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► **To cite this version:**

G rard Gouesbet. Optical forces exerted by on-axis Bessel beams on Rayleigh particles in the framework of generalized Lorenz-Mie theory. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 2021, 260, pp.107471. 10.1016/j.jqsrt.2020.107471 . hal-03109123

HAL Id: hal-03109123

<https://hal.science/hal-03109123v1>

Submitted on 2 Jan 2023

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Optical forces exerted by on-axis Bessel beams on Rayleigh particles in the framework of generalized Lorenz-Mie theory

G erard Gouesbet¹

1.CORIA-UMR 6614- Normandie Universit e
CNRS-Universit e et INSA de Rouen
Campus Universitaire du Madrillet
76800, Saint-Etienne-du Rouvray, France.
Corresponding author: Gouesbet@coria.fr

November 6, 2020

Abstract

The study of optical forces exerted by off-axis Bessel beams on Rayleigh particles in the framework of generalized Lorenz-Mie theory demonstrated the existence of extra-forces named axicon forces, complementing the classical scattering and gradient forces already discussed extensively in the literature. The present paper revisits the same issue but in the more restricted case of on-axis beams. Results obtained in the case of off-axis beams are then confirmed but, computations being carried out in an easier framework, their meaning is displayed more clearly. Beside, we may easily distinguish between dark and non dark beams, a distinction which was not carried out in the off-axis study. It is concluded that the generalized Lorenz-Mie theory for Rayleigh particle does not identify with the usual dipole theory.

Keywords : optical forces ; generalized Lorenz-Mie theory ; Bessel beams ; Rayleigh particles

1 Introduction.

Recent papers have been devoted to the study of longitudinal and transverse optical forces exerted by circularly symmetric Bessel beams illuminating Rayleigh particles in an off-axis configuration, in the framework of generalized Lorenz-Mie theory [1], [2], [3]. For the generalized Lorenz-Mie framework, the reader may refer to [4], [5], [6] and to [7] for a recent review. For a collection of reviews

devoted to optical forces, the reader may refer to [8], [9], [10], [11], [12], [13], [14], [15], [16]. In particular, we are told from the dipole theory that optical forces maybe decomposed into two kinds of forces, namely scattering and gradient forces that we shall name standard forces, e.g. [17], [18]. In contrast, the aforementioned works devoted to off-axis Bessel beams revealed the existence of a new kind of forces that have been named axicon forces, which may also be called non-standard forces. The present paper is devoted to a complementary investigation of the issue in which an on-axis configuration is investigated rather than an off-axis configuration. Because on-axis computations are far more easier than off-axis computations, the present paper provides complementary insights on the origin and behavior of non-standard forces. Furthermore, it provides the opportunity to compare the behaviors of dark and non dark beams. It is for the sake of providing such complementary insights in the framework of easier computations that calculations on the on-axis configuration presented in this paper are obtained by starting again from the beginning rather than taking the on-axis limit of the off-axis results.

The paper is organized as follows. Section 2 deals with prerequisites to the next sections, namely the description of the illuminating Bessel beams considered in the paper and the way to handle Rayleigh particles, complemented with a few mathematical expressions required in the sequel. Section 3 deals with longitudinal forces. Section 4 deals with transverse forces. Section 5 is devoted to a complementary discussion. Section 6 is a conclusion.

Within the context of such an organization of the paper, the main results are as follows. In section 3.3., we exhibit on-axis longitudinal optical forces which are proportional to α^6 (in which α is the size parameter), some of them being genuine scattering forces as expected while others, although they could have been expected to be scattering forces as well, are actually non standard forces called axicon forces. In section 3.4, it is found that the on-axis longitudinal gradient optical forces, which are proportional to α^3 are zero. In section 4.1, it is demonstrated that both scattering and gradient transverse forces are zero, as we may have expected for on-axis axisymmetric beams. The fact that transverse scattering forces are zero is in agreement with results concerning the transverse components of the Poynting vector, which are found to be zero as well, derived in section 4.2, while the fact that transverse gradient forces are zero is in agreement with results concerning the transverse gradients of the electric intensity, which are found to be zero as well, derived in section 4.3.

2 Prerequisites: Bessel beams and Rayleigh particles.

2.1 Bessel beams.

Bessel beams have been introduced by Durnin and co-workers [19], [20]. They possess the appealing properties of being self-healing and non diffracting and, more important in the context of the present paper, they furthermore possess a propagation invariance property, namely the intensity of the beam is constant along the direction of propagation. There exist an infinity of kinds of Bessel beams, in particular depending on the value given to an arbitrary function $g(\alpha_0)$, in which α_0 is the axicon angle (or half-cone angle) of the beam, with different linear and circular polarizations [21], [22], [23], [24]. Let us assume that we are considering beams propagating along the z -direction toward the positive z 's, then all of these Bessel beams generically exhibit the following structure:

$$K_i = k_i(x, y)e^{-ik_z z}, \quad i = x, y, z, \quad K = E \text{ or } H \quad (1)$$

in which $k_z = k \cos \alpha_0$ is the longitudinal wave number.

Among them, we shall specifically consider circularly symmetric Bessel beams, e.g. [22], [23], [24]. By definition of the expression "circularly symmetric", the longitudinal component S_z and the transverse component $S_t = (S_x^2 + S_y^2)^{1/2}$ of the Poynting vector of such beams do not depend on the azimuthal angle φ . In the terminology of [25], [26], they are axisymmetric beams as well in the sense that the longitudinal component S_z of the Poynting vector does not depend on φ (this definition of axisymmetric beams does not say anything concerning the properties of the transverse component S_t).

In the present work, these beams are encoded by their beam shape coefficients (BSCs) $g_{n, TM}^m$ and $g_{n, TE}^m$ (TM for "Transverse Magnetic", TE for "Transverse Electric"), the encoding being carried out either in terms of scalar potentials, e.g. [6] or in terms of vector spherical wave functions [27]. In an off-axis configuration, the BSCs read as [24]:

$$g_{n, TM}^m = -g(\alpha_0)e^{ik_z z_0} [i^{l-m+1} e^{i(l-m+1)\phi_0} A_n^m + i^{l-m-1} e^{i(l-m-1)\phi_0} B_n^m] \quad (2)$$

$$g_{n, TE}^m = ig(\alpha_0)e^{ik_z z_0} [i^{l-m+1} e^{i(l-m+1)\phi_0} A_n^m - i^{l-m-1} e^{i(l-m-1)\phi_0} B_n^m] \quad (3)$$

in which we isolated coefficients A_n^m and B_n^m which are real numbers according to:

$$A_n^m = (-1)^{(m-|m|)/2} \frac{(n-m)!}{(n+|m|)!} J_{l-m+1}(\sigma_0) [\tau_n^m(\cos \alpha_0) + m\pi_n^m(\cos \alpha_0)] \quad (4)$$

$$B_n^m = (-1)^{(m-|m|)/2} \frac{(n-m)!}{(n+|m|)!} J_{l-m-1}(\sigma_0) [\tau_n^m(\cos \alpha_0) - m\pi_n^m(\cos \alpha_0)] \quad (5)$$

in which l is the order of the Bessel beam under consideration, $\sigma_0 = k_t \rho_0$, $k_t = k \sin \alpha_0$ is the transverse wave number, $\rho_0 = (x_0^2 + y_0^2)^{1/2}$, $\phi_0 = \tan^{-1}(y_0/x_0)$, (x_0, y_0, z_0) denotes the location of the beam origin with respect to the origin of the particle coordinate system at which the scatterer is located, and therefore specifies the off-axis location of the beam, and $J_k(\cdot)$ is the k -order Bessel function of the first kind. When $g(\alpha_0) = (1 + \cos \alpha_0)/4$, the beam reduces to a Davis circularly symmetric Bessel beam as discussed in [21], [28]. When $g(\alpha_0) = 1/2$, the beam reduces to another kind of Bessel beam discussed in [29], [30], [31]. Furthermore τ_n^m and π_n^m are generalized Legendre functions defined as:

$$\tau_n^m(\cos \mu) = \frac{dP_n^m(\cos \mu)}{d\mu} \quad (6)$$

$$\pi_n^m(\cos \mu) = \frac{P_n^m(\cos \mu)}{\sin \mu} \quad (7)$$

in which P_n^m are associated Legendre functions here defined according to Hobson's notation [32]:

$$P_n^{|m|}(\cos \mu) = (-1)^{|m|} (\sin \mu)^{|m|} \frac{d^{|m|} P_n(\cos \mu)}{(d \cos \mu)^{|m|}} \quad (8)$$

in which $P_n(\cos \mu)$ are the usual Legendre polynomials. Also, for negative superscripts, we may use [33]:

$$P_n^{-m}(\cos \mu) = (-1)^m \frac{(n-m)!}{(n+m)!} P_n^m(\cos \mu) \quad (9)$$

More specifically, we shall restrict our attention to the case of on-axis configurations for which $\sigma_0 = 0$. Then, because $J_k(0) = \delta_{k0}$ [34], the coefficients

A_n^m and B_n^m are zero excepted for $m = l \pm 1$. Therefore, the non zero BSCs are those for which $m = l \pm 1$. They are readily found to read as:

$$g_{n, TM}^{l+1} = -g(\alpha_0) \exp(ik_z z_0) A_n^{l+1} \quad (10)$$

$$g_{n, TM}^{l-1} = -g(\alpha_0) \exp(ik_z z_0) B_n^{l-1} \quad (11)$$

$$g_{n, TE}^{l+1} = ig(\alpha_0) \exp(ik_z z_0) A_n^{l+1} \quad (12)$$

$$g_{n, TE}^{l-1} = -ig(\alpha_0) \exp(ik_z z_0) B_n^{l-1} \quad (13)$$

in which:

$$A_n^{l+1} = (-1)^{(l+1-|l+1|)/2} \frac{(n-l-1)!}{(n+|l+1|)!} [r_n^{l+1}(\cos \alpha_0) + (l+1)\pi_n^{l+1}(\cos \alpha_0)] \quad (14)$$

$$B_n^{l-1} = (-1)^{(l-1-|l-1|)/2} \frac{(n-l+1)!}{(n+|l-1|)!} [r_n^{l-1}(\cos \alpha_0) - (l-1)\pi_n^{l-1}(\cos \alpha_0)] \quad (15)$$

When dealing with Rayleigh particles, as we do in the present paper, only terms associated with partial waves for which $n = 1$ have to be retained. Furthermore, as indicated by the GLMT framework, the superscript m ranges from $(-n)$ to $(+n)$, that is to say from (-1) to $(+1)$. The fact that for on-axis situations, the only acceptable values of m are $l \pm 1$ then has a few consequences as follows.

