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► **To cite this version:**

Jennifer Blondel, Sébastien Joannès, Eveline Hervé-Luanco. Analytical modelling of the effect of morphological fluctuations on the transverse elastic behaviour of unidirectional fibre reinforced composites. *International Journal of Solids and Structures*, 2020, 206, pp.436-455. 10.1016/j.ijstr.2020.06.001 . hal-03109172

HAL Id: hal-03109172

<https://hal.science/hal-03109172>

Submitted on 24 Oct 2022

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1 Analytical modelling of the effect of morphological
2 fluctuations on the transverse elastic behaviour of
3 unidirectional fibre reinforced composites

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12 **Abstract**

This paper proposes to take into account the influence of some morphological fluctuations – often observed in unidirectional composite materials – on the modelling of the elastic behaviour of such materials. This work relies on a Generalized Self-Consistent Scheme coupled with a Morphologically Representative Pattern based approach. An analytical model is proposed to deal with “non-percolated” or trapped matrix regions. Closed-form analytical expressions are provided to investigate different kinds of morphological fluctuation effects on the effective transverse elastic behaviour. Finally some examples are given to illustrate the effectiveness of this approach.

13 *Keywords:* Homogenization, Fibre reinforced composites,
14 Micromechanical models, Morphologically Representative Patterns,
15 Generalized Self-Consistent Scheme, Fluctuations of morphology

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Preprint submitted to International Journal of Solids and Structures May 5, 2020

List of Acronyms & Symbols

17 Acronyms

18	EHM	Equivalent Homogeneous Medium.....	15
19	FEM	Finite Element Method.....	8
20	GHW	Refers to the results published in Gusev et al. (2000).....	30
21	GSCS	Generalized Self-Consistent Scheme.....	10
22	MRP	Morphologically Representative Pattern.....	10
23	RVE	Representative Volume Element.....	11
24	UD	Unidirectional (fibre-reinforced composites).....	7
25	ZC	Refers to the results published in Zimmer and Cost (1970)	30

26 Subscripts, superscripts and brackets

27	•	Represents an arbitrary (0, 1, 2)-tensor field or a matrix variable	
28			
29	•	Defines a first-order tensor (vector)	
30	•	Defines a second-order tensor	
31	•	Defines a fourth-order tensor	
32	• ^(i_λ) or • _{i_λ}	Phase index inside the pattern λ, i _λ ∈ {1, ..., n _λ }	

- 33 $\bullet^{(k)}$ or \bullet_k The letter k is used generically to designate any phase (k). It
34 can be the phase index $\bullet^{(i)}$ or \bullet_i , $i \in \{1, 2, \dots, n + 1\}$ or the
35 phase index inside the pattern λ , $i_\lambda \in \{1, \dots, n_\lambda\}$
- 36 $\bullet^{[j]}$ An estimate of \bullet during the iteration process
- 37 \bullet^f , \bullet^m or \bullet^{eff} Related to “fibre”, “matrix” or “effective” properties
- 38 $\bullet^{\textcircled{\lambda}}$ or $\bullet_{\textcircled{\lambda}}$ Indicates the pattern dependency
- 39 $\bullet^{\overline{\lambda}}$ or $\bullet_{\overline{\lambda}}$ Identity transformation defined for two patterns and two phases,
40 $\overline{\lambda} \equiv \text{Id}(\lambda)$ with $\overline{1} = 1$ and $\overline{2} = 2$
- 41 $\bullet^{\underline{\lambda}}$ or $\bullet_{\underline{\lambda}}$ Complementary transformation defined for two patterns and
42 two phases,
43 $\underline{\lambda} \equiv \text{Cp}(\lambda)$ with $\underline{1} = 2$ and $\underline{2} = 1$
- 44 $\langle \bullet \rangle_{\Omega_\square}$ Average volume fraction of any \bullet expression over the domain Ω_\square
- 45 $\left\{ \underset{\subseteq \bullet}{A_p} \right\}$ When \bullet needs to be expanded, \bullet can be, partially or completely,
46 split into several parts: $\left\{ \underset{\subseteq \bullet}{A_1} \right\}, \left\{ \underset{\subseteq \bullet}{A_2} \right\}, \dots; \left\{ \underset{\subseteq \bullet}{A_p} \right\}, p \in \mathbb{N}$, represents
47 all these parts. When \bullet can be written in different forms, $\left\{ \underset{\subseteq \bullet}{B_p} \right\},$
48 $\left\{ \underset{\subseteq \bullet}{C_p} \right\}, \dots$ are used. When only one part is needed, index is
49 omitted

50 Material constants and parameters

- 51 μ_{23} Shear modulus
- 52 ν Poison’s ratio
- 53 E Young’s modulus

54	k_{23}	Transverse bulk modulus
55	$\underset{\approx}{\mathcal{C}}$	Elastic stiffness tensor
56	Other Symbols	
57	\underline{x}	Position vector, i.e. $\underline{x} = x_1\underline{e}_1 + x_2\underline{e}_2 + x_3\underline{e}_3$ or $\underline{x} = r\underline{e}_r + x_1\underline{e}_1$
58	$(\underline{e}_1, \underline{e}_2, \underline{e}_3)$	Cartesian orthonormal basis set
59	$(\underline{e}_r, \underline{e}_\theta, \underline{e}_1)$	Cylindrical orthonormal basis set
60	(x_1, x_2, x_3)	Cartesian coordinates of \underline{x} in the basis $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$
61	(r, θ, x_1)	Cylindrical coordinates of \underline{x} in the basis $(\underline{e}_r, \underline{e}_\theta, \underline{e}_1)$
62	Ω	Domain of material whose effective properties are sought
63	Ω_λ	Domain of the pattern λ , partition of Ω
64	Ω_i	Domain of phase (i) , partition of Ω
65	$\Omega_{i\lambda}$	Domain of phase (i) inside the pattern λ , partition of Ω_i
66	n	Number of phases
67	N_λ	Number of “Morphological Representative Pattern” families
68	n_λ	Number of phases inside pattern λ
69	f	Volume fraction of phase (1), in the case $N_\lambda = 2$ and $n_\lambda = 2$
70		for each pattern; i.e. $f = f_1$
71	f_i	Volume fraction of phase (i)

72	m_λ	Volume fraction of pattern λ
73	m	Volume fraction of the “direct” pattern, in the case $N_\lambda = 2$
74		and $n_\lambda = 2$ for each pattern; i.e. $m = m_1$
75	$c_{i\lambda}$	Volume fraction of phase (i) inside pattern λ
76	$c_{i\otimes}$	Volume fraction of the “ith” phase inside pattern λ
77	c	Volume fraction of phase (1) inside the “direct” pattern, in the
78		case $N_\lambda = 2$ and $n_\lambda = 2$ for each pattern; i.e. $c = c_{1_1} = c_{1\textcircled{1}}$
79	R_i	Outer radius of phase (i) lying within the radii R_{i-1} and R_i
80	\underline{T}	Stress vector
81	\underline{T}^0	Applied stress vector
82	T_i	Stress vector component along \underline{e}_i
83	\underline{u}	Displacement vector
84	\underline{u}^0	Displacement vector applied to each pattern, homogeneous con-
85		ditions
86	\underline{u}^g	“Given” displacement vector applied to the equivalent homoge-
87		neous configuration, homogeneous conditions
88	u_i	Displacement vector component along \underline{e}_i
89	$\underline{\sigma}$	Stress tensor
90	$\underline{\varepsilon}$	Strain tensor

91	$\underline{\underline{\varepsilon}}^0$	Strain tensor such that $\underline{u}^0 = \underline{\underline{\varepsilon}}^0 \cdot \underline{x}$ is applied to each pattern
92	\underline{E}	Strain tensor such that $\underline{u}^g = \underline{E} \cdot \underline{x}$ is applied to the equivalent
93		homogeneous configuration
94	\mathcal{A}_{i_λ}	Average intensity concentration for phase (i_λ)
95	$\underline{\underline{\mathcal{A}}}_i$	Average intensity concentration tensor for phase (i)
96	Bold	Bold notation for rectangular arrays
97		i.e. rectangular arrays such as \mathbf{V} , \mathbf{J} , \mathbf{N} or \mathbf{Q} with components
98		denoted Q_{kl} for example
99	\mathbf{Q} or \mathbf{Q}^*	Transfer matrices
100	A_k, B_k, C_k and D_k	Integration constants for each phase (k) in the case of
101		an in-plane transverse shear mode
102	$\underline{\underline{A}}_{i_\lambda}$ and $\underline{\underline{D}}_{i_\lambda}$	Constants for phase (i_λ)

103 **1. Introduction**

104 *1.1. Evaluating the transverse elastic properties of unidirectional compos-* 105 *ites*

106 Knowing and anticipating how a structure will deform when subjected
107 to a load is a daily challenge for engineers. Depending on their anisotropy,
108 composite materials can exhibit a strong dependence on the direction in
109 which loads are applied. It is clear that a single unidirectional (UD) ply,
110 for example reinforcing the rubber of a tire, will exhibit a considerable re-
111 sistance to deformation if the load is applied along the direction of the
112 reinforcement. In contrast, this same ply will undergo a large deformation
113 if the load is now applied in a non-longitudinal direction. To design effi-
114 cient composite structures and take advantage of the anisotropy, engineers
115 need to characterize or predict the load-deformation material response for
116 all the directions in which loads may occur. Regarding transverse proper-
117 ties, experimental characterization is always a tedious procedure and result
118 uncertainties are often large. These complications provide a strong practical
119 motivation for the development of models which can be used to predict the
120 material performance.

121 However, this does not mean that models remove the underlying difficul-
122 ties. If the prediction of UD longitudinal properties is rarely complicated,
123 transverse properties are more affected by interactions between the con-
124 stituents and predicting accurately the transverse properties is always chal-
125 lenging. Models taking into account the morphological distribution of the
126 phases, in addition to their volume fractions, are required to obtain realistic
127 predictions for the transverse properties of such UD composite materials.

128 Morphological variabilities and fibre packing strongly influence the trans-
129 verse linear elastic behaviour of these composites. Numerous and varied
130 methods have already been proposed in order to predict effective proper-
131 ties of such a multi-phases medium. Existing solutions include, on the one
132 hand, full-field approaches – as the Finite Element Method (FEM) for ex-
133 ample – requiring to model the microstructure in details. Such methods
134 require complex¹ mesh operations, especially when the fibres get very close
135 or when functionally graded materials are considered. On the other hand,
136 mean-field approaches are based on a single inclusion problem and unlike
137 full-field approaches, only statistical information about the microstructure
138 is needed. Both approaches have their advantages and this work is focused
139 on an efficient, fast and cost-effective analytical mean-field model to take
140 into account morphological fluctuations at the microscopic scale, in order to
141 predict the transverse linear elastic shear and bulk moduli of unidirectional
142 composites.

143 *1.2. A rational microstructure model*

144 As explained previously, this work is devoted to the study of the elastic
145 behaviour of fibre-reinforced composites with different distributions of fi-
146 bres. Restricting the attention to cylindrical shape and transverse isotropic
147 elasticity, we can refer to Hashin and Rosen’s (1964) composite assem-
148 blage or to Christensen and Lo’s (1979) three-phase model which deal
149 with two-phases coaxial cylinder for a two-phases material. A model has

¹The two-dimensional models dealt with in the present investigation are nevertheless far more simple than the three-dimensional ones require in the case of macroscopically isotropic particulate composites, such as in Bardella et al. (2018).

150 been developed in Hervé and Zaoui (1995) to predict the elastic behaviour
151 of heterogeneous fibre-reinforced composites including the case of coated
152 fibre-reinforced composites, usually referred to as the $(n + 1)$ -phase model².
153 This work has also provided the elastic strain and stress fields in an infinite
154 medium constituted of an n -layered transversely isotropic cylindrical inclu-
155 sion, surrounded by a transversely isotropic cylindrical matrix subjected to
156 uniform conditions at infinity. These fields have been given in a simpler
157 form in Hervé-Luanco (2020). From that, we can study the transverse elas-
158 tic behaviour of more complex morphologies regarding the way the fibres
159 are distributed inside the matrix and by also accounting for fibres that can
160 have a functionally graded behaviour. These concerns lead directly, on the
161 one hand, to more than one pattern³, and, on the other hand to more than
162 two phases in each pattern.

163 The way used here to take into account complex morphologies is to
164 add as many elementary patterns as needed like in Marcadon et al. (2007)
165 or in Majewski et al. (2017) in the case of particle-reinforced composites
166 having size effects, or in Diani and Gilormini (2014) in the case of linear
167 viscoelasticity of nano-reinforced polymers with an interphase, or in Bil-
168 ger et al. (2007), Bardella and Genna (2001) and Bardella et al. (2018)
169 in the case of porous materials. This latter case, dealing with macroscopi-
170 cally isotropic particulate composites, is particularly relevant to show that

²It should be noted that the three-phase model proposed in Christensen and Lo (1979) corresponds also to the $(n + 1)$ -phase model developed in Hervé and Zaoui (1995) in the particular case of $n = 2$.

³A “pattern” contains a “representative disposition of some of the constitutive mechanical phases”, as defined by Bornert et al. (1996).

171 the present study makes sense. Although the materials are very different,
172 in Bardella et al.’s paper (2018) it is shown that the method adopted in the
173 present investigation, using representative patterns to account for complex
174 morphologies, works very well even with an extremely large filler volume
175 fraction.

176 If a biphasic matrix-inclusion material is considered, at low volume frac-
177 tion of inclusions, there is no doubt that the matrix is a continuous and
178 percolating phase as opposed to inclusions which are distributed discontin-
179 uously. If the volume fraction of inclusions increases sufficiently a network
180 of percolating aggregates can be formed. The matrix is then trapped by ag-
181 gregates which makes it lose its continuity and leads to a “phase inversion”
182 phenomenon as discussed in Alb erola et al.’s paper (1994). In this specific
183 case of two patterns with two phases in each pattern, using an “inverse”
184 pattern – where the inclusion surrounds the matrix – allows to model this
185 phenomenon (Alb erola et al., 1994; M el e et al., 2005).

186 In this work, the Generalized Self-Consistent Scheme (GSCS) used in Herv e
187 and Zaoui (1995) and in Herv e-Luanco (2020) is coupled with the Mor-
188 phologically Representative Pattern (MRP) based approach developed by
189 Bornert et al. (1996) in order to take into account local morphological fluc-
190 tuations and heterogeneous distribution of fibres. Analytical estimates are
191 provided for both effective transverse shear and bulk moduli. In this pa-
192 per, the approach followed by Marcadon et al. (2007) for particle-reinforced
193 composites in the case of two patterns with two phases has been generalized
194 in the case of: 1) fibre-reinforced composites 2) for any number of patterns
195 with any number of phases inside each pattern like in Herv e-Luanco and
196 Joann es (2016a). In this latter paper a model has been developed to predict

197 the transverse properties of a fibre-reinforced composite in the case of trans-
198 port phenomena. In a second article (Joannès and Hervé-Luanco, 2016b),
199 the authors have used their model to study the influence of fibre packing
200 effects on the transverse properties of the composites regarding transport
201 phenomena.

202 In the present paper, a MRP-based approach is coupled to the $(n + 1)$ -
203 phase model thanks to a rearrangement of the transfer matrices in terms
204 of their dependence on the behaviour of a given phase as shown in Hervé-
205 Luanco (2020). It is organized as follows: Section 2 is devoted to the de-
206 velopment of the model. Closed-form equations are given for the transverse
207 shear and bulk moduli. In section 3, the model is applied in the case of two
208 patterns with two phases inside each pattern. An example of application is
209 given to highlight the possibilities of the presented model. Two comparisons
210 with experimental data are presented in section 4.

211 **2. Extension of the $(n + 1)$ -phase model by using a MRP-based** 212 **approach**

213 *2.1. Introduction*

214 Let us consider the configuration defined in Hervé and Zaoui (1995)
215 where the elementary pattern representing the microstructure is a n -phase
216 cylindrical inclusion which is embedded in an infinite matrix subjected to
217 homogeneous boundary conditions at infinity (Figure 1). Each phase is
218 homogeneous, linearly elastic and transversely isotropic with the axis of
219 transverse isotropy along the direction of the fibre.

220 Let us assume first that the Representative Volume Element (RVE) of
221 the studied microstructure is a Hashin assemblage of cylindrical domains

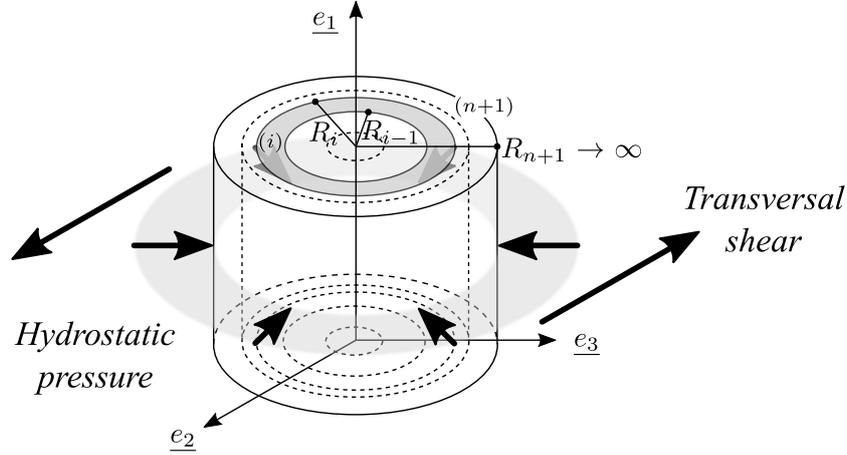


Figure 1: One single elementary pattern made of a n -layered cylindrical inclusion embedded in an infinite matrix, i.e. phase $(n + 1)$.

222 consisting of a set of N_λ families (see Figure 2) of homothetic identical fi-
 223 nite composite domains whose material content is known. These families are
 224 called here “pattern”, they are made of different phases and this microstruc-
 225 ture can be referred to as a generalized Hashin’s assemblage of patterns. The
 226 configuration under study is referred to a Cartesian rectangular coordinate
 227 system in the basis $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ thus allowing to locate any point from the
 228 origin O by its position vector $\underline{x} = x_1\underline{e}_1 + x_2\underline{e}_2 + x_3\underline{e}_3$. Direction given by
 229 \underline{e}_1 is parallel to the fibre longitudinal axis while $(O, \underline{e}_2, \underline{e}_3)$ define the trans-
 230 verse plane. A cylindrical coordinates system with the same origin and
 231 the orthonormal basis set $(\underline{e}_r, \underline{e}_\theta, \underline{e}_1)$ can also be used. In this cylindrical
 232 representation, coordinates of the position vector are denoted (r, θ, x_1) . As
 233 in Hervé and Zaoui (1995), let phase (i) , for $i \in \{1, 2, \dots, n + 1\}$, lie within

234 the shell limited by the inner radius R_{i-1} and the outer radius R_i (see Fig-
 235 ure 1). It should be noted that we consider $R_0 = 0$ and $R_{n+1} \rightarrow \infty$. The
 236 interfaces between the different phases are supposed to be perfect leading
 237 to the continuity of the displacement vector \underline{u} and of the stress vector \underline{T} at
 238 each interface $r = R_i$. Let $k_{23}^{(i)}$ and $\mu_{23}^{(i)}$ be respectively the transverse bulk
 239 and shear moduli of phase (i) and $\underline{\varepsilon}$ and $\underline{\sigma}$ be respectively the strain and
 240 stress tensors. A cylindrical symmetry behaviour is considered where $k_{23}^{(i)}$
 241 and $\mu_{23}^{(i)}$ depend only on r . In order to determine the effective transverse
 242 bulk and shear moduli of the assemblage (i.e. k_{23}^{eff} and μ_{23}^{eff}), two elementary
 243 loadings modes are studied: an in-plane hydrostatic mode and an in-plane
 244 transverse shear one.

245 2.2. Methodology

246 In the two studied modes the same methodology has been used in order
 247 to derive closed-form solutions of the transverse bulk and shear moduli. The
 248 local fields inside each phase of each pattern λ with $\lambda \in \{1, 2, \dots, N_\lambda\}$ are
 249 here expressed thanks to the model developed in Hervé and Zaoui (1995)
 250 and simplified recently in Hervé-Luanco (2020). Let n_λ be the number of
 251 phases inside pattern λ and m_λ the volume fraction of pattern λ . Phase
 252 (i_λ) corresponds to the part of phase (i) present in pattern λ . Depend-
 253 ing on the context, i_λ can take an explicit form⁴ or represents the position
 254 of phase (i) inside the pattern λ using an integer to indicate the ranking.
 255 This is particularly the case if calculations are carried out with i_λ such
 256 as, for example, $i_\lambda - 1$. In that latter case, i_λ may take the following val-
 257 ues $\{1, 2, \dots, n_\lambda\}$. Consider that the studied material occupies a volume Ω

⁴For instance 1_2 , designating phase (1) in pattern 2.

258 and that the volume corresponding to pattern λ is denoted Ω_λ . We suppose
 259 that $\Omega = \cup_{\lambda=1}^{N_\lambda} \Omega_\lambda$ and that $\cap_{\lambda=1}^{N_\lambda} \Omega_\lambda = \emptyset$. We also denote by Ω_i the vol-
 260 ume corresponding to all the parts of phase (i) present in several patterns
 261 and consider that $\Omega_i = \cup_{\lambda=1}^{N_\lambda} \Omega_{i_\lambda}$ and $\cap_{\lambda=1}^{N_\lambda} \Omega_{i_\lambda} = \emptyset$ where Ω_{i_λ} is the volume
 262 corresponding to the phase (i) in pattern λ . The particular case of two
 263 patterns and two phases (section 3) makes it possible to grasp more easily
 264 the relationship between (i) and (i_λ).
 265 Overall properties are defined through consideration of a boundary value
 266 problem. The determination of the effective behaviour is derived from first,
 267 the solution of the elementary problem of each pattern embedded in an in-
 268 finite homogeneous elastic matrix with adequate moduli subjected to the
 269 same uniform strain at infinity and secondly, with some adequate average
 270 operation leading to the determination of the effective elastic stiffness ten-
 271 sor $\underline{\underline{C}}^{\text{eff}}$ (transverse moduli in this paper). The first step is to consider that
 272 the same kind of boundary conditions are imposed on each pattern as for
 273 the classical GSCS model (Hervé-Luanco, 2020). The terminal sections of
 274 the composite are subjected to the following boundary condition:

$$\left. \begin{aligned} u_1^0 &= 0 \\ T_2^0 &= 0 \\ T_3^0 &= 0 \end{aligned} \right\} \quad (1)$$

275 and the lateral surface, i.e. $r = R_{n+1}$, to the following ones:

$$\underline{u}^g = \underline{E} \cdot \underline{x} \quad (2)$$

276 On the terminal section of each pattern:

$$\left. \begin{aligned} u_1^0 &= 0 \\ T_2^0 &= 0 \\ T_3^0 &= 0 \end{aligned} \right\} \quad (3)$$

277 and on the lateral surface of each pattern:

$$\underline{u}^0 = \underline{\varepsilon}^0 \cdot \underline{x} \quad (4)$$

278 where $\underline{\varepsilon}^0$ in Eq. (4) depends on the considered mode (in-plane hydrostatic or in-plane transverse shear mode).

