^\nu_k| \le C|k|^{-1}$ for all k,

then the assertion of Theorem 3.3 holds with a suitable C and a sufficiently big M (certainly same is true if u is a random field).

It is very likely (but we have not checked this) that, on the contrary, the assertion of Theorem 3.3 jointly with relation (2.3) (or (2.1)), which should be understood as in 2) above, imply the validity of (3.2) for $|l| \in [c\nu, c_1]$ with suitable $c, c_1 > 0$, and for $p = 2$ (probably also for all $p > 0$).

Much more interesting and more involved is the relation between the 2/3-law (3.5) with $|r| \in I^{xK}_{inert}$ and Kolmogorov–Obukhov law (3.12) with $|k| \in I^{K}_{inert}$, for any 3d random field $u(x)$, depending on a parameter ν. On a physical level of rigour it is explained on pp. 134-135 of [20] that the two laws are equivalent for sufficiently general fields u, but for a mathematical reader this explanation seems rather insufficient. [4] In [21, §21.4] (also see [11, Section 4.5]), assuming that $u(x)$ is an homogeneous and isotropic random field on $\mathbb{R}^3$, the equivalence of the two laws is established by a formal calculation, based on the spectral representation for $u(x)$ (see [21, §11.2]). By analogy with what was said above concerning the two laws of burgulence, it seems that this calculation cannot be rigorously justified without imposing additional restrictions on $u(x)$ (and/or on its Fourier transform), cf. above assumption 1). So we think that without referring to some new properties of fluid's flow with large Reynolds number (e.g. without evoking a new estimate for solutions of the 3d Navier–Stokes system), the two laws of turbulence should be regarded not as the same assertion, written in the x- and in Fourier presentations, but rather as two different (although related) statements.

To find a natural sufficient condition which would guarantee for a vector field on $\mathbb{T}^3$ (or for a stationary field on $\mathbb{R}^3$, or on $\mathbb{T}^3$) equivalence of the two laws of the K41 theory, or at least that one of them implies another, is an interesting open question. The technique, developed to prove the equivalence of the two laws under the hidden condition above may allow to calculate the asymptotic of $S^{\|}_{p,r}(u)$ for $|r|$ in the inertial range, and thus to correct relation (3.7), which most likely is wrong for large p.

4 The mixing

The mixing in eq. (1.1) means that in a function space H^m, $m \ge 1$, where we study the equation, there exists a unique Borel measure μ_ν, such that for any

[4] The corresponding argument was added by E. Lifschitz to the third Russian edition of the book, after L. Landau passed away. In that version of the book (which corresponds to the second English edition [20]) the part, dedicated to the theory of turbulence, was significantly edited and enlarged.

"reasonable" functional f on H^m and for any solution $u(t,x;u_0)$, $u_0 \in H^1$, we have

$$\mathbb{E} f(u(t;u_0)) \to \int_{H^m} f(u)\, \mu_\nu(du) \quad \text{as} \quad t \to \infty. \tag{4.1}$$

The measure μ_ν is called the stationary measure for eq. (1.1). If u_0 is a r.v., distributed as μ_ν, then $u(t;u_0) =: u^{st}(t)$ is a stationary solution: $\mathcal{D}(u^{st}(t)) \equiv \mu_\nu$.

It may be derived from a general theory that the mixing holds for eq. (1.1), but then the rate of convergence in (4.1) would depend on ν. In the same time, in the theory of turbulence the rate of convergence to a statistical equilibrium should not depend on the viscosity (see [2], e.g. pages 6-7 and 109), and for solutions of (1.1) it does not:

Theorem 4.1. *If the functional $f(u)$ is continuous in some L_p–norm, $p < \infty$, and $|f(u)| \le C|u|^N_{L_p}$ for suitable $C > 0$ and $N \in \mathbb{N}$, then (4.1) holds. The rate of convergence is at least $(\ln t)^{-\kappa_p}$, for some $\kappa_p > 0$.*

The proof follows from the results in Section 2, basic methods to prove the mixing in stochastic PDEs, and from another remarkable feature of the Burgers equation: if $\omega \in \Omega$ belongs to the full-measure set for which the force $\xi^\omega(t,x)$ is well defined, then for any u_0 and u_1,

$$\big|u^\omega(t;u_0) - u^\omega(t;u_1)\big|_{L_1} \le |u_0 - u_1|_{L_1} \quad \text{for every} \;\; t \ge 0.$$

If in (1.2) $b_s \equiv b_{-s}$, then the random field $\xi(t,x)$ is homogeneous in x. In this case the measure μ_ν also is homogeneous, and the stationary solution u^{st} is stationary in t, homogeneous in x. All results in Section 3 remain true for this solution, describing the stationary and space-homogeneous burgulence.