For $l = 0$, the values $m = \pm 1$ have to be retained. They both satisfy $m = l \pm 1$ and indeed pertain to the range $[-1, +1]$. For $l = 1$, among the values $m = 2, 0$, only $m = 0$ is to be retained. For $l = -1$, among the values $m = 0, -2$, only $m = 0$ is again to be retained. For $l = +2$, among the values $m = 3, 1$, only $m = 1$ is to be retained. For $l = -2$, among the values $m = -1, -3$, only $m = -1$ is to be retained. For $|l| > 2$, all BSCs have to be discarded since the corresponding values of m lie outside of the range $[-1, +1]$. Therefore, this case $|l| > 2$, being trivial, is not discussed in the present paper.

We now recall that a beam is called dark (on the z -axis) iff (if and only if) the longitudinal component $S_z(\theta = 0)$ of the Poynting vector \mathbf{S} taken on the

axis, i.e. for $\theta = 0$ is zero, e.g.[26], [35]. This component $S_z(\theta = 0)$ reads as, e.g. Eq.(33) in [35]:

$$(S_z)_{\theta=0} = \frac{-E_0 H_0^*}{r^2} \text{Re} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} i c_n^{pw} c_p^{pw*} \Omega_n \Omega_p \quad (16)$$

$$\{[g_{n, TM}^1 \psi_n'(kr) + g_{n, TE}^1 \psi_n(kr)][g_{p, TE}^{1*} \psi_p'(kr) - g_{p, TM}^{1*} \psi_p(kr)]$$

$$- [g_{n, TM}^{-1} \psi_n'(kr) - g_{n, TE}^{-1} \psi_n(kr)][g_{p, TM}^{-1*} \psi_p(kr) + g_{p, TE}^{-1*} \psi_p'(kr)]\},$$

in which $\Omega_n = -n(n+1)/2$ and the coefficients c_n^{pw} are typical of the classical Lorenz-Mie theory expressed in the Bromwich method, e.g. [36], and read as:

$$c_n^{pw} = \frac{1}{ik} (-i)^n \frac{2n+1}{n(n+1)} \quad (17)$$

We then observe that the cases $l = 0$ and ± 2 correspond to non dark beams while $l = \pm 1$ (and $|l| > 2$) are dark beams. We know that such dark beams for $l = \pm 1$ (and $|l| > 2$) are vortex beams, i.e. the wave front has a helical structure and the phase is indeterminate on the beam axis producing a phase singularity and implying the darkness of the beam. *A contrario*, the beams for $l = 0$ and ± 2 are not vortex beams. The fact that beams with $l = 0$ are non vortex beams is known since a long time. The fact that the beams with $l = \pm 2$ are non vortex beams as well is fairly new and has been extensively discussed in [35], together with a discussion of these beams expressed in terms of coordinates instead of being expressed only in terms of BSCs as done in the present paper.

On the one hand, BSCs of dark beams, whether axisymmetric or not, must satisfy, e.g. Eq.(96) in [26]:

$$g_{TM}^{\pm 1} = g_{TE}^{\pm 1} = 0 \quad (18)$$

which is indeed the case for $l = \pm 1$ (and $|l| > 2$). On the other hand, BSCs of axisymmetric non dark beams in which all BSCs $g_{TM}^{\pm 1}$ and $g_{TE}^{\pm 1}$ are different from zero must satisfy, e.g. Eq.(101) in [26]:

$$\frac{g_n}{2} = g_{n, TM}^1 = \frac{1}{K} g_{n, TM}^{-1} = -i\varepsilon g_{n, TE}^1 = \frac{i\varepsilon}{K} g_{n, TE}^{-1} \quad (19)$$

in which $\varepsilon = \pm 1$, and which introduces a set of uni-index BSCs g_n . Using Eqs.10-13, this equation is found to be satisfied for $l = 0$ with:

$$g_n = -2g(\alpha_0) \exp(ik_z z_0) \frac{\tau_n^1(\cos \alpha_0) + \pi_n^1(\cos \alpha_0)}{n(n+1)} \quad (20)$$

$$K = -n(n+1) \frac{\tau_n^{-1}(\cos \alpha_0) + \pi_n^{-1}(\cos \alpha_0)}{\tau_n^1(\cos \alpha_0) + \pi_n^1(\cos \alpha_0)} \quad (21)$$

$$\varepsilon = -1 \quad (22)$$

Eq.19 is not satisfied for the other axisymmetric non dark beams with $l = \pm 2$ because some of the BSCs $g_{TM}^{\pm 1}$ and $g_{TE}^{\pm 1}$ are zero, e.g. the remark between Eqs.(65) and (66) in [26] in association with the fact that, under such circumstances, G_{np}^1 in Eq.(64) and G_{np}^2 in Eq.(65) become identical to 0.

2.2 Rayleigh particles.

The GLMT formalism involves Mie coefficients a_n and b_n which are exactly those of the classical Lorenz-Mie theory. In the case of Rayleigh particles, these Mie coefficients read as, see [18] and [37], pp. 143-144:

$$a_1 = \frac{2i}{3} \frac{m^2 - 1}{m^2 + 2} \alpha^3 + O(i\alpha^5) + \frac{4}{9} \left(\frac{m^2 - 1}{m^2 + 2} \right)^2 \alpha^6 \quad (23)$$

$$b_1 = O(i\alpha^5) \quad (24)$$

in which m is the refractive index (here taken to be real, not to be confused with the superscript m used in the BSCs) with respect to the surrounding medium and α the size parameter $\pi d/\lambda$. The other coefficients a_n and b_n ($n > 1$) involves still higher powers of α . Real parts are then proportional to α^6 and imaginary parts are proportional to α^3 while higher powers are discarded and we therefore retain only:

$$\text{Im}(a_1) = \frac{2}{3} \frac{m^2 - 1}{m^2 + 2} \alpha^3 \quad (25)$$

$$\text{Re}(a_1) = \frac{4}{9} \left(\frac{m^2 - 1}{m^2 + 2} \right)^2 \alpha^6 \quad (26)$$

In other words, only partial waves with $n = 1$ are to be retained in the GLMT formulation. However, due to the existence of coupling terms in the GLMT formulation, as we shall see, this does not mean that the only BSCs which intervene in the computations have $n = 1$. This remark will appear to be a key point to understand some results which will be obtained in the present paper (and which has been obtained in previous papers in the same series).

2.3 Mathematical expressions.

Ricatti-Bessel functions $\psi_n(kr)$ may be expressed in terms of spherical Bessel functions $j_n(kr)$ according to [34]:

$$\psi_n(x) = xj_n(x) \tag{27}$$

Furthermore, we may establish, see again [34]:

$$\psi'_n(x) = (n + 1)j_n(x) - xj_{n+1}(x) \tag{28}$$

and we have [34]:

$$\lim_{x \rightarrow 0} j_n(x) = \frac{x^n}{(2n + 1)!!} \tag{29}$$

We may then establish:

$$\left[\frac{\psi_n(kr)}{kr} \right]_{r=0} = 0 \tag{30}$$

$$\left[\frac{\psi'_n(kr)}{kr} \right]_{r=0} = \frac{2}{3} \delta_{n1} \tag{31}$$

Furthermore, we have [34]:

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x} \tag{32}$$

Also, using:

$$\left(\frac{d^2}{dr^2} + k^2\right)rj_n(kr) = \frac{n(n+1)}{r}j_n(kr) \quad (33)$$

and Eq.27, and again Eq.(11.144) from [34], we establish:

$$[\psi_1''(kr) + \psi_1(kr)]_{kr=0} = \frac{2}{3} \quad (34)$$

3 Longitudinal optical forces.

3.1 Cross-section.

In the GLMT framework, we express the optical force components F_i by using pressure radiation cross-section components $C_{pr,i}$ ($i = x, y, z$). In the present section, we deal with the longitudinal cross-section (along z -direction) reading as:

$$\begin{aligned} C_{pr,z} = & \frac{\lambda^2}{\pi} \sum_{n=1}^{\infty} \sum_{m=-n}^n \left\{ \frac{1}{(n+1)^2} \frac{(n+1+|m|)!}{(n-|m|)!} \right. \\ & \text{Re}[(a_n + a_{n+1}^* - 2a_n a_{n+1}^*)g_{n,TM}^m g_{n+1,TM}^{m*} \\ & + (b_n + b_{n+1}^* - 2b_n b_{n+1}^*)g_{n,TE}^m g_{n+1,TE}^{m*}] \\ & + m \frac{2n+1}{n^2(n+1)^2} \frac{(n+|m|)!}{(n-|m|)!} \\ & \left. \text{Re}[i(2a_n b_n^* - a_n - b_n^*)g_{n,TM}^m g_{n,TE}^{m*}] \right\} \end{aligned} \quad (35)$$

with notations being the ones of [4]. We used the normalization condition $E_0 H_0^*/2 = 1$, which was used in original works, e.g. Eq.(3.106) in [6]. When using this normalization condition, the forces are related to the cross-sections according to $F_i = C_{pr,i}/c$ ([37], p.14). The time-dependence of the wave is $\exp(+i\omega t)$ which is the usual choice in GLMT. Also, note that the denominator $1/\pi$ was misspelled to $1/2\pi$ in [1]. Furthermore, a_n and b_n are the usual Mie coefficients of the usual Lorenz-Mie theory, the star denotes a complex conjugation and λ is the wavelength. In the sequel, we may possibly use a metonymic abuse of language in which cross-sections will be called forces.