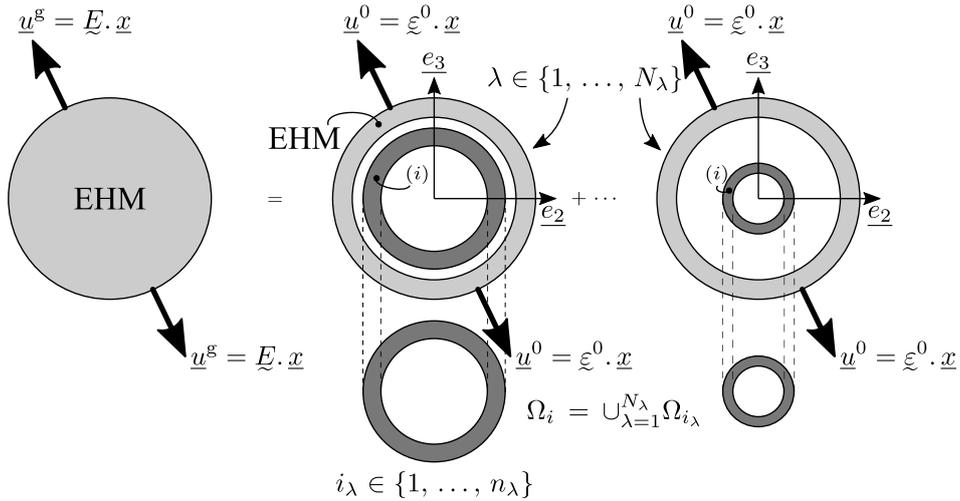


Figure 2: MRP approach with N_λ patterns made of a n_λ -layered cylindrical inclusion embedded in an infinite medium, i.e. phase $(n+1)$, also corresponding to the Equivalent Homogeneous Medium (EHM).

279

280 The linear constitutive relation for elasticity will be denoted by:

$$\underline{\sigma} = \underline{\mathcal{C}} : \underline{\varepsilon} \quad (5)$$

281 Since we disregard eigenstrains the average strain tensor in any phase (i) is
 282 written as follows:

$$\langle \underline{\varepsilon} \rangle_{\Omega_i} = \underline{\underline{\mathcal{A}}}_i : \underline{E} \quad (6)$$

283 where $\underline{\underline{\mathcal{A}}}_i$ is the average intensity concentration tensor regarding phase (i).
 284 The effective elastic moduli tensor $\underline{\underline{C}}^{\text{eff}}$ is derived from the following rela-
 285 tions:

$$\underline{\Sigma} = \langle \underline{\sigma} \rangle_{\Omega} = \underline{\underline{C}}^{\text{eff}} : \langle \underline{\varepsilon} \rangle_{\Omega} = \underline{\underline{C}}^{\text{eff}} : \underline{E} \quad (7)$$

286 It comes immediately from Eqs. (5), (Eq. (6)) and (Eq. (7)) that:

$$\underline{\Sigma} = \langle \underline{\underline{C}} : \underline{\underline{\mathcal{A}}} \rangle_{\Omega} : \underline{E} = \sum_{i=1}^n f_i \underline{\underline{C}}^{(i)} : \underline{\underline{\mathcal{A}}}_i : \underline{E} \quad (8)$$

287 where f_i is the volume fraction of phase (i). Consequently, as defined by Hill
 288 (1963):

$$\underline{\underline{C}}^{\text{eff}} = \sum_{i=1}^n f_i \underline{\underline{C}}^{(i)} : \underline{\underline{\mathcal{A}}}_i \quad (9)$$

289 2.3. In-plane hydrostatic mode

290 In the case where an in-plane hydrostatic mode is considered, following
 291 Hervé and Zaoui (1995), $\underline{\varepsilon}^0$ in Eq. (4) is chosen as:

$$\underline{\varepsilon}^0 = \beta^0 \left[(\underline{e}_2 \otimes \underline{e}_2) + (\underline{e}_3 \otimes \underline{e}_3) \right] \quad (10)$$

292 Thanks to the $(n+1)$ -phase problem solved in Hervé and Zaoui (1995) (Eq.⁹⁵ (37)),
 293 with $\epsilon = 0$, as typographically defined in Hervé and Zaoui's paper (1995),
 294 we know the average strain tensor $\langle \underline{\varepsilon} \rangle_{\Omega_{i_\lambda}}$ in each phase (i_λ) of any pattern λ :

$$\langle \underline{\varepsilon} \rangle_{\Omega_{i_\lambda}} = \frac{Q_{11}^{(i_\lambda-1)}}{Q_{11}^{(n_\lambda)}} \beta^0 \left[(\underline{e}_2 \otimes \underline{e}_2) + (\underline{e}_3 \otimes \underline{e}_3) \right] \quad (11)$$

295 From Eqs. (10) and (Eq. (11)) it is worth noticing that $\langle \underline{\varepsilon} \rangle_{\Omega_{i_\lambda}}$ can be written
 296 under the following form:

$$\langle \underline{\varepsilon} \rangle_{\Omega_{i_\lambda}} = \mathcal{A}_{i_\lambda} \underline{\varepsilon}^0 \quad (12)$$

297 where \mathcal{A}_{i_λ} is a scalar defined by:

$$\mathcal{A}_{i_\lambda} = \frac{Q_{11}^{(i_\lambda-1)}}{Q_{11}^{(n_\lambda)}} \quad (13)$$

298 and where \mathbf{Q} is the transfer matrix defined in Hervé and Zaoui (1995). It
 299 is now possible to write the strain average of whole phase (i) (m_λ is the
 300 volume fraction of pattern λ):

$$\langle \underline{\varepsilon} \rangle_{\Omega_i} = \sum_{\lambda=1}^{N_\lambda} m_\lambda \langle \underline{\varepsilon} \rangle_{\Omega_{i_\lambda}} = \sum_{\lambda=1}^{N_\lambda} m_\lambda \mathcal{A}_{i_\lambda} \underline{\varepsilon}^0 \quad (14)$$

301 Therefore:

$$\underline{E} = \langle \underline{\varepsilon} \rangle_{\Omega} = \sum_{i=1}^n f_i \langle \underline{\varepsilon} \rangle_{\Omega_i} = \left[\sum_{i=1}^n f_i \left(\sum_{\lambda=1}^{N_\lambda} m_\lambda \mathcal{A}_{i_\lambda} \right) \right] \underline{\varepsilon}^0 \quad (15)$$

302 It is worth noticing that consequently:

$$\left. \begin{aligned} \underline{E} &= E \left[(\underline{e}_2 \otimes \underline{e}_2) + (\underline{e}_3 \otimes \underline{e}_3) \right] \\ \text{with } E &= \left[\sum_{i=1}^n f_i \left(\sum_{\lambda=1}^{N_\lambda} m_\lambda \mathcal{A}_{i_\lambda} \right) \right] \beta^0 \end{aligned} \right\} \quad (16)$$

303 From Eqs.⁹⁵ (39) and (40) of Hervé and Zaoui (1995), with $\epsilon = 0$ and using
 304 Eq. (A.1) from Appendix A (which summarizes some basic elastic relations
 305 applied to transversely isotropic systems), we get the average stress tensor in
 306 phase (i_λ) when each pattern λ is subjected to boundary conditions defined
 307 in Eq. (4) with $\underline{\varepsilon}^0$ given by Eq. (10):

$$\left. \begin{aligned} \langle \sigma_{11} \rangle_{\Omega_{i_\lambda}} &= 2 C_{12}^{(i_\lambda)} \mathcal{A}_{i_\lambda} \beta^0 \\ \langle \sigma_{22} (\underline{e}_2 \otimes \underline{e}_2) + \sigma_{33} (\underline{e}_3 \otimes \underline{e}_3) \rangle_{\Omega_{i_\lambda}} &= 2 k_{23}^{(i_\lambda)} \mathcal{A}_{i_\lambda} \underline{\varepsilon}^0 \end{aligned} \right\} \quad (17)$$

308 leading to:

$$\left. \begin{aligned} \langle \sigma_{11} \rangle_{\Omega} &= \left[\sum_{i=1}^n f_i \left(\sum_{\lambda=1}^{N_{\lambda}} m_{\lambda} 2 C_{12}^{(i_{\lambda})} \mathcal{A}_{i_{\lambda}} \right) \right] \beta^0 \\ \langle \sigma_{22} (\underline{e}_2 \otimes \underline{e}_2) + \sigma_{33} (\underline{e}_3 \otimes \underline{e}_3) \rangle_{\Omega} &= \\ & \left[\sum_{i=1}^n f_i \left(\sum_{\lambda=1}^{N_{\lambda}} m_{\lambda} 2 k_{23}^{(i_{\lambda})} \mathcal{A}_{i_{\lambda}} \right) \right] \underline{\varepsilon}^0 \end{aligned} \right\} \quad (18)$$

309 The effective elastic moduli tensor $\underline{\underline{C}}^{\text{eff}}$ is derived from relation (Eq. (7))
 310 and Eq. (A.1) applied to the equivalent homogeneous medium:

$$\left. \begin{aligned} \Sigma_{11} &= 2 C_{12}^{\text{eff}} E = \langle \sigma_{11} \rangle_{\Omega} \\ \Sigma_{22} (\underline{e}_2 \otimes \underline{e}_2) + \Sigma_{33} (\underline{e}_3 \otimes \underline{e}_3) &= 2 k_{23}^{\text{eff}} \underline{E} \\ &= \langle \sigma_{22} (\underline{e}_2 \otimes \underline{e}_2) + \sigma_{33} (\underline{e}_3 \otimes \underline{e}_3) \rangle_{\Omega} \end{aligned} \right\} \quad (19)$$

311 Comparison between Eq. (19) and Eq. (18) where \underline{E} and E are given by
 312 Eq. (16) implies that:

$$\left. \begin{aligned} C_{12}^{\text{eff}} &= \frac{\sum_{i=1}^n f_i \left(\sum_{\lambda=1}^{N_{\lambda}} m_{\lambda} C_{12}^{(i_{\lambda})} \mathcal{A}_{i_{\lambda}} \right)}{\sum_{i=1}^n f_i \left(\sum_{\lambda=1}^{N_{\lambda}} m_{\lambda} \mathcal{A}_{i_{\lambda}} \right)} \\ k_{23}^{\text{eff}} &= \frac{\sum_{i=1}^n f_i \left(\sum_{\lambda=1}^{N_{\lambda}} m_{\lambda} k_{23}^{(i_{\lambda})} \mathcal{A}_{i_{\lambda}} \right)}{\sum_{i=1}^n f_i \left(\sum_{\lambda=1}^{N_{\lambda}} m_{\lambda} \mathcal{A}_{i_{\lambda}} \right)} \end{aligned} \right\} \quad (20)$$

313 The effective transverse bulk modulus is finally obtained by substituting $\mathcal{A}_{i_{\lambda}}$
 314 from Eq. (13) in Eq. (20):

$$k_{23}^{\text{eff}} = \frac{\sum_{i=1}^n f_i k_{23}^{(i)} \sum_{\lambda=1}^{N_{\lambda}} m_{\lambda} \frac{Q_{11}^{(i_{\lambda}-1)}}{Q_{11}^{(n_{\lambda})}}}{\sum_{i=1}^n f_i \sum_{\lambda=1}^{N_{\lambda}} m_{\lambda} \frac{Q_{11}^{(i_{\lambda}-1)}}{Q_{11}^{(n_{\lambda})}}} \quad (21)$$

315 and we get the effective value of C_{12} in the same manner:

$$C_{12}^{\text{eff}} = \frac{\sum_{i=1}^n f_i C_{12}^{(i)} \sum_{\lambda=1}^{N_\lambda} m_\lambda \frac{Q_{11}^{(i_\lambda-1)}}{Q_{11}^{(n_\lambda)}}}{\sum_{i=1}^n f_i \sum_{\lambda=1}^{N_\lambda} m_\lambda \frac{Q_{11}^{(i_\lambda-1)}}{Q_{11}^{(n_\lambda)}}} \quad (22)$$

316 Eq. (22) will not be considered in the following because only the elastic
317 transverse behaviour is addressed in this paper.

318 It should be noted that k_{23}^{eff} , given by Eq. (21), depends on the volume frac-
319 tion of each phase in the composite, on the volume fraction of each pattern,
320 on the distribution of the different phases inside each pattern and also on
321 the transverse properties (bulk and shear moduli) of each phase.

322 Such a coupling between the effective moduli is totally similar to that no-
323 ticed for macroscopically isotropic particulate composites by Bardella and
324 Genna (2001).

325 It has been shown in Hervé-Luanco (2020) (following Eq. (23)) that $Q_{11}^{(n_\lambda)}$
326 depends on k_{23}^{eff} and on μ_{23}^{eff} :

$$\mathbf{Q}^{(n_\lambda)} = \mathbf{J}_{n_\lambda+1}^{-1} (R_{n_\lambda}) \mathbf{Q}^{*(n_\lambda)} \quad (23)$$

327 with

$$\mathbf{J}_{n_\lambda+1}^{-1} (R_{n_\lambda}) = \frac{1}{2 R_{n_\lambda} (k_{23}^{(n_\lambda+1)} + \mu_{23}^{(n_\lambda+1)})} \begin{pmatrix} 2 \mu_{23}^{(n_\lambda+1)} & R_{n_\lambda} \\ 2 k_{23}^{(n_\lambda+1)} & -R_{n_\lambda} \end{pmatrix} \quad (24)$$

328 leading immediately to:

$$Q_{11}^{(n_\lambda)} = \frac{2 \mu_{23}^{(n_\lambda+1)} Q_{11}^{*(n_\lambda)} + R_{n_\lambda} Q_{21}^{*(n_\lambda)}}{2 R_{n_\lambda} (k_{23}^{(n_\lambda+1)} + \mu_{23}^{(n_\lambda+1)})} \quad (25)$$

329 k_{23}^{eff} can be rearranged by substituting Eq. (25) into Eq. (21):

$$k_{23}^{\text{eff}} = \frac{\sum_{i=1}^n f_i k_{23}^{(i)} \sum_{\lambda=1}^{N_\lambda} m_\lambda \frac{Q_{11}^{(i_\lambda-1)} R_{n_\lambda}}{2 \mu_{23}^{\text{eff}} Q_{11}^{*(n_\lambda)} + R_{n_\lambda} Q_{21}^{*(n_\lambda)}}}{\sum_{i=1}^n f_i \sum_{\lambda=1}^{N_\lambda} m_\lambda \frac{Q_{11}^{(i_\lambda-1)} R_{n_\lambda}}{2 \mu_{23}^{\text{eff}} Q_{11}^{*(n_\lambda)} + R_{n_\lambda} Q_{21}^{*(n_\lambda)}}} \quad (26)$$

330 where $(k_{23}^{(n_\lambda+1)} + \mu_{23}^{(n_\lambda+1)})$ has been removed because this expression is the
 331 same in all the patterns (same infinite medium).

332 The different matrices \mathbf{Q} and \mathbf{Q}^* present in Eq. (26) depend all on the
 333 pattern λ they refer to. For this reason and in order to more easily evalu-
 334 ate k_{23}^{eff} , we will write it in the following form⁵ using the symbol \otimes to denote
 335 the pattern dependency:

$$k_{23}^{\text{eff}} = \frac{\sum_{i=1}^n f_i k_{23}^{(i)} \sum_{\lambda=1}^{N_\lambda} m_\lambda \frac{Q_{11}^{(i_\lambda-1)\otimes} R_{n_\lambda\otimes}}{2 \mu_{23}^{\text{eff}} Q_{11}^{*\otimes} + R_{n_\lambda\otimes} Q_{21}^{*\otimes}}}{\sum_{i=1}^n f_i \sum_{\lambda=1}^{N_\lambda} m_\lambda \frac{Q_{11}^{(i_\lambda-1)\otimes} R_{n_\lambda\otimes}}{2 \mu_{23}^{\text{eff}} Q_{11}^{*\otimes} + R_{n_\lambda\otimes} Q_{21}^{*\otimes}}} \quad (27)$$

336 For $N_\lambda > 1$ (more than one pattern), it is worth noting that the effec-
 337 tive transverse bulk modulus k_{23}^{eff} depends on the effective transverse shear
 338 modulus μ_{23}^{eff} .

339 2.4. In-plane transverse shear mode

340 In the case where an in-plane transverse shear mode is considered, fol-
 341 lowing Hervé and Zaoui (1995), $\underline{\varepsilon}^0$ in Eq. (4) is chosen as:

$$\underline{\varepsilon}^0 = \gamma^0 \left[(\underline{e}_2 \otimes \underline{e}_2) - (\underline{e}_3 \otimes \underline{e}_3) \right] \quad (28)$$

⁵As \mathbf{Q}^* is only written for n_λ and n_λ being perfectly defined for each pattern, $\mathbf{Q}^{*(n_\lambda)}$ will be equivalently written $\mathbf{Q}^{*\otimes}$.

342 The same methodology as the one presented in the case of an in-plane
 343 hydrostatic mode is used.

344 From Eq.⁹⁵ (74) published in Hervé and Zaoui (1995), the average strain
 345 tensor $\langle \underline{\varepsilon} \rangle_{\Omega_{i_\lambda}}$ in each phase (i_λ) of any pattern λ is given by Eq. (29),
 346 where A_{i_λ} – not to be confused with \mathcal{A}_{i_λ} – and D_{i_λ} are defined constants
 347 for phase (i_λ) (see Hervé and Zaoui (1995)):

$$\langle \underline{\varepsilon} \rangle_{\Omega_{i_\lambda}} = \frac{1}{D_{n_\lambda+1}} \left[D_{i_\lambda} - 3 A_{i_\lambda} \frac{(R_{i_\lambda}^4 - R_{i_\lambda-1}^4)}{R_{i_\lambda}^2 (R_{i_\lambda}^2 - R_{i_\lambda-1}^2)} \right] \underline{\varepsilon}^0 = \mathcal{A}_{i_\lambda} \underline{\varepsilon}^0 \quad (29)$$

348 leading to:

$$\mathcal{A}_{i_\lambda} = \frac{1}{D_{n_\lambda+1}} \left[D_{i_\lambda} - 3 A_{i_\lambda} \frac{(R_{i_\lambda}^4 - R_{i_\lambda-1}^4)}{R_{i_\lambda}^2 (R_{i_\lambda}^2 - R_{i_\lambda-1}^2)} \right] \quad (30)$$

349 It is easy to deduce that we have still:

$$\underline{E} = \langle \underline{\varepsilon} \rangle_\Omega = \sum_{i=1}^n f_i \langle \underline{\varepsilon} \rangle_{\Omega_i} = \left[\sum_{i=1}^n f_i \left(\sum_{\lambda=1}^{N_\lambda} m_\lambda \mathcal{A}_{i_\lambda} \right) \right] \underline{\varepsilon}^0 \quad (31)$$

350 with \mathcal{A}_{i_λ} given now by Eq. (30) and consequently:

$$\left. \begin{aligned} \underline{E} &= E [(e_2 \otimes e_2) - (e_3 \otimes e_3)] \\ \text{with } E &= \left[\sum_{i=1}^n f_i \left(\sum_{\lambda=1}^{N_\lambda} m_\lambda \mathcal{A}_{i_\lambda} \right) \right] \gamma^0 \end{aligned} \right\} \quad (32)$$

351 Using Eq. (A.1) we get the average stress tensor in phase (i_λ) when each
 352 pattern λ is subjected to boundary conditions defined in Eq. (4) with $\underline{\varepsilon}^0$
 353 given by Eq. (28):

$$\langle \underline{\sigma} \rangle_{\Omega_{i_\lambda}} = 2 \mu_{23}^{(i_\lambda)} \mathcal{A}_{i_\lambda} \underline{\varepsilon}^0 \quad (33)$$

354 leading to:

$$\langle \underline{\sigma} \rangle_\Omega = \left[\sum_{i=1}^n f_i \left(\sum_{\lambda=1}^{N_\lambda} m_\lambda 2 \mu_{23}^{(i_\lambda)} \mathcal{A}_{i_\lambda} \right) \right] \underline{\varepsilon}^0 \quad (34)$$

355 The effective elastic stiffness tensor $\underline{\underline{C}}^{\text{eff}}$ is derived from relations (Eq. (7))
 356 and (Eq. (A.1)) applied to the equivalent homogeneous medium:

$$\underline{\underline{\Sigma}} = 2 \mu_{23}^{\text{eff}} \underline{\underline{E}} = \langle \underline{\underline{\sigma}} \rangle_{\Omega} \quad (35)$$

357 Comparison between Eq. (34) and Eq. (35) (with $\underline{\underline{\varepsilon}}^0$ given by Eq. (28)),
 358 where $\underline{\underline{E}}$ and E are given by Eq. (32), implies that:

$$\mu_{23}^{\text{eff}} = \frac{\sum_{i=1}^n f_i \left(\sum_{\lambda=1}^{N_\lambda} m_\lambda \mu_{23}^{(i_\lambda)} \mathcal{A}_{i_\lambda} \right)}{\sum_{i=1}^n f_i \left(\sum_{\lambda=1}^{N_\lambda} m_\lambda \mathcal{A}_{i_\lambda} \right)} \quad (36)$$

359 Substituting Eq. (30) into Eq. (36) yields the following expression of μ_{23}^{eff} :

$$\mu_{23}^{\text{eff}} = \frac{\sum_{i=1}^n f_i \mu_{23}^{(i)} \sum_{\lambda=1}^{N_\lambda} m_\lambda \frac{1}{D_{n_\lambda+1}} \left[D_{i_\lambda} - 3 A_{i_\lambda} \frac{(R_{i_\lambda}^4 - R_{i_\lambda-1}^4)}{R_{i_\lambda}^2 (R_{i_\lambda}^2 - R_{i_\lambda-1}^2)} \right]}{\sum_{i=1}^n f_i \sum_{\lambda=1}^{N_\lambda} m_\lambda \frac{1}{D_{n_\lambda+1}} \left[D_{i_\lambda} - 3 A_{i_\lambda} \frac{(R_{i_\lambda}^4 - R_{i_\lambda-1}^4)}{R_{i_\lambda}^2 (R_{i_\lambda}^2 - R_{i_\lambda-1}^2)} \right]} \quad (37)$$

