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Axicon terms associated with gradient optical forces in generalized Lorenz-Mie theory.

G erard Gouesbet¹, Leonardo Andr e Ambrosio²

1.CORIA-UMR 6614- Normandie Universit e
CNRS-Universit e et INSA de Rouen
Campus Universitaire du Madrillet
76800, Saint-Etienne-du Rouvray, France.

2.Department of Electrical and Computer Engineering
S ao Carlos School of Engineering, University of S ao Paulo
400 Trabalhador s ao-carlense Ave., S ao Paulo, SP 13566-590, Brazil.
Corresponding author : gouesbet@coria.fr

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Abstract

In a recent paper devoted to the study of longitudinal optical forces exerted by off-axis Bessel beams in the Rayleigh regime in generalized Lorenz-Mie theory, new kind of optical forces are discovered. They are different than the well established scattering and gradient forces, and called axicon forces. In the present paper, within the same framework, we demonstrate the existence of a new kind of axicon terms, associated with gradient forces, which are zero both when the axicon angle is zero and when an on-axis configuration is considered.

Keywords : generalized Lorenz-Mie theory; Bessel beams; optical forces; scattering forces; gradient forces; axicon forces.

1 Introduction.

This paper is the continuation of a previous paper [1] which was devoted to the study of longitudinal optical forces exerted by off-axis Bessel beams in the Rayleigh regime in the framework of generalized Lorenz-Mie theory, see [2], [3], [4], [5], [6], [7], [8], [9] for a collection of reviews devoted to optical forces, and [10] for a recent review written on the occasion of Arthur Ashkin's receipt of the 2018 Nobel prize in physics for his pioneering work in optical levitation and

manipulation. In [1], beside classical scattering and gradient forces, a new kind of optical forces associated with scattering forces, named axicon (scattering) forces, has been discovered. Concerning gradient forces, they have been found to be zero in agreement with the fact the longitudinal gradients of (i) the electric intensity (ii) the energy density and (iii) the longitudinal component of the Poynting vector are zero for Bessel beams. In the present paper, we proceed to a refined investigation of the gradient forces identified in [1] and demonstrate that they are actually accompanied by axicon terms which are zero if the axicon angle is zero and/or if the configuration is an on-axis configuration.

Before proceeding further, it is important to insist on the meaning of new expressions introduced in the present paper and in the previous paper [1]. The new terminology introduces "axicon forces" and "axicon terms". Axicon forces are forces which are deduced from the expressions for pressure radiation cross-sections (or forces) but which have no counterpart in expressions dealing with scattering expressions, e.g. such as expressed by the Poynting vector, which may be called axicon scattering forces as examined in [1]. Conversely, axicon terms are extra-terms which do not arise from the expressions for pressure radiation cross-sections (or forces) but which arise from gradient expressions. Axicon terms associated with gradient expressions are examined in the present paper. .

The paper is organized as follows. Section 2 presents a background already presented in [1] but which is repeated here for the convenience of the reader, complemented by recalling a few results taken from [1] but useful for the present paper. Section 3 deals with the longitudinal gradient of the electric intensity. Section 4 deals with the longitudinal gradient of the time averaged energy density. Section 5 deals with the longitudinal gradient of the longitudinal component of the Poynting vector. In these sections 3, 4, and 5, it is demonstrated that some axicon terms must be associated with the gradient force. Section 6 provides an extra-discussion of some issues enlightening the results obtained in the present paper and in the previous paper [1]. Section 7 is a conclusion.

2 Physical and mathematical background.

The GLMT *stricto sensu* describes the interaction between an arbitrary (structured) shaped beam and a homogeneous spherical particle defined by its diameter and its complex refractive index, e.g. [11], [12], [13], and references therein. Also, see reviews [14], [15], in particular for other GLMTs devoted to the interaction between laser beams and other kinds of scatterers. In the present paper, we consider a Rayleigh particle located at the origin O_P of a Cartesian coordinate system O_Pxyz illuminated by an off-axis Bessel beam propagating along the z -direction.

2.1 Optical forces.

In the GLMT framework, we express the optical force components F_i by using (unnormalized) pressure radiation cross-section components $C_{pr,i}$ ($i = x, y, z$). In the present paper, we deal only with the longitudinal cross-section (along z -direction) reading as:

$$\begin{aligned}
C_{pr,z} = & E_0 H_0^* \frac{\lambda^2}{2\pi} \sum_{n=1}^{\infty} \sum_{m=-n}^n \left\{ \frac{1}{(n+1)^2} \frac{(n+1+|m|)!}{(n-|m|)!} \right. & (1) \\
& \text{Re}[(a_n + a_{n+1}^* - 2a_n a_{n+1}^*) g_{n,TM}^m g_{n+1,TM}^{m*} \\
& + (b_n + b_{n+1}^* - 2b_n b_{n+1}^*) g_{n,TE}^m g_{n+1,TE}^{m*}] \\
& + m \frac{2n+1}{n^2(n+1)^2} \frac{(n+|m|)!}{(n-|m|)!} \\
& \left. \text{Re}[i(2a_n b_n^* - a_n - b_n^*) g_{n,TM}^m g_{n,TE}^{m*}] \right\}
\end{aligned}$$

with notations being the ones of [11]. However, note that, strictly speaking, $C_{pr,z}$ of Eq.1 would be a pressure radiation cross-section, i.e. would be homogeneous to an area, if we used the normalization condition $E_0 H_0^*/2 = 1$ (which was used in original works, e.g. Eq.(3.106) in [13]). Here, we did not use this condition for the sake of consistency with later expressions concerning the Poynting vector. When using the normalization condition, the forces are related to the cross-sections according to $F_i = C_{pr,i}/c$ ([16], p.14). The time-dependence of the wave is $\exp(+i\omega t)$ which is the usual choice in GLMT. Also, note that the denominator $1/(2\pi)$ was misspelled to $1/(4\pi)$ in [1].

Furthermore, specifically, a_n and b_n are the usual Mie coefficients of the usual Lorenz-Mie theory, $g_{n,TM}^m$ and $g_{n,TE}^m$, with TM and TE standing for "Transverse Magnetic" and "Transverse Electric" respectively, are the beam shape coefficients (BSCs) encoding the description of the beam, either in terms of scalar potentials, e.g. [13] or in terms of vector spherical wave functions [17], the star denotes a complex conjugation and λ is the wavelength. In the present paper, we shall consider only the longitudinal force of Eq.1 because it will lead to simpler computations than for the two other force components along x and y , and, more important, because we expect a natural separation between gradient and scattering forces along the z -direction.