For Rayleigh particles, when we retain only the ($n = 1$)-partial waves and the Mie coefficient a_1 , the longitudinal cross-section $C_{pr,z}$ reduces to:

$$\widetilde{C}_{pr,z} = \frac{3\lambda^2}{2\pi} \text{Re}(a_1 G) \quad (36)$$

in which the tilde denotes the restriction to $(n = 1)$ -partial waves and:

$$G = g_{1,TM}^{-1}(g_{2,TM}^{-1*} + ig_{1,TE}^{-1*}) + g_{1,TM}^1(g_{2,TM}^{1*} - ig_{1,TE}^{1*}) + \frac{1}{3}g_{1,TM}^0 g_{2,TM}^{0*} \quad (37)$$

As previously announced, the expression for $C_{pr,z}$ not only contains BSCs with a subscript $n = 1$ associated with $(n = 1)$ partial waves, but it contains as well BSCs with a subscript $n = 2$. The occurrence of such higher-multipole terms is the consequence of the coupling terms $g_{n,TM}^m g_{n+1,TM}^{m*}$ and $g_{n,TE}^m g_{n+1,TE}^{m*}$ in Eq.35. They will generate what can be called axicon forces. For use in the sequel, we now use Eqs.25-26 to separate in Eq.36 the term depending on $\text{Re}(G)$ from the one depending on $\text{Im}(G)$, leading to:

$$\widetilde{C}_{pr,z}^{\text{Re}} = \frac{2\lambda^2}{3\pi} \left(\frac{m^2 - 1}{m^2 + 2} \right)^2 \alpha^6 \text{Re}(G) \quad (38)$$

$$\widetilde{C}_{pr,z}^{\text{Im}} = \frac{-\lambda^2}{\pi} \frac{m^2 - 1}{m^2 + 2} \alpha^3 \text{Im}(G) \quad (39)$$

We recall that, according to the literature, e.g. [18], [38], [39], [40], [41], and many other papers including my three recent papers on longitudinal and transverse optical forces in an off-axis configuration [1], [2], [3], the α^6 -term and the α^3 -term should correspond to the scattering and the gradient forces respectively. This expectation will not be perfectly fulfilled. Now, depending on the order l of the beam, due to the facts that $m = l \pm 1$ and that m pertains to the range $[-1, +1]$, only certain BSCs have to be retained in the expression of G . Here is the resulting list of the expressions of G depending on the value of m .

$$\left. \begin{aligned} G_0 &= g_{1,TM}^{-1}(g_{2,TM}^{-1*} + ig_{1,TE}^{-1*}) + g_{1,TM}^1(g_{2,TM}^{1*} - ig_{1,TE}^{1*}), \quad l = 0 \\ G_{\pm 1} &= \frac{1}{3}g_{1,TM}^0 g_{2,TM}^{0*}, \quad l = \pm 1 \\ G_{+2} &= g_{1,TM}^1(g_{2,TM}^{1*} - ig_{1,TE}^{1*}), \quad l = +2 \\ G_{-2} &= g_{1,TM}^{-1}(g_{2,TM}^{-1*} + ig_{1,TE}^{-1*}), \quad l = -2 \end{aligned} \right\} \quad (40)$$

Next, replacing the BSCs by their values from Eqs.10-13, being careful with the fact that the meaning of BSCs depend on the order, the expressions of G_l are found to be:

$$\left. \begin{aligned}
G_0 &= |g(\alpha_0)|^2 [A_1^1(A_1^1 + A_2^1) + B_1^{-1}(B_1^{-1} + B_2^{-1})] \\
G_{+1} &= \frac{1}{3} |g(\alpha_0)|^2 B_1^0 B_2^0 \\
G_{-1} &= \frac{1}{3} |g(\alpha_0)|^2 A_1^0 A_2^0 \\
G_{+2} &= |g(\alpha_0)|^2 B_1^1 (B_2^1 - B_1^1) \\
G_{-2} &= |g(\alpha_0)|^2 A_1^{-1} (A_2^{-1} - A_1^{-1})
\end{aligned} \right\} \quad (41)$$

3.2 Poynting vector.

To interpret the cross sections $\widetilde{C}_{pr,z}^{\text{Re}}$, i.e. the associated G_i 's, we need to express the longitudinal component of the Poynting restricted to ($n = 1$) partial waves, taken at the location of the particle. The longitudinal component S_z in general reads as, e.g. Eqs.(35)-(37) in [26]:

$$S_z = \frac{-E_0 H_0^*}{2r^2} \text{Re} \sum_{n=1}^{\infty} \sum_{m=-n}^{+n} \sum_{p=1}^{\infty} \sum_{q=-p}^{+p} i c_n^{pw} c_p^{pw*} e^{i(m-q)\varphi} \quad (42)$$

$$(\sin \theta S_{np}^{mq} + \cos \theta C_{np}^{mq})$$

in which (r, θ, φ) are spherical coordinates attached to the Cartesian coordinates (x, y, z) , and :

$$\begin{aligned}
S_{np}^{mq} &= kr [-g_{n, TM}^m g_{p, TM}^{q*} \psi_p(\psi_n + \psi_n'') P_n^{|m|} \tau_p^{|q|} \\
&\quad + g_{n, TE}^m g_{p, TE}^{q*} \psi_n(\psi_p + \psi_p'') P_p^{|q|} \tau_n^{|m|} \\
&\quad + q g_{n, TM}^m g_{p, TE}^{q*} \psi_p'(\psi_n + \psi_n'') P_n^{|m|} \pi_p^{|q|} \\
&\quad + m g_{n, TM}^m g_{p, TE}^{q*} \psi_n'(\psi_p + \psi_p'') P_p^{|q|} \pi_n^{|m|}] \quad (43)
\end{aligned}$$

$$\begin{aligned}
C_{np}^{mq} &= -g_{n, TM}^m g_{p, TM}^{q*} \psi_p \psi_n' (\tau_n^{|m|} \tau_p^{|q|} + m q \pi_n^{|m|} \pi_p^{|q|}) \\
&\quad + g_{n, TM}^m g_{p, TE}^{q*} \psi_n \psi_p' (m \pi_n^{|m|} \tau_p^{|q|} + q \pi_p^{|q|} \tau_n^{|m|}) \\
&\quad - g_{n, TE}^m g_{p, TM}^{q*} \psi_p \psi_n (m \pi_n^{|m|} \tau_p^{|q|} + q \pi_p^{|q|} \tau_n^{|m|}) \\
&\quad + g_{n, TE}^m g_{p, TE}^{q*} \psi_n \psi_p' (m q \pi_n^{|m|} \pi_p^{|q|} + \tau_n^{|m|} \tau_p^{|q|}) \quad (44)
\end{aligned}$$

in which k is the wave number, $\psi_n = \psi_n(kr)$ are Ricatti-Bessel functions, and a prime denotes a derivative with respect to the argument (and a double

prime a second derivative). In Eqs.42-44, and many other equations in the sequel, arguments are omitted for convenience. The location of the particle may be implemented by taking $\theta = 0$ and, thereafter, $r = 0$. Furthermore, we have to retain only ($n = 1$) partial waves in the summations of Eq.42. Denoting again (with a tilde) \widetilde{S}_z the component S_z restricted to ($n = 1$) partial waves, we may deal with the following equation, valid whatever the order of the beam:

$$(\widetilde{S}_z)_{\theta=0} = \frac{-9}{4} \operatorname{Re} \sum_{m=-1}^{+1} \sum_{q=-1}^{+1} i \exp[i(m-q)\varphi] \left(\frac{C^{mq}}{k^2 r^2}\right)_{\theta=0} \quad (45)$$

Let us consider the case $l = 0$. From Eq.44, we may establish $C_{11}^{1-1} = C_{11}^{-11} = 0$, so that Eq.45 leads to:

$$(\widetilde{S}_z)_P = \frac{-9}{4} \operatorname{Re} i \left(\frac{C^{11} + C^{-1-1}}{k^2 r^2}\right)_P \quad (46)$$

in which the subscript P means that \widetilde{S}_z is now evaluated at the particle location ($\theta = 0, r = 0$). Using Eqs.30-31, we then evaluate:

$$\left(\frac{C^{11}}{k^2 r^2}\right)_P = \frac{8}{9} g_{1, TM}^1 g_{1, TE}^{1*} \quad (47)$$

$$\left(\frac{C^{-1-1}}{k^2 r^2}\right)_P = \frac{-8}{9} g_{1, TM}^{-1} g_{1, TE}^{-1*} \quad (48)$$

Then, evaluating the BSCs using Eqs10-15, we obtain, for $l = 0$:

$$(\widetilde{S}_z)_{P, l=0} = |g(\alpha_0)|^2 (C + 1)^2 \quad (49)$$

We similarly obtain:

$$(\widetilde{S}_z)_{P, l=+1} = (\widetilde{S}_z)_{P, l=-1} = 0 \quad (50)$$

$$(\widetilde{S}_z)_{P, l=+2} = (\widetilde{S}_z)_{P, l=-2} = \frac{-1}{2} |g(\alpha_0)|^2 (C - 1)^2 \quad (51)$$