360 In Hervé and Zaoui (1995), the form of the solution of the displacement
 361 field is expressed in terms of four constants (A_k , B_k , C_k , D_k) in the case
 362 of a transverse shear mode and \mathbf{V}_k denotes the matrix with these constants
 363 as components:

$$\mathbf{V}_k = \begin{pmatrix} A_k \\ B_k \\ C_k \\ D_k \end{pmatrix} \quad (38)$$

364 Here A_{i_λ} and D_{i_λ} correspond respectively to A_k and D_k and are given
 365 in Eq. (39) where phase (k) will be replaced by phase (i_λ) present in pat-
 366 tern λ and n will be replaced by n_λ (see Eq.⁹⁵ (72) in Hervé and Zaoui
 22

367 (1995):

$$\frac{\mathbf{V}_k}{D_{n+1}} = \frac{1}{Q_{44}^{(n)} Q_{11}^{(n)} - Q_{41}^{(n)} Q_{14}^{(n)}} \begin{pmatrix} Q_{11}^{(k-1)} & Q_{12}^{(k-1)} & Q_{13}^{(k-1)} & Q_{14}^{(k-1)} \\ Q_{21}^{(k-1)} & Q_{22}^{(k-1)} & Q_{23}^{(k-1)} & Q_{24}^{(k-1)} \\ Q_{31}^{(k-1)} & Q_{32}^{(k-1)} & Q_{33}^{(k-1)} & Q_{34}^{(k-1)} \\ Q_{41}^{(k-1)} & Q_{42}^{(k-1)} & Q_{43}^{(k-1)} & Q_{44}^{(k-1)} \end{pmatrix} \begin{pmatrix} -Q_{14}^{(n)} \\ 0 \\ 0 \\ Q_{11}^{(n)} \end{pmatrix} \quad (39)$$

368 leading to:

$$\left. \begin{aligned} \frac{A_k}{D_{n+1}} &= \frac{-Q_{11}^{(k-1)} Q_{14}^{(n)} + Q_{14}^{(k-1)} Q_{11}^{(n)}}{Q_{44}^{(n)} Q_{11}^{(n)} - Q_{41}^{(n)} Q_{14}^{(n)}} \\ \frac{D_k}{D_{n+1}} &= \frac{-Q_{14}^{(n)} Q_{41}^{(k-1)} + Q_{44}^{(k-1)} Q_{11}^{(n)}}{Q_{44}^{(n)} Q_{11}^{(n)} - Q_{41}^{(n)} Q_{14}^{(n)}} \end{aligned} \right\} \quad (40)$$

It is important to highlight that A_k , i.e. A_{i_λ} and D_k , i.e. D_{i_λ} , depend on μ_{23}^{eff} through the different components of $\mathbf{Q}^{(n_\lambda)}$. In order to show the dependence on μ_{23}^{eff} in Eq. (37), $\mathbf{Q}^{(n_\lambda)}$ is expressed thanks to Eq. (23) with $\mathbf{J}_{n+1}^{-1}(R_n)$ (see Hervé-Luanco (2020)) given in the transverse shear mode by:

$$\mathbf{J}_{n+1}^{-1}(R_n) = \frac{1}{\nu_{n+1} - 1} \begin{pmatrix} -\frac{R_{n+1}^2}{8 R_n^3} & -\frac{R_{n+1}^2}{8 R_n^3} & -\frac{R_{n+1}^2}{48 \mu_{23}^{(n+1)} R_n^2} & -\frac{R_{n+1}^2}{48 \mu_{23}^{(n+1)} R_n^2} \\ 0 & -\frac{R_n^3}{4 R_{n+1}^4} & -\frac{\nu_{n+1} R_n^4}{12 \mu_{23}^{(n+1)} R_{n+1}^4} & -\frac{(2 \nu_{n+1} - 3) R_n^4}{24 \mu_{23}^{(n+1)} R_{n+1}^4} \\ \frac{R_n}{8 R_{n+1}^2} & \frac{R_n}{8 R_{n+1}^2} & \frac{R_n^2}{16 \mu_{23}^{(n+1)} R_{n+1}^2} & -\frac{R_n^2}{16 \mu_{23}^{(n+1)} R_{n+1}^2} \\ -\frac{1}{2 R_n} & -\frac{1}{4 R_n} & \frac{\nu_{n+1} - 1}{4 \mu_{23}^{(n+1)}} & -\frac{2 \nu_{n+1} - 1}{8 \mu_{23}^{(n+1)}} \end{pmatrix} \quad (41)$$

369 After tedious calculations (all the details are given in Appendix B) the final
 370 equation providing the effective shear modulus is:

$$\mu_{23}^{\text{eff}} = \frac{\sum_{i=1}^n f_i \mu_{23}^{(i)} \sum_{\lambda=1}^{N_\lambda} \frac{m_\lambda R_{n_\lambda \otimes} \left(6 \mu_{23}^{\text{eff}} \alpha_{\otimes}^{(i)} + R_{n_\lambda \otimes} \beta_{\otimes}^{(i)} \right)}{A_{\otimes} \mu_{23}^{\text{eff}^2} + B_{\otimes} \mu_{23}^{\text{eff}} + C_{\otimes}}}{\sum_{i=1}^n f_i \sum_{\lambda=1}^{N_\lambda} \frac{m_\lambda R_{n_\lambda \otimes} \left(6 \mu_{23}^{\text{eff}} \alpha_{\otimes}^{(i)} + R_{n_\lambda \otimes} \beta_{\otimes}^{(i)} \right)}{A_{\otimes} \mu_{23}^{\text{eff}^2} + B_{\otimes} \mu_{23}^{\text{eff}} + C_{\otimes}}} \quad (42)$$

371 with

$$\begin{aligned} \alpha_{\otimes}^{(i)} = & Q_{44}^{(i_\lambda-1)\otimes} \left(Q_{11}^{*\otimes} + Q_{21}^{*\otimes} \right) \\ & - Q_{41}^{(i_\lambda-1)\otimes} \left(Q_{14}^{*\otimes} + Q_{24}^{*\otimes} \right) \\ & - 3 \frac{\left(R_{i_\lambda \otimes}^4 - R_{i_\lambda-1 \otimes}^4 \right)}{R_{i_\lambda \otimes}^2 \left(R_{i_\lambda \otimes}^2 - R_{i_\lambda-1 \otimes}^2 \right)} \\ & \times \left[Q_{14}^{(i_\lambda-1)\otimes} \left(Q_{11}^{*\otimes} + Q_{21}^{*\otimes} \right) \right. \\ & \left. - Q_{11}^{(i_\lambda-1)\otimes} \left(Q_{14}^{*\otimes} + Q_{24}^{*\otimes} \right) \right] \end{aligned} \quad (43)$$

372 and

$$\begin{aligned} \beta_{\otimes}^{(i)} = & Q_{44}^{(i_\lambda-1)\otimes} \left(Q_{31}^{*\otimes} + Q_{41}^{*\otimes} \right) \\ & - Q_{41}^{(i_\lambda-1)\otimes} \left(Q_{34}^{*\otimes} + Q_{44}^{*\otimes} \right) \\ & - 3 \frac{\left(R_{i_\lambda \otimes}^4 - R_{i_\lambda-1 \otimes}^4 \right)}{R_{i_\lambda \otimes}^2 \left(R_{i_\lambda \otimes}^2 - R_{i_\lambda-1 \otimes}^2 \right)} \\ & \times \left[Q_{14}^{(i_\lambda-1)\otimes} \left(Q_{31}^{*\otimes} + Q_{41}^{*\otimes} \right) \right. \\ & \left. - Q_{11}^{(i_\lambda-1)\otimes} \left(Q_{34}^{*\otimes} + Q_{44}^{*\otimes} \right) \right] \end{aligned} \quad (44)$$

373 $A_{\textcircled{\lambda}}$, $B_{\textcircled{\lambda}}$ and $C_{\textcircled{\lambda}}$ have been determined in Eq. (B.14) of Appendix B:

$$\left. \begin{aligned} A_{\textcircled{\lambda}} &= 12 Z_{12}^{\textcircled{\lambda}} + 6 \frac{R_{n_{\lambda}\textcircled{\lambda}}}{k_{23}^{\text{eff}}} \left(Z_{14}^{\textcircled{\lambda}} + Z_{32}^{\textcircled{\lambda}} + Z_{24}^{\textcircled{\lambda}} + Z_{31}^{\textcircled{\lambda}} \right) \\ B_{\textcircled{\lambda}} &= \frac{2 Z_{34}^{\textcircled{\lambda}} R_{n_{\lambda}\textcircled{\lambda}}^2}{k_{23}^{\text{eff}}} + 2 R_{n_{\lambda}\textcircled{\lambda}} \left(2 Z_{14}^{\textcircled{\lambda}} + 2 Z_{32}^{\textcircled{\lambda}} + Z_{24}^{\textcircled{\lambda}} + Z_{31}^{\textcircled{\lambda}} \right) \\ C_{\textcircled{\lambda}} &= Z_{34}^{\textcircled{\lambda}} R_{n_{\lambda}\textcircled{\lambda}}^2 \end{aligned} \right\} \quad (45)$$

374 where $Z_{ij}^{\textcircled{\lambda}}$ denotes:

$$Z_{ij}^{\textcircled{\lambda}} \stackrel{\text{def}}{=} Q_{i4}^{*\textcircled{\lambda}} Q_{j1}^{*\textcircled{\lambda}} - Q_{j4}^{*\textcircled{\lambda}} Q_{i1}^{*\textcircled{\lambda}} \quad (46)$$

375 It should be noted (see Eqs. (42) and (Eq. (45))) that the transverse shear
 376 modulus depends on the transverse bulk modulus. Finally both moduli k_{23}^{eff}
 377 and μ_{23}^{eff} are linked together. In addition, the radius $R_{n_{\lambda}\textcircled{\lambda}}$ disappears in
 378 both final equations (Eq. (27)) and (Eq. (42)) for particular cases of N_{λ}
 379 and n_{λ} (see section 3 for $N_{\lambda} = 2$ and $n_{\lambda} = 2$ for each pattern).

380 **3. Particular case of two patterns with two phases in each pat-** 381 **tern ($N_{\lambda} = n_{\lambda} = 2$)**

382 In this section we consider one “direct” pattern and one “inverse” pattern
 383 as drawn in Figure 3.

384 The first pattern ($\lambda = 1$) is called the “direct” pattern and consists in two
 385 concentric cylinders where the internal phase is made of the fibre mate-
 386 rial (1) and the external phase of the pure matrix material (2). The second
 387 pattern ($\lambda = 2$) is called the “inverse” pattern and, on the opposite, the
 388 internal phase is made now of the pure matrix material (2) and the external
 389 phase of the fibre material (1). Let f be the overall volume fraction of fibre,

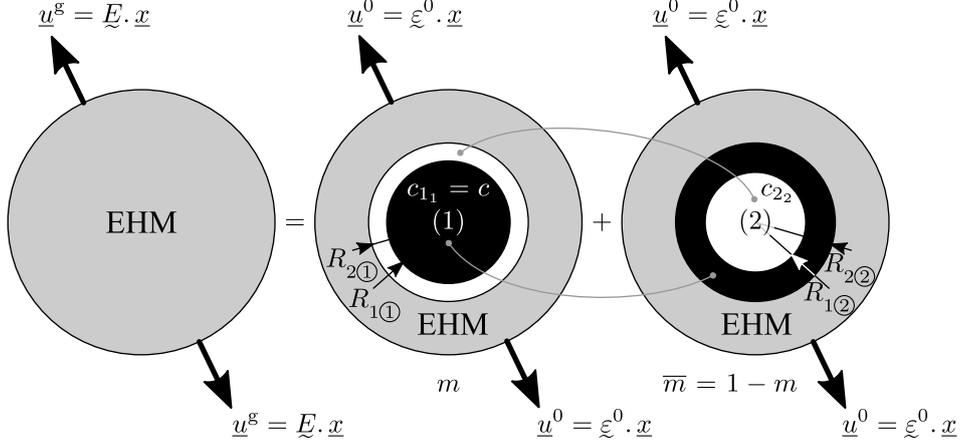


Figure 3: Two morphologically representative patterns.

390 i.e. $f = f_1$ and let m be the volume fraction of the first pattern and c the
391 volume fraction of fibre inside this first pattern.
392 Let phase (i) ($i \in \{1, 2\}$) lie in each pattern λ within the shell limited by
393 the two concentric cylinders with the radii $R_{i-1\lambda}$ and $R_{i\lambda}$ for $\lambda \in \{1, 2\}$.
394 In this studied configuration, phase (1) represents the fibres and is split
395 into phase (1_1) inside the direct pattern 1 and phase (1_2) inside the inverse
396 pattern 2. As shown in Figure 3, phase (1_1) corresponds to the first phase,
397 i.e. the internal phase of pattern 1, and lies between radii $R_{0\textcircled{1}} = 0$ and $R_{1\textcircled{1}}$.
398 Still considering the fibres, phase (1_2) corresponds to the outer phase, i.e.
399 the second phase of pattern 2, and lies between radii $R_{1\textcircled{2}}$ and $R_{2\textcircled{2}}$. It
400 is important not to confuse the phase number and its indexing number
401 within each pattern. This is expressed as follows for the volume fractions
402 but also applies to radii: $c_{i\lambda}$ corresponds to the volume fraction of phase (i)
403 inside pattern λ whereas $c_{i\textcircled{\lambda}}$ is the volume fraction of the “ith” phase inside

404 pattern λ ; c_{2_2} and $c_{2_2\textcircled{2}}$ are thus not equivalent. The parameters presented
 405 in Figure 3 are linked by the following relations:

$$\left. \begin{aligned} c_{1_1} = c_{1\textcircled{1}} = c = R_{1\textcircled{1}}^2 / R_{2\textcircled{1}}^2 \quad \text{and} \quad c_{2_1} = c_{2\textcircled{1}} = 1 - c \\ c_{1_2} = c_{2\textcircled{1}} = 1 - c_{2_2} \quad \text{and} \quad c_{2_2} = c_{1\textcircled{2}} = R_{1\textcircled{2}}^2 / R_{2\textcircled{2}}^2 \\ f_1 = mc_{1_1} + (1 - m)c_{1_2} = mc + (1 - m)(1 - c_{2_2}) = f \\ f_2 = 1 - f \end{aligned} \right\} \quad (47)$$

406 It should be noted that f , m and c are the three independent parameters of
 407 the model and all these data allow to express c_{2_2} as $c_{2_2} = [(1 - f) + m(c - 1)] / (1 - m)$.
 408 Let also $k_{23}^{(i)}$ and $\mu_{23}^{(i)}$ denote respectively the plane strain bulk modulus and
 409 transverse shear modulus of phase (i).

410 Eq. (27) and Eq. (42) have been particularized in this particular case of
 411 two patterns with two phases in each pattern. For this purpose, the different
 412 transfer matrices \mathbf{Q} and \mathbf{Q}^* have been determined in Appendix C in the
 413 case of an hydrostatic pressure loading and in the case of a transverse shear
 414 loading. These developments lead to the following solution for the effective
 415 transverse modulus (see details of calculation in Appendix C.1).

$$k_{23}^{\text{eff}} \stackrel{\text{def}}{=} \frac{\underbrace{\{\mathbf{C}_1\}}_{\subseteq k_{23}^{\text{eff}}} \mu_{23}^{\text{eff}} + \underbrace{\{\mathbf{C}_2\}}_{\subseteq k_{23}^{\text{eff}}}}{\underbrace{\{\mathbf{C}_3\}}_{\subseteq k_{23}^{\text{eff}}} \mu_{23}^{\text{eff}} + \underbrace{\{\mathbf{C}_4\}}_{\subseteq k_{23}^{\text{eff}}}} \quad (48)$$

416 where coefficients $\underbrace{\{\mathbf{C}_p\}}_{\subseteq k_{23}^{\text{eff}}}$ for $p \in \llbracket 1, 4 \rrbracket$ are known in terms of the three in-
 417 dependent parameters f , m , and c and in terms of moduli of each phase
 418 inside the two patterns (see Eq. (C.18)). With more tedious calculations
 419 the effective transverse shear modulus has been determined in Appendix

420 C.2 as the solution of the following equation:

$$\mu_{23}^{\text{eff}} = \mu_{23}^{\text{eff}} \left(f, m, c, k_{23}^{\text{eff}}, \mu_{23}^{\text{eff}}, k_{23}^{(i)}, \mu_{23}^{(i)} \right) \stackrel{\text{def}}{=} \frac{\{\mathbf{A}_1\}}{\{\mathbf{A}_2\}} \stackrel{\text{def}}{=} \frac{\subseteq \mu_{23}^{\text{eff}}}{\subseteq \mu_{23}^{\text{eff}}} \quad (49)$$


421

422 where coefficients $\{\mathbf{A}_p\}$, $p \in \llbracket 1, 2 \rrbracket$, function of μ_{23}^{eff} , are defined by Eq. (C.34)
 423 and Eqs. (C.36) to (Eq. (C.44)).

424 **4. Highlighting the usefulness of the “two patterns - two phases”** 425 **approach**

426 The specific case of two patterns with two phases in each pattern is par-
 427 ticularly suitable for taking into account local morphological fluctuations,
 428 such as trapped matrix regions induced by the heterogeneous distribution of
 429 fibres. The purpose of this section is to illustrate this ability and to justify
 430 the addition of an inverse pattern to the classical GSCS.

431 In order to check the accuracy of any model, a sufficient amount of re-
 432 liable “reference data” must be available. Initial resources are generally
 433 coming from experimental characterization tests and results from the liter-
 434 ature.

435 A very large majority of experimental studies reported in the literature
 436 are based on “mechanical” testing techniques. Although the mechanical
 437 characterization of longitudinal and transverse Young’s moduli does not,
 438 generally, provide any difficulties, this is not as simple for the characteri-
 439 zation of the transverse bulk and shear moduli which are considered in the
 440 present paper. These moduli are usually missing from the publications and
 441 in the best case, are estimated indirectly from elastic constants easier to

442 obtain. To facilitate the characterization of these transverse elastic constants
443 and reduce the inherent propagation of experimental uncertainties,
444 alternative testing methods have been proposed.

445 Ultrasonic methods offer such an alternative and have been developed
446 since the 1970's, initiated by Markham (1970) and Zimmer and Cost (1970).
447 The principle is based on the propagation of ultrasonic waves through a
448 sample and the phase velocity of these propagating plane waves is measured.
449 An appropriate geometry thus makes it possible to propagate the waves in
450 the desired directions and to estimate more easily elastic constants using the
451 well-known Christoffel's equations. A recent review of ultrasonic methods
452 written by Paterson et al. (2018) can be consulted for further details.

453 Beyond experimental results on the transverse elastic behaviour, the
454 comparison with a micromechanical model, as presented in the present paper,
455 requires to know the properties of the constituents but also the morphology
456 of the microstructure. This is the case for Zimmer and Cost's paper
457 (1970) which will allow – in a first step – to highlight that the introduction
458 of the inverse pattern leads to better predictions of the “average”
459 experimental values. If ultrasonic methods make it possible to limit exper-
460 imental uncertainties, it nonetheless remains true that these uncertainties
461 on average properties remain large. In addition, other problems are raised
462 when using experimental results as reference data, even when increasing the
463 number of tests.

464 Experimental results are indeed affected by behaviours unaccounted for
465 by the linear elastic homogenization procedure, such as nonlinearities, un-
466 certainties on the actual moduli of the employed phases, or even the pres-
467 ence of interphases of unknown properties between the reinforcement and

468 the matrix. The use of numerical experiments makes it possible to get rid
 469 of these issues. In a second step, results obtained with full-field finite ele-
 470 ment simulations are thus employed to test the present approach. Gusev
 471 et al.’s paper (2000) has been selected because it combines both experimen-
 472 tal results – using ultrasonic techniques – and numerical ones for the desired
 473 elastic properties.

474 Material properties related to the two above-mentioned examples cho-
 475 sen to illustrate the present approach are gathered in Table 1, where “ZC”
 476 refers to Zimmer and Cost (1970), “GHW” to Gusev et al. (2000), and the
 superscripts “f” and “m” to the fibre and the matrix respectively. As sug-

Property & Notation		ZC	GHW	Units
Overall volume fraction of fibres	$\langle f \rangle$	0.49	0.54	-
Young’s modulus of the fibre	$\langle E^f \rangle$	72.4	72.5	GPa
Poisson’s ratio of the fibre	$\langle \nu^f \rangle$	0.20	0.20	-
Young’s modulus of the matrix	$\langle E^m \rangle$	4.34	5.32	GPa
Poisson’s ratio of the matrix	$\langle \nu^m \rangle$	0.36	0.365	-

Table 1: Materials data as provided by Zimmer and Cost (1970) (referred as ZC, E-glass fibre and ScotchplyTM 1002 epoxy matrix) and by Gusev et al. (2000) (referred as GHW, E-glass fibre and 913 epoxy matrix).

477
 478 gested by Zimmer and Cost, due to the viscoelasticity of the resin and the
 479 ultrasonic frequencies chosen, the Young’s modulus of the matrix, $\langle E^m \rangle_{ZC}$,
 480 is presented with a 40 % increase over the measured value for modelling pur-
 481 poses; which gave, in their situation, the best matches between experiences

482 and reference models. We place ourselves under the same conditions.

483 The primary objective of this paper is clearly the development of the
484 model, while emphasizing its effectiveness by experimental and numerical
485 examples ⁶ .

486 *4.1. Proposed methodology*

487 Reference data, i.e. transverse bulk and shear moduli, are firstly ex-
488 tracted from the two publications mentioned above (Zimmer and Cost, 1970;
489 Gusev et al., 2000) and the uncertainties on the average values are evalu-
490 ated. These results are intended to be compared with those obtained by the
491 model presented in Section 3. For this purpose, material data from Table 1
492 are used as input data for the model. The transverse moduli of the uni-
493 directional composites are derived by using an iterative algorithm – based

⁶A work in progress will complement the illustrative nature of this section and will offer to scan, different morphologies and fibre volume fractions for validation and parameter calibration purposes.