Energy spectrum of the stationary measure μ_ν is $E_k(\mu_\nu) = \int e_k(u)\mu_\mu(du)$, where e_k is as in (3.9). Obviously,

$$E_k(\mu_\nu) = \langle\!\langle e_k(u^{st}(t)) \rangle\!\rangle = \mathbb{E} e_k(u^{st}(t)).$$

Since $\langle\!\langle e_k(u^{st}(t)) \rangle\!\rangle$ satisfies the spectral power law, then $E_k(\mu_\nu)$ also does:

$$E_k(\mu_\nu) \sim k^{-2} \quad \text{for } 1 \le k \le C\nu^{-1}.$$

Due to Theorem 4.1 the instant energy spectrum of every solution converges to that of μ_ν:

$$\mathbb{E} e_k(u(t;u_0)) \to E_k(\mu) \qquad \forall\, u_0 \in H^1,\ \forall\, k \in \mathbb{N},$$

uniformly in ν.

Similarly the structure function of μ_ν, defined as $S_{p,l}(\mu_\nu) = \int_{H^m} |u(\cdot + l) - u(\cdot)|^p_{L_p} \mu_\nu(du)$, satisfies (3.2) and (3.3) for l in the inertial and dissipation ranges, correspondingly. As above, the instant structure function of every solution converges, as time grows, to $S_{p,l}(\mu_\nu)$ for for all p and l, uniformly in ν (and in l). If $b_s \equiv b_{-s}$, then the measure μ_ν is homogeneous and then

$$S_{p,l}(\mu_\nu) = \mathbb{E}|u^{st}(t,x+l) - u^{st}(t,x)|^p = \int_{H^m} |u(x+l) - u(x)|^p \mu_\nu(du) \quad \text{for any} \;\; t, x.$$

The results in this section are in line with the general theory of turbulence which postulates that statistical characteristics of turbulent flows converge, as time grows, to a universal statistical equilibrium. They also are in the spirit of K41, where the velocity field of a fluid is assumed to be stationary in t and homogeneous in x.

5 Inviscid limit

Another remarkable feature of the Burgers equation (1.1) is that, as $\nu \to 0$ (so the Reynolds number of the corresponding "1d fluid" grows to infinity) the solutions of the equation converge to inviscid limits:

$$u^\nu(t,\cdot;u_0) \to u^0(t,\cdot;u_0) \quad \text{in} \;\; L_p(S^1) \;\; \forall\, t \geq 0, \;\; \text{a.s.},$$

for every $p < \infty$ and every u_0. This result is due to Lax–Oleinik (1957). The limit $u^0(t,x;u_0)$ is called an "inviscid solution", or an "entropy solution" of equation (1.1) with $\nu = 0$. The limiting function $u^0(t,x)$ is bounded in x for every t, but in general is not continuous. Still its structure function and spectral energy are well defined and inherit the laws, proved for u^ν with $\nu > 0$. Now the laws are valid with $\nu = 0$:

Theorem 5.1. *For each entropy solution u^0,*
1) $E_k(u^0) \sim k^{-2}$ for all k;
2) $S_{p,l}(u^0) \sim |l|^{\min(p,1)}$ if $p > 0$ and $|l| \leq c_1$.

Since the spectral power law for $E_k(u^0)$ holds for all $k \geq 1$, then the dissipation range of u^0 is empty. Its inertial range in the Fourier presentation is all of $\mathbb{N}$, and in x it is the whole interval $[0, c_1]$. The inviscid solutions define in the space $L_1 = \{u \in L_1(S^1) : \int u\,dx = 0\}$ a mixing Markov process, whose stationary measure is supported by the space $L_1 \cap \big(\cap_{p<\infty} L_p(S^1)\big)$.

These results describe the *inviscid burgulence*. They have no analogy in the K41 theory since there the Reynolds number *Rey* of fluid's flow is a fixed finite quantity, and since on the mathematical side of the question, behaviour of solutions of the 3d hydrodynamical equations on time-intervals of order $\gtrsim 1$, when $Rey \to \infty$, is a completely open problem.

6 Conclusions

The stochastic Burgers equation (1.1) with small viscosity makes a consistent model of 1d turbulence. Its rigorously proved statistical properties make natural and close analogies for the main laws of the K41 theory of turbulence. This, once again, supports the belief that the K41 theory is "close to the truth".

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