3.3 Scattering forces.

To deal with the scattering forces, it is more convenient to split G_0 as the sum of two terms according to:

$$G_0 = G_{01} + G_{02} \quad (52)$$

$$G_{01} = i(g_{1,TM}^{-1}g_{1,TE}^{-1*} - g_{1,TM}^1g_{1,TE}^{1*}) \quad (53)$$

$$G_{02} = g_{1,TM}^{-1}g_{2,TM}^{-1*} + g_{1,TM}^1g_{2,TM}^{+1*} \quad (54)$$

in which G_{01} only contains ($n = 1$)-BSCs while G_0^2 involves coupling terms between ($n = 1$) and ($n = 2$)-BSCs. Similarly, we express G_{+2} and G_{-2} as the sum of two terms, according to:

$$G_2 = G_{21} + G_{22} \quad (55)$$

$$G_{21} = -ig_{1,TM}^1g_{1,TE}^{1*} \quad (56)$$

$$G_{22} = g_{1,TM}^1g_{2,TM}^{1*} \quad (57)$$

$$G_{-2} = G_{-21} + G_{-22} \quad (58)$$

$$G_{-21} = ig_{1,TM}^{-1}g_{1,TE}^{-1*} \quad (59)$$

$$G_{-22} = g_{1,TM}^{-1}g_{2,TM}^{-1*} \quad (60)$$

Expressing the BSCs of Eqs.53-54 in terms of A_n^m 's and B_n^m 's using Eqs.10-13, we obtain:

$$G_{01} = |g(\alpha_0)|^2 [(B_1^{-1})^2 + (A_1^1)^2] \quad (61)$$

$$G_{02} = |g(\alpha_0)|^2 (B_1^{-1} B_2^{-1} + A_1^1 A_2^1) \quad (62)$$

$$G_{21} = -|g(\alpha_0)|^2 (B_1^1)^2 \quad (63)$$

$$G_{22} = |g(\alpha_0)|^2 B_1^1 B_2^1 \quad (64)$$

$$G_{-21} = -|g(\alpha_0)|^2 (A_1^{-1})^2 \quad (65)$$

$$G_{-22} = |g(\alpha_0)|^2 A_1^{-1} A_2^{-1} \quad (66)$$

allowing one to complement Eq.41. Expressing the A_n^m 's and B_n^m 's using Eqs.14-15, evaluating the involved generalized Legendre functions τ_n^m and π_n^m using Eqs.6-9, and introducing the notations $C = \cos \alpha_0$ and $S = \sin \alpha_0$, we then may explicitly express the G_i 's as:

$$\left. \begin{aligned} G_{01} &= \frac{1}{2} |g(\alpha_0)|^2 (C+1)^2 \\ G_{02} &= \frac{1}{2} |g(\alpha_0)|^2 (C+1)(2C^2 + C - 1) \\ G_{+1} &= |g(\alpha_0)|^2 CS^2 \\ G_{-1} &= |g(\alpha_0)|^2 CS^2 \\ G_{21} &= \frac{-1}{4} |g(\alpha_0)|^2 (C-1)^2 \\ G_{22} &= \frac{1}{4} |g(\alpha_0)|^2 (C-1)(2C^2 - C - 1) \\ G_{-21} &= \frac{-1}{4} |g(\alpha_0)|^2 (C-1)^2 \\ G_{-22} &= \frac{1}{4} |g(\alpha_0)|^2 (C-1)(2C^2 - C - 1) \end{aligned} \right\} \quad (67)$$

from which we may readily express the corresponding optical forces using Eq.38, noting again that all G_i 's are real numbers (the list of such cross sections, proportional to α^6 , is not provided because it is obtained in a quite trivial way using Eq.38 and would add nothing significant to the present paper).

The interpretation of the corresponding optical forces is now carried out using the expressions for the Poynting vector given previously in the previous subsection. We then observe that the optical forces associated with the dark beams $l = \pm 1$ are proportional to CS^2 , see lines 3 and 4 in Eq.67. Comparing with Eq.50 for $(\widetilde{S}_z)_{P,l=+1}$ and $(\widetilde{S}_z)_{P,l=-1}$, we then conclude that these optical forces are not scattering forces. They are axicon forces, i.e. non standard forces. They however disappear if the axicon angle is zero ($S = 0$) then justifying the name of axicon forces.

For non dark beams, we observe (first line of 67 and Eq.49) that G_{01} for $l = 0$ is associated with scattering forces. Comparing lines 5 and 7 of Eq.67 and Eq.51, we similarly observe that G_{21} and G_{-21} for $l = \pm 2$ are associated with scattering forces as well. Therefore optical forces associated with G_{02} , G_{22} and G_{-22} are not scattering forces but non standard forces which may be called axicon forces as well. The behavior of these forces is however different whether $l = 0$ or $l = \pm 2$. For $l = \pm 2$, the axicon forces of G_{22} and G_{-22} vanish if the axicon angle is zero (then $C = 1$), but the scattering forces of G_{21} and G_{-21} vanish as well. Conversely, for $l = 0$, both the axicon force and the scattering force do not vanish when the axicon angle is zero, but they become equal to $2|g(\alpha_0)|^2$, that is to say the axicon force becomes a scattering force in the limit $\alpha_0 = 0$. The same may actually be said for $l = \pm 2$ with however the difference that the axicon forces and the scattering forces both vanish in this limit.

3.4 Gradient forces.

Since all the G_i 's in Eq.41 are real numbers, Eq.39 implies that $\widetilde{C_{pr,z}^{lm}}$ is zero, in all cases, i.e. that gradient forces are zero. This is coherent with Eq.1 which immediately implies that $\mathbf{E} \cdot \mathbf{E}^*$ does not depend on z , i.e. $\nabla_z(\mathbf{E} \cdot \mathbf{E}^*)$ is zero. Let us interestingly note that $[\varepsilon_0 \mathbf{E} \cdot \mathbf{E}^* + \mu_0 \mathbf{H} \cdot \mathbf{H}^*]$ and the Poynting vector $(\mathbf{E} \times \mathbf{H}^*)$ do not depend as well on z , meaning that their z -gradient components are both 0 as well, i.e. the time-averaged energy density and the Poynting vector as well have no gradient in the z -direction.

However, the Rayleigh particle is sensitive only to $(n = 1)$ -partial waves. We must then check whether the above comments are still valid for the restricted tilde-values. Let us then examine the longitudinal gradient $\nabla_z(\widetilde{\mathbf{E} \cdot \mathbf{E}^*})$ in which the tilde again denotes a restriction to $(n = 1)$ -partial waves. When restricted to the $(n = 1)$ -partial waves, the electric components in spherical coordinates read as (e.g. [6], pp.55-56):

$$\widetilde{E}_r = kE_0 c_1^{pw} (\psi_1'' + \psi_1) \sum_{m=-1}^{+1} g_{1,TM}^m P_1^{|m|} e^{im\varphi} \quad (68)$$

$$\widetilde{E}_\theta = \frac{E_0}{r} c_1^{pw} \sum_{m=-1}^{+1} [g_{1,TM}^m \psi_1' \tau_1^{|m|} + m g_{1,TE}^m \psi_1 \pi_1^{|m|}] e^{im\varphi} \quad (69)$$

$$\widetilde{E}_\varphi = \frac{iE_0}{r} c_1^{pw} \sum_{m=-1}^{+1} [m g_{1,TM}^m \psi_1' \pi_1^{|m|} + g_{1,TE}^m \psi_1 \tau_1^{|m|}] e^{im\varphi} \quad (70)$$

in which, once again, arguments are conveniently omitted. For $l = 0$, we only have to retain $m = \pm 1$, leading to:

$$\widetilde{E}_r = kE_0c_1^{pw}(\psi_1'' + \psi_1)(g_{1,TM}^1 e^{i\varphi} + g_{1,TM}^{-1} e^{-i\varphi})P_1^1 \quad (71)$$

$$\widetilde{E}_\theta = \frac{E_0}{r}c_1^{pw}[(g_{1,TM}^{-1}\psi_1'\tau_1^1 - g_{1,TE}^{-1}\psi_1\pi_1^1)e^{-i\varphi} + (g_{1,TM}^1\psi_1'\tau_1^1 + g_{1,TE}^1\psi_1\pi_1^1)e^{i\varphi}] \quad (72)$$

$$\widetilde{E}_\varphi = \frac{iE_0}{r}c_1^{pw}[(-g_{1,TM}^{-1}\psi_1'\pi_1^1 + g_{1,TE}^{-1}\psi_1\tau_1^1)e^{-i\varphi} + (g_{1,TM}^1\psi_1'\pi_1^1 + g_{1,TE}^1\psi_1\tau_1^1)e^{i\varphi}] \quad (73)$$

From these equations, we evaluate the electric intensity $\widetilde{I}_E = \widetilde{\mathbf{E}} \cdot \widetilde{\mathbf{E}}^* = \widetilde{E}_r E_r^* + \widetilde{E}_\theta E_\theta^* + \widetilde{E}_\varphi E_\varphi^*$ which, after a significant amount of calculations, becomes:

$$\widetilde{I}_E = \widetilde{I}_E^{-2} + \widetilde{I}_E^0 + \widetilde{I}_E^2 \quad (74)$$

in which:

$$\begin{aligned} \widetilde{I}_E^{-2} &= \widetilde{I}_E^{2*} = \frac{9|E_0|^2}{4k^2} e^{-2i\varphi} \sin^2 \theta \{ [k^2(\psi_1'' + \psi_1)^2 - \frac{\psi_1'\psi_1'}{r^2}] g_{1,TM}^{-1} g_{1,TM}^{1*} \\ &\quad - \frac{\psi_1\psi_1}{r^2} g_{1,TE}^{-1} g_{1,TE}^{1*} \} \end{aligned} \quad (75)$$

$$\begin{aligned} \widetilde{I}_E^0 &= \frac{9|E_0|^2}{4k^2} e^{0\varphi} \{ [k^2 \sin^2 \theta (\psi_1'' + \psi_1)^2 + \cos^2 \theta \frac{\psi_1'\psi_1'}{r^2}] \\ &\quad \times (g_{1,TM}^{-1} g_{1,TM}^{-1*} + g_{1,TM}^1 g_{1,TM}^{1*}) \\ &\quad + 2 \cos \theta \frac{\psi_1'\psi_1}{r^2} (g_{1,TM}^1 g_{1,TE}^{1*} - g_{1,TM}^{-1} g_{1,TE}^{-1*} + g_{1,TE}^1 g_{1,TM}^{1*} - g_{1,TE}^{-1} g_{1,TM}^{-1*}) \\ &\quad + (\cos^2 \theta + 1) \frac{\psi_1\psi_1}{r^2} (g_{1,TE}^1 g_{1,TE}^{1*} + g_{1,TE}^{-1} g_{1,TE}^{-1*}) \\ &\quad + \frac{\psi_1'\psi_1'}{r^2} (g_{1,TM}^{-1} g_{1,TM}^{-1*} + g_{1,TM}^1 g_{1,TM}^{1*}) \} \end{aligned} \quad (76)$$