494 on Eqs. (C.17) and (C.33) – which can be described by Eq. (50)

$$\left. \begin{aligned}
 k_{23}^{\text{eff}} &\approx k_{23}^{[j+1]} \stackrel{\text{def}}{=} \frac{\{\mathbf{C}_1\}\mu_{23}^{[j]} + \{\mathbf{C}_2\}}{\{\mathbf{C}_3\}\mu_{23}^{[j]} + \{\mathbf{C}_4\}} \\
 &\quad \substack{\subseteq k_{23}^{\text{eff}} \\ \subseteq k_{23}^{\text{eff}}} \\
 &\text{with } \{\mathbf{C}_p\}, p \in \llbracket 1, 4 \rrbracket, \text{ depending on } (f, m, c, k_{23}^{(i)}, \mu_{23}^{(i)}) \\
 &\quad \substack{\subseteq k_{23}^{\text{eff}} \\ \subseteq k_{23}^{\text{eff}}} \\
 \mu_{23}^{\text{eff}} &\approx \mu_{23}^{[j+1]} \stackrel{\text{def}}{=} \frac{\{\mathbf{A}_1\}}{\{\mathbf{A}_2\}} \\
 &\quad \substack{\subseteq \mu_{23}^{\text{eff}} \\ \subseteq \mu_{23}^{\text{eff}}} \\
 &\text{with } \{\mathbf{A}_p\}, p \in \llbracket 1, 2 \rrbracket, \text{ depending on } (f, m, c, k_{23}^{(i)}, \mu_{23}^{(i)}) \\
 &\quad \substack{\subseteq \mu_{23}^{\text{eff}} \\ \subseteq \mu_{23}^{\text{eff}}} \\
 &k_{23}^{\text{eff}} \approx k_{23}^{[j]}, \mu_{23}^{\text{eff}} \approx \mu_{23}^{[j]}
 \end{aligned} \right\} \quad (50)$$

495 and where $k_{23}^{[j]}$ and $\mu_{23}^{[j]}$ are respectively estimates of the effective plane-
 496 strain bulk and transverse shear moduli at step $[j]$, $j \in \mathbb{N}$, of the iteration
 497 process (see Figure 4). The initial estimates, $k_{23}^{[0]}$ and $\mu_{23}^{[0]}$, can be chosen
 498 as the plane-strain bulk and transverse shear moduli of the matrix or the
 499 ones of the fibre leading respectively to the lower or to the upper estimates.
 500 The iteration process is stopped when the relative error defined by Eq. (51)
 501 is lower than 10^{-6} .

$$\text{Err} = \frac{\sqrt{(\mu_{23}^{[j+1]} - \mu_{23}^{[j]})^2 + (k_{23}^{[j+1]} - k_{23}^{[j]})^2}}{\sqrt{(\mu_{23}^{[0]})^2 + (k_{23}^{[0]})^2}} \quad (51)$$

502 Figure 5 illustrates the iteration process for two isotropic but contrasted
 503 phases as in Hervé et al. (1991). Convergence is obtained within only a few
 504 iterations and in a fraction of a second on a personal computer.

505 The input parameters in Eq. (50) are the elastic behaviour of each phase
 506 and the following three morphological parameters: f , the volume fraction

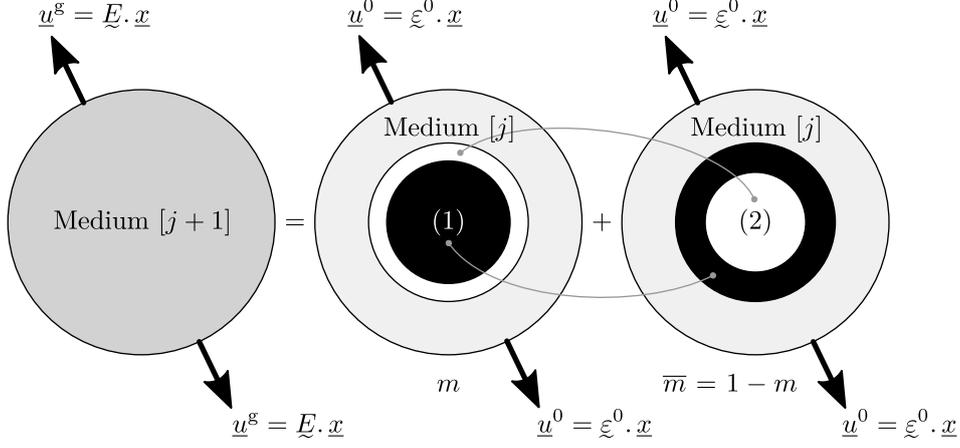


Figure 4: Recursive algorithm to get the effective transverse behaviour.

507 of fibre, m , the volume fraction of the “direct” pattern and c , the volume
 508 fraction of fibre inside the “direct” pattern. If the values of m and c were
 509 chosen arbitrarily in the example given in Figure 5, it is necessary to proceed
 510 differently for the comparison with Zimmer and Cost (1970) and Gusev
 511 et al. (2000) papers. Let us recall that m and c must satisfy, respectively,
 512 Eqs. (52) and (53) as in Joannès and Hervé-Luanco (2016b) .

$$m \geq \frac{1}{2} + \left| f - \frac{1}{2} \right| \quad (52)$$

513

$$0 \leq c_{\min} = 1 + \frac{f-1}{m} < c < c_{\max} = \frac{f}{m} \leq 1 \quad (53)$$

514 Knowing f from Table 1, the range of variation of m and c is determined
 515 and possible pairs (m, c) – distributed over the entire parameter space – are
 516 selected to assess the domain of the response of the model in terms of
 517 effective moduli k_{23}^{eff} and μ_{23}^{eff} . Plotting the response domain of the model
 518 makes it possible to compare it with the reference data.

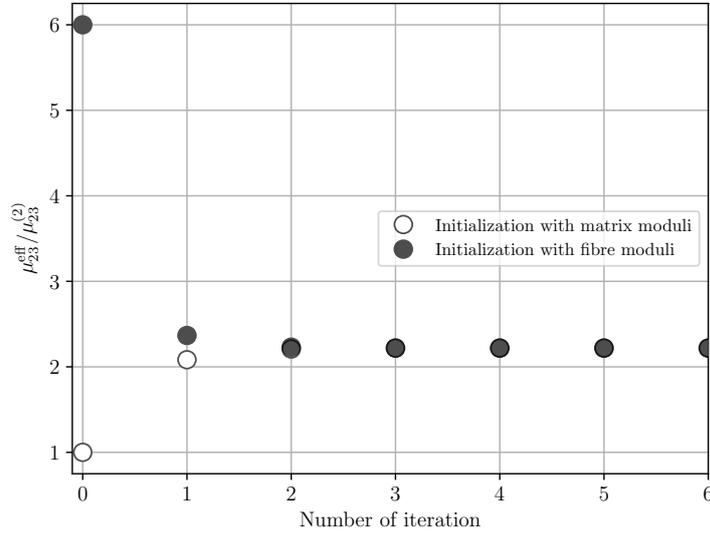


Figure 5: Effective shear modulus of a fibre-reinforced composite yielded by an iterative algorithm and normalized by the shear modulus of the matrix (phase (2)). The fibre (phase (1)) volume fraction, f , is equal to 0.5, as to m and c . Shear moduli are $\mu^{(1)} = 6$ and $\mu^{(2)} = 1$ with the Poisson's coefficients $\nu^{(1)} = 0$ and $\nu^{(2)} = 0.45$.

519 *4.2. Results and discussion*

520 As previously mentioned, Zimmer and Cost's paper (1970) is particularly
521 suitable to highlight the interest of the present model. Zimmer and Cost
522 have considered the ultrasonic characterization of a glass-epoxy composite:
523 E-glass fibres reinforcing a ScotchplyTM 1002 epoxy matrix (see Table 1).
524 The following results for C_{22} and C_{23} (see Eq. (54)) have been reported with
525 their associated uncertainties.

$$\left. \begin{aligned} \langle C_{22} \rangle_{\text{ZC}} &= (17.79 \pm 1.03) \text{ GPa} \\ \langle C_{23} \rangle_{\text{ZC}} &= (9.79 \pm 1.52) \text{ GPa} \end{aligned} \right\} \quad (54)$$

526 It is worth noting that the transverse moduli C_{22} and C_{23} are related to the
 527 effective transverse plane-strain bulk and shear moduli through Eq. (A.2).
 528 Experimental average values from Zimmer and Cost (1970) are finally given
 529 as follows:

$$\left. \begin{aligned} \langle k_{23}^{\text{eff}} \rangle_{\text{ZC}} &= (13.79 \pm 1.28) \text{ GPa} \\ \langle \mu_{23}^{\text{eff}} \rangle_{\text{ZC}} &= (4.00 \pm 1.28) \text{ GPa} \end{aligned} \right\} \quad (55)$$

530 $\langle k_{23}^{\text{eff}} \rangle_{\text{ZC}}$ is provided with an estimated uncertainty of slightly more than 9 %
 531 and $\langle \mu_{23}^{\text{eff}} \rangle_{\text{ZC}}$ with an estimated uncertainty of 32 %.

532

533 By following the iterative process described in § 4.1, and by varying m
 534 and c , it is possible to build gradually the domain of the response of the
 535 model corresponding to Zimmer and Cost’s experimental results (1970).
 536 The data of Table 1 being given, for each investigated pair, (m, c) , a “point”
 537 of the response domain is obtained. In order to illustrate the way the error
 538 defined in Eq. (51) decreases during the iterative algorithm, its value has
 539 been plotted in Figure 6 (top and bottom), respectively when the initial
 540 medium corresponds to the fibre (glass) and to the matrix (epoxy).

541 It is worth noticing that the speed of convergence, obviously for one
 542 set of parameters, is very fast; namely the number of iterations necessary
 543 to reach the desired relative error (10^{-6}) is low, i.e. 9 iterations for the
 544 conditions considered here. We have chosen to keep in Figure 6 only five
 545 different values for m with $c = f = 0.49$. This is also the case for the
 546 response domain plotted in Figure 7, even if, for this figure more values
 547 of m and c are necessary to draw the envelop curves.

548 Figure 7 requires some explanations; the values of k_{23}^{eff} or μ_{23}^{eff} are placed
 549 on the y-axis and evolve as a function of c (placed on the x-axis) and of m

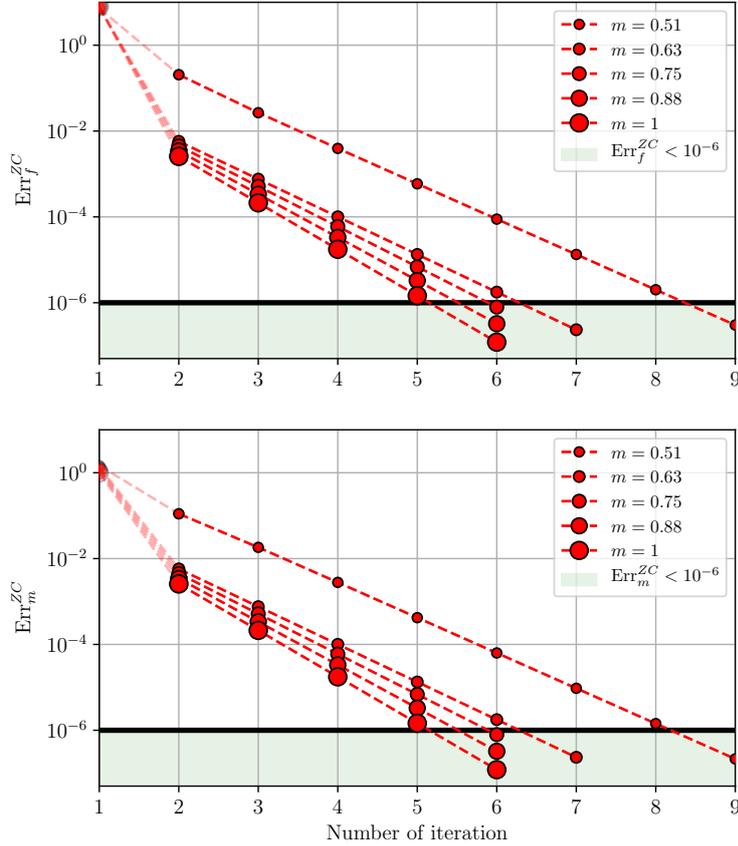


Figure 6: Error minimization during the iteration process corresponding to Zimmer and Cost's experimental conditions (1970), in the particular case where $c = f = 0.49$ and where medium [0] corresponds to the fibre (top) or to the matrix (bottom).

550 represented with dotted lines. The value of m is between 0.51 and 1, this
 551 latter case corresponding to the classical three-phase model (represented by
 552 the “ \odot ” symbol), and the limit values for c are indicated by left and right
 553 triangles. What appears first is the relatively large area related to Zimmer

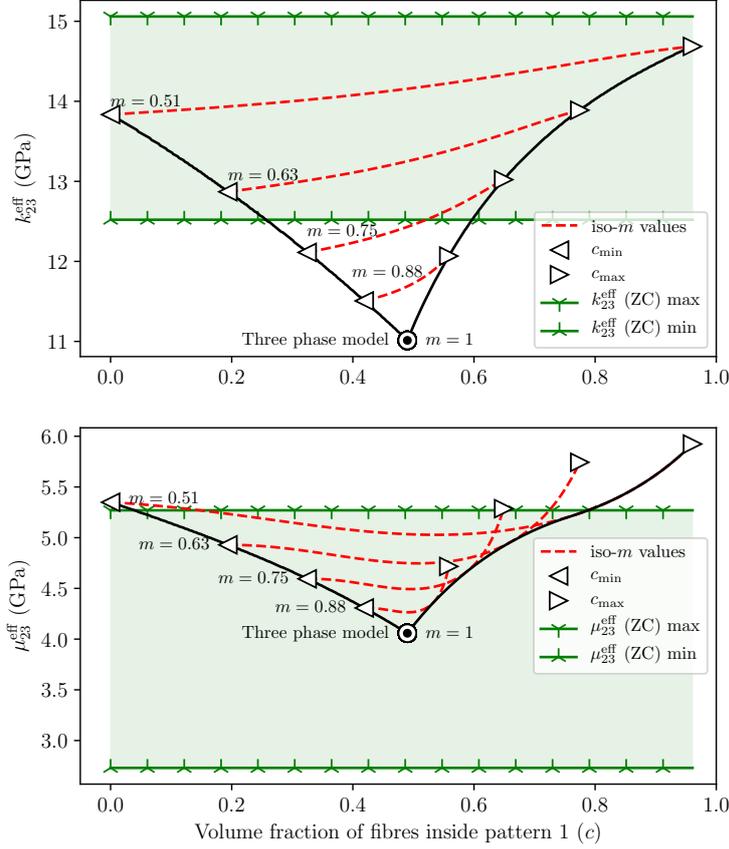


Figure 7: Analytical results, obtained with the present model, are compared to Zimmer and Cost's experimental data (1970).

554 and Cost's results (1970) inherent to the experimental uncertainties. If the
 555 GSCS predicted value for μ_{23}^{eff} is very close to that obtained experimentally
 556 by Zimmer and Cost, it is not the same for k_{23}^{eff} , which is far below the
 557 expected value. The response domain of the proposed model is located be-

558 tween the solid line curves (envelope curves). The response domain is quite
559 large and allows to reach the plain-strain bulk modulus obtained experi-
560 mentally. An infinity of pairs (m, c) achieves this goal but the experimental
561 uncertainty is really too broad to provide further conclusions. These are
562 the limits of this first comparison step based on experimental results.

563 As already stated, the second set of data is taken from Gusev et al.
564 (2000) where a glass-epoxy composite is also considered: E-glass fibres re-
565 inforcing a 913 epoxy matrix. In Gusev et al.’s work (2000), the ultrasonic
566 experimental results are compared to finite element simulations making it
567 possible to overcome a large part of the uncertainty problems raised above.
568 Elastic constants “are calculated numerically based on periodic Monte Carlo
569 realizations with unit cells comprising a random dispersion of 100 non-
570 overlapping fibres”. According to the authors, this process is sufficient to
571 reduce the uncertainty on the average values of elastic constants to less
572 than 1%. They show a good agreement between the experimental and nu-
573 merical results. Thanks to their numerical approach, they also show “that
574 the randomness of the composite microstructure had a significant influence
575 on the transverse composite elastic constants”; what is now well known and
576 justifies the use of the MRP approach which offers great flexibility. Gusev
577 et al.’s numerical results (2000) are used in this second comparison step
578 and C_{22} and C_{23} are calculated from the compliance matrix provided in
579 their paper. The following results have been found while considering an
580 uncertainty of 1%:

$$\left. \begin{aligned} \langle C_{22} \rangle_{\text{GWH}} &= (22.38 \pm 0.22) \text{ GPa} \\ \langle C_{23} \rangle_{\text{GWH}} &= (10.09 \pm 0.10) \text{ GPa} \end{aligned} \right\} \quad (56)$$

581 which leads to Eq. (57) (using Eq. (A.2)) :

$$\left. \begin{aligned} \langle k_{23}^{\text{eff}} \rangle_{\text{GWH}} &= (16.23 \pm 0.16) \text{ GPa} \\ \langle \mu_{23}^{\text{eff}} \rangle_{\text{GWH}} &= (6.14 \pm 0.16) \text{ GPa} \end{aligned} \right\} \quad (57)$$

582 It is necessary to say that the value of μ_{23}^{eff} provided in Gusev et al.'s pa-
583 per (2000) is slightly below (about 1 %) the value calculated here, which
584 ensures the transverse isotropy.

585 As for the first step, the domain of the response of the proposed model
586 is plotted in Figure 8. It can be seen that the analytical results are in good
587 agreement and encompass the numerical values. However Gusev et al. do
588 not give precise estimates of the uncertainties of their simulations, which
589 in all cases are negligible compared to the range of values represented and
590 are not shown in Figure 8. The three-phase model corresponding to $m = 1$
591 is the lowest value of the predicted moduli in Figure 7 and Figure 8, the
592 predicted moduli are far below the expected values for the materials that
593 are considered in this paper. This can be explained by the fact that, in the
594 studied microstructures, a part of the matrix is trapped by the fibres (see
595 for instance an image⁷ of the transverse cross-section of Gusev et al.'s ma-
596 terial (2000) in Figure 9).

597 Figure 7 and Figure 8 clearly show that the introduction of an inverse
598 pattern makes it possible to considerably extend the prediction range of the
599 three phase model. The reference values considered in the two comparison
600 steps are reached without any difficulty by varying m and c . These last two
601 figures provide a wealth of results and other analyzes can be drawn from
602 them:

⁷A similar micrograph is presented in Zimmer and Cost's publication (1970).

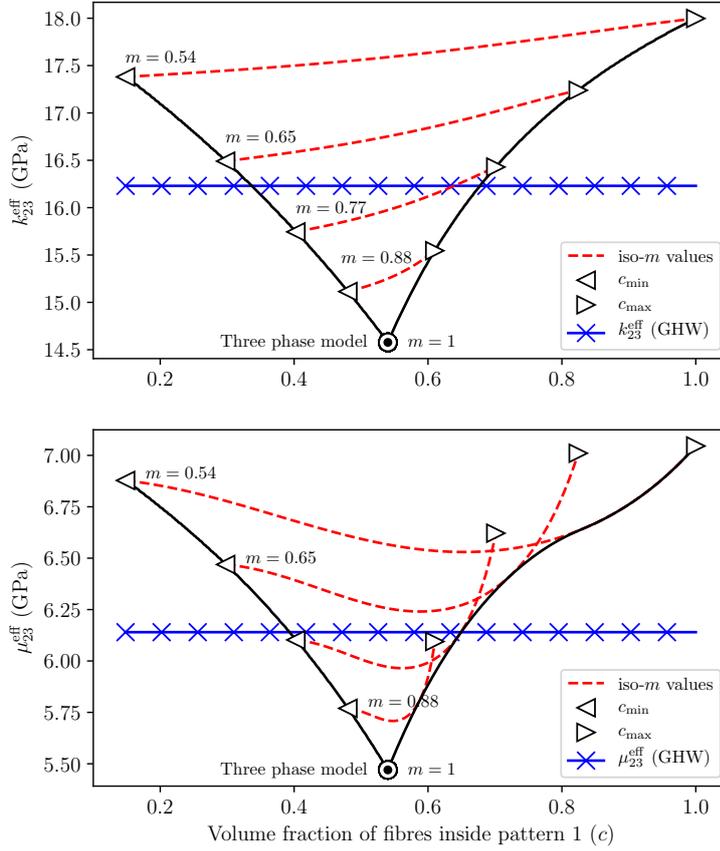


Figure 8: Analytical predicted transverse moduli compared to Gusev et al.'s numerical results (2000).

- 603 • Even considering relatively low contributions of the inverse pattern,
 604 it is possible to increase the transverse moduli by several percent.
 605 This is “graphically” highlighted in Figure 7 where the experimental
 606 uncertainty range is plotted. In the studied configuration and as a

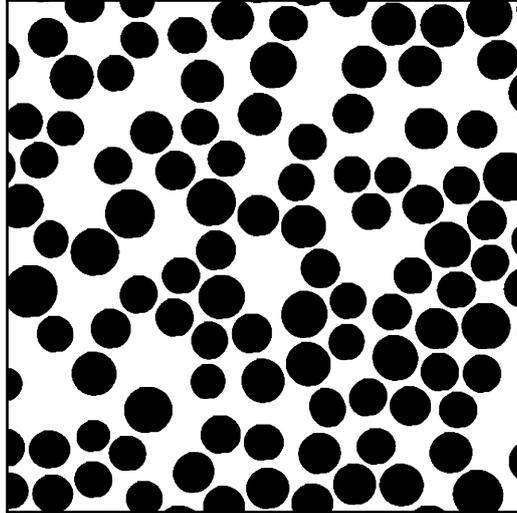


Figure 9: Example of a transverse cross-section microstructure of a unidirectional composite taken from Gusev et al. (2000). Glass fibres appears in black and the epoxy matrix in white, trapped matrix regions are clearly visible.

607 first approximation, considering a third of reverse pattern and two-
 608 thirds of direct pattern allows about 20% variation on the values of k_{23}^{eff}
 609 or μ_{23}^{eff} .

- 610 • In the case of the plain strain bulk modulus, for a given $m \neq 1$, vary-
 611 ing c does not offer much “modulation” of the predicted values.
- 612 • The effect of c is much more visible when considering the transverse
 613 shear modulus. Two areas appear in the k_{23}^{eff} graphs of Figure 7
 614 and Figure 8: “ $c < f$ ” and “ $c > f$ ”. When $c < f$, the sensitivity
 615 to c is very low as in the case of μ_{23}^{eff} analysed above. When $c > f$,
 616 the sensitivity to c becomes very large and tiny variations are enough
 617 to increase the predicted values by a few percent. This highly sensi-
 618 tive area is probably difficult to use, while providing very little added

619 value in terms of the predicted range.

