



A simple proof of the Fundamental Theorem of Algebra

Ricardo Pérez-Marco

► To cite this version:

| Ricardo Pérez-Marco. A simple proof of the Fundamental Theorem of Algebra. 2021. hal-03109761v1

HAL Id: hal-03109761

<https://hal.science/hal-03109761v1>

Preprint submitted on 13 Jan 2021 (v1), last revised 15 Jan 2021 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A SIMPLE PROOF OF THE FUNDAMENTAL THEOREM OF ALGEBRA

RICARDO PÉREZ-MARCO

ABSTRACT. We present a simple short proof of the Fundamental Theorem of Algebra, with minimal use of topology.

1. STATEMENT.

Theorem 1.1. *A non constant polynomial $P(z) \in \mathbb{C}[z]$ with complex coefficients has a root.*

The proof is based on the following elementary facts:

- A polynomial has at most a finite number of roots.
- A polynomial is proper.
- The Inverse Function Theorem.
- Removing from \mathbb{C} a finite number of points leaves a connected space.

2. THE PROOF.

It is enough to consider a monic polynomial P . We denote by $\mathcal{C} = (P')^{-1}(0)$ the finite set of critical points of P , and by $\mathcal{D} = P(\mathcal{C})$ is the finite set of critical values of P .

- Let $R = \{c \in \mathbb{C}; \text{ the polynomial } P(z) - c \text{ has a simple root and no double roots}\}$.
- $R \subset \mathbb{C} - \mathcal{D}$. This is because if $c \in \mathcal{D}$, then $c = P(z_0)$ for some critical point $z_0 \in \mathcal{C}$, hence $P'(z_0) = 0$ and $P(z) - c = 0$ has a double root at z_0 . Note that $\mathbb{C} - \mathcal{D}$ is connected (\mathcal{D} being finite).
- R is open. By application of the Inverse Function Theorem at the simple roots of $P(z) - c$ we can locally follow each root for c' near c and they remain simple. Moreover, by properness of P the number of roots remains locally constant and we have don't have any other roots (in particular double ones) for c' near c .
- R is closed in $\mathbb{C} - \mathcal{D}$. Since P is monic, if c is uniformly bounded then any root of $P(z) - c$ is uniformly bounded. We can take a subsequence of $c_n \rightarrow c_\infty \in \mathbb{C} - \mathcal{D}$ such that the limit of roots of $P(z) - c_n$ converge to roots of $P(z) - c_\infty$. We have at least one such limit so $P(z) - c_\infty$ has roots. Moreover, all roots of $P(z) - c_\infty$ are all simple since $c_\infty \in \mathbb{C} - \mathcal{D}$.
- R is non-empty. For any $a \in \mathbb{C}$ we have that for $c = P(a)$, $P(z) - c$ has at least $z = a$ as root. If we choose $a \in \mathbb{C} - P^{-1}(\mathcal{D})$, then for any root z_0 of $P(z) - c$ with $c = P(a)$, we have $P(z_0) = P(a) \notin \mathcal{D}$, so $z_0 \notin P^{-1}(\mathcal{D})$, but $\mathcal{C} \subset P^{-1}(\mathcal{D})$, and $z_0 \notin \mathcal{C}$, and the root z_0 is simple.

2010 *Mathematics Subject Classification.* 30C15, 12D10.

Key words and phrases. Roots, complex polynomials, fundamental theorem of algebra.

The above proves that $R = \mathbb{C} - \mathcal{D}$. Now, if $0 \in \mathcal{D}$, then $0 = P(z_0)$ for a critical point z_0 of P that is also a root of P . If $0 \notin \mathcal{D}$, then $0 \in R = \mathbb{C} - \mathcal{D}$ and the equation $P(z) - 0 = 0$ has a simple root. In all cases P has a root. \diamond

3. COMMENT.

The above proof is inspired from a beautiful proof by Daniel Litt [1]. He works in the global space of monic polynomials of degree $n \geq 1$ (biholomorphic to \mathbb{C}^n), and removes the algebraic set \mathcal{D}_n , defined by the discriminant, of polynomials with a double root. He uses that the complement of an algebraic variety in \mathbb{C}^n is connected. Essentially the proof above achieves the same goal in a more elementary way working with $n = 1$. In particular, we only need the simpler fact that the complement of a finite set in the plane is connected (which for $n = 1$ is the same as the connectedness of the complement of an algebraic variety in \mathbb{C}^n). We also avoid the use of discriminants.

Acknowledgment. I am grateful to my friends Marie-Claude Arnaud, Kingshook Biswas and Alain Chenciner for their comments and suggestions to improve the presentation. In particular, to Kingshook that proposed a simplification of a first draft.

REFERENCES

- [1] LITT, D.; *Yet another proof of the Fundamental Theorem of Algebra*, Manuscript, 2011.
(www.daniellitt.com/blog/2016/10/6/a-minimal-proof-of-the-fundamental-theorem-of-algebra)

CNRS, IMJ-PRG, UNIVERSITÉ DE PARIS, BOÎTE COURRIER 7012, 75005 PARIS CEDEX 13, FRANCE

Email address: `ricardo.perez.marco@gmail.com`