We are now interested in the calculation of the derivative of the (restricted) electric intensity with respect to z , taken at $z = 0$, i.e. at the place where the particle is located. This requires in a first step to express $\cos \theta$,

$\sin \theta$ and the radial coordinate r in terms of Cartesian coordinates, a trivial task indeed. Furthermore, in evaluating these derivatives, and other derivatives in subsequent subsections, derivatives with respect to φ may be omitted. In a few words, this comes from the fact that z -dependence is only encoded in the spherical coordinates r and θ , but not in φ . The suspicious reader may have a direct check by using the equation $\partial f(r, \theta) \exp(\pm ik\varphi) / \partial z = \exp(\pm ik\varphi) [\partial f(r, \theta) / \partial z] + f(r, \theta) [\partial \exp(\pm ik\varphi) / \partial z]$, leaving the derivatives evaluated below unchanged since $f(r, \theta)$ is proportional to $\sin \theta$ which is 0 at the particle location ($\theta = 0$).

Then, we consider $\widetilde{I_E^{-2}}$ and $\widetilde{I_E^2}$, and evaluate:

$$\left\{ \frac{\partial}{\partial z} \sin^2 \theta \left[k^2 (\psi_1'' + \psi_1)^2 - \frac{\psi_1' \psi_1'}{r^2} \right] \right\}_{z=0} = 0 \quad (77)$$

$$\left(\frac{\partial}{\partial z} \sin^2 \theta \frac{\psi_1 \psi_1}{r^2} \right)_{z=0} = 0 \quad (78)$$

in which we expressed the terms dependent on r and θ in terms of x, y, z and recalled that, using Eqs.27 and 32:

$$\psi_1(kr) = kr \left[\frac{\sin(kr)}{(kr)^2} - \frac{\cos(kr)}{kr} \right] \quad (79)$$

Hence:

$$\left(\frac{\partial \widetilde{I_E^{-2}}}{\partial z} \right)_{z=0} = \left(\frac{\partial \widetilde{I_E^2}}{\partial z} \right)_{z=0} = 0 \quad (80)$$

For $(\partial \widetilde{I_E^0} / \partial z)_{z=0}$, all derivatives with respect to z , taken at $z = 0$, of the different terms involved in Eq.76 are found to be 0, excepted one of them, namely:

$$\Omega = \left(\frac{\partial}{\partial z} \cos \theta \frac{\psi_1' \psi_1'}{r^2} \right)_{z=0} \neq 0 \quad (81)$$

leading to:

$$\left(\frac{\partial \widetilde{I_E^0}}{\partial z} \right)_{z=0} = \frac{9}{2} \frac{|E_0|^2}{k^2} \gamma \Omega \quad (82)$$

in which:

$$\begin{aligned}\gamma &= g_{1,TM}^1 g_{1,TE}^{1*} - g_{1,TM}^{-1} g_{1,TE}^{-1*} + g_{1,TE}^1 g_{1,TM}^{1*} - g_{1,TE}^{-1} g_{1,TM}^{-1*} \quad (83) \\ &= 2 \operatorname{Re}(g_{1,TM}^1 g_{1,TE}^{1*} - g_{1,TM}^{-1} g_{1,TE}^{-1*})\end{aligned}$$

We then evaluate γ in terms of A_n^m 's and B_n^m 's using Eqs10-13, and we obtain, remembering that A_n^m 's and B_n^m 's are real quantities:

$$\gamma = 2 |g(\alpha_0)|^2 \operatorname{Re} i[(A_1^1)^2 + (B_1^{-1})^2] = 0 \quad (84)$$

so that the derivative of the electric intensity taken at the location of the particle is indeed zero, coherently with the fact that longitudinal gradient forces have been found to be 0. Since, gradient forces are expressed in terms of the gradient of the electric intensity, we do not need to examine the gradient of the magnetic intensity (which would be similarly found to be zero) nor the gradient of the Poynting vector. This result for on-axis gradient forces is however in contrast with we have found for the off-axis case in which the gradient for the restricted electric intensity was not zero but contains what has been called axicon terms [2].

We proceed similar for the other orders l still to be investigated. For $l = \pm 1$, when only the value $m = 0$ is to be retained in Eqs.68-70, we obtain:

$$\begin{aligned}\widetilde{I}_E &= \frac{9 |E_0|^2}{4k^2} \{ [k^2 \cos^2 \theta (\psi_1'' + \psi_1)^2 + \sin^2 \theta \frac{\psi_1' \psi_1'}{r^2}] g_{1,TM}^0 g_{1,TM}^{0*} \quad (85) \\ &\quad + \sin^2 \theta \frac{\psi_1 \psi_1}{r^2} g_{1,TE}^0 g_{1,TE}^{0*} \}\end{aligned}$$

whose derivative with respect to z taken at the location of the particle is found to be 0 (without having to specify the values of the BSCs).

For $l = +2$, when only the value $m = 1$ is to be retained in Eqs.68-70, we obtain:

$$\begin{aligned}\widetilde{I}_E &= \frac{9 |E_0|^2}{4k^2} \{ [k^2 \sin^2 \theta (\psi_1'' + \psi_1)^2 + \cos^2 \theta \frac{\psi_1' \psi_1'}{r^2}] g_{1,TM}^1 g_{1,TM}^{1*} \quad (86) \\ &\quad + 2 \cos \theta \frac{\psi_1' \psi_1}{r^2} (g_{1,TM}^1 g_{1,TE}^{1*} + g_{1,TE}^1 g_{1,TM}^{1*}) \\ &\quad + (\cos^2 \theta + 1) \frac{\psi_1 \psi_1}{r^2} g_{1,TE}^1 g_{1,TE}^{1*} + \frac{\psi_1' \psi_1'}{r^2} g_{1,TM}^1 g_{1,TM}^{1*} \}\end{aligned}$$

Among the various derivatives with respect to z , taken at the particle location, involved in Eq.86, only the term Ω of Eq.81 is non zero, so that we obtain:

$$\left(\frac{\partial \widetilde{I}_E}{\partial z}\right)_{z=0} = \frac{9|E_0|^2}{2k^2} \Omega \operatorname{Re}(g_{1,TM}^1 g_{1,TE}^{1*}) = 0 \quad (87)$$

which is zero because $g_{1,TM}^1 g_{1,TE}^{1*}$ is found to be imaginary.

For $l = -2$, when only the value $m = -1$ is to be retained in Eqs.68-70, we obtain:

$$\begin{aligned} \widetilde{I}_E &= \frac{9|E_0|^2}{4k^2} \{ [k^2 \sin^2 \theta (\psi_1'' + \psi_1)^2 + \cos^2 \theta \frac{\psi_1' \psi_1'}{r^2}] g_{1,TM}^{-1} g_{1,TE}^{-1*} \\ &\quad - 2 \cos \theta \frac{\psi_1' \psi_1}{r^2} (g_{1,TM}^{-1} g_{1,TE}^{-1*} + g_{1,TE}^{-1} g_{1,TM}^{-1*}) \\ &\quad + (\cos^2 \theta + 1) \frac{\psi_1 \psi_1}{r^2} g_{1,TE}^{-1} g_{1,TE}^{-1*} + \frac{\psi_1' \psi_1'}{r^2} g_{1,TM}^{-1} g_{1,TM}^{-1*} \} \end{aligned} \quad (88)$$

which, similarly as for $l = 2$ leads to:

$$\left(\frac{\partial \widetilde{I}_E}{\partial z}\right)_{z=0} = \frac{-9|E_0|^2}{2k^2} \Omega \operatorname{Re}(g_{1,TM}^{-1} g_{1,TE}^{-1*}) = 0 \quad (89)$$

Therefore, in all cases, we find that the derivative of \widetilde{I}_E with respect to z , taken at the particle location is zero, which was already the case for the derivative of I_E . This is in deep contrast with the case of off-axis configurations where, although $\partial I_E / \partial z$ for $z = 0$ was still 0, it was not the case for $\partial \widetilde{I}_E / \partial z$ which was not 0. These non zero terms were called axicon terms [2].