- 620 • As a first approximation, setting $c = f$ seems to be a good modelling
621 compromise. This allows a wide modulus range to be obtained while
622 ensuring stability and smoothness regarding predicted values. More-
623 over, it seems that (m, c) solutions pairs for k_{23}^{eff} and μ_{23}^{eff} are quite
624 close⁸.

625 5. Conclusion

626 In this paper, a pattern-based method has been introduced to take into
627 account the effect of the morphological fluctuations on the transverse elastic
628 behaviour of fibre reinforced composites. For that purpose a Generalized
629 Self-Consistent Scheme based on N_λ patterns with n_λ phases each has been
630 proposed. The particular case of two patterns with two phases has been
631 completely developed, the first pattern is a “direct” one (fibre embedded
632 in the matrix) and the other is an “inverted” one (matrix embedded in the
633 phase fibre). This second pattern allows us to take into account the role the
634 non-percolated matrix part plays on the transverse elastic behaviour. Two
635 additional morphological parameters (m , the volume fraction of the direct
636 pattern in the microstructure and c , the volume fraction of fibre inside the
637 second pattern) have been introduced.

638 As it has been shown in this paper, these two parameters make it pos-
639 sible to widely extend the predictive capacity of the classical three phase
640 model. For a given microstructure, several pairs (m, c) seem to be able

⁸This analysis, dealing with the uniqueness of solution pairs, constitutes a work in progress that goes beyond the scope of the present paper.

641 to accurately describe the experimental results. This means that from an
642 appropriately identified pair (m, c) , it is possible to predict the transverse
643 behaviour of a composite microstructure with trapped or non-percolated
644 matrix areas. This identification, which is not the subject of this paper,
645 can be carried out in different ways. An image analysis procedure, derived
646 from a covariogram analysis can for example be applied and is under devel-
647 opment by the authors. Moreover, a work in progress aims to show that for
648 a given microstructure, a couple (m, c) can be chosen quasi-independently
649 of the phase contrast and the phase properties in order to predict effective
650 transverse elastic constants.

651 **Acknowledgement**

652 This study has been supported by Michelin Company. The authors
653 are indebted to R. Bruant and A. Mbiakop-Ngassa for fruitful discussions
654 during this work.

655 **Appendix A. Elastic behaviour of transversely isotropic systems**

656 The linear constitutive relation, for elasticity is written here in the form:

$$\begin{pmatrix} \sigma_{11} \equiv \sigma_1 \\ \sigma_{22} \equiv \sigma_2 \\ \sigma_{33} \equiv \sigma_3 \\ \sigma_{23} \equiv \sigma_4 \\ \sigma_{31} \equiv \sigma_5 \\ \sigma_{12} \equiv \sigma_6 \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{12} & C_{23} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{55} \end{pmatrix} \begin{pmatrix} \varepsilon_1 \equiv \varepsilon_{11} \\ \varepsilon_2 \equiv \varepsilon_{22} \\ \varepsilon_3 \equiv \varepsilon_{33} \\ \varepsilon_4 \equiv 2 \varepsilon_{23} \\ \varepsilon_5 \equiv 2 \varepsilon_{31} \\ \varepsilon_6 \equiv 2 \varepsilon_{12} \end{pmatrix} \tag{A.1}$$

657 In the equivalent homogeneous material we will use C_{ij}^{eff} and in phase (k)
658 $C_{ij}^{(k)}$ to describe their respective elastic behaviour. By combining C_{22} and
659 C_{23} , k_{23} and μ_{23} can be calculated as follows:

$$\begin{aligned} k_{23} &= \frac{C_{22} + C_{23}}{2} \\ \mu_{23} &= \frac{C_{22} - C_{23}}{2} \end{aligned} \quad (\text{A.2})$$

660 **Appendix B. Determination of μ_{23}^{eff} in the case of N_λ patterns**
661 **with n_λ phases**

662 In section 2.4, the homogenisation approach has led to the following
663 effective transverse shear modulus (Eq. (37)):

$$\mu_{23}^{\text{eff}} = \frac{\sum_{i=1}^n f_i \mu_{23}^{(i)} \sum_{\lambda=1}^{N_\lambda} m_\lambda \frac{1}{D_{n_\lambda+1}} \left[D_{i_\lambda} - 3 A_{i_\lambda} \frac{(R_{i_\lambda}^4 - R_{i_\lambda-1}^4)}{R_{i_\lambda}^2 (R_{i_\lambda}^2 - R_{i_\lambda-1}^2)} \right]}{\sum_{i=1}^n f_i \sum_{\lambda=1}^{N_\lambda} m_\lambda \frac{1}{D_{n_\lambda+1}} \left[D_{i_\lambda} - 3 A_{i_\lambda} \frac{(R_{i_\lambda}^4 - R_{i_\lambda-1}^4)}{R_{i_\lambda}^2 (R_{i_\lambda}^2 - R_{i_\lambda-1}^2)} \right]} \quad (\text{B.1})$$

664 $A_{i_\lambda}/D_{n_\lambda+1}$ and $D_{i_\lambda}/D_{n_\lambda+1}$ in Eq. (B.1) are given by Eq. (40) which is
665 written in the following form:

$$\left. \begin{aligned} \frac{A_{i_\lambda}}{D_{n_\lambda+1}} &= \frac{Q_{14}^{(i_\lambda-1)} Q_{11}^{(n_\lambda)} - Q_{11}^{(i_\lambda-1)} Q_{14}^{(n_\lambda)}}{Q_{44}^{(n_\lambda)} Q_{11}^{(n_\lambda)} - Q_{41}^{(n_\lambda)} Q_{14}^{(n_\lambda)}} \\ \frac{D_{i_\lambda}}{D_{n_\lambda+1}} &= \frac{Q_{44}^{(i_\lambda-1)} Q_{11}^{(n_\lambda)} - Q_{14}^{(n_\lambda)} Q_{41}^{(i_\lambda-1)}}{Q_{44}^{(n_\lambda)} Q_{11}^{(n_\lambda)} - Q_{41}^{(n_\lambda)} Q_{14}^{(n_\lambda)}} \end{aligned} \right\} \quad (\text{B.2})$$

666 It should be noted that A_{i_λ} and D_{i_λ} depend on μ_{23}^{eff} through the different
667 components of $\mathbf{Q}^{(n_\lambda)}$. In order to exhibit the dependance on μ_{23}^{eff} in Eq. (B.1),
668 $\mathbf{Q}^{(n_\lambda)}$ is expressed thanks to Eq. (B.3) where $\mathbf{Q}^{*(n_\lambda)}$ does not depend on μ_{23}^{eff} :

$$\mathbf{Q}^{(n_\lambda)} = \mathbf{J}_{n_\lambda+1}^{-1} (R_{n_\lambda}) \mathbf{Q}^{*(n_\lambda)} \quad (\text{B.3})$$

669 with $(\nu_{n_\lambda+1} - 1) \mathbf{J}_{n_\lambda+1}^{-1}(R_{n_\lambda})$ (see Hervé-Luanco (2020)) given in the trans-
 670 verse shear mode by:

$$\left(\begin{array}{cccc} -\frac{R_{n_\lambda+1}^2}{8 R_{n_\lambda}^3} & -\frac{R_{n_\lambda+1}^2}{8 R_{n_\lambda}^3} & -\frac{R_{n_\lambda+1}^2}{48 \mu_{23}^{(n_\lambda+1)} R_{n_\lambda}^2} & -\frac{R_{n_\lambda+1}^2}{48 \mu_{23}^{(n_\lambda+1)} R_{n_\lambda}^2} \\ 0 & -\frac{R_{n_\lambda}^3}{4 R_{n_\lambda+1}^4} & -\frac{\nu_{n_\lambda+1} R_{n_\lambda}^4}{12 \mu_{23}^{(n_\lambda+1)} R_{n_\lambda+1}^4} & -\frac{(2 \nu_{n_\lambda+1} - 3) R_{n_\lambda}^4}{24 \mu_{23}^{(n_\lambda+1)} R_{n_\lambda+1}^4} \\ -\frac{R_{n_\lambda}}{8 R_{n_\lambda+1}^2} & -\frac{R_{n_\lambda}}{8 R_{n_\lambda+1}^2} & -\frac{R_{n_\lambda}^2}{16 \mu_{23}^{(n_\lambda+1)} R_{n_\lambda+1}^2} & -\frac{R_{n_\lambda}^2}{16 \mu_{23}^{(n_\lambda+1)} R_{n_\lambda+1}^2} \\ -\frac{1}{2 R_{n_\lambda}} & -\frac{1}{4 R_{n_\lambda}} & -\frac{\nu_{n_\lambda+1} - 1}{4 \mu_{23}^{(n_\lambda+1)}} & -\frac{2 \nu_{n_\lambda+1} - 1}{8 \mu_{23}^{(n_\lambda+1)}} \end{array} \right) \quad (\text{B.4})$$

671 To facilitate reading, n_λ being perfectly defined for each pattern, $\mathbf{Q}^{*(n_\lambda)}$
 672 will be noted $\mathbf{Q}^{*(\otimes)}$ in the following equations where \otimes denotes the pattern
 673 dependency. It is worth taking into account that phase $(n_\lambda + 1)$ is the
 674 same phase in all the patterns and corresponds to the effective medium. To
 675 determine k_{23}^{eff} and μ_{23}^{eff} , we will thus consider that $k_{23}^{\text{eff}} = k_{23}^{(n_\lambda+1)}$ and $\mu_{23}^{\text{eff}} =$
 676 $\mu_{23}^{(n_\lambda+1)}$. Coefficient $\nu_{n_\lambda+1}$ can be calculated by using its definition given
 677 in Hervé and Zaoui (1995):

$$\nu_{n_\lambda+1} = \frac{C_{23}^{(n_\lambda+1)}}{C_{23}^{(n_\lambda+1)} + C_{22}^{(n_\lambda+1)}} \quad (\text{B.5})$$

678 By using Eq. (A.2):

$$\left. \begin{array}{l} k_{23}^{(n_\lambda+1)} = \frac{C_{22}^{(n_\lambda+1)} + C_{23}^{(n_\lambda+1)}}{2} \\ \mu_{23}^{(n_\lambda+1)} = \frac{C_{22}^{(n_\lambda+1)} - C_{23}^{(n_\lambda+1)}}{2} \end{array} \right\} \quad (\text{B.6})$$

679 $\nu_{n_\lambda+1}$ can then equivalently be rewritten as:

$$\nu_{n_\lambda+1} = \frac{k_{23}^{(n_\lambda+1)} - \mu_{23}^{(n_\lambda+1)}}{2 k_{23}^{(n_\lambda+1)}} \quad (\text{B.7})$$

680 Or more generally as:

$$\nu_k = \frac{k_{23}^{(k)} - \mu_{23}^{(k)}}{2 k_{23}^{(k)}} \quad (\text{B.8})$$

Let us now calculate the denominator of equation (Eq. (B.2)):

$$\underbrace{\{\mathbf{A}\}}_{\subseteq/D_{n_\lambda+1}} \stackrel{\text{def}}{=} Q_{44}^{(n_\lambda)} Q_{11}^{(n_\lambda)} - Q_{41}^{(n_\lambda)} Q_{14}^{(n_\lambda)} \quad (\text{B.9})$$

681 All of the components of $\mathbf{Q}^{(n_\lambda)}$ present in the previous equation can be
 682 determined by introducing Eq. (B.4) in (Eq. (B.3)):

$$\left. \begin{aligned} Q_{44}^{(n_\lambda)} &= \frac{1}{\nu_{n_\lambda+1} - 1} \left[-\frac{Q_{14}^{*\Lambda}}{2 R_{n_\lambda}} - \frac{Q_{24}^{*\Lambda}}{4 R_{n_\lambda}} \right. \\ &\quad \left. + \frac{Q_{34}^{*\Lambda}}{4 \mu_{23}^{(n_\lambda+1)}} (\nu_{n_\lambda+1} - 1) - \frac{Q_{44}^{*\Lambda}}{8 \mu_{23}^{(n_\lambda+1)}} (2 \nu_{n_\lambda+1} - 1) \right] \\ Q_{11}^{(n_\lambda)} &= \frac{1}{\nu_{n_\lambda+1} - 1} \left(\frac{R_{n_\lambda+1}}{R_{n_\lambda}} \right)^2 \left[-\frac{Q_{11}^{*\Lambda}}{8 R_{n_\lambda}} \right. \\ &\quad \left. - \frac{Q_{21}^{*\Lambda}}{8 R_{n_\lambda}} - \frac{Q_{31}^{*\Lambda}}{48 \mu_{23}^{(n_\lambda+1)}} - \frac{Q_{41}^{*\Lambda}}{48 \mu_{23}^{(n_\lambda+1)}} \right] \\ Q_{41}^{(n_\lambda)} &= \frac{1}{\nu_{n_\lambda+1} - 1} \left[-\frac{Q_{11}^{*\Lambda}}{2 R_{n_\lambda}} - \frac{Q_{21}^{*\Lambda}}{4 R_{n_\lambda}} \right. \\ &\quad \left. + \frac{Q_{31}^{*\Lambda}}{4 \mu_{23}^{(n_\lambda+1)}} (\nu_{n_\lambda+1} - 1) - \frac{Q_{41}^{*\Lambda}}{8 \mu_{23}^{(n_\lambda+1)}} (2 \nu_{n_\lambda+1} - 1) \right] \\ Q_{14}^{(n_\lambda)} &= \frac{1}{\nu_{n_\lambda+1} - 1} \left(\frac{R_{n_\lambda+1}}{R_{n_\lambda}} \right)^2 \left[-\frac{Q_{14}^{*\Lambda}}{8 R_{n_\lambda}} \right. \\ &\quad \left. - \frac{Q_{24}^{*\Lambda}}{8 R_{n_\lambda}} - \frac{Q_{34}^{*\Lambda}}{48 \mu_{23}^{(n_\lambda+1)}} - \frac{Q_{44}^{*\Lambda}}{48 \mu_{23}^{(n_\lambda+1)}} \right] \end{aligned} \right\} \quad (\text{B.10})$$

683 Using Eqs. (B.7) and (Eq. (B.10)), $\underbrace{\{\mathbf{A}\}}_{\subseteq/D_{n_\lambda+1}}$ can consequently be rewritten

684 as:

$$\underbrace{\{\mathbf{A}\}}_{\subseteq/D_{n_\lambda+1}} = \frac{4 R_{n_\lambda+1}^2}{R_{n_\lambda}^2 \left(1 + \frac{\mu_{23}^{(n_\lambda+1)}}{k_{23}^{(n_\lambda+1)}} \right)^2} \left(\underbrace{\{\mathbf{B}_1\}}_{\subseteq/D_{n_\lambda+1}} \underbrace{\{\mathbf{B}_2\}}_{\subseteq/D_{n_\lambda+1}} - \underbrace{\{\mathbf{B}_3\}}_{\subseteq/D_{n_\lambda+1}} \underbrace{\{\mathbf{B}_4\}}_{\subseteq/D_{n_\lambda+1}} \right) \quad (\text{B.11})$$

685 with $\{\mathbf{B}_p\}_{\subseteq/D_{n_\lambda+1}}$, $p \in \llbracket 1, 4 \rrbracket$, given by:

$$\left. \begin{aligned} \{\mathbf{B}_1\}_{\subseteq/D_{n_\lambda+1}} &\stackrel{\text{def}}{=} -\frac{Q_{14}^{*\langle\lambda\rangle}}{2 R_{n_\lambda}} - \frac{Q_{24}^{*\langle\lambda\rangle}}{4 R_{n_\lambda}} + \frac{Q_{34}^{*\langle\lambda\rangle}}{4 \mu_{23}^{(n_\lambda+1)}} (\nu_{n_\lambda+1} - 1) \\ &\quad - \frac{Q_{44}^{*\langle\lambda\rangle}}{8 \mu_{23}^{(n_\lambda+1)}} (2 \nu_{n_\lambda+1} - 1) \\ \{\mathbf{B}_2\}_{\subseteq/D_{n_\lambda+1}} &\stackrel{\text{def}}{=} -\frac{Q_{11}^{*\langle\lambda\rangle}}{8 R_{n_\lambda}} - \frac{Q_{21}^{*\langle\lambda\rangle}}{8 R_{n_\lambda}} - \frac{Q_{31}^{*\langle\lambda\rangle}}{48 \mu_{23}^{(n_\lambda+1)}} - \frac{Q_{41}^{*\langle\lambda\rangle}}{48 \mu_{23}^{(n_\lambda+1)}} \\ \{\mathbf{B}_3\}_{\subseteq/D_{n_\lambda+1}} &\stackrel{\text{def}}{=} -\frac{Q_{11}^{*\langle\lambda\rangle}}{2 R_{n_\lambda}} - \frac{Q_{21}^{*\langle\lambda\rangle}}{4 R_{n_\lambda}} + \frac{Q_{31}^{*\langle\lambda\rangle}}{4 \mu_{23}^{(n_\lambda+1)}} (\nu_{n_\lambda+1} - 1) \\ &\quad - \frac{Q_{41}^{*\langle\lambda\rangle}}{8 \mu_{23}^{(n_\lambda+1)}} (2 \nu_{n_\lambda+1} - 1) \\ \{\mathbf{B}_4\}_{\subseteq/D_{n_\lambda+1}} &\stackrel{\text{def}}{=} -\frac{Q_{14}^{*\langle\lambda\rangle}}{8 R_{n_\lambda}} - \frac{Q_{24}^{*\langle\lambda\rangle}}{8 R_{n_\lambda}} - \frac{Q_{34}^{*\langle\lambda\rangle}}{48 \mu_{23}^{(n_\lambda+1)}} - \frac{Q_{44}^{*\langle\lambda\rangle}}{48 \mu_{23}^{(n_\lambda+1)}} \end{aligned} \right\} \quad (\text{B.12})$$

By rearranging (Eq. (B.12)) thanks to $Z_{ij}^{\langle\lambda\rangle} \stackrel{\text{def}}{=} Q_{i4}^{*\langle\lambda\rangle} Q_{j1}^{*\langle\lambda\rangle} - Q_{j4}^{*\langle\lambda\rangle} Q_{i1}^{*\langle\lambda\rangle}$ with $i, j \in \{1, 4\}$ (Eq. (46)) and by using (Eq. (B.7)) it follows that:

$$384 R_{n_\lambda}^2 \mu_{23}^{(n_\lambda+1)^2} \left(\begin{array}{cc} \{\mathbf{B}_1\}_{\subseteq/D_{n_\lambda+1}} & \{\mathbf{B}_2\}_{\subseteq/D_{n_\lambda+1}} \\ - & - \end{array} \begin{array}{cc} \{\mathbf{B}_3\}_{\subseteq/D_{n_\lambda+1}} & \{\mathbf{B}_4\}_{\subseteq/D_{n_\lambda+1}} \end{array} \right) = A_{\langle\lambda\rangle} \mu_{23}^{(n_\lambda+1)^2} + B_{\langle\lambda\rangle} \mu_{23}^{(n_\lambda+1)} + C_{\langle\lambda\rangle} \quad (\text{B.13})$$

686 with $A_{\langle\lambda\rangle}$, $B_{\langle\lambda\rangle}$ and $C_{\langle\lambda\rangle}$ given by:

$$\left. \begin{aligned} A_{\langle\lambda\rangle} &\stackrel{\text{def}}{=} 12 Z_{12}^{\langle\lambda\rangle} + 6 \frac{R_{n_\lambda \langle\lambda\rangle}}{k_{23}^{\text{eff}}} \left(Z_{14}^{\langle\lambda\rangle} + Z_{32}^{\langle\lambda\rangle} + Z_{24}^{\langle\lambda\rangle} + Z_{31}^{\langle\lambda\rangle} \right) \\ B_{\langle\lambda\rangle} &\stackrel{\text{def}}{=} \frac{2 Z_{34}^{\langle\lambda\rangle} R_{n_\lambda \langle\lambda\rangle}^2}{k_{23}^{\text{eff}}} + 2 R_{n_\lambda \langle\lambda\rangle} \left(2 Z_{14}^{\langle\lambda\rangle} + 2 Z_{32}^{\langle\lambda\rangle} + Z_{24}^{\langle\lambda\rangle} + Z_{31}^{\langle\lambda\rangle} \right) \\ C_{\langle\lambda\rangle} &\stackrel{\text{def}}{=} Z_{34}^{\langle\lambda\rangle} R_{n_\lambda \langle\lambda\rangle}^2 \end{aligned} \right\} \quad (\text{B.14})$$

It is important to notice that $A_{\textcircled{\lambda}}$ and $B_{\textcircled{\lambda}}$ depend on k_{23}^{eff} .