2.2 Poynting vector.

For the same reasons, we shall only consider the z -component S_z of the Poynting vector defined as:

$$S_z = \frac{1}{2} \text{Re}(E_x H_y^* - E_y H_x^*) \quad (2)$$

which has already been evaluated to [18], [19]:

$$S_z = \frac{-E_0 H_0^*}{2r^2} \text{Re} \sum_{n=1}^{\infty} \sum_{m=-n}^{+n} \sum_{p=1}^{\infty} \sum_{q=-p}^{+p} i c_n^{pw} c_p^{pw*} e^{i(m-q)\varphi} \quad (3)$$

$$(\sin \theta S_{np}^{mq} + \cos \theta C_{np}^{mq})$$

in which (r, θ, φ) are spherical coordinates attached to the Cartesian coordinates (x, y, z) , c_n^{pw} are prefactors appearing in the Bromwich formulation of the Lorenz-Mie theory [20], reading as:

$$c_n^{pw} = \frac{1}{ik} (-i)^n \frac{2n+1}{n(n+1)} \quad (4)$$

and :

$$S_{np}^{mq} = kr [-g_{n, TM}^m g_{p, TM}^{q*} \psi_p (\psi_n + \psi_n'') P_n^{|m|} \tau_p^{|q|} \quad (5)$$

$$+ g_{n, TE}^m g_{p, TE}^{q*} \psi_n (\psi_p + \psi_p'') P_p^{|q|} \tau_n^{|m|}$$

$$+ q g_{n, TM}^m g_{p, TE}^{q*} \psi_p' (\psi_n + \psi_n'') P_n^{|m|} \pi_p^{|q|}$$

$$+ m g_{n, TM}^m g_{p, TE}^{q*} \psi_n' (\psi_p + \psi_p'') P_p^{|q|} \pi_n^{|m|}]$$

$$C_{np}^{mq} = -g_{n, TM}^m g_{p, TM}^{q*} \psi_p \psi_n' (\tau_n^{|m|} \tau_p^{|q|} + m q \pi_n^{|m|} \pi_p^{|q|}) \quad (6)$$

$$+ g_{n, TM}^m g_{p, TE}^{q*} \psi_n' \psi_p' (m \pi_n^{|m|} \tau_p^{|q|} + q \pi_p^{|q|} \tau_n^{|m|})$$

$$- g_{n, TE}^m g_{p, TM}^{q*} \psi_p \psi_n (m \pi_n^{|m|} \tau_p^{|q|} + q \pi_p^{|q|} \tau_n^{|m|})$$

$$+ g_{n, TE}^m g_{p, TE}^{q*} \psi_n \psi_p' (m q \pi_n^{|m|} \pi_p^{|q|} + \tau_n^{|m|} \tau_p^{|q|})$$

in which k is the wavenumber, $P_n^{|m|} = P_n^{|m|}(\cos \theta)$ are associated Legendre functions, $\tau_n^{|m|} = \tau_n^{|m|}(\cos \theta)$ and $\pi_n^{|m|} = \pi_n^{|m|}(\cos \theta)$ are generalized Legendre functions, and $\psi_n = \psi_n(kr)$ are Ricatti-Bessel functions, while a prime denotes a derivative with respect to the argument (and a double prime a second derivative). In Eqs.5-6, arguments are omitted for convenience. The omission of arguments will be recurrent in the sequel whenever we find it convenient.

2.3 Mathematical background.

Associated Legendre functions are defined according to Hobson's notation [21]:

$$P_n^{|m|}(\cos \theta) = (-1)^{|m|}(\sin \theta)^{|m|} \frac{d^{|m|} P_n(\cos \theta)}{(d \cos \theta)^{|m|}} \quad (7)$$

in which $P_n(\cos \theta)$ are the usual Legendre polynomials. Generalized Legendre functions may then be evaluated according to:

$$\tau_n^m(\cos \theta) = \frac{dP_n^m(\cos \theta)}{d\theta} \quad (8)$$

$$\pi_n^m(\cos \theta) = \frac{P_n^m(\cos \theta)}{\sin \theta} \quad (9)$$

Ricatti-Bessel functions $\psi_n(kr)$ may be expressed in terms of spherical Bessel functions $j_n(kr)$ according to:

$$\psi_n(x) = x j_n(x) \quad (10)$$

Furthermore, we have [22]:

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x} \quad (11)$$

2.4 Bessel beams.

Bessel beams have been introduced by Durnin and co-workers [23], [24]. They possess the appealing property of being self-healing and non diffracting, and, more important in the context of the present paper, they furthermore possess a propagation invariance property, namely the intensity of the beam is constant along the direction of propagation. There exist an infinity of kinds of Bessel beams, in particular depending on the value given to an arbitrary function $g(\alpha_0)$, in which α_0 is the axicon angle (or half-cone angle) of the beam, with different linear and circular polarizations [25], [26], [27], [28]. All of them generically exhibit the following structure:

$$K_i = k_i(x, y)e^{-ik_z z}, \quad i = x, y, z, \quad K = E \text{ or } H \quad (12)$$

in which $k_z = k \cos \alpha_0$ is the longitudinal wavenumber. Among all of them, we shall pay a particular attention to circularly symmetric Bessel beams of arbitrary order whose BSCs in an off-axis configuration read as [28]:

$$g_{n, TM}^m = -g(\alpha_0)e^{ik_z z_0} [i^{l-m+1}e^{i(l-m+1)\phi_0} A_n^m + i^{l-m-1}e^{i(l-m-1)\phi_0} B_n^m] \quad (13)$$

$$g_{n, TE}^m = ig(\alpha_0)e^{ik_z z_0} [i^{l-m+1}e^{i(l-m+1)\phi_0} A_n^m - i^{l-m-1}e^{i(l-m-1)\phi_0} B_n^m] \quad (14)$$

in which:

$$A_n^m = (-1)^{(m-|m|)/2} \frac{(n-m)!}{(n+|m|)!} J_{l-m+1}(\sigma_0) [\tau_n^m(\cos \alpha_0) + m\pi_n^m(\cos \alpha_0)] \quad (15)$$

$$B_n^m = (-1)^{(m-|m|)/2} \frac{(n-m)!}{(n+|m|)!} J_{l-m-1}(\sigma_0) [\tau_n^m(\cos \alpha_0) - m\pi_n^m(\cos \alpha_0)] \quad (16)$$

in which l is the order of the Bessel beam under consideration, $\sigma_0 = k_t \rho_0$, $k_t = k \sin \alpha_0$ is the transverse wavenumber, $\rho_0 = (x_0^2 + y_0^2)^{1/2}$, $\phi_0 = \tan^{-1}(y_0/x_0)$, (x_0, y_0, z_0) denotes the location of the beam origin with respect to the origin of the particle coordinate system at which the scatterer is located, and therefore specifies the off-axis location of the beam, and $J_k(\cdot)$ is the k -order Bessel function of the first kind. When $g(\alpha_0) = (1 + \cos \alpha_0)/4$, the beam reduces to a Davis circularly symmetric Bessel beam as discussed in [25], [29]. When $g(\alpha_0) = 1/2$, the beam reduces to another kind of Bessel beam discussed in [30], [31], [32].