4 Transverse optical forces.

4.1 Cross-sections.

In the present section, we deal with the transverse cross-sections (along x - and y -directions) reading as, e.g. Eqs.(3.181) and (3.185) in [6], with an "obvious" typo in Eq. (3.178) in which the last term should involve $g_{n,TE}^p g_{m,TM}^{p+1*}$ instead of $g_{n,TE}^p g_{m,TE}^{p+1*}$:

$$\begin{aligned}
C_{pr,x} &= \frac{\lambda^2}{2\pi} \sum_{p=1}^{\infty} \sum_{n=p}^{\infty} \sum_{m=p-1 \neq 0}^{\infty} \frac{(n+p)!}{(n-p)!} \\
&\times [\text{Re}(S_{mn}^{p-1} + S_{nm}^{-p} - 2U_{mn}^{p-1} - 2U_{nm}^{-p}) (\frac{\delta_{m,n+1}}{m^2} - \frac{\delta_{n,m+1}}{n^2}) \\
&+ \frac{2n+1}{n^2(n+1)^2} \delta_{nm} \text{Re}(T_{mn}^{p-1} - T_{nm}^{-p} - 2V_{mn}^{p-1} + 2V_{nm}^{-p})]
\end{aligned} \tag{90}$$

in which:

$$S_{nm}^p = (a_n + a_m^*) g_{n, TM}^p g_{m, TM}^{p+1*} + (b_n + b_m^*) g_{n, TE}^p g_{m, TE}^{p+1*} \tag{91}$$

$$T_{nm}^p = -i(a_n + b_m^*) g_{n, TM}^p g_{m, TE}^{p+1*} + i(b_n + a_m^*) g_{n, TE}^p g_{m, TM}^{p+1*} \tag{92}$$

$$U_{nm}^p = a_n a_m^* g_{n, TM}^p g_{m, TM}^{p+1*} + b_n b_m^* g_{n, TE}^p g_{m, TE}^{p+1*} \tag{93}$$

$$V_{nm}^p = i b_n a_m^* g_{n, TE}^p g_{m, TM}^{p+1*} - i a_n b_m^* g_{n, TM}^p g_{m, TE}^{p+1*} \tag{94}$$

The y -component $C_{pr,y}$ is obtained from the x -component by changing Re to Im . To achieve the calculations, we recommend to split $C_{pr,x}$ into three terms according to $C_{pr,x} = C_{pr,x}^\alpha + C_{pr,x}^\beta + C_{pr,x}^\gamma$ in which:

$$\begin{aligned}
C_{pr,x}^\alpha &= \frac{\lambda^2}{2\pi} \sum_{p=1}^{\infty} \sum_{n=p}^{\infty} \sum_{m=p-1 \neq 0}^{\infty} \frac{(n+p)!}{(n-p)!} \text{Re}(S_{mn}^{p-1} + S_{nm}^{-p} - 2U_{mn}^{p-1} - 2U_{nm}^{-p}) \frac{\delta_{m,n+1}}{m^2} \\
\end{aligned} \tag{95}$$

$$\begin{aligned}
C_{pr,x}^\beta &= \frac{-\lambda^2}{2\pi} \sum_{p=1}^{\infty} \sum_{n=p}^{\infty} \sum_{m=p-1 \neq 0}^{\infty} \frac{(n+p)!}{(n-p)!} \text{Re}(S_{mn}^{p-1} + S_{nm}^{-p} - 2U_{mn}^{p-1} - 2U_{nm}^{-p}) \frac{\delta_{n,m+1}}{n^2} \\
\end{aligned} \tag{96}$$

$$\begin{aligned}
C_{pr,x}^\gamma &= \frac{\lambda^2}{2\pi} \sum_{p=1}^{\infty} \sum_{n=p}^{\infty} \sum_{m=p-1 \neq 0}^{\infty} \frac{(n+p)!}{(n-p)!} \frac{2n+1}{n^2(n+1)^2} \delta_{nm} \text{Re}(T_{mn}^{p-1} - T_{nm}^{-p} - 2V_{mn}^{p-1} + 2V_{nm}^{-p}) \\
\end{aligned} \tag{97}$$

The corresponding reduced partial pressure radiation cross sections along the x -direction, retaining only the ($n = 1$)-partial waves, i.e. the Mie coefficient a_1 , then read as.

$$\widetilde{C}_{pr,x}^{\alpha} = \frac{\lambda^2}{4\pi} \text{Re}[a_1 g_{2,TM}^{0*} (g_{1,TM}^1 + g_{1,TM}^{-1})] \quad (98)$$

$$\begin{aligned} \widetilde{C}_{pr,x}^{\beta} = & \frac{-3\lambda^2}{4\pi} \text{Re}[a_1 (g_{1,TM}^0 g_{2,TM}^{1*} + g_{1,TM}^0 g_{2,TM}^{-1*} \\ & + 4g_{1,TM}^1 g_{2,TM}^{2*} + 4g_{1,TM}^{-1} g_{2,TM}^{-2*})] \end{aligned} \quad (99)$$

$$\begin{aligned} \widetilde{C}_{pr,x}^{\gamma} = & \frac{3\lambda^2}{4\pi} \text{Re}\{ia_1 [g_{1,TM}^0 (g_{1,TE}^{-1*} - g_{1,TE}^{1*}) \\ & + g_{1,TE}^{0*} (g_{1,TM}^{-1} - g_{1,TM}^1)]\} \end{aligned} \quad (100)$$

Assembling everything, we then obtain:

$$\widetilde{C}_{pr,x} = \frac{\lambda^2}{4\pi} \text{Re}(a_1 H) \quad (101)$$

in which:

$$\begin{aligned} H = & g_{2,TM}^{0*} (g_{1,TM}^1 + g_{1,TM}^{-1}) - 3g_{1,TM}^0 (g_{2,TM}^{1*} + g_{2,TM}^{-1*}) \\ & - 12(g_{1,TM}^1 g_{2,TM}^{2*} + g_{1,TM}^{-1} g_{2,TM}^{-2*}) \\ & + 3i[g_{1,TM}^0 (g_{1,TE}^{-1*} - g_{1,TE}^{1*}) + g_{1,TE}^{0*} (g_{1,TM}^{-1} - g_{1,TM}^1)] \end{aligned} \quad (102)$$

Similarly, the reduced y -component reads as:

$$\widetilde{C}_{pr,y} = \frac{\lambda^2}{4\pi} \text{Im}(a_1 H) \quad (103)$$

Using Eqs.25-26, we may rewrite $\widetilde{C}_{pr,x}$ as the sum of two terms, one $\widetilde{C}_{pr,x}^{\text{Re}}$ proportional to α^6 and the other one $\widetilde{C}_{pr,x}^{\text{Im}}$ proportional to α^3 , according to:

$$\widetilde{C}_{pr,x}^{\text{Re}} = \frac{\lambda^2}{9\pi} \left(\frac{m^2 - 1}{m^2 + 2} \right)^2 \alpha^6 \text{Re}(H) \quad (104)$$

$$\widetilde{C}_{pr,x}^{\text{Im}} = \frac{-\lambda^2}{6\pi} \frac{m^2 - 1}{m^2 + 2} \alpha^3 \text{Im}(H) \quad (105)$$

Similarly, $\widetilde{C}_{pr,y}$ of Eq.103 may be rewritten as the sum of two terms reading as:

$$\widetilde{C}_{pr,y}^{\text{Re}} = \frac{\lambda^2}{6\pi} \frac{m^2 - 1}{m^2 + 2} \alpha^3 \text{Re}(H) \quad (106)$$

$$\widetilde{C}_{pr,y}^{\text{Im}} = \frac{\lambda^2}{9\pi} \left(\frac{m^2 - 1}{m^2 + 2} \right)^2 \alpha^6 \text{Im}(H) \quad (107)$$

We now recall (e.g. previous section) that the α^6 -term and the α^3 -term should correspond to the scattering and the gradient forces respectively.

For $l = 0$, we only retain $m = \pm 1$ meaning that all BSCs of the kind g_n^0 and of the kind g_n^2 have to be discarded. Therefore, we have $(H)_{l=0} = 0$ and :

$$\widetilde{C}_{pr,x}^{\text{Re}} = \widetilde{C}_{pr,x}^{\text{Im}} = \widetilde{C}_{pr,y}^{\text{Re}} = \widetilde{C}_{pr,y}^{\text{Im}} = 0 \quad (108)$$

meaning that transverse scattering and gradient forces are zero. With similar arguments, it is found as well that $H = 0$ and therefore that Eq.108 still holds for $l = \pm 1$ and $l = \pm 2$. Therefore we conclude that, whatever the order l , scattering and gradient forces are zero which is actually what we should have expected for axisymmetric (actually circularly symmetric) beams, whether it is dark or not dark.

4.2 Poynting vector.

The transverse components of the Poynting vector are found to read as [25], [26]:

$$\begin{aligned} S_x &= \frac{1}{2} E_0 H_0^* \text{Re} \sum_{n=1}^{\infty} \sum_{m=-n}^{+n} \sum_{p=1}^{\infty} \sum_{q=-p}^{+p} c_n^{pw} c_p^{pw*} e^{i(m-q)\varphi} \\ &\times \left[\frac{k \sin \varphi}{r} (\psi_n'' + \psi_n) A_{nmpq} + \frac{ik \cos \theta \cos \varphi}{r} (\psi_n'' + \psi_n) B_{nmpq} + \frac{i \sin \theta \cos \varphi}{r^2} C_{nmpq} \right] \end{aligned} \quad (109)$$

$$\begin{aligned}
S_y &= \frac{1}{2} E_0 H_0^* \operatorname{Re} \sum_{n=1}^{\infty} \sum_{m=-n}^{+n} \sum_{p=1}^{\infty} \sum_{q=-p}^{+p} c_n^{pw} c_p^{pw*} e^{i(m-q)\varphi} \\
&\times \left[-\frac{k \cos \varphi}{r} (\psi_n'' + \psi_n) A_{nmpq} + \frac{ik \cos \theta \sin \varphi}{r} (\psi_n'' + \psi_n) B_{nmpq} + \frac{i \sin \theta \sin \varphi}{r^2} C_{nmpq} \right]
\end{aligned} \tag{110}$$

in which :

$$A_{nmpq} = [\alpha_{nmpq} \psi_p' \tau_p^{|q|} + q \beta_{nmpq} \psi_p \pi_p^{|q|}] P_n^{|m|} \tag{111}$$

$$B_{nmpq} = [-q \alpha_{nmpq} \psi_p' \pi_p^{|q|} - \beta_{nmpq} \psi_p \tau_p^{|q|}] P_n^{|m|} \tag{112}$$

$$\begin{aligned}
C_{nmpq} &= \alpha_{nmpq} (m \psi_n' \psi_p' \pi_n^{|m|} \tau_p^{|q|} + q \psi_n \psi_p \tau_n^{|m|} \pi_p^{|q|}) \\
&+ \beta_{nmpq} (mq \psi_n' \psi_p \pi_n^{|m|} \pi_p^{|q|} - \psi_n \psi_p' \tau_n^{|m|} \tau_p^{|q|})
\end{aligned} \tag{113}$$

in which :

$$\alpha_{nmpq} = g_{p, TM}^{q*} g_{n, TE}^m - g_{n, TM}^m g_{p, TE}^{q*} \tag{114}$$

$$\beta_{nmpq} = g_{n, TM}^m g_{p, TM}^{q*} + g_{p, TE}^{q*} g_{n, TE}^m \tag{115}$$