Finally:

$$\begin{aligned} \{A\}_{\subseteq/D_{n_{\lambda}+1}} &= \frac{R_{n_{\lambda}+1}^2}{96 R_{n_{\lambda}}^4 \left(1 + \frac{\mu_{23}^{(n_{\lambda}+1)}}{k_{23}^{(n_{\lambda}+1)}}\right)^2 \mu_{23}^{(n_{\lambda}+1)2}} \\ &\quad \times \left(A_{\textcircled{\lambda}} \mu_{23}^{(n_{\lambda}+1)2} + B_{\textcircled{\lambda}} \mu_{23}^{(n_{\lambda}+1)} + C_{\textcircled{\lambda}} \right) \end{aligned} \quad (\text{B.15})$$

687 Eq. (B.2) takes the following form and consequently can be calculated

688 thanks to Eq. (B.15) and to Eq. (B.10):

$$\left. \begin{aligned} \frac{A_{i_{\lambda}}}{D_{n_{\lambda}+1}} &= \frac{Q_{14}^{(i_{\lambda}-1)} Q_{11}^{(n_{\lambda})} - Q_{11}^{(i_{\lambda}-1)} Q_{14}^{(n_{\lambda})}}{\{A\}_{\subseteq/D_{n_{\lambda}+1}}} \\ \frac{D_{i_{\lambda}}}{D_{n_{\lambda}+1}} &= \frac{Q_{44}^{(i_{\lambda}-1)} Q_{11}^{(n_{\lambda})} - Q_{14}^{(n_{\lambda})} Q_{41}^{(i_{\lambda}-1)}}{\{A\}_{\subseteq/D_{n_{\lambda}+1}}} \end{aligned} \right\} \quad (\text{B.16})$$

689 leading to:

$$\frac{A_{i_{\lambda}}}{D_{n_{\lambda}+1}} = \frac{4 R_{n_{\lambda}} \left(1 + \frac{\mu_{23}^{(n_{\lambda}+1)}}{k_{23}^{(n_{\lambda}+1)}}\right) \left(6 \mu_{23}^{(n_{\lambda}+1)2} \alpha_{A_{i_{\lambda}}} + R_{n_{\lambda}} \mu_{23}^{(n_{\lambda}+1)} \beta_{A_{i_{\lambda}}}\right)}{A_{\textcircled{\lambda}} \mu_{23}^{(n_{\lambda}+1)2} + B_{\textcircled{\lambda}} \mu_{23}^{(n_{\lambda}+1)} + C_{\textcircled{\lambda}}} \quad (\text{B.17})$$

690 with

$$\left. \begin{aligned} \alpha_{A_{i_{\lambda}}} &\stackrel{\text{def}}{=} Q_{14}^{(i_{\lambda}-1)} \left(Q_{11}^{*\textcircled{\lambda}} + Q_{21}^{*\textcircled{\lambda}} \right) - Q_{11}^{(i_{\lambda}-1)} \left(Q_{14}^{*\textcircled{\lambda}} + Q_{24}^{*\textcircled{\lambda}} \right) \\ \beta_{A_{i_{\lambda}}} &\stackrel{\text{def}}{=} Q_{14}^{(i_{\lambda}-1)} \left(Q_{31}^{*\textcircled{\lambda}} + Q_{41}^{*\textcircled{\lambda}} \right) - Q_{11}^{(i_{\lambda}-1)} \left(Q_{34}^{*\textcircled{\lambda}} + Q_{44}^{*\textcircled{\lambda}} \right) \end{aligned} \right\} \quad (\text{B.18})$$

691 and

$$\frac{D_{i_{\lambda}}}{D_{n_{\lambda}+1}} = \frac{4 R_{n_{\lambda}} \left(1 + \frac{\mu_{23}^{(n_{\lambda}+1)}}{k_{23}^{(n_{\lambda}+1)}}\right) \left[6 \mu_{23}^{(n_{\lambda}+1)2} \alpha_{D_{i_{\lambda}}} + R_{n_{\lambda}} \mu_{23}^{(n_{\lambda}+1)} \beta_{D_{i_{\lambda}}}\right]}{A_{\textcircled{\lambda}} \mu_{23}^{(n_{\lambda}+1)2} + B_{\textcircled{\lambda}} \mu_{23}^{(n_{\lambda}+1)} + C_{\textcircled{\lambda}}} \quad (\text{B.19})$$

692 with

$$\left. \begin{aligned} \alpha_{D_{i_\lambda}} &\stackrel{\text{def}}{=} Q_{44}^{(i_\lambda-1)} \left(Q_{11}^{*\langle \lambda \rangle} + Q_{21}^{*\langle \lambda \rangle} \right) - Q_{41}^{(i_\lambda-1)} \left(Q_{14}^{*\langle \lambda \rangle} + Q_{24}^{*\langle \lambda \rangle} \right) \\ \beta_{D_{i_\lambda}} &\stackrel{\text{def}}{=} Q_{44}^{(i_\lambda-1)} \left(Q_{31}^{*\langle \lambda \rangle} + Q_{41}^{*\langle \lambda \rangle} \right) - Q_{41}^{(i_\lambda-1)} \left(Q_{34}^{*\langle \lambda \rangle} + Q_{44}^{*\langle \lambda \rangle} \right) \end{aligned} \right\} \quad (\text{B.20})$$

693 Replacing $A_{i_\lambda}/D_{n_\lambda+1}$ and $D_{i_\lambda}/D_{n_\lambda+1}$ respectively with the expressions
694 given by Eq. (B.17) and by Eq. (B.19) in Eq. (B.1), μ_{23}^{eff} becomes:

$$\mu_{23}^{\text{eff}} = \frac{\sum_{i=1}^n f_i \mu_{23}^{(i)} \sum_{\lambda=1}^{N_\lambda} \frac{4 m_\lambda R_{n_\lambda} \left(1 + \frac{\mu_{23}^{(n_\lambda+1)}}{k_{23}^{(n_\lambda+1)}} \right) \mu_{23}^{(n_\lambda+1)} \left(6 \mu_{23}^{(n_\lambda+1)} \alpha_{\langle \lambda \rangle}^{(i)} + R_{n_\lambda} \beta_{\langle \lambda \rangle}^{(i)} \right)}{A_{\langle \lambda \rangle} \mu_{23}^{(n_\lambda+1)^2} + B_{\langle \lambda \rangle} \mu_{23}^{(n_\lambda+1)} + C_{\langle \lambda \rangle}}}{\sum_{i=1}^n f_i \sum_{\lambda=1}^{N_\lambda} \frac{4 m_\lambda R_{n_\lambda} \left(1 + \frac{\mu_{23}^{(n_\lambda+1)}}{k_{23}^{(n_\lambda+1)}} \right) \mu_{23}^{(n_\lambda+1)} \left(6 \mu_{23}^{(n_\lambda+1)} \alpha_{\langle \lambda \rangle}^{(i)} + R_{n_\lambda} \beta_{\langle \lambda \rangle}^{(i)} \right)}{A_{\langle \lambda \rangle} \mu_{23}^{(n_\lambda+1)^2} + B_{\langle \lambda \rangle} \mu_{23}^{(n_\lambda+1)} + C_{\langle \lambda \rangle}}} \quad (\text{B.21})$$

695 with

$$\left. \begin{aligned} \alpha_{\langle \lambda \rangle}^{(i)} &\stackrel{\text{def}}{=} \alpha_{D_{i_\lambda}} - 3 \frac{R_{i_\lambda}^4 - R_{i_\lambda-1}^4}{R_{i_\lambda}^2 (R_{i_\lambda}^2 - R_{i_\lambda-1}^2)} \alpha_{A_{i_\lambda}} \\ \beta_{\langle \lambda \rangle}^{(i)} &\stackrel{\text{def}}{=} \beta_{D_{i_\lambda}} - 3 \frac{R_{i_\lambda}^4 - R_{i_\lambda-1}^4}{R_{i_\lambda}^2 (R_{i_\lambda}^2 - R_{i_\lambda-1}^2)} \beta_{A_{i_\lambda}} \end{aligned} \right\} \quad (\text{B.22})$$

696 $\mu_{23}^{(n_\lambda+1)}$ and $k_{23}^{(n_\lambda+1)}$ do not depend on the pattern they are attached to (see
697 Figure 2) consequently they do not depend on λ and Eq. (B.21) can be
698 rewritten as:

$$\mu_{23}^{\text{eff}} = \frac{\sum_{i=1}^n f_i \mu_{23}^{(i)} \sum_{\lambda=1}^{N_\lambda} \frac{m_\lambda R_{n_\lambda} \left(6 \mu_{23}^{\text{eff}} \alpha_{\langle \lambda \rangle}^{(i)} + R_{n_\lambda} \beta_{\langle \lambda \rangle}^{(i)} \right)}{A_{\langle \lambda \rangle} \mu_{23}^{\text{eff}2} + B_{\langle \lambda \rangle} \mu_{23}^{\text{eff}} + C_{\langle \lambda \rangle}}}{\sum_{i=1}^n f_i \sum_{\lambda=1}^{N_\lambda} \frac{m_\lambda R_{n_\lambda} \left(6 \mu_{23}^{\text{eff}} \alpha_{\langle \lambda \rangle}^{(i)} + R_{n_\lambda} \beta_{\langle \lambda \rangle}^{(i)} \right)}{A_{\langle \lambda \rangle} \mu_{23}^{\text{eff}2} + B_{\langle \lambda \rangle} \mu_{23}^{\text{eff}} + C_{\langle \lambda \rangle}}} \quad (\text{B.23})$$

699 where

$$\left. \begin{aligned}
 \alpha_{\mathbb{A}}^{(i)} &= Q_{44}^{(i_{\lambda}-1)} \left(Q_{11}^{*\mathbb{A}} + Q_{21}^{*\mathbb{A}} \right) \\
 &\quad - Q_{41}^{(i_{\lambda}-1)} \left(Q_{14}^{*\mathbb{A}} + Q_{24}^{*\mathbb{A}} \right) \\
 &\quad - 3 \frac{\left(R_{i_{\lambda}}^4 - R_{i_{\lambda}-1}^4 \right)}{R_{i_{\lambda}}^2 \left(R_{i_{\lambda}}^2 - R_{i_{\lambda}-1}^2 \right)} \\
 &\quad \times \left[Q_{14}^{(i_{\lambda}-1)} \left(Q_{11}^{*\mathbb{A}} + Q_{21}^{*\mathbb{A}} \right) \right. \\
 &\quad \left. - Q_{11}^{(i_{\lambda}-1)} \left(Q_{14}^{*\mathbb{A}} + Q_{24}^{*\mathbb{A}} \right) \right] \\
 \beta_{\mathbb{A}}^{(i)} &= Q_{44}^{(i_{\lambda}-1)} \left(Q_{31}^{*\mathbb{A}} + Q_{41}^{*\mathbb{A}} \right) \\
 &\quad - Q_{41}^{(i_{\lambda}-1)} \left(Q_{34}^{*\mathbb{A}} + Q_{44}^{*\mathbb{A}} \right) \\
 &\quad - 3 \frac{\left(R_{i_{\lambda}}^4 - R_{i_{\lambda}-1}^4 \right)}{R_{i_{\lambda}}^2 \left(R_{i_{\lambda}}^2 - R_{i_{\lambda}-1}^2 \right)} \\
 &\quad \times \left[Q_{14}^{(i_{\lambda}-1)} \left(Q_{31}^{*\mathbb{A}} + Q_{41}^{*\mathbb{A}} \right) \right. \\
 &\quad \left. - Q_{11}^{(i_{\lambda}-1)} \left(Q_{34}^{*\mathbb{A}} + Q_{44}^{*\mathbb{A}} \right) \right]
 \end{aligned} \right\} \quad (\text{B.24})$$

700 It is worth remembering that Eq. (B.23) is an implicit equation depending
 701 both on μ_{23}^{eff} and on k_{23}^{eff} through $A_{\mathbb{A}}$ and $B_{\mathbb{A}}$. In the different applications
 702 we will use the following expressions to take into account the fact that the
 703 different radii and the matrices \mathbf{Q} and \mathbf{Q}^* present in Eq. (B.23) depend on
 704 the pattern λ they are attached to:

$$\mu_{23}^{\text{eff}} = \frac{\sum_{i=1}^n f_i \mu_{23}^{(i)} \sum_{\lambda=1}^{N_{\lambda}} \frac{m_{\lambda} R_{n_{\lambda} \mathbb{A}} \left(6 \mu_{23}^{\text{eff}} \alpha_{\mathbb{A}}^{(i)} + R_{n_{\lambda} \mathbb{A}} \beta_{\mathbb{A}}^{(i)} \right)}{A_{\mathbb{A}} \mu_{23}^{\text{eff}2} + B_{\mathbb{A}} \mu_{23}^{\text{eff}} + C_{\mathbb{A}}}}{\sum_{i=1}^n f_i \sum_{\lambda=1}^{N_{\lambda}} \frac{m_{\lambda} R_{n_{\lambda} \mathbb{A}} \left(6 \mu_{23}^{\text{eff}} \alpha_{\mathbb{A}}^{(i)} + R_{n_{\lambda} \mathbb{A}} \beta_{\mathbb{A}}^{(i)} \right)}{A_{\mathbb{A}} \mu_{23}^{\text{eff}2} + B_{\mathbb{A}} \mu_{23}^{\text{eff}} + C_{\mathbb{A}}}} \quad (\text{B.25})$$

705 where finally

$$\left. \begin{aligned}
\alpha_{\mathbb{A}}^{(i)} &= Q_{44}^{(i_{\lambda}-1)\mathbb{A}} \left(Q_{11}^{*\mathbb{A}} + Q_{21}^{*\mathbb{A}} \right) \\
&\quad - Q_{41}^{(i_{\lambda}-1)\mathbb{A}} \left(Q_{14}^{*\mathbb{A}} + Q_{24}^{*\mathbb{A}} \right) \\
&\quad - 3 \frac{\left(R_{i_{\lambda}\mathbb{A}}^4 - R_{i_{\lambda}-1\mathbb{A}}^4 \right)}{R_{i_{\lambda}\mathbb{A}}^2 \left(R_{i_{\lambda}\mathbb{A}}^2 - R_{i_{\lambda}-1\mathbb{A}}^2 \right)} \\
&\quad \times \left[Q_{14}^{(i_{\lambda}-1)\mathbb{A}} \left(Q_{11}^{*\mathbb{A}} + Q_{21}^{*\mathbb{A}} \right) \right. \\
&\quad \left. - Q_{11}^{(i_{\lambda}-1)\mathbb{A}} \left(Q_{14}^{*\mathbb{A}} + Q_{24}^{*\mathbb{A}} \right) \right] \\
\beta_{\mathbb{A}}^{(i)} &= Q_{44}^{(i_{\lambda}-1)\mathbb{A}} \left(Q_{31}^{*\mathbb{A}} + Q_{41}^{*\mathbb{A}} \right) \\
&\quad - Q_{41}^{(i_{\lambda}-1)\mathbb{A}} \left(Q_{34}^{*\mathbb{A}} + Q_{44}^{*\mathbb{A}} \right) \\
&\quad - 3 \frac{\left(R_{i_{\lambda}\mathbb{A}}^4 - R_{i_{\lambda}-1\mathbb{A}}^4 \right)}{R_{i_{\lambda}\mathbb{A}}^2 \left(R_{i_{\lambda}\mathbb{A}}^2 - R_{i_{\lambda}-1\mathbb{A}}^2 \right)} \\
&\quad \times \left[Q_{14}^{(i_{\lambda}-1)\mathbb{A}} \left(Q_{31}^{*\mathbb{A}} + Q_{41}^{*\mathbb{A}} \right) \right. \\
&\quad \left. - Q_{11}^{(i_{\lambda}-1)\mathbb{A}} \left(Q_{34}^{*\mathbb{A}} + Q_{44}^{*\mathbb{A}} \right) \right]
\end{aligned} \right\} \quad (\text{B.26})$$

706 **Appendix C. Details of the calculations made in the particular**
707 **case of 2 patterns with 2 inverted phases**

708 In this section a configuration with two patterns is considered, one is a
709 “direct” pattern and the other is an “inverse” one, each one being made of
710 two phases ($n = 2$, $N_{\lambda} = 2$ and $n_{\lambda} = 2$ for each pattern) see description in
711 section 3 and Figure 3.

712 In the following, we will use the functions and parameters defined below:

$$\begin{aligned}
\text{Id} : \{1, 2\}_{\subseteq \Omega} &\rightarrow \{1, 2\}_{\subseteq \mathbb{N}} \\
\lambda &\mapsto \text{Id}(\lambda) \equiv \boxed{\lambda} = \begin{cases} 1 & \text{if } \lambda = 1 \\ 2 & \text{if } \lambda = 2 \end{cases} \quad (\text{C.1})
\end{aligned}$$

$$\begin{aligned} \text{Cp} : \{1, 2\}_{\subseteq \Omega} &\rightarrow \{1, 2\}_{\subseteq \mathbb{N}} \\ \lambda &\mapsto \text{Cp}(\lambda) \equiv \mathbf{\lambda} = \begin{cases} 1 & \text{if } \lambda = 2 \\ 2 & \text{if } \lambda = 1 \end{cases} \end{aligned} \quad (\text{C.2})$$

713 where the codomain $\{1, 2\}_{\subseteq \mathbb{N}}$ of Id and Cp transformations have totally lost
714 any “pattern” reference. It means that images $\text{Id}(\lambda)$, i.e. $\overline{\lambda}$, and $\text{Cp}(\lambda)$,
715 i.e. $\mathbf{\lambda}$, are just natural numbers and can, for example, describe phase num-
716 bers if they are placed in parentheses or in indices as for $\bullet^{(k)}$ or \bullet_k . So,
717 taken from Hervé and Zaoui (1995):

$$\left. \begin{aligned} \rho_{\overline{\lambda}} &\stackrel{\text{def}}{=} \mu_{23}^{(\overline{\lambda})} / \mu_{23}^{(\mathbf{\lambda})} \\ a_{\overline{\lambda}} &\stackrel{\text{def}}{=} \rho_{\overline{\lambda}} + (3 - 4 \nu_{\overline{\lambda}}) \\ b_{\overline{\lambda}} &\stackrel{\text{def}}{=} (3 - 2 \nu_{\overline{\lambda}}) + \rho_{\overline{\lambda}} (2 \nu_{\mathbf{\lambda}} - 3) \\ c_{\overline{\lambda}} &\stackrel{\text{def}}{=} 1 + \rho_{\overline{\lambda}} (3 - 4 \nu_{\mathbf{\lambda}}) \\ d_{\overline{\lambda}} &\stackrel{\text{def}}{=} 2 \nu_{\overline{\lambda}} - 1 + \rho_{\overline{\lambda}} (1 - 2 \nu_{\mathbf{\lambda}}) \\ e_{\overline{\lambda}} &\stackrel{\text{def}}{=} 1 - \rho_{\overline{\lambda}} \end{aligned} \right\} \quad (\text{C.3})$$

718 see Eq. (B.8) for $\nu_{\overline{\lambda}}$ and $\nu_{\mathbf{\lambda}}$. And:

$$q_{\overline{\lambda}}^2 \stackrel{\text{def}}{=} R_{1\overline{\lambda}}^2 / R_{2\overline{\lambda}}^2 \Rightarrow \begin{cases} q_{\textcircled{1}}^2 = c \\ q_{\textcircled{2}}^2 = c_{2_2} \end{cases} \quad (\text{C.4})$$

719 To calculate $\alpha_{\overline{\lambda}}^{(i)}$ and $\beta_{\overline{\lambda}}^{(i)}$ in the case of the determination of the effective
720 transverse shear modulus it may be noted that:

$$c_{\overline{\lambda}} = a_{\overline{\lambda}} + 2 d_{\overline{\lambda}} \quad (\text{C.5})$$

721 *Appendix C.1. Development of k_{23}^{eff}*

722 In the case of two patterns with two phases each, Eq. (27) becomes:

$$k_{23}^{\text{eff}} = \frac{\sum_{i=1}^2 f_i k_{23}^{(i)} \sum_{\lambda=1}^2 m_{\lambda} \frac{Q_{11}^{(i_{\lambda}-1)\otimes} R_{2\otimes}}{2 \mu_{23}^{\text{eff}} Q_{11}^{*\otimes} + R_{2\otimes} Q_{21}^{*\otimes}}}{\sum_{i=1}^2 f_i \sum_{\lambda=1}^2 m_{\lambda} \frac{Q_{11}^{(i_{\lambda}-1)\otimes} R_{2\otimes}}{2 \mu_{23}^{\text{eff}} Q_{11}^{*\otimes} + R_{2\otimes} Q_{21}^{*\otimes}}} \quad (\text{C.6})$$

723 Substituting the different components of the Q^{\otimes} matrices (see Hervé and
724 Zaoui (1995)) (for $\lambda \in \{1, 2\}$) in Eq. (C.6) gives:

$$k_{23}^{\text{eff}} = \frac{f k_{23}^{(1)} \{A_1\} + (1-f) k_{23}^{(2)} \{A_2\}}{f \{A_1\} + (1-f) \{A_2\}} \stackrel{\text{def}}{=} \frac{\{B_1\}}{\{B_2\}} \quad (\text{C.7})$$

725 where

$$\left. \begin{aligned} \{A_1\}_{\subseteq k_{23}^{\text{eff}}} &\stackrel{\text{def}}{=} \frac{m Q_{11}^{(0)\otimes} R_{2\otimes} + (1-m) Q_{11}^{(1)\otimes} R_{2\otimes}}{2 \mu_{23}^{\text{eff}} Q_{11}^{*\otimes} + R_{2\otimes} Q_{21}^{*\otimes}} + \frac{(1-m) Q_{11}^{(1)\otimes} R_{2\otimes}}{2 \mu_{23}^{\text{eff}} Q_{11}^{*\otimes} + R_{2\otimes} Q_{21}^{*\otimes}} \\ \{A_2\}_{\subseteq k_{23}^{\text{eff}}} &\stackrel{\text{def}}{=} \frac{m Q_{11}^{(1)\otimes} R_{2\otimes} + (1-m) Q_{11}^{(0)\otimes} R_{2\otimes}}{2 \mu_{23}^{\text{eff}} Q_{11}^{*\otimes} + R_{2\otimes} Q_{21}^{*\otimes}} + \frac{(1-m) Q_{11}^{(0)\otimes} R_{2\otimes}}{2 \mu_{23}^{\text{eff}} Q_{11}^{*\otimes} + R_{2\otimes} Q_{21}^{*\otimes}} \end{aligned} \right\} \quad (\text{C.8})$$

726 The Q^{\otimes} matrices are determined from Hervé and Zaoui (1995) for $\lambda \in$
727 $\{1, 2\}$:

$$Q^{(0)\otimes} = I \implies Q_{11}^{(0)\otimes} = 1 \quad (\text{C.9})$$

728

$$Q^{(1)\otimes} = N^{\lambda(1)} = \frac{1}{\mu_{23}^{(\mathbf{A})} + k_{23}^{(\mathbf{A})}} \begin{pmatrix} (\mu_{23}^{(\mathbf{A})} + k_{23}^{(\mathbf{A})}) & R_{0\otimes}^2 (\mu_{23}^{(\mathbf{A})} - \mu_{23}^{(\mathbf{A})}) \\ (k_{23}^{(\mathbf{A})} - k_{23}^{(\mathbf{A})}) & R_{1\otimes}^2 (\mu_{23}^{(\mathbf{A})} + k_{23}^{(\mathbf{A})}) \end{pmatrix} \quad (\text{C.10})$$

729 $R_{0\otimes}^2 = 0$ by definition leading to:

730

$$Q_{11}^{(1)\otimes} = \frac{\mu_{23}^{(\mathbf{A})} + k_{23}^{(\mathbf{A})}}{\mu_{23}^{(\mathbf{A})} + k_{23}^{(\mathbf{A})}} \quad (\text{C.11})$$

731 In order to determine k_{23}^{eff} in terms of μ_{23}^{eff} from Eq. (C.6) the \mathbf{Q} matrices
 732 have to be calculated by using the \mathbf{Q}^* matrices like in Hervé-Luanco (2020).