2.5 Summary of some previous results.

In this subsection, we recall a few results obtained in [1] related to gradient forces. In the Rayleigh regime which was considered in [1] and which is still considered in the present paper, only the $(n = 1)$ -partial waves are to be considered and we only have to retain, in the expansions, the Mie coefficient a_1 which simplifies to :

$$a_1 = \frac{2i}{3} \frac{m^2 - 1}{m^2 + 2} \alpha^3 + O(i\alpha^5) + \frac{4}{9} \left(\frac{m^2 - 1}{m^2 + 2} \right)^2 \alpha^6 \quad (17)$$

$$b_1 = O(i\alpha^5) \quad (18)$$

in which m is the refractive index (here taken to be real) with respect to the surrounding medium and α the size parameter $\pi d/\lambda$. The other coefficients a_n and b_n ($n > 1$) involves still higher powers of α . Real parts are then proportional to α^6 and imaginary parts are proportional to α^3 while higher powers are discarded and we therefore retain only:

$$\text{Im}(a_1) = \frac{2}{3} \frac{m^2 - 1}{m^2 + 2} \alpha^3 \quad (19)$$

$$\text{Re}(a_1) = \frac{4}{9} \left(\frac{m^2 - 1}{m^2 + 2} \right)^2 \alpha^6 \quad (20)$$

The scattering force represented by $C_{pr,z}^s$ and the gradient force represented by $C_{pr,z}^g$ are proportional to α^6 and α^3 respectively. It is then demonstrated that, by evaluating $C_{pr,z}^g$ with the proviso that the Rayleigh particle is only sensitive to the $(n = 1)$ - partial waves, we have:

$$C_{pr,z}^g = 0 \quad (21)$$

The nullity of this term identified as a gradient term is consistent with the fact that, due to Eq.12, the longitudinal gradients of the electric intensity $\mathbf{E} \cdot \mathbf{E}^*$, of the time-averaged energy density $(\varepsilon_0 \mathbf{E} \cdot \mathbf{E}^* + \mu_0 \mathbf{H} \cdot \mathbf{H}^*)$ and of the z -component of the Poynting vector are zero. However, the Rayleigh particle does not perceive all the partial waves involved in Eq.12 but only the $(n = 1)$ -partial waves. This implies that a refinement of the analysis is required in which the electric intensity, the time-averaged energy density and the z -component of the Poynting vector would be evaluated by only considering the contributions of the $(n = 1)$ -partial waves. The aim of the present paper is to proceed to this refined analysis. Quantities restricted to $(n = 1)$ partial waves are decorated with a tilde.

3 Electric intensity.

When restricted to the ($n = 1$)-partial waves, the electric components in spherical coordinates read as (e.g. [13], pp.55-56):

$$\widetilde{E}_r = kE_0c_1^{pw}(\psi_1'' + \psi_1) \sum_{m=-1}^{+1} g_{1,TM}^m P_1^{|m|} e^{im\varphi} \quad (22)$$

$$\widetilde{E}_\theta = \frac{E_0}{r} c_1^{pw} \sum_{m=-1}^{+1} [g_{1,TM}^m \psi_1' \tau_1^{|m|} + mg_{1,TE}^m \psi_1 \pi_1^{|m|}] e^{im\varphi} \quad (23)$$

$$\widetilde{E}_\varphi = \frac{iE_0}{r} c_1^{pw} \sum_{m=-1}^{+1} [mg_{1,TM}^m \psi_1' \pi_1^{|m|} + g_{1,TE}^m \psi_1 \tau_1^{|m|}] e^{im\varphi} \quad (24)$$

From these equations, we may evaluate:

$$\begin{aligned} \widetilde{E}_r \widetilde{E}_r^* &= \frac{9}{4} \frac{|E_0|^2}{k^2} k^2 (\psi_1'' + \psi_1)^2 [e^{-2i\varphi} g_{1,TM}^{-1} g_{1,TM}^{1*} P_1^1 P_1^1 \\ &+ e^{-i\varphi} (g_{1,TM}^{-1} g_{1,TM}^{0*} + g_{1,TM}^0 g_{1,TM}^{1*}) P_1^1 P_1^0 \\ &+ e^{0\varphi} (g_{1,TM}^{-1} g_{1,TM}^{-1*} P_1^1 P_1^1 + g_{1,TM}^0 g_{1,TM}^{0*} P_1^0 P_1^0 + g_{1,TM}^1 g_{1,TM}^{1*} P_1^1 P_1^1] \\ &+ e^{i\varphi} (g_{1,TM}^0 g_{1,TM}^{-1*} + g_{1,TM}^1 g_{1,TM}^{0*}) P_1^1 P_1^0 \\ &+ e^{2i\varphi} g_{1,TM}^1 g_{1,TM}^{-1*} P_1^1 P_1^1] \end{aligned} \quad (25)$$

in which we have used:

$$c_1^{pw} c_1^{pw*} = \frac{9}{4k^2} \quad (26)$$

and:

$$\begin{aligned}
\widetilde{E_\theta E_\theta^*} &= \frac{9}{4} \frac{|E_0|^2}{k^2} \left\{ \frac{\psi_1' \psi_1'}{r^2} [e^{-2i\varphi} g_{1, TM}^{-1} g_{1, TM}^{1*} \tau_1^1 \tau_1^1 \right. \\
&+ e^{-i\varphi} (g_{1, TM}^{-1} g_{1, TM}^{0*} + g_{1, TM}^0 g_{1, TM}^{1*}) \tau_1^1 \tau_1^0 \\
&+ e^{0\varphi} (g_{1, TM}^{-1} g_{1, TM}^{-1*} \tau_1^1 \tau_1^1 + g_{1, TM}^0 g_{1, TM}^{0*} \tau_1^0 \tau_1^0 + g_{1, TM}^1 g_{1, TM}^{1*} \tau_1^1 \tau_1^1) \\
&+ e^{i\varphi} (g_{1, TM}^{-1*} g_{1, TM}^0 + g_{1, TM}^{0*} g_{1, TM}^1) \tau_1^1 \tau_1^0 \\
&+ e^{2i\varphi} g_{1, TM}^{-1*} g_{1, TM}^1 \tau_1^1 \tau_1^1] \\
&+ \frac{\psi_1' \psi_1}{r^2} [e^{-2i\varphi} (g_{1, TM}^{-1} g_{1, TE}^{1*} - g_{1, TE}^{-1} g_{1, TM}^{1*}) \pi_1^1 \tau_1^1 \\
&+ e^{-i\varphi} (g_{1, TM}^0 g_{1, TE}^{1*} - g_{1, TE}^{-1} g_{1, TM}^{0*}) \pi_1^1 \tau_1^0 \\
&+ e^{0\varphi} (g_{1, TE}^1 g_{1, TM}^{1*} - g_{1, TM}^{-1} g_{1, TE}^{-1*} + g_{1, TM}^1 g_{1, TE}^{1*} - g_{1, TE}^{-1} g_{1, TM}^{-1*}) \pi_1^1 \tau_1^1 \\
&+ e^{i\varphi} (g_{1, TE}^1 g_{1, TM}^{0*} - g_{1, TM}^0 g_{1, TE}^{-1*}) \pi_1^1 \tau_1^0 \\
&+ e^{2i\varphi} (g_{1, TE}^1 g_{1, TM}^{-1*} - g_{1, TM}^1 g_{1, TE}^{-1*}) \pi_1^1 \tau_1^1 \\
&- \frac{\psi_1 \psi_1}{r^2} [e^{-2i\varphi} g_{1, TE}^{-1} g_{1, TE}^{1*} - e^{0\varphi} (g_{1, TE}^{-1} g_{1, TE}^{-1*} + g_{1, TE}^1 g_{1, TE}^{1*}) + e^{2i\varphi} g_{1, TE}^1 g_{1, TE}^{-1*}] \pi_1^1 \pi_1^1
\end{aligned} \tag{27}$$

$$\begin{aligned}
\widetilde{E_\varphi E_\varphi^*} &= \frac{9}{4} \frac{|E_0|^2}{k^2} \left\{ \frac{\psi_1' \psi_1'}{r^2} [-e^{-2i\varphi} g_{1, TM}^{-1} g_{1, TM}^{1*} \pi_1^1 \pi_1^1 \right. \\
&+ e^{0\varphi} (g_{1, TM}^{-1} g_{1, TM}^{-1*} + g_{1, TM}^1 g_{1, TM}^{1*}) \pi_1^1 \pi_1^1 \\
&- e^{2i\varphi} g_{1, TM}^1 g_{1, TM}^{-1*} \pi_1^1 \pi_1^1] \\
&+ \frac{\psi_1' \psi_1}{r^2} [e^{-2i\varphi} (g_{1, TE}^{-1} g_{1, TM}^{1*} - g_{1, TM}^{-1} g_{1, TE}^{1*}) \pi_1^1 \tau_1^1 \\
&+ e^{-i\varphi} (g_{1, TE}^0 g_{1, TM}^{1*} - g_{1, TM}^{-1} g_{1, TE}^{0*}) \pi_1^1 \tau_1^0 \\
&+ e^{0\varphi} (g_{1, TE}^{1*} g_{1, TM}^1 - g_{1, TM}^{-1} g_{1, TE}^{-1*} + g_{1, TE}^1 g_{1, TM}^{1*} - g_{1, TE}^{-1} g_{1, TM}^{-1*}) \pi_1^1 \tau_1^1 \\
&+ e^{i\varphi} (g_{1, TM}^1 g_{1, TE}^{0*} - g_{1, TE}^0 g_{1, TM}^{-1*}) \pi_1^1 \tau_1^0 \\
&+ e^{2i\varphi} (g_{1, TM}^1 g_{1, TE}^{-1*} - g_{1, TE}^1 g_{1, TM}^{-1*}) \pi_1^1 \tau_1^1 \\
&+ \frac{\psi_1 \psi_1}{r^2} [e^{-2i\varphi} g_{1, TE}^{-1} g_{1, TE}^{1*} \tau_1^1 \tau_1^1 + e^{-i\varphi} (g_{1, TE}^{-1} g_{1, TE}^{0*} + g_{1, TE}^0 g_{1, TE}^{1*}) \tau_1^1 \tau_1^0 \\
&+ e^{0\varphi} (g_{1, TE}^{-1} g_{1, TE}^{-1*} \tau_1^1 \tau_1^1 + g_{1, TE}^0 g_{1, TE}^{0*} \tau_1^0 \tau_1^0 + g_{1, TE}^1 g_{1, TE}^{1*} \tau_1^1 \tau_1^1) \\
&+ e^{i\varphi} (g_{1, TE}^0 g_{1, TE}^{-1*} + g_{1, TE}^1 g_{1, TE}^{0*}) \tau_1^1 \tau_1^0 + e^{2i\varphi} g_{1, TE}^1 g_{1, TE}^{-1*} \tau_1^1 \tau_1^1]
\end{aligned} \tag{28}$$

We may then evaluate the corresponding electric intensity according to:

$$\widetilde{I_E} = \widetilde{E_r E_r^*} + \widetilde{E_\theta E_\theta^*} + \widetilde{E_\varphi E_\varphi^*} \tag{29}$$

which is found to read as:

$$\widetilde{I}_E = \sum_{k=-2}^{+2} \widetilde{I}_E^k \quad (30)$$

Specifying the values of associated Legendre functions and of generalized Legendre functions according to:

$$P_0^0 = P_0^0(\cos \theta) = 1 \quad (31)$$

$$P_1^0 = P_1^0(\cos \theta) = -\tau_1^1(\cos \theta) = \cos \theta \quad (32)$$

$$P_1^1 = P_1^1(\cos \theta) = \tau_1^0(\cos \theta) = -\sin \theta \quad (33)$$

$$\pi_1^1 = \pi_1^1(\cos \theta) = -1 \quad (34)$$

we obtain:

$$\widetilde{I}_E^{-2} = \widetilde{I}_E^{2*} = \frac{9}{4} \frac{|E_0|^2}{k^2} e^{-2i\varphi} \sin^2 \theta \left\{ [k^2(\psi_1'' + \psi_1)^2 - \frac{\psi_1' \psi_1'}{r^2}] g_{1, TM}^{-1} g_{1, TM}^{1*} + \frac{\psi_1 \psi_1}{r^2} g_{1, TE}^{-1} g_{1, TE}^{1*} \right\} \quad (35)$$

$$\begin{aligned} \widetilde{I}_E^{-1} &= \widetilde{I}_E^{1*} = \frac{9}{4} \frac{|E_0|^2}{k^2} e^{-i\varphi} \sin \theta \left\{ \cos \theta \left[\frac{\psi_1' \psi_1'}{r^2} - k^2(\psi_1'' + \psi_1)^2 \right] (g_{1, TM}^{-1} g_{1, TM}^{0*} + g_{1, TM}^0 g_{1, TM}^{1*}) \right. \\ &\quad + \frac{\psi_1' \psi_1}{r^2} (g_{1, TM}^0 g_{1, TE}^{1*} - g_{1, TE}^{-1} g_{1, TM}^{0*} - g_{1, TM}^{-1} g_{1, TE}^{0*} + g_{1, TE}^0 g_{1, TM}^{1*}) \\ &\quad \left. + \cos \theta \frac{\psi_1 \psi_1}{r^2} (g_{1, TE}^{-1} g_{1, TE}^{0*} + g_{1, TE}^0 g_{1, TE}^{1*}) \right\} \quad (36) \end{aligned}$$

$$\begin{aligned}
\widetilde{I}_E^0 &= \frac{9}{4} \frac{|E_0|^2}{k^2} e^{0\varphi} \{ [k^2 \sin^2 \theta (\psi_1'' + \psi_1)^2 + \cos^2 \theta \frac{\psi_1' \psi_1'}{r^2}] (g_{1, TM}^{-1} g_{1, TM}^{-1*} + g_{1, TM}^1 g_{1, TM}^{1*}) \\
&\quad + [k^2 \cos^2 \theta (\psi_1'' + \psi_1)^2 + \sin^2 \theta \frac{\psi_1' \psi_1'}{r^2}] g_{1, TM}^0 g_{1, TM}^{0*} \\
&\quad + 2 \cos \theta \frac{\psi_1' \psi_1}{r^2} (g_{1, TM}^1 g_{1, TE}^{1*} - g_{1, TM}^{-1} g_{1, TE}^{-1*} + g_{1, TE}^1 g_{1, TM}^{1*} - g_{1, TE}^{-1} g_{1, TM}^{-1*}) \\
&\quad + (\cos^2 \theta + 1) \frac{\psi_1 \psi_1}{r^2} (g_{1, TE}^1 g_{1, TE}^{1*} + g_{1, TE}^{-1} g_{1, TE}^{-1*}) \\
&\quad + \sin^2 \theta \frac{\psi_1 \psi_1}{r^2} g_{1, TE}^0 g_{1, TE}^{0*} + \frac{\psi_1' \psi_1'}{r^2} (g_{1, TM}^{-1} g_{1, TM}^{-1*} + g_{1, TM}^1 g_{1, TM}^{1*}) \}
\end{aligned} \tag{37}$$

We are now interested in the calculation of the derivative of the (restricted) electric intensity with respect to z , taken at $z = 0$, i.e. at the place where the particle is located. This requires in a first step to express $\cos \theta$, $\sin \theta$ and the radial coordinate r in terms of Cartesian coordinates, a trivial task indeed. Furthermore, in evaluating these derivatives, and other derivatives in subsequent subsections, derivatives with respect to φ may be omitted. In a few words, this comes from the fact that z -dependence is only encoded in the spherical coordinates r and θ , but not in φ . The suspicious reader may have a direct check by using the equation $\partial f(r, \theta) \exp(\pm ik\varphi) / \partial z = \exp(\pm ik\varphi) \partial f(r, \theta) / \partial z + f(r, \theta) \partial \exp(\pm ik\varphi) / \partial z$, leaving the derivatives evaluated below unchanged.

Then, we consider \widetilde{I}_E^{-2} and \widetilde{I}_E^2 , and evaluate:

$$\left\{ \frac{\partial}{\partial z} \sin^2 \theta [k^2 (\psi_1'' + \psi_1)^2 - \frac{\psi_1' \psi_1'}{r^2}] \right\}_{z=0} = 0 \tag{38}$$

$$\left(\frac{\partial}{\partial z} \sin^2 \theta \frac{\psi_1 \psi_1}{r^2} \right)_{z=0} = 0 \tag{39}$$

in which we expressed the terms dependent on r and θ in terms of x, y, z and recalled that, using Eqs.10 and 11:

$$\psi_1(kr) = kr \left[\frac{\sin(kr)}{(kr)^2} - \frac{\cos(kr)}{kr} \right] \tag{40}$$

Hence:

$$\left(\frac{\partial \widetilde{I}_E^{-2}}{\partial z} \right)_{z=0} = \left(\frac{\partial \widetilde{I}_E^2}{\partial z} \right)_{z=0} = 0 \tag{41}$$

For $\widetilde{I_E^{-1}}$ and $\widetilde{I_E^{-1}}$, we have:

$$\Omega_1 = \left\{ \frac{\partial}{\partial z} \sin \theta \cos \theta \left[\frac{\psi_1' \psi_1'}{r^2} - k^2 (\psi_1'' + \psi_1)^2 \right] \right\}_{z=0} \neq 0 \quad (42)$$

$$\left(\frac{\partial}{\partial z} \sin \theta \frac{\psi_1' \psi_1'}{r^2} \right)_{z=0} = 0 \quad (43)$$

$$\Omega_2 = \left(\frac{\partial}{\partial z} \sin \theta \cos \theta \frac{\psi_1 \psi_1}{r^2} \right)_{z=0} \neq 0 \quad (44)$$

so that we obtain:

$$\left(\frac{\partial \widetilde{I_E^{-1}}}{\partial z} \right)_{z=0} = \left[\left(\frac{\partial \widetilde{I_E^{-1}}}{\partial z} \right)_{z=0} \right]^* = \frac{9 |E_0|^2}{4 k^2} e^{-i\varphi} (\alpha \Omega_1 + \beta \Omega_2) \quad (45)$$

in which α (not to be confused with the size parameter) and β are given by:

$$\alpha = g_{1,TM}^{-1} g_{1,TM}^{0*} + g_{1,TM}^0 g_{1,TM}^{1*} \quad (46)$$

$$\beta = g_{1,TE}^{-1} g_{1,TE}^{0*} + g_{1,TE}^0 g_{1,TE}^{1*} \quad (47)$$

For $(\partial \widetilde{I_E^0} / \partial z)_{z=0}$, all derivatives with respect to z , taken at $z = 0$, of the different terms involved in Eq.37 are found to be 0, excepted one of them, namely:

$$\Omega_3 = \left(\frac{\partial}{\partial z} \cos \theta \frac{\psi_1' \psi_1'}{r^2} \right)_{z=0} \neq 0 \quad (48)$$

leading to:

$$\left(\frac{\partial \widetilde{I_E^0}}{\partial z} \right)_{z=0} = \frac{9 |E_0|^2}{2 k^2} \gamma \Omega_3 \quad (49)$$

in which:

$$\begin{aligned}
\gamma &= g_{1,TM}^1 g_{1,TE}^{1*} - g_{1,TM}^{-1} g_{1,TE}^{-1*} + g_{1,TE}^1 g_{1,TM}^{1*} - g_{1,TE}^{-1} g_{1,TM}^{-1*} \\
&= 2 \operatorname{Re}(g_{1,TM}^1 g_{1,TE}^{1*} - g_{1,TM}^{-1} g_{1,TE}^{-1*})
\end{aligned} \tag{50}$$

We may then evaluate α , β and γ using Eqs.(15)-(18) of [1]. In terms of A_n^m and B_n^m , we obtain:

$$\begin{aligned}
\alpha &= i |g(\alpha_0)|^2 \{e^{i\phi_0} [A_1^0 (A_1^{-1} + A_1^1) + B_1^0 (B_1^{-1} + B_1^1)] \\
&\quad - e^{-i\phi_0} (B_1^{-1} A_1^0 + B_1^0 A_1^1) - e^{3i\phi_0} (A_1^{-1} B_1^0 + A_1^0 B_1^1)\}
\end{aligned} \tag{51}$$

$$\begin{aligned}
\beta &= i |g(\alpha_0)|^2 \{e^{i\phi_0} [A_1^0 (A_1^{-1} + A_1^1) + B_1^0 (B_1^{-1} + B_1^1)] \\
&\quad + e^{-i\phi_0} (B_1^{-1} A_1^0 + B_1^0 A_1^1) + e^{3i\phi_0} (A_1^{-1} B_1^0 + A_1^0 B_1^1)\}
\end{aligned} \tag{52}$$

$$\begin{aligned}
\gamma &= 2 \operatorname{Re}\{i |g(\alpha_0)|^2 [(A_1^1)^2 + (B_1^{-1})^2 - (B_1^1)^2 - (A_1^{-1})^2 \\
&\quad + e^{2i\phi_0} (A_1^1 B_1^1 - A_1^{-1} B_1^{-1}) + e^{-2i\phi_0} (A_1^{-1} B_1^{-1} - A_1^1 B_1^1)]\}
\end{aligned} \tag{53}$$

Recalling that the A_n^m 's and B_n^m 's are real numbers, Eq.53 simplifies to:

$$\gamma = -4 |g(\alpha_0)|^2 \sin(2\phi_0) (A_1^1 B_1^1 - A_1^{-1} B_1^{-1}) \tag{54}$$

Next, we express the A_n^m 's and B_n^m 's using expressions (17)-(18) of [1], in terms of generalized Legendre functions π_n^m and τ_n^m , and, expressing these Legendre functions themselves, we obtain:

$$A_1^{-1} = -J_{l+2}(\tau_1^{-1} - \pi_1^{-1}) = -\frac{1}{2} J_{l+2}(C - 1) \tag{55}$$

$$A_1^0 = J_{l+1} \tau_1^0 = -J_{l+1} S \tag{56}$$

$$A_1^1 = \frac{1}{2} J_l(\tau_1^1 + \pi_1^1) = \frac{-1}{2} J_l(C + 1) \tag{57}$$

$$B_1^{-1} = -J_l(\tau_1^{-1} + \pi_1^{-1}) = \frac{-1}{2}J_l(C + 1) \quad (58)$$

$$B_1^0 = J_{l-1}\tau_1^0 = -J_{l-1}S \quad (59)$$

$$B_1^1 = \frac{1}{2}J_{l-2}(\tau_1^1 - \pi_1^1) = \frac{-1}{2}J_{l-2}(C - 1) \quad (60)$$

in which $C = \cos \alpha_0$, $S = \sin \alpha_0$ and in which we have used $\tau_1^{-1} = C/2$, $\tau_1^0 = -S$, $\tau_1^1 = -C$, $\pi_1^{-1} = 1/2$ and $\pi_1^1 = -1$. Furthermore the argument σ_0 of the Bessel functions is omitted. We then obtain:

$$\begin{aligned} \alpha = & \frac{iS}{2} |g(\alpha_0)|^2 \{ e^{i\phi_0} [(C-1)(J_{l+1}J_{l+2} + J_{l-2}J_{l-1}) + (C+1)(J_{l-1}J_l + J_lJ_{l+1})] \\ & - e^{-i\phi_0} (C+1)(J_lJ_{l+1} + J_{l-1}J_l) \\ & - e^{3i\phi_0} (C-1)(J_{l-1}J_{l+2} + J_{l-2}J_{l+1}) \} \end{aligned} \quad (61)$$

$$\begin{aligned} \beta = & \frac{iS}{2} |g(\alpha_0)|^2 \{ e^{i\phi_0} [(C-1)(J_{l+1}J_{l+2} + J_{l-2}J_{l-1}) + (C+1)(J_{l-1}J_l + J_lJ_{l+1})] \\ & + e^{-i\phi_0} (C+1)(J_lJ_{l+1} + J_{l-1}J_l) \\ & + e^{3i\phi_0} (C-1)(J_{l-1}J_{l+2} + J_{l-2}J_{l+1}) \} \end{aligned} \quad (62)$$

$$\gamma = -|g(\alpha_0)|^2 (C-1)(C+1)(J_{l-2}J_l - J_lJ_{l+2}) \sin 2\phi_0 \quad (63)$$

Therefore, although $(\partial I_E / \partial z)_{z=0} = 0$, it happens that $(\partial \tilde{I}_E / \partial z)_{z=0} \neq 0$ and contains extra axicon terms. However these extra-terms are zero whenever (i) the axicon angle is zero, i.e. $C = 1$ and $S = 0$ (ii) the configuration is an on-axis configuration. The fact that the extra-terms are zero for an on-axis configuration results from the mathematical property that $J_k(0) = 0$ excepted for $k = 0$, and from the physical property that $\sigma_0 = 0$ for an on-axis configuration.

4 Energy density.

To deal with the energy density $\varepsilon_0 \mathbf{E} \cdot \mathbf{E}^* + \mu_0 \mathbf{H} \cdot \mathbf{H}^*$, we have to evaluate the magnetic intensity $\widetilde{I}_H = \mathbf{H} \cdot \mathbf{H}^*$. Instead of Eqs.22-24, we now have to deal with (see again [13], pp.55-56):

$$\widetilde{H}_r = kH_0 c_1^{pw} (\psi_1'' + \psi_1) \sum_{m=-1}^{+1} g_{1,TE}^m P_1^{|m|} e^{im\varphi} \quad (64)$$

$$\widetilde{H}_\theta = \frac{H_0}{r} c_1^{pw} \sum_{m=-1}^{+1} [g_{1,TE}^m \psi_1' \tau_1^{|m|} - m g_{1,TM}^m \psi_1 \pi_1^{|m|}] e^{im\varphi} \quad (65)$$

$$\widetilde{H}_\varphi = \frac{iH_0}{r} c_1^{pw} \sum_{m=-1}^{+1} [m g_{1,TE}^m \psi_1' \pi_1^{|m|} - g_{1,TM}^m \psi_1 \tau_1^{|m|}] e^{im\varphi} \quad (66)$$

We then observe that the magnetic intensity is deduced directly from the electric intensity by carrying out the following changes:

$$E_0 \rightarrow H_0, g_{1,TM}^m \rightarrow g_{1,TE}^m, g_{1,TE}^m \rightarrow -g_{1,TM}^m \quad (67)$$

so that we directly obtain:

$$\left(\frac{\partial \widetilde{I}_H^{-2}}{\partial z}\right)_{z=0} = \left(\frac{\partial \widetilde{I}_H^2}{\partial z}\right)_{z=0} = 0 \quad (68)$$

$$\left(\frac{\partial \widetilde{I}_H^{-1}}{\partial z}\right)_{z=0} = \left[\left(\frac{\partial \widetilde{I}_H^1}{\partial z}\right)_{z=0}\right]^* = \frac{9}{4} \frac{|H_0|^2}{k^2} e^{-i\varphi} (\beta\Omega_1 + \alpha\Omega_2) \quad (69)$$

$$\left(\frac{\partial \widetilde{I}_H^0}{\partial z}\right)_{z=0} = \frac{-9}{2} \frac{|H_0|^2}{k^2} \gamma\Omega_3 \quad (70)$$

Therefore, although $(\partial \widetilde{I}_H / \partial z)_{z=0} = 0$, it happens that $(\partial \widetilde{I}_H / \partial z)_{z=0} \neq 0$ and contains extra axicon terms. However, once again, these extra-terms are zero whenever (i) the axicon angle is zero, i.e. $C = 1$ and $S = 0$ (ii) the configuration is an on-axis configuration. Because this is the same conclusion that for the

electric intensity, it immediately extends to the time averaged energy density $\varepsilon_0 \widetilde{\mathbf{E} \cdot \mathbf{E}^*} + \mu_0 \widetilde{\mathbf{H} \cdot \mathbf{H}^*}$.

More specifically, adding $\varepsilon_0 \widetilde{\mathbf{E} \cdot \mathbf{E}^*}$ and $\mu_0 \widetilde{\mathbf{H} \cdot \mathbf{H}^*}$, and using $H_0/E_0 = \sqrt{\varepsilon_0/\mu_0}$, we may establish that, after simplifying and rearranging:

$$\left[\frac{\partial}{\partial z} (\varepsilon_0 \widetilde{\mathbf{E} \cdot \mathbf{E}^*} + \mu_0 \widetilde{\mathbf{H} \cdot \mathbf{H}^*}) \right]_{z=0} = 2 \operatorname{Re}(\Gamma e^{-i\varphi}) \Omega_4 \quad (71)$$

in which:

$$\Omega_4 = \Omega_1 + \Omega_2 = \left\{ \frac{\partial}{\partial z} \sin \theta \cos \theta \left[\frac{(\psi_1)^2 + (\psi_1')^2}{r^2} - k^2 (\psi_1'' + \psi_1)^2 \right] \right\}_{z=0} \neq 0 \quad (72)$$

and:

$$\Gamma = \alpha + \beta = iS |g(\alpha_0)|^2 e^{i\phi_0} [(C-1)(J_{l+1}J_{l+2} + J_{l-2}J_{l-1}) + (C+1)((J_{l-1}J_l) + J_l J_{l+1})] \quad (73)$$

to which the above comments still apply.

5 Poynting vector.

Finally, we have to evaluate and discuss $(\partial \widetilde{S}_z / \partial z)_{z=0}$. We start from Eq.(3) in [1] and specify it to the Rayleigh situation when only the partial waves with $n = 1$ are to be retained, that is to say we now consider:

$$\left(\frac{\partial \widetilde{S}_z}{\partial z} \right)_{z=0} = \frac{-9E_0 H_0^*}{8k^2} \operatorname{Re} \sum_{m=-1}^{+1} \sum_{q=-1}^{+1} i e^{i(m-q)\varphi} \left[\frac{\partial}{\partial z} \frac{1}{r^2} (\sin \theta S_{11}^{mq} + \cos \theta C_{11}^{mq}) \right]_{z=0} \quad (74)$$

in which we have used Eq.26. The coefficients S_{11}^{mq} and C_{11}^{mq} are given by Eqs.5-6. Each of them contains four terms. Therefore we have $9 \times 2 \times 4 = 72$ derivatives to evaluate with the fortunate fact, however, that there are many redundant terms. Once these derivatives are done, the only nonzero derivatives are found to be:

$$\left[\frac{\partial}{\partial z} \frac{1}{r} \sin \theta \psi_1 (\psi_1'' + \psi_1) P_1^1 \tau_1^1\right]_{z=0} = \left[\frac{\partial}{\partial z} \frac{1}{r} \sin^2 \theta \cos \theta \psi_1 (\psi_1'' + \psi_1)\right]_{z=0} \quad (75)$$

$$\left[\frac{\partial}{\partial z} \frac{1}{r} \sin \theta \psi_1' (\psi_1'' + \psi_1) P_1^1 \pi_1^0\right]_{z=0} = -\left[\frac{\partial}{\partial z} \frac{1}{r} \sin \theta \cos \theta \psi_1' (\psi_1'' + \psi_1)\right]_{z=0} \quad (76)$$

$$\left[\frac{\partial}{\partial z} \frac{1}{r} \sin \theta \psi_1' (\psi_1'' + \psi_1) P_1^0 \pi_1^1\right]_{z=0} = -\left[\frac{\partial}{\partial z} \frac{1}{r} \sin \theta \cos \theta \psi_1' (\psi_1'' + \psi_1)\right]_{z=0} \quad (77)$$

$$\left[\frac{\partial}{\partial z} \frac{1}{r^2} \cos \theta \psi_1 \psi_1' (\tau_1^1 \tau_1^1 + \pi_1^1 \pi_1^1)\right]_{z=0} = \left[\frac{\partial}{\partial z} \frac{1}{r^2} \cos \theta (\cos^2 \theta + 1) \psi_1 \psi_1'\right]_{z=0} \quad (78)$$

$$\left[\frac{\partial}{\partial z} \frac{1}{r^2} \cos \theta \psi_1 \psi_1' (\tau_1^1 \tau_1^1 - \pi_1^1 \pi_1^1)\right]_{z=0} = \left[\frac{\partial}{\partial z} \frac{1}{r^2} \cos \theta (\cos^2 \theta - 1) \psi_1 \psi_1'\right]_{z=0} \quad (79)$$

$$\left[\frac{\partial}{\partial z} \frac{1}{r^2} \cos \theta \psi_1' \psi_1' \pi_1^1 \tau_1^0\right]_{z=0} = \left[\frac{\partial}{\partial z} \frac{1}{r^2} \sin \theta \cos \theta \psi_1' \psi_1'\right]_{z=0} \quad (80)$$

$$\left[\frac{\partial}{\partial z} \frac{1}{r^2} \cos \theta \psi_1 \psi_1 \pi_1^1 \tau_1^0\right]_{z=0} = \left[\frac{\partial}{\partial z} \frac{1}{r^2} \sin \theta \cos \theta \psi_1 \psi_1\right]_{z=0} \quad (81)$$

$$\left[\frac{\partial}{\partial z} \frac{1}{r^2} \cos \theta \psi_1 \psi_1' \tau_1^0 \tau_1^0\right]_{z=0} = \left[\frac{\partial}{\partial z} \frac{1}{r^2} \sin^2 \theta \cos \theta \psi_1 \psi_1'\right]_{z=0} \quad (82)$$

These quantities appear in S_{11}^{mq} with $mq = -1 - 1, 11, -11, 1 - 1, -10, 10, 0 - 1, 01, 00$ and in C_{11}^{mq} with the same values of mq . Inserting these nonzero values in Eq.74 and rearranging we obtain, after a fairly lengthy and tedious computation:

$$\begin{aligned} \left(\frac{\partial \widetilde{S}_z}{\partial z}\right)_{z=0} &= \frac{-9E_0 H_0^*}{8k^2} \operatorname{Re} i\{(\cos \varphi G_1 + i \sin \varphi G_2) \\ &\quad \left[\frac{\partial}{\partial z} \frac{1}{r^2} \sin \theta \cos \theta (kr \psi_1' (\psi_1'' + \psi_1) - \psi_1' \psi_1')\right]_{z=0} \\ &\quad + (\cos \varphi G_1^* - i \sin \varphi G_2^*) \\ &\quad \left[\frac{\partial}{\partial z} \frac{1}{r^2} \sin \theta \cos \theta \psi_1 \psi_1\right]_{z=0} \end{aligned} \quad (83)$$

in which we used the fact that $\text{Re } i(z + z^*) = 0$, and in which:

$$G_1 = g_{1,TE}^{0*}(g_{1,TM}^{-1} - g_{1,TM}^1) + g_{1,TM}^0(g_{1,TE}^{-1*} - g_{1,TE}^{1*}) \quad (84)$$

$$G_2 = -g_{1,TE}^{0*}(g_{1,TM}^1 + g_{1,TM}^{-1}) + g_{1,TM}^0(g_{1,TE}^{-1*} + g_{1,TE}^{1*}) \quad (85)$$

G_1 and G_2 are now evaluated using the now usual manner, leading to:

$$G_1 = |g(\alpha_0)|^2 (G_1^{-1}e^{-i\phi_0} - G_1^1e^{i\phi_0} - G_1^{-3}e^{-3i\phi_0} - G_1^3e^{3i\phi_0}) \quad (86)$$

in which:

$$G_1^{-1} = S\left[\frac{C-1}{2}(J_{l+1}J_{l+2} + J_{l-2}J_{l-1}) - (C+1)J_{l-1}J_l\right] \quad (87)$$

$$G_1^1 = S\left[\frac{C-1}{2}(J_{l+1}J_{l+2} + J_{l-2}J_{l-1}) - (C+1)J_lJ_{l+1}\right] \quad (88)$$

$$G_1^{-3} = G_1^3 = \frac{S(C-1)}{2}(J_{l-1}J_{l+2} - J_{l-2}J_{l+1}) \quad (89)$$

Therefore, $G_1 = 0$ if (i) the axicon angle is zero ($S = 0$, $C = 1$) and (ii) if the off-axis configuration becomes an on-axis configuration. As for G_2 , we find:

$$G_2 = |g(\alpha_0)|^2 (G_2^{-1}e^{-i\phi_0} - G_2^1e^{i\phi_0} - G_2^{-3}e^{-3i\phi_0} + G_2^3e^{3i\phi_0}) \quad (90)$$

in which:

$$G_2^{-1} = G_1^1, G_2^1 = -G_1^{-1}, G_2^{-3} = G_1^{-3}, G_2^3 = G_1^3 \quad (91)$$

which is commented similarly as G_1 .

Therefore, as a whole, although $(\partial S_z / \partial z)_{z=0}$ is equal to zero, it happens that $(\partial \widetilde{S}_z / \partial z)_{z=0}$ is in general not equal to 0, although it is zero (i) if the axicon angle is zero (ii) for an on-axis configuration.

6 Discussions related to axicon forces and axicon terms.

The contents of this section arises from reviewer comments and from mail exchanges with Pr. James Lock from Cleveland University, USA.

6.1 Relationship with the standard theory of scattering and optical forces.

The previous paper [1] introduces new axicon forces which have no counterpart in the usual standard theory of optical forces expressed in terms of scattering and of gradient forces. We begin with axicon forces discussed in [1]. We may then observe that the newly discussed axicon forces indeed arise from higher