We now evaluate the reduced Poynting vector components, at the location of the particle (which is located at the origin of the coordinates), starting from Eqs.109-110. The location of the particle is going to be implemented using two steps (i) considering a location on the axis $\theta = 0$ and (ii) considering the origin of the coordinates on the axis $z = r = 0$. The x -component of the reduced Poynting vector on the axis then reads as:

$$[\widetilde{S}_x]_{\theta=0} = \frac{9}{4} \frac{\psi_1'' + \psi_1}{kr} \operatorname{Re} \sum_{m=-1}^{+1} \sum_{q=-1}^{+1} e^{i(m-q)\varphi} [\sin \varphi A_{1m1q} + i \cos \varphi B_{1m1q}] \tag{116}$$

in which we have used $c_1^{pw} c_1^{pw*} = 9/(4k^2)$. This may be expanded as:

$$[\widetilde{S}_x]_{\theta=0} = \frac{9}{8} \frac{\psi_1'' + \psi_1}{kr} \operatorname{Re} i \sum_{j=-3}^{+3} a^j e^{ij\varphi} \quad (117)$$

in which:

$$a^{-3} = A_{1-111} + B_{1-111} \quad (118)$$

$$a^{-2} = A_{1-110} + B_{1-110} + A_{1011} + B_{1011} \quad (119)$$

$$a^{-1} = A_{1-11-1} + B_{1-11-1} - A_{1-111} + B_{1-111} + A_{1010} + B_{1010} + A_{1111} + B_{1111} \quad (120)$$

$$a^0 = -A_{1-110} + B_{1-110} + A_{101-1} + B_{101-1} - A_{1011} + B_{1011} + A_{1110} + B_{1110} \quad (121)$$

$$a^1 = -A_{1-11-1} + B_{1-11-1} - A_{1010} + B_{1010} + A_{111-1} + B_{111-1} - A_{1111} + B_{1111} \quad (122)$$

$$a^2 = -A_{101-1} + B_{101-1} - A_{1110} + B_{1110} \quad (123)$$

$$a^3 = -A_{111-1} + B_{111-1} \quad (124)$$

Coefficients A_{nmpq} and B_{nmpq} involved in Eqs.118-124 are afterward evaluated using Eqs.111-112 and expressing generalized Legendre functions and associated Legendre functions using Eqs.6-9 leading to:

$$a^{\pm 3} = a^{\pm 2} = a^{\pm 1} = 0 \quad (125)$$

and to:

$$a^0 = 2[\psi_1'(\alpha_{1011} - \alpha_{101-1}) + \psi_1(\beta_{1011} + \beta_{101-1})] \quad (126)$$

The reduced x -component of the Poynting vector on the axis then becomes:

$$[\widetilde{S}_x]_{\theta=0} = \frac{9}{4} \frac{\psi_1'' + \psi_1}{kr} [\psi_1' \operatorname{Re} i(\alpha_{1011} - \alpha_{101-1}) + \psi_1 \operatorname{Re} i(\beta_{1011} + \beta_{101-1})] \quad (127)$$

in which the coefficients α_{nmpq} and β_{nmpq} do not depend on the radial coordinate (see Eqs.114-115). When evaluating the component S_x of the Poynting vector, the terms $(\psi_1'' + \psi_1)$ on the one hand and $\psi_1'/(kr)$, $\psi_1/(kr)$ on the other hand came from different components of the electric and magnetic fields as we can see by using $S_x = \operatorname{Re}(E_y H_z^* - E_z H_y^*)/2$ and referring to Eqs.(3.39)-(3-50) in [6]. We may therefore consider separately the limits of these terms when $z = r = 0$. We may then use Eqs.30, 31, and 34 to convert Eq.127 to:

$$[\widetilde{S}_x]_{\theta=r=0} = \operatorname{Re} i(\alpha_{1011} - \alpha_{101-1}) \quad (128)$$

Using Eq.114, $[\widetilde{S}_x]_{\theta=r=0}$ may be eventually expressed in terms of BSCs as:

$$[\widetilde{S}_x]_{\theta=r=0} = \operatorname{Re} i[g_{1, TM}^0(g_{1, TE}^{-1*} - g_{1, TE}^{1*}) + g_{1, TE}^0(g_{1, TM}^{1*} - g_{1, TM}^{-1*})] \quad (129)$$

Let us then start with $l = 0$ for which $m = \pm 1$, so that the BSCs g_n^0 have to be discarded. Then, we have $[\widetilde{S}_x]_{\theta=r=0} = 0$. The same result is obtained for $l = \pm 1$ and $l = \pm 2$. The component $[\widetilde{S}_y]_{\theta=r=0}$ is treated similarly with the same null result. This is coherent with the fact that the transverse scattering forces have been found to be 0. It is however in contrast with the case of an off-axis configuration in which non-standard axicon scattering forces were complementing the usual scattering forces [3].

4.3 On transverse gradient forces.

We now return to Eqs.75-76 and evaluate the derivatives of various x - and y -dependent quantities involved in these expressions, taken at the location of the particle. In doing so, we do not take into account the φ -dependent terms occurring in $I_E^{\pm 2}$. Again, being however said in a slightly different way, the reason why is that the terms to be derived are of the form $\sin \theta f(r, \theta)g(\varphi)$ whose derivative may be written as $[\partial \sin \theta f(r, \theta)/\partial x]g(\varphi) + \sin \theta f(r, \theta)\partial g(\varphi)/\partial x$. Having

to take the derivative at the location of the particle, the second term of the summation $\sin \theta f(r, \theta) \partial g(\varphi) / \partial x$ is always zero because the particle is located at $\theta = 0$, so that we are left with the derivative of the first term. Another point of view might be to say that the derivative with respect to x is taken along the x -axis, i.e. for a fixed value of $\varphi = 0^\circ$ (and similarly for the derivative with respect to y taken along the y -axis, i.e. for $\varphi = 90^\circ$). We obtain:

$$\left\{ \frac{\partial}{\partial x} \sin^2 \theta \left[k^2 (\psi_1'' + \psi_1)^2 - \frac{\psi_1' \psi_1'}{r^2} \right] \right\}_{x=y=0} = \left\{ \frac{\partial}{\partial y} \sin^2 \theta \left[k^2 (\psi_1'' + \psi_1)^2 - \frac{\psi_1' \psi_1'}{r^2} \right] \right\}_{x=y=0} = 0 \quad (130)$$

$$\left(\frac{\partial}{\partial x} \sin^2 \theta \frac{\psi_1 \psi_1}{r^2} \right)_{x=y=0} = \left(\frac{\partial}{\partial y} \sin^2 \theta \frac{\psi_1 \psi_1}{r^2} \right)_{x=y=0} = 0 \quad (131)$$

$$\left\{ \frac{\partial}{\partial x} \left[k^2 \sin^2 \theta (\psi_1'' + \psi_1)^2 + \cos^2 \theta \frac{\psi_1' \psi_1'}{r^2} \right] \right\}_{x=y=0} = \left\{ \frac{\partial}{\partial y} \left[k^2 \sin^2 \theta (\psi_1'' + \psi_1)^2 + \cos^2 \theta \frac{\psi_1' \psi_1'}{r^2} \right] \right\}_{x=y=0} = 0 \quad (132)$$

$$\left(\frac{\partial}{\partial x} \cos \theta \frac{\psi_1' \psi_1}{r^2} \right)_{x=y=0} = \left(\frac{\partial}{\partial y} \cos \theta \frac{\psi_1' \psi_1}{r^2} \right)_{x=y=0} = 0 \quad (133)$$

$$\left[\frac{\partial}{\partial x} (\cos^2 \theta + 1) \frac{\psi_1 \psi_1}{r^2} \right]_{x=y=0} = \left[\frac{\partial}{\partial y} (\cos^2 \theta + 1) \frac{\psi_1 \psi_1}{r^2} \right]_{x=y=0} = 0 \quad (134)$$

$$\left(\frac{\partial}{\partial x} \frac{\psi_1' \psi_1'}{r^2} \right)_{x=y=0} = \left(\frac{\partial}{\partial y} \frac{\psi_1' \psi_1'}{r^2} \right)_{x=y=0} = 0 \quad (135)$$

This implies $\partial \widetilde{I}_E / \partial x = \partial \widetilde{I}_E / \partial y = 0$ (as we should have expected for the symmetric beams under study) and therefore that the gradient forces along the x - and y -directions should be expected to be zero. This is indeed what we found in subsection 4.1. This is again in deep contrast with the case of an off-axis configuration in which non standard axicon gradient forces were complementing the usual gradient forces.

5 Complementary discussion.

The results obtained in the present paper and in previous papers on the same issue ask the question to know what is the relationship between the Rayleigh limit of the GLMT and the classical dipole theory of forces. In a first attempt to state about this question, we might say that both approaches do not exactly identify, insofar as the axicon forces occurring in the former are not expressed in terms of scattering or gradient forces expressed in the latter. However, recent extensive comparisons between these approaches have been numerically achieved and it has been found that they agree up to 1,000 decimal places for off-axis longitudinal forces in the case of illuminating circularly symmetric Bessel beams [42] and up to 3,000 decimal places in the case of off-axis transverse forces [43]. From these results, we may put forward a very strong conjecture that the Rayleigh limit of the GLMT actually identifies with the dipole theory of forces, but that some kind of dictionary between both approaches is required to examine how the vocabulary "axicon forces" in the GLMT approach matches the vocabulary "scattering/gradient" forces in the dipole theory approach. This conjecture, and the construction of a dictionary, are currently under examination from a formal point of view.