$$\mathbf{Q}^{*\langle\lambda\rangle} = \mathbf{J}_2^{\langle\lambda\rangle} (R_{2\langle\lambda\rangle}) \mathbf{Q}^{(1)\langle\lambda\rangle} \quad (\text{C.12})$$

733 with

$$\mathbf{J}_2^{\langle\lambda\rangle} (R_{2\langle\lambda\rangle}) = \begin{pmatrix} R_{2\langle\lambda\rangle} & \frac{R_{1\langle\lambda\rangle}^2}{R_{2\langle\lambda\rangle}} \\ 2 k_{23}^{(\mathbf{A})} & -2 \frac{R_{1\langle\lambda\rangle}^2}{R_{2\langle\lambda\rangle}^2} \mu_{23}^{(\mathbf{A})} \end{pmatrix} \quad (\text{C.13})$$

734 It can easily be shown that:

$$\left. \begin{aligned} Q_{11}^{*\langle\lambda\rangle} &= \frac{R_{2\langle\lambda\rangle} \left(\mu_{23}^{(\mathbf{A})} + k_{23}^{(\langle\lambda\rangle)} \right) + \frac{R_{1\langle\lambda\rangle}^2}{R_{2\langle\lambda\rangle}} \left(k_{23}^{(\mathbf{A})} - k_{23}^{(\langle\lambda\rangle)} \right)}{\mu_{23}^{(\mathbf{A})} + k_{23}^{(\mathbf{A})}} \\ Q_{21}^{*\langle\lambda\rangle} &= \frac{2 k_{23}^{(\mathbf{A})} \left(\mu_{23}^{(\mathbf{A})} + k_{23}^{(\langle\lambda\rangle)} \right) - 2 \frac{R_{1\langle\lambda\rangle}^2}{R_{2\langle\lambda\rangle}^2} \mu_{23}^{(\mathbf{A})} \left(k_{23}^{(\mathbf{A})} - k_{23}^{(\langle\lambda\rangle)} \right)}{\mu_{23}^{(\mathbf{A})} + k_{23}^{(\mathbf{A})}} \end{aligned} \right\} \quad (\text{C.14})$$

735 $\{\mathbf{A}_p\}$, $p \in \llbracket 1, 2 \rrbracket$, in Eq. (C.8) are then written as:

$$\left. \begin{aligned} \left\{ \mathbf{A}_1 \right\}_{\subseteq k_{23}^{\text{eff}}} &= \frac{m \left(\mu_{23}^{(2)} + k_{23}^{(2)} \right)}{2 \left(\mu_{23}^{(2)} + k_{23}^{(1)} \right) \left(\mu_{23}^{\text{eff}} + k_{23}^{(2)} \right) + 2 c \left(k_{23}^{(2)} - k_{23}^{(1)} \right) \left(\mu_{23}^{\text{eff}} - \mu_{23}^{(2)} \right)} \\ &\quad + \frac{(1-m) \left(\mu_{23}^{(1)} + k_{23}^{(2)} \right)}{2 \left(\mu_{23}^{(1)} + k_{23}^{(2)} \right) \left(\mu_{23}^{\text{eff}} + k_{23}^{(1)} \right) + 2 c_{22} \left(k_{23}^{(1)} - k_{23}^{(2)} \right) \left(\mu_{23}^{\text{eff}} - \mu_{23}^{(1)} \right)} \\ \left\{ \mathbf{A}_2 \right\}_{\subseteq k_{23}^{\text{eff}}} &= \frac{m \left(\mu_{23}^{(2)} + k_{23}^{(1)} \right)}{2 \left(\mu_{23}^{(2)} + k_{23}^{(1)} \right) \left(\mu_{23}^{\text{eff}} + k_{23}^{(2)} \right) + 2 c \left(k_{23}^{(2)} - k_{23}^{(1)} \right) \left(\mu_{23}^{\text{eff}} - \mu_{23}^{(2)} \right)} \\ &\quad + \frac{(1-m) \left(\mu_{23}^{(1)} + k_{23}^{(1)} \right)}{2 \left(\mu_{23}^{(1)} + k_{23}^{(2)} \right) \left(\mu_{23}^{\text{eff}} + k_{23}^{(1)} \right) + 2 c_{22} \left(k_{23}^{(1)} - k_{23}^{(2)} \right) \left(\mu_{23}^{\text{eff}} - \mu_{23}^{(1)} \right)} \end{aligned} \right\} \quad (\text{C.15})$$

736 By reintroducing $\{\mathbf{A}_1\}$ and $\{\mathbf{A}_2\}$ from Eq. (C.15) in Eq. (C.7) we obtain:

$$\left. \begin{aligned}
\{\mathbf{B}_1\}_{\subseteq k_{23}^{\text{eff}}} &= \frac{fk_{23}^{(1)}m(\mu_{23}^{(2)} + k_{23}^{(2)}) + (1-f)k_{23}^{(2)}m(\mu_{23}^{(2)} + k_{23}^{(1)})}{(\mu_{23}^{(2)} + k_{23}^{(1)})(\mu_{23}^{\text{eff}} + k_{23}^{(2)}) + c(k_{23}^{(2)} - k_{23}^{(1)})(\mu_{23}^{\text{eff}} - \mu_{23}^{(2)})} \\
&+ \frac{fk_{23}^{(1)}(1-m)(\mu_{23}^{(1)} + k_{23}^{(2)}) + (1-f)k_{23}^{(2)}(1-m)(\mu_{23}^{(1)} + k_{23}^{(1)})}{(\mu_{23}^{(1)} + k_{23}^{(2)})(\mu_{23}^{\text{eff}} + k_{23}^{(1)}) + c_{22}(k_{23}^{(1)} - k_{23}^{(2)})(\mu_{23}^{\text{eff}} - \mu_{23}^{(1)})} \\
\{\mathbf{B}_2\}_{\subseteq k_{23}^{\text{eff}}} &= \frac{fm(\mu_{23}^{(2)} + k_{23}^{(2)}) + (1-f)m(\mu_{23}^{(2)} + k_{23}^{(1)})}{(\mu_{23}^{(2)} + k_{23}^{(1)})(\mu_{23}^{\text{eff}} + k_{23}^{(2)}) + c(k_{23}^{(2)} - k_{23}^{(1)})(\mu_{23}^{\text{eff}} - \mu_{23}^{(2)})} \\
&+ \frac{f(1-m)(\mu_{23}^{(1)} + k_{23}^{(2)}) + (1-f)(1-m)(\mu_{23}^{(1)} + k_{23}^{(1)})}{(\mu_{23}^{(1)} + k_{23}^{(2)})(\mu_{23}^{\text{eff}} + k_{23}^{(1)}) + c_{22}(k_{23}^{(1)} - k_{23}^{(2)})(\mu_{23}^{\text{eff}} - \mu_{23}^{(1)})}
\end{aligned} \right\} \quad (\text{C.16})$$

737 k_{23}^{eff} can finally be developed in terms of μ_{23}^{eff} :

$$k_{23}^{\text{eff}} = \frac{\{\mathbf{C}_1\}_{\subseteq k_{23}^{\text{eff}}}\mu_{23}^{\text{eff}} + \{\mathbf{C}_2\}_{\subseteq k_{23}^{\text{eff}}}}{\{\mathbf{C}_3\}_{\subseteq k_{23}^{\text{eff}}}\mu_{23}^{\text{eff}} + \{\mathbf{C}_4\}_{\subseteq k_{23}^{\text{eff}}}} \quad (\text{C.17})$$

738 with

$$\left.
\begin{aligned}
\{\mathbf{C}_1\}_{\subseteq k_{23}^{\text{eff}}} &\stackrel{\text{def}}{=} \left[f k_{23}^{(1)} m \left(\mu_{23}^{(2)} + k_{23}^{(2)} \right) + (1-f) k_{23}^{(2)} m \left(\mu_{23}^{(2)} + k_{23}^{(1)} \right) \right] \\
&\times \left[c_{22} \left(k_{23}^{(1)} - k_{23}^{(2)} \right) + \left(\mu_{23}^{(1)} + k_{23}^{(2)} \right) \right] \\
&+ \left[f k_{23}^{(1)} (1-m) \left(\mu_{23}^{(1)} + k_{23}^{(2)} \right) + (1-f) k_{23}^{(2)} (1-m) \left(\mu_{23}^{(1)} + k_{23}^{(1)} \right) \right] \\
&\times \left[c \left(k_{23}^{(2)} - k_{23}^{(1)} \right) + \left(\mu_{23}^{(2)} + k_{23}^{(1)} \right) \right] \\
\{\mathbf{C}_2\}_{\subseteq k_{23}^{\text{eff}}} &\stackrel{\text{def}}{=} \left[f k_{23}^{(1)} m \left(\mu_{23}^{(2)} + k_{23}^{(2)} \right) + (1-f) k_{23}^{(2)} m \left(\mu_{23}^{(2)} + k_{23}^{(1)} \right) \right] \\
&\times \left[\left(\mu_{23}^{(1)} + k_{23}^{(2)} \right) k_{23}^{(1)} - c_{22} \left(k_{23}^{(1)} - k_{23}^{(2)} \right) \mu_{23}^{(1)} \right] \\
&+ \left[f k_{23}^{(1)} (1-m) \left(\mu_{23}^{(1)} + k_{23}^{(2)} \right) + (1-f) k_{23}^{(2)} (1-m) \left(\mu_{23}^{(1)} + k_{23}^{(1)} \right) \right] \\
&\times \left[\left(\mu_{23}^{(2)} + k_{23}^{(1)} \right) k_{23}^{(2)} - c \left(k_{23}^{(2)} - k_{23}^{(1)} \right) \mu_{23}^{(2)} \right] \\
\{\mathbf{C}_3\}_{\subseteq k_{23}^{\text{eff}}} &\stackrel{\text{def}}{=} \left[f m \left(\mu_{23}^{(2)} + k_{23}^{(2)} \right) + (1-f) m \left(\mu_{23}^{(2)} + k_{23}^{(1)} \right) \right] \\
&\times \left[c_{22} \left(k_{23}^{(1)} - k_{23}^{(2)} \right) + \left(\mu_{23}^{(1)} + k_{23}^{(2)} \right) \right] \\
&+ \left[f (1-m) \left(\mu_{23}^{(1)} + k_{23}^{(2)} \right) + (1-f) (1-m) \left(\mu_{23}^{(1)} + k_{23}^{(1)} \right) \right] \\
&\times \left[c \left(k_{23}^{(2)} - k_{23}^{(1)} \right) + \left(\mu_{23}^{(2)} + k_{23}^{(1)} \right) \right] \\
\{\mathbf{C}_4\}_{\subseteq k_{23}^{\text{eff}}} &\stackrel{\text{def}}{=} \left[f m \left(\mu_{23}^{(2)} + k_{23}^{(2)} \right) + (1-f) m \left(\mu_{23}^{(2)} + k_{23}^{(1)} \right) \right] \\
&\times \left[\left(\mu_{23}^{(1)} + k_{23}^{(2)} \right) k_{23}^{(1)} - c_{22} \left(k_{23}^{(1)} - k_{23}^{(2)} \right) \mu_{23}^{(1)} \right] \\
&+ \left[f (1-m) \left(\mu_{23}^{(1)} + k_{23}^{(2)} \right) + (1-f) (1-m) \left(\mu_{23}^{(1)} + k_{23}^{(1)} \right) \right] \\
&\times \left[\left(\mu_{23}^{(2)} + k_{23}^{(1)} \right) k_{23}^{(2)} - c \left(k_{23}^{(2)} - k_{23}^{(1)} \right) \mu_{23}^{(2)} \right]
\end{aligned}
\right\} \tag{C.18}$$

739 *Appendix C.2. Development of μ_{23}^{eff}*

740 In the context of two patterns with two phases Eq. (42) becomes:

$$\mu_{23}^{\text{eff}} = \frac{\sum_{i=1}^2 f_i \mu_{23}^{(i)} \sum_{\lambda=1}^2 \frac{m_\lambda R_{2\ominus} (6 \mu_{23}^{\text{eff}} \alpha_{\ominus}^{(i)} + R_{2\ominus} \beta_{\ominus}^{(i)})}{A_{\ominus} \mu_{23}^{\text{eff}2} + B_{\ominus} \mu_{23}^{\text{eff}} + C_{\ominus}}}{\sum_{i=1}^2 f_i \sum_{\lambda=1}^2 \frac{m_\lambda R_{2\ominus} (6 \mu_{23}^{\text{eff}} \alpha_{\ominus}^{(i)} + R_{2\ominus} \beta_{\ominus}^{(i)})}{A_{\ominus} \mu_{23}^{\text{eff}2} + B_{\ominus} \mu_{23}^{\text{eff}} + C_{\ominus}}} \quad (\text{C.19})$$

741 In order to calculate $A_{\ominus} \mu_{23}^{\text{eff}2} + B_{\ominus} \mu_{23}^{\text{eff}} + C_{\ominus}$ let us use the following equa-
742 tions:

$$\left. \begin{aligned} A_{\ominus} &= 12 Z_{12}^{\ominus} + 6 \frac{R_{2\ominus}}{k_{23}^{\text{eff}}} \left(Z_{14}^{\ominus} + Z_{32}^{\ominus} + Z_{24}^{\ominus} + Z_{31}^{\ominus} \right) \\ B_{\ominus} &= \frac{2 Z_{34}^{\ominus} R_{2\ominus}^2}{k_{23}^{\text{eff}}} + 2 R_{2\ominus} \left(2 Z_{14}^{\ominus} + 2 Z_{32}^{\ominus} + Z_{24}^{\ominus} + Z_{31}^{\ominus} \right) \\ C_{\ominus} &= Z_{34}^{\ominus} R_{2\ominus}^2 \end{aligned} \right\} \quad (\text{C.20})$$

743 where all the Z_{ij} (see Eq.⁹⁵ (81) in Hervé and Zaoui (1995)) are already
744 defined by:

$$Z_{ij}^{\ominus} \stackrel{\text{def}}{=} Q_{i4}^{*\ominus} Q_{j1}^{*\ominus} - Q_{j4}^{*\ominus} Q_{i1}^{*\ominus} \quad (\text{C.21})$$

745 with:

$$\mathbf{Q}^{*\ominus} = \mathbf{J}_2^{\ominus} (R_{2\ominus}) \mathbf{Q}^{(1)\ominus} \quad (\text{C.22})$$

746 From Hervé and Zaoui (1995), in the case of a transverse shear loading,

747 $\mathbf{J}_2^{(\lambda)}(R_{2\lambda})$ is given by:

$$\mathbf{J}_2^{(\lambda)}(R_{2\lambda}) = \begin{pmatrix} -4 \nu_{\mathbf{A}} R_{2\lambda} & R_{2\lambda} & 4(1 - \nu_{\mathbf{A}}) R_{2\lambda} & R_{2\lambda} \\ (6 - 4 \nu_{\mathbf{A}}) R_{2\lambda} & R_{2\lambda} & -(2 - 4 \nu_{\mathbf{A}}) R_{2\lambda} & -R_{2\lambda} \\ 0 & -6 \mu_{23}^{(\mathbf{A})} & -8 \mu_{23}^{(\mathbf{A})} & 2 \mu_{23}^{(\mathbf{A})} \\ 12 \mu_{23}^{(\mathbf{A})} & -6 \mu_{23}^{(\mathbf{A})} & -4 \mu_{23}^{(\mathbf{A})} & -2 \mu_{23}^{(\mathbf{A})} \end{pmatrix} \quad (\text{C.23})$$

and $\mathbf{Q}^{(1)\lambda}$ by:

$$\mathbf{Q}^{(1)\lambda} = \frac{1}{4(1 - \nu_{\mathbf{A}})} \times \begin{pmatrix} \frac{a_{\mathbf{A}}}{q_{\lambda}^2} & \frac{1 - \rho_{\mathbf{A}}}{q_{\lambda}^2} & \frac{1 - \rho_{\mathbf{A}}}{q_{\lambda}^2} & 0 \\ 2 q_{\lambda}^4 b_{\mathbf{A}} & q_{\lambda}^4 c_{\mathbf{A}} & 2 q_{\lambda}^4 d_{\mathbf{A}} & q_{\lambda}^4 (\rho_{\mathbf{A}} - 1) \\ -3 q_{\lambda}^2 (1 - \rho_{\mathbf{A}}) & 0 & q_{\lambda}^2 a_{\mathbf{A}} & q_{\lambda}^2 (1 - \rho_{\mathbf{A}}) \\ -6 d_{\mathbf{A}} & 3(1 - \rho_{\mathbf{A}}) & 2 b_{\mathbf{A}} & c_{\mathbf{A}} \end{pmatrix} \quad (\text{C.24})$$

748 The different $\mathbf{Q}^{*\lambda}$ matrices are calculated from Eq. (C.22) with the help

749 of Eq. (C.23) and Eq. (C.24):

$$\left. \begin{aligned}
Q_{11}^{*\langle \lambda \rangle} &= \frac{R_{2\langle \lambda \rangle}}{4(1-\nu_{\langle \lambda \rangle})} \left[-4\nu_{\langle \lambda \rangle} \frac{a_{\langle \lambda \rangle}}{q_{\langle \lambda \rangle}^2} + 2q_{\langle \lambda \rangle}^4 b_{\langle \lambda \rangle} \right. \\
&\quad \left. -12(1-\nu_{\langle \lambda \rangle}) q_{\langle \lambda \rangle}^2 e_{\langle \lambda \rangle} - 6d_{\langle \lambda \rangle} \right] \\
Q_{21}^{*\langle \lambda \rangle} &= \frac{R_{2\langle \lambda \rangle}}{4(1-\nu_{\langle \lambda \rangle})} \left[(6-4\nu_{\langle \lambda \rangle}) \frac{a_{\langle \lambda \rangle}}{q_{\langle \lambda \rangle}^2} + 2q_{\langle \lambda \rangle}^4 b_{\langle \lambda \rangle} \right. \\
&\quad \left. +3(2-4\nu_{\langle \lambda \rangle}) q_{\langle \lambda \rangle}^2 e_{\langle \lambda \rangle} + 6d_{\langle \lambda \rangle} \right] \\
Q_{31}^{*\langle \lambda \rangle} &= \frac{2\mu_{23}^{(\langle \lambda \rangle)}}{4(1-\nu_{\langle \lambda \rangle})} \left[-6q_{\langle \lambda \rangle}^4 b_{\langle \lambda \rangle} + 12q_{\langle \lambda \rangle}^2 e_{\langle \lambda \rangle} - 6d_{\langle \lambda \rangle} \right] \\
Q_{41}^{*\langle \lambda \rangle} &= \frac{2\mu_{23}^{(\langle \lambda \rangle)}}{4(1-\nu_{\langle \lambda \rangle})} \left[6\frac{a_{\langle \lambda \rangle}}{q_{\langle \lambda \rangle}^2} - 6q_{\langle \lambda \rangle}^4 b_{\langle \lambda \rangle} + 6q_{\langle \lambda \rangle}^2 e_{\langle \lambda \rangle} + 6d_{\langle \lambda \rangle} \right] \\
Q_{14}^{*\langle \lambda \rangle} &= \frac{R_{2\langle \lambda \rangle}}{4(1-\nu_{\langle \lambda \rangle})} \left[-q_{\langle \lambda \rangle}^4 e_{\langle \lambda \rangle} + 4(1-\nu_{\langle \lambda \rangle}) q_{\langle \lambda \rangle}^2 e_{\langle \lambda \rangle} + c_{\langle \lambda \rangle} \right] \\
Q_{24}^{*\langle \lambda \rangle} &= \frac{R_{2\langle \lambda \rangle}}{4(1-\nu_{\langle \lambda \rangle})} \left[-q_{\langle \lambda \rangle}^4 e_{\langle \lambda \rangle} - (2-4\nu_{\langle \lambda \rangle}) q_{\langle \lambda \rangle}^2 e_{\langle \lambda \rangle} - c_{\langle \lambda \rangle} \right] \\
Q_{34}^{*\langle \lambda \rangle} &= \frac{2\mu_{23}^{(\langle \lambda \rangle)}}{4(1-\nu_{\langle \lambda \rangle})} \left[3q_{\langle \lambda \rangle}^4 e_{\langle \lambda \rangle} - 4q_{\langle \lambda \rangle}^2 e_{\langle \lambda \rangle} + c_{\langle \lambda \rangle} \right] \\
Q_{44}^{*\langle \lambda \rangle} &= \frac{2\mu_{23}^{(\langle \lambda \rangle)}}{4(1-\nu_{\langle \lambda \rangle})} \left[3q_{\langle \lambda \rangle}^4 e_{\langle \lambda \rangle} - 2q_{\langle \lambda \rangle}^2 e_{\langle \lambda \rangle} - c_{\langle \lambda \rangle} \right]
\end{aligned} \right\} \quad (\text{C.25})$$

750 The different components $Z_{ij}^{\langle \lambda \rangle}$ present in Eq. (C.20) can then be determined

751 thanks to Eq. (C.21) with the previous values of $\mathbf{Q}^{*\langle \lambda \rangle}$ leading to:

$$\begin{aligned}
Z_{12}^{\langle \lambda \rangle} &= \frac{R_{2\langle \lambda \rangle}^2}{16(1-\nu_{\langle \lambda \rangle})^2} \left[-6q_{\langle \lambda \rangle}^2 a_{\langle \lambda \rangle} e_{\langle \lambda \rangle} - 6q_{\langle \lambda \rangle}^2 e_{\langle \lambda \rangle} c_{\langle \lambda \rangle} + 12q_{\langle \lambda \rangle}^2 d_{\langle \lambda \rangle} e_{\langle \lambda \rangle} \right. \\
&\quad +4q_{\langle \lambda \rangle}^6 b_{\langle \lambda \rangle} e_{\langle \lambda \rangle} (3-4\nu_{\langle \lambda \rangle}) + 6q_{\langle \lambda \rangle}^6 e_{\langle \lambda \rangle}^2 (4\nu_{\langle \lambda \rangle} - 3) + 4q_{\langle \lambda \rangle}^4 b_{\langle \lambda \rangle} c_{\langle \lambda \rangle} \\
&\quad \left. -12q_{\langle \lambda \rangle}^4 e_{\langle \lambda \rangle} d_{\langle \lambda \rangle} + \frac{2a_{\langle \lambda \rangle} c_{\langle \lambda \rangle}}{q_{\langle \lambda \rangle}^2} (3-4\nu_{\langle \lambda \rangle}) + 8a_{\langle \lambda \rangle} e_{\langle \lambda \rangle} (3-6\nu_{\langle \lambda \rangle} + 4\nu_{\langle \lambda \rangle}^2) \right]
\end{aligned} \quad (\text{C.26})$$