6 Conclusion.

The present paper is in the continuation of a series of papers [1], [2], [3] which, when studying optical forces exerted by Bessel beams on Rayleigh particles in the generalized Lorenz-Mie theory framework in an off-axis configuration, exhibited the occurrence of non standard forces, named axicon forces, which were complementing the usual scattering and gradient optical forces put forward by Ashkin. In order to gain a better understanding of the situation, the present paper has been devoted to a similar study however carried out in the case of an on-axis configuration. On-axis calculations being much simpler than off-axis calculations, it was intended that it would be possible to gain a better understanding of the origin of the non standard forces. In an on-axis configuration, non standard transverse forces do not occur, a fact which may be viewed as the result of the symmetries displayed by the beam under study (they are both axisymmetric and circularly symmetry). However, non standard longitudinal forces do occur. Their origin is due to the fact that the Rayleigh limit of the generalized Lorenz-Mie theory is more general than the Rayleigh dipole theory, because it exhibits coupling terms between BSCs of order $n = 1$ and BSCs of order $n = 2$, i.e. with higher-multipole BSCs.

References

- [1] G. Gouesbet. Gradient, scattering and other kinds of longitudinal optical forces exerted by off-axis Bessel beams in the Rayleigh regime in the framework of generalized Lorenz-Mie theory. *Journal of Quantitative and Spectroscopy Transfer*, 246:Article 106913, 7 pages, 2020.
- [2] G. Gouesbet and L.A. Ambrosio. Axicon terms associated with gradient optical forces in generalized Lorenz-Mie theory. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 257:Paper 107260, 8 pages, 2020.
- [3] G. Gouesbet and L.A. Ambrosio. Axicon optical forces and other kinds of transverse optical forces exerted by off-axis Bessel beams in the Rayleigh regime in the framework of generalized Lorenz-Mie theory. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 258:Paper 107356, 9 pages, 2021.
- [4] G. Gouesbet, B. Maheu, and G. Gréhan. Light scattering from a sphere arbitrarily located in a Gaussian beam, using a Bromwich formulation. *Journal of the Optical Society of America A*, 5,9:1427–1443, 1988.
- [5] G. Gouesbet, G. Gréhan, and B. Maheu. *Combustion measurements, edited by N. Chigier*, chapter : Generalized Lorenz-Mie theory and applications to optical sizing, pages 339–384. Hemisphere Publishing Corporation, New-York, USA, 1991.
- [6] G. Gouesbet and G. Gréhan. *Generalized Lorenz-Mie theories, second edition*. Springer International Publishing AG, 2017.
- [7] G. Gouesbet. Van de Hulst Essay: A review on generalized Lorenz-Mie theories with wow stories and epistemological discussion. *Journal of Quantitative Spectroscopy and Radiative Transfer*, Paper 107117, 35 pages, 2020.
- [8] E.M. Furst. Interactions, structure, and microscopic response : complex fluid rheology using laser tweezers. *Soft Materials*, 1,2:167–185, 2003.
- [9] V.A. Soifer, V.V. Kotlyar, and S.N. Khonina. Optical microparticle manipulation : Advances and new possibilities created by diffractive optics. *Physics of Particles and Nuclei*, 35,6:733–766, 2004.
- [10] K.C. Neuman and S.M. Block. Optical trapping. *Review of Scientific Instruments*, 75,9:2787–2809, 2004.
- [11] T.A. Nieminen, G. Knoener, N.R. Heckenberg, and H. Rubinsztein-Dunlop. Physics of optical tweezers. *Laser Manipulation of Cells and Tissues*, 82:207–236, 2007.
- [12] M. Dienerowitz, M. Mazilu, and K. Dholakia. Optical manipulation of nanoparticles: a review. *Journal of nanophotonics*, 2:Paper 021875, 32 pages, 2008.

- [13] A. Jonas and P. Zemanek. Light at work: The use of optical forces for particle manipulation, sorting and analysis. *Electrophoresis*, 29:4813–4851, 2008.
- [14] T.A. Nieminen, N. du Preez-Wilkinson, A.B. Stilgoe, V.L.Y. Loke, A.A.M. Bui, and H. Rubinsztein-Dunlop. Optical tweezers: Theory and modelling. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 146:59–80, 2014.
- [15] A.-I. Bunea and J. Glückstad. Strategies for optical trapping in biological samples: Aiming at microrobotic surgeons. *Laser photonics Rev.*, Paper 1800227:17 pages, 2019.
- [16] G. Gouesbet. Generalized Lorenz-Mie theories and mechanical effects of laser light, on the occasion of Arthur Ashkin’s receipt of the 2018 Nobel prize in physics for his pioneering work in optical levitation and manipulation: A review. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 225:258–277, 2019.
- [17] A. Ashkin, J.M. Dziedzic, J.E. Bjorkholm, and S. Chu. Observation of a single beam gradient force optical trap for dielectric particles. *Optics Letters*, 11, 5:288–290, 1986.
- [18] J.A. Lock. Calculation of the radiation trapping force for laser tweezers by use of generalized Lorenz-Mie theory. II. On-axis trapping force. *Applied Optics*, 43,12:2545–2554, 2004.
- [19] J. Durnin, Jr. J.J. Miceli, and J.H. Eberly. Diffraction-free beams. *Physical Review Letters*, 58, 15:1499–1501, 1987.
- [20] J. Durnin. Exact solutions for nondiffracting beams. I. The scalar theory. *Journal of the Optical Society of America A*, 4, 4:651–654, 1987.
- [21] J.A. Lock. Angular spectrum and localized model of Davis-type beam. *Journal of the Optical Society of America A*, 30, 3:489–500, 2013.
- [22] J.J. Wang, T. Wriedt, J.A. Lock, and L. Mädler. General description of circularly symmetric Bessel beams of arbitrary order. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 184:218–232, 2016.
- [23] J.J. Wang, T. Wriedt, J.A. Lock, and Y.C. Jiao. General description of transverse mode Bessel beams and construction of basis Bessel fields. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 195:8–17, 2017.
- [24] J.J. Wang, T. Wriedt, L.Mädler, Y.P. Han, and P. Hartmann. Multipole expansion of circularly Bessel beams of arbitrary order for scattering calculations. *Optics Communications*, 387:102–109, 2017.
- [25] G. Gouesbet. Partial wave expansions and properties of axisymmetric light beams. *Applied Optics*, 35,9:1543–1555, 1996.

- [26] G. Gouesbet. Poynting theorem in terms of beam shape coefficients and applications to axisymmetric, dark and non-dark, vortex and non-vortex beams. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 201:184–196, 2017.
- [27] G. Gouesbet. T-matrix formulation and generalized Lorenz-Mie theories in spherical coordinates. *Optics Communications*, 283, 4:517–521, 2010.
- [28] S.R. Mishra. A vector wave analysis of a Bessel beam. *Optics Communications*, 85:159–161, 1991.
- [29] T. Cizmar, V. Kollarova, Z. Bouchal, and P. Zemanek. Sub-micron particle organization by self-imaging of non-diffracting beams. *New Journal of Physics*, 8 (3), 43, 2006.
- [30] J.M. Taylor and G.D. Love. Multipole expansion of Bessel and Gaussian beams for Mie scattering calculations. *Journal of the Optical Society of America A*, 26, 2:278–282, 2009.
- [31] J. Chen, J. Ng, P. Wang, and Z. Lin. Analytical partial wave expansion of vector Bessel beam and its application to optical binding. *Optics Letters*, 35, 10:1674–1676, 2010.
- [32] L. Robin. *Fonctions sphériques de Legendre et fonctions sphéroïdales. Volumes 1, 2, 3.* Gauthier-Villars, Paris, 1957.
- [33] Z.X. Wang and D.R. Guo. *Special functions.* World Scientific, Singapore, 1989.
- [34] G.B. Arfken and H.J. Weber. *Mathematical methods for physicists, sixth edition.* Elsevier Academic Press, Amsterdam, 2005.
- [35] G. Gouesbet and J.A. Lock. A darkness theorem for the beam shape coefficients and its relationship to higher-order non vortex Bessel beams. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 201:229–235, 2017.
- [36] G. Gouesbet and G. Gréhan. Sur la généralisation de la théorie de Lorenz-Mie. *Journal of Optics*, 13,2:97–103, 1982.
- [37] H.C. van de Hulst. *Light scattering by small particles.* Wiley, New York, 1957.
- [38] Y. Harada and T. Asakura. Radiation forces on a dielectric sphere in the Rayleigh scattering regime. *Optics Communications*, 124:529–541, 1996.
- [39] E.R. Dufresne and D.G. Grier. Optical tweezer arrays and optical substrates created with diffractive optics. *Review of scientific instruments*, 69,5:1974–1977, 1998.
- [40] C.H. Chen, P.T. Tai, and W.F. Hsieh. Bottle beam from a bare laser for single-beam trapping. *Applied Optics*, 43,32:6001–6006, 2004.

- [41] J. Chen, J. Ng, S. Liu, and Z. Lin. Analytical calculation of axial optical force on a Rayleigh particle illuminated by Gaussian beams beyond the paraxial approximation. *Physical Review E*, 80:Paper 026607, 1–9, 2009.
- [42] L.A. Ambrosio and G. Gouesbet. On longitudinal radiation pressure cross section in the generalized Lorenz-Mie theory and its relationship with the dipole theory of forces. *submitted to Journal of Optical Society of America*.
- [43] L.A. Ambrosio and G. Gouesbet. On transverse radiation pressure cross-sections in the generalized Lorenz-Mie theory and their relationship with the dipole theory of forces. *Submitted to Journal of Quantitative Spectroscopy and Radiative Transfer*.