752

$$\begin{aligned}
Z_{31}^{\otimes} &= \frac{2 \mu_{23}^{(\mathbf{A})} R_{2\otimes}}{16 (1 - \nu_{\mathbf{A}})^2} \left[8 q_{\otimes}^4 b_{\mathbf{A}} c_{\mathbf{A}} + 8 q_{\otimes}^6 e_{\mathbf{A}} b_{\mathbf{A}} (2 - 3 \nu_{\mathbf{A}}) \right. \\
&\quad - 12 q_{\otimes}^2 e_{\mathbf{A}} c_{\mathbf{A}} (2 - \nu_{\mathbf{A}}) - 12 q_{\otimes}^2 a_{\mathbf{A}} e_{\mathbf{A}} \nu_{\mathbf{A}} + 12 q_{\otimes}^6 e_{\mathbf{A}}^2 (3 \nu_{\mathbf{A}} - 2) \\
&\quad \left. + 24 q_{\otimes}^2 e_{\mathbf{A}} d_{\mathbf{A}} (2 - \nu_{\mathbf{A}}) - \frac{4 a_{\mathbf{A}} c_{\mathbf{A}} \nu_{\mathbf{A}}}{q_{\otimes}^2} - 24 q_{\otimes}^4 d_{\mathbf{A}} e_{\mathbf{A}} + 16 \nu_{\mathbf{A}} e_{\mathbf{A}} a_{\mathbf{A}} \right] \quad (\text{C.27})
\end{aligned}$$

753

$$\begin{aligned}
Z_{32}^{\otimes} &= \frac{2 \mu_{23}^{(\mathbf{A})} R_{2\otimes}}{16 (1 - \nu_{\mathbf{A}})^2} \left[3 (6 - 4 \nu_{\mathbf{A}}) q_{\otimes}^2 a_{\mathbf{A}} e_{\mathbf{A}} + 6 q_{\otimes}^6 e_{\mathbf{A}}^2 (5 - 6 \nu_{\mathbf{A}}) \right. \\
&\quad - 8 (3 - 2 \nu_{\mathbf{A}}) a_{\mathbf{A}} e_{\mathbf{A}} - 4 (5 - 6 \nu_{\mathbf{A}}) q_{\otimes}^6 e_{\mathbf{A}} b_{\mathbf{A}} \\
&\quad - 12 (3 - 2 \nu_{\mathbf{A}}) q_{\otimes}^2 e_{\mathbf{A}} d_{\mathbf{A}} + \frac{2 a_{\mathbf{A}} c_{\mathbf{A}}}{q_{\otimes}^2} (3 - 2 \nu_{\mathbf{A}}) \\
&\quad \left. - 4 q_{\otimes}^4 b_{\mathbf{A}} c_{\mathbf{A}} + 6 q_{\otimes}^2 c_{\mathbf{A}} e_{\mathbf{A}} (3 - 2 \nu_{\mathbf{A}}) + 12 q_{\otimes}^4 e_{\mathbf{A}} d_{\mathbf{A}} \right] \quad (\text{C.28})
\end{aligned}$$

754

$$\begin{aligned}
Z_{14}^{\otimes} &= \frac{2 \mu_{23}^{(\mathbf{A})} R_{2\otimes}}{16 (1 - \nu_{\mathbf{A}})^2} \left[6 (2 \nu_{\mathbf{A}} - 1) q_{\otimes}^2 a_{\mathbf{A}} e_{\mathbf{A}} + 6 q_{\otimes}^6 e_{\mathbf{A}}^2 (5 - 6 \nu_{\mathbf{A}}) \right. \\
&\quad + 12 q_{\otimes}^4 e_{\mathbf{A}} d_{\mathbf{A}} + 8 (3 - 4 \nu_{\mathbf{A}}) a_{\mathbf{A}} e_{\mathbf{A}} \\
&\quad + 4 q_{\otimes}^6 b_{\mathbf{A}} e_{\mathbf{A}} (6 \nu_{\mathbf{A}} - 5) + 12 (1 - 2 \nu_{\mathbf{A}}) q_{\otimes}^2 d_{\mathbf{A}} e_{\mathbf{A}} \\
&\quad \left. + \frac{2 a_{\mathbf{A}} c_{\mathbf{A}}}{q_{\otimes}^2} (3 - 2 \nu_{\mathbf{A}}) - 4 q_{\otimes}^4 b_{\mathbf{A}} c_{\mathbf{A}} + 6 (2 \nu_{\mathbf{A}} - 1) q_{\otimes}^2 e_{\mathbf{A}} c_{\mathbf{A}} \right] \quad (\text{C.29})
\end{aligned}$$

755

$$\begin{aligned}
Z_{24}^{\otimes} &= \frac{2 \mu_{23}^{(\mathbf{A})} R_{2\otimes}}{16 (1 - \nu_{\mathbf{A}})^2} \left[12 (\nu_{\mathbf{A}} - 2) q_{\otimes}^2 a_{\mathbf{A}} e_{\mathbf{A}} - 12 q_{\otimes}^6 e_{\mathbf{A}}^2 (2 - 3 \nu_{\mathbf{A}}) \right. \\
&\quad - 24 q_{\otimes}^4 d_{\mathbf{A}} e_{\mathbf{A}} + 16 \nu_{\mathbf{A}} a_{\mathbf{A}} e_{\mathbf{A}} + 8 (2 - 3 \nu_{\mathbf{A}}) q_{\otimes}^6 e_{\mathbf{A}} b_{\mathbf{A}} \\
&\quad \left. + 24 q_{\otimes}^2 e_{\mathbf{A}} d_{\mathbf{A}} \nu_{\mathbf{A}} - \frac{4 \nu_{\mathbf{A}} a_{\mathbf{A}} c_{\mathbf{A}}}{q_{\otimes}^2} + 8 q_{\otimes}^4 b_{\mathbf{A}} c_{\mathbf{A}} - 12 \nu_{\mathbf{A}} q_{\otimes}^2 e_{\mathbf{A}} c_{\mathbf{A}} \right] \quad (\text{C.30})
\end{aligned}$$

756

$$Z_{34}^{\otimes} = \frac{4 \mu_{23}^{(\otimes)^2}}{16 (1 - \nu_{\otimes})^2} \left[18 q_{\otimes}^2 a_{\otimes} e_{\otimes} - 18 q_{\otimes}^6 e_{\otimes}^2 + 36 q_{\otimes}^4 e_{\otimes} d_{\otimes} - 24 a_{\otimes} e_{\otimes} \right. \\ \left. + 12 q_{\otimes}^6 e_{\otimes} b_{\otimes} - 36 q_{\otimes}^2 e_{\otimes} d_{\otimes} + \frac{6 a_{\otimes} c_{\otimes}}{q_{\otimes}^2} - 12 q_{\otimes}^4 b_{\otimes} c_{\otimes} + 18 q_{\otimes}^2 e_{\otimes} c_{\otimes} \right] \quad (\text{C.31})$$

757 and consequently to $\alpha_{\otimes}^{(i)}$, A_{\otimes} , B_{\otimes} , and C_{\otimes} , which can be factorized by
 758 respect to the radius $R_{2\otimes}$. Let us introduce $\alpha_{R\otimes}^{(i)}$, $A_{R\otimes}$, $B_{R\otimes}$, $C_{R\otimes}$ defined
 759 as:

$$\left. \begin{aligned} \alpha_{R\otimes}^{(i)} &\stackrel{\text{def}}{=} \alpha_{\otimes}^{(i)} / R_{2\otimes} \\ A_{R\otimes} &\stackrel{\text{def}}{=} A_{\otimes} / R_{2\otimes}^2 \\ B_{R\otimes} &\stackrel{\text{def}}{=} B_{\otimes} / R_{2\otimes}^2 \\ C_{R\otimes} &\stackrel{\text{def}}{=} C_{\otimes} / R_{2\otimes}^2 \end{aligned} \right\} \quad (\text{C.32})$$

760 such that μ_{23}^{eff} in Eq. (C.19) writes now:

$$\mu_{23}^{\text{eff}} = \frac{\sum_{i=1}^2 f_i \mu_{23}^{(i)} \sum_{\lambda=1}^2 \frac{m_{\lambda} (6 \mu_{23}^{\text{eff}} \alpha_{R\otimes}^{(i)} + \beta_{\otimes}^{(i)})}{A_{R\otimes} \mu_{23}^{\text{eff}^2} + B_{R\otimes} \mu_{23}^{\text{eff}} + C_{R\otimes}}}{\sum_{i=1}^2 f_i \sum_{\lambda=1}^2 \frac{m_{\lambda} (6 \mu_{23}^{\text{eff}} \alpha_{R\otimes}^{(i)} + \beta_{\otimes}^{(i)})}{A_{R\otimes} \mu_{23}^{\text{eff}^2} + B_{R\otimes} \mu_{23}^{\text{eff}} + C_{R\otimes}}} \stackrel{\text{def}}{=} \frac{\{\mathbf{A}_1\}}{\subseteq \mu_{23}^{\text{eff}}} \stackrel{\text{def}}{=} \frac{\{\mathbf{A}_2\}}{\subseteq \mu_{23}^{\text{eff}}} \quad (\text{C.33})$$

761 with $\{\mathbf{A}_p\}$, $p \in \llbracket 1, 2 \rrbracket$, given by:

$$\left. \begin{aligned}
\{\mathbf{A}_1\}_{\subseteq \mu_{23}^{\text{eff}}} &= \frac{m \left[f \mu_{23}^{(1)} \left(6 \alpha_{\mathcal{R}1}^{(1)} \mu_{23}^{\text{eff}} + \beta_{\textcircled{1}}^{(1)} \right) + (1-f) \mu_{23}^{(2)} \left(6 \alpha_{\mathcal{R}1}^{(2)} \mu_{23}^{\text{eff}} + \beta_{\textcircled{1}}^{(2)} \right) \right]}{A_{\mathcal{R}1} \mu_{23}^{\text{eff}^2} + B_{\mathcal{R}1} \mu_{23}^{\text{eff}} + C_{\mathcal{R}1}} \\
&+ \frac{(1-m) \left[f \mu_{23}^{(1)} \left(6 \alpha_{\mathcal{R}2}^{(1)} \mu_{23}^{\text{eff}} + \beta_{\textcircled{2}}^{(1)} \right) + (1-f) \mu_{23}^{(2)} \left(6 \alpha_{\mathcal{R}2}^{(2)} \mu_{23}^{\text{eff}} + \beta_{\textcircled{2}}^{(2)} \right) \right]}{A_{\mathcal{R}2} \mu_{23}^{\text{eff}^2} + B_{\mathcal{R}2} \mu_{23}^{\text{eff}} + C_{\mathcal{R}2}} \\
\{\mathbf{A}_2\}_{\subseteq \mu_{23}^{\text{eff}}} &= \frac{m \left[f \left(6 \alpha_{\mathcal{R}1}^{(1)} \mu_{23}^{\text{eff}} + \beta_{\textcircled{1}}^{(1)} \right) + (1-f) \left(6 \alpha_{\mathcal{R}1}^{(2)} \mu_{23}^{\text{eff}} + \beta_{\textcircled{1}}^{(2)} \right) \right]}{A_{\mathcal{R}1} \mu_{23}^{\text{eff}^2} + B_{\mathcal{R}1} \mu_{23}^{\text{eff}} + C_{\mathcal{R}1}} \\
&+ \frac{(1-m) \left[f \left(6 \alpha_{\mathcal{R}2}^{(1)} \mu_{23}^{\text{eff}} + \beta_{\textcircled{2}}^{(1)} \right) + (1-f) \left(6 \alpha_{\mathcal{R}2}^{(2)} \mu_{23}^{\text{eff}} + \beta_{\textcircled{2}}^{(2)} \right) \right]}{A_{\mathcal{R}2} \mu_{23}^{\text{eff}^2} + B_{\mathcal{R}2} \mu_{23}^{\text{eff}} + C_{\mathcal{R}2}}
\end{aligned} \right\} \quad (\text{C.34})$$

762 It is worth noticing that we have used the fact that $A_{\mathcal{R}\lambda}$, $B_{\mathcal{R}\lambda}$ and $C_{\mathcal{R}\lambda}$
763 only depend on pattern λ but not on phase (i) in Eq. (C.33).

764 We can now get the final expressions of the eight terms $\alpha_{\mathcal{R}\lambda}^{(i)}$ and $\beta_{\lambda}^{(i)}$ (Eq. (B.26))
765 because $\mathbf{Q}^{(0)\lambda} = \mathbf{I}$, an identity array, and because $\mathbf{Q}^{(1)\lambda}$ can be determined
766 by the help of Eq.⁹⁵ (69) in Hervé and Zaoui (1995):

$$\left. \begin{aligned}
Q_{14}^{(1)\lambda} &= 0 \\
Q_{11}^{(1)\lambda} &= \frac{a_{\lambda}}{4 (1 - \nu_{\lambda}) q_{\lambda}^2} \\
Q_{44}^{(1)\lambda} &= \frac{c_{\lambda}}{4 (1 - \nu_{\lambda})} \\
Q_{41}^{(1)\lambda} &= \frac{-6d_{\lambda}}{4 (1 - \nu_{\lambda})}
\end{aligned} \right\} \quad (\text{C.35})$$

767 leading finally to:

$$\left. \begin{aligned} \alpha_{\mathcal{R}(\lambda)}^{(\lambda)} &= \frac{2}{4(1-\nu_{\lambda})} \left[\frac{a_{\lambda}}{q_{\lambda}^2} (3-4\nu_{\lambda}) + q_{\lambda}^4 (2b_{\lambda} - 3e_{\lambda}) \right] \\ \alpha_{\mathcal{R}(\lambda)}^{(\lambda)} &= \frac{2}{16(1-\nu_{\lambda})^2} \left[3a_{\lambda}e_{\lambda}(1-q_{\lambda}^2) \right. \\ &\quad \left. + c_{\lambda}q_{\lambda}^4(2b_{\lambda} - 3e_{\lambda}) + \frac{a_{\lambda}c_{\lambda}}{q_{\lambda}^2}(3-4\nu_{\lambda}) \right] \end{aligned} \right\} \quad (\text{C.36})$$

768 and to:

$$\left. \begin{aligned} \beta_{\lambda}^{(\lambda)} &= \frac{12\mu_{23}^{(\lambda)}}{4(1-\nu_{\lambda})} \left[\frac{a_{\lambda}}{q_{\lambda}^2} - 2q_{\lambda}^4 b_{\lambda} + 3q_{\lambda}^4 e_{\lambda} \right] \\ \beta_{\lambda}^{(\lambda)} &= \frac{12\mu_{23}^{(\lambda)}}{16(1-\nu_{\lambda})^2} \left[-2c_{\lambda}b_{\lambda}q_{\lambda}^4 + 3a_{\lambda}e_{\lambda}q_{\lambda}^4 \right. \\ &\quad \left. + 3a_{\lambda}e_{\lambda}(q_{\lambda}^2 - 1) + 6e_{\lambda}d_{\lambda}q_{\lambda}^4 + \frac{a_{\lambda}c_{\lambda}}{q_{\lambda}^2} \right] \end{aligned} \right\} \quad (\text{C.37})$$

769 We can also get the final expressions of $A_{\mathcal{R}(\lambda)}$, $B_{\mathcal{R}(\lambda)}$, $C_{\mathcal{R}(\lambda)}$.

770 $A_{\mathcal{R}(\lambda)}$ is given by:

$$A_{\mathcal{R}(\lambda)} \stackrel{\text{def}}{=} \underbrace{\{A_1\}}_{\subseteq A_{\mathcal{R}(\lambda)}} + \frac{\{A_2\}}{k_{23}^{\text{eff}}} \quad (\text{C.38})$$

771 where

$$\begin{aligned} \underbrace{\{A_1\}}_{\subseteq A_{\mathcal{R}(\lambda)}} &= \frac{12Z_{12}^{(\lambda)}}{R_{2(\lambda)}^2} \\ &= \frac{12}{4(1-\nu_{\lambda})^2} \left[-6q_{\lambda}^2 a_{\lambda}e_{\lambda} - 6q_{\lambda}^2 e_{\lambda}c_{\lambda} + 12q_{\lambda}^2 d_{\lambda}e_{\lambda} \right. \\ &\quad + 4q_{\lambda}^6 b_{\lambda}e_{\lambda}(3-4\nu_{\lambda}) + 6q_{\lambda}^6 e_{\lambda}^2(4\nu_{\lambda}-3) + 4q_{\lambda}^4 b_{\lambda}c_{\lambda} \\ &\quad \left. - 12q_{\lambda}^4 e_{\lambda}d_{\lambda} + \frac{2a_{\lambda}c_{\lambda}}{q_{\lambda}^2}(3-4\nu_{\lambda}) + 8a_{\lambda}e_{\lambda}(3-6\nu_{\lambda}+4\nu_{\lambda}^2) \right] \end{aligned} \quad (\text{C.39})$$

772 and

$$\begin{aligned}
\{\mathbf{A}_2\}_{\subseteq A_{\mathcal{R}\mathcal{O}}} &= \frac{6}{R_{2\mathcal{A}}} \left(Z_{14}^{\mathcal{A}} + Z_{32}^{\mathcal{A}} + Z_{24}^{\mathcal{A}} + Z_{31}^{\mathcal{A}} \right) \\
&= \frac{12 \mu_{23}^{(\mathbf{A})}}{16 (1 - \nu_{\mathbf{A}})^2} \left[-12 q_{\mathcal{A}}^2 a_{\mathcal{A}} e_{\mathcal{A}} + 12 q_{\mathcal{A}}^6 e_{\mathcal{A}}^2 - 24 q_{\mathcal{A}}^4 d_{\mathcal{A}} e_{\mathcal{A}} \right. \\
&\quad + 16 \nu_{\mathbf{A}} a_{\mathcal{A}} e_{\mathcal{A}} - 8 q_{\mathcal{A}}^6 b_{\mathcal{A}} e_{\mathcal{A}} + 24 q_{\mathcal{A}}^2 d_{\mathcal{A}} e_{\mathcal{A}} \\
&\quad \left. + \frac{4 a_{\mathcal{A}} c_{\mathcal{A}}}{q_{\mathcal{A}}^2} (3 - 4 \nu_{\mathbf{A}}) + 8 q_{\mathcal{A}}^4 b_{\mathcal{A}} c_{\mathcal{A}} - 12 q_{\mathcal{A}}^2 e_{\mathcal{A}} c_{\mathcal{A}} \right]
\end{aligned} \tag{C.40}$$

773 $B_{\mathcal{R}\mathcal{A}}$ is given by:

$$B_{\mathcal{R}\mathcal{A}} \stackrel{\text{def}}{=} \{\mathbf{A}_1\}_{\subseteq B_{\mathcal{R}\mathcal{O}}} + \frac{\{\mathbf{A}_2\}_{\subseteq B_{\mathcal{R}\mathcal{O}}}}{k_{23}^{\text{eff}}} \tag{C.41}$$

774 where

$$\begin{aligned}
\{\mathbf{A}_1\}_{\subseteq B_{\mathcal{R}\mathcal{O}}} &= \frac{2}{R_{2\mathcal{A}}} \left(2Z_{14}^{\mathcal{A}} + 2Z_{32}^{\mathcal{A}} + Z_{24}^{\mathcal{A}} + Z_{31}^{\mathcal{A}} \right) \\
&= \frac{8 \mu_{23}^{(\mathbf{A})}}{16 (1 - \nu_{\mathbf{A}})^2} \left[36 (1 - \nu_{\mathbf{A}}) q_{\mathcal{A}}^6 e_{\mathcal{A}}^2 \right. \\
&\quad \left. + 24 (\nu_{\mathbf{A}} - 1) q_{\mathcal{A}}^6 e_{\mathcal{A}} b_{\mathcal{A}} + 12 (1 - \nu_{\mathbf{A}}) \frac{a_{\mathcal{A}} c_{\mathcal{A}}}{q_{\mathcal{A}}^2} \right]
\end{aligned} \tag{C.42}$$

775 and

$$\begin{aligned}
\{\mathbf{A}_2\}_{\subseteq B_{\mathcal{R}\mathcal{O}}} &= 2 Z_{34}^{\mathcal{A}} \\
&= \frac{8 \mu_{23}^{(\mathbf{A})^2}}{16 (1 - \nu_{\mathbf{A}})^2} \left[18 q_{\mathcal{A}}^2 a_{\mathcal{A}} e_{\mathcal{A}} - 18 q_{\mathcal{A}}^6 e_{\mathcal{A}}^2 + 36 q_{\mathcal{A}}^4 e_{\mathcal{A}} d_{\mathcal{A}} \right. \\
&\quad - 24 a_{\mathcal{A}} e_{\mathcal{A}} + 12 q_{\mathcal{A}}^6 e_{\mathcal{A}} b_{\mathcal{A}} - 36 q_{\mathcal{A}}^2 e_{\mathcal{A}} d_{\mathcal{A}} \\
&\quad \left. + \frac{6 a_{\mathcal{A}} c_{\mathcal{A}}}{q_{\mathcal{A}}^2} - 12 q_{\mathcal{A}}^4 b_{\mathcal{A}} c_{\mathcal{A}} + 18 q_{\mathcal{A}}^2 e_{\mathcal{A}} c_{\mathcal{A}} \right]
\end{aligned} \tag{C.43}$$

776 $C_{R(\lambda)}$ is given by:

$$\begin{aligned}
 C_{R(\lambda)} &= Z_{34}^{(\lambda)} \\
 &= \frac{4 \mu_{23}^{(\lambda)^2}}{16 (1 - \nu_{\lambda})^2} \left[18 q_{(\lambda)}^2 a_{\lambda} e_{\lambda} - 18 q_{(\lambda)}^6 e_{\lambda}^2 + 36 q_{(\lambda)}^4 e_{\lambda} d_{\lambda} \right. \\
 &\quad \left. - 24 a_{\lambda} e_{\lambda} + 12 q_{(\lambda)}^6 e_{\lambda} b_{\lambda} - 36 q_{(\lambda)}^2 e_{\lambda} d_{\lambda} \right. \\
 &\quad \left. + \frac{6 a_{\lambda} c_{\lambda}}{q_{(\lambda)}^2} - 12 q_{(\lambda)}^4 b_{\lambda} c_{\lambda} + 18 q_{(\lambda)}^2 e_{\lambda} c_{\lambda} \right] \tag{C.44}
 \end{aligned}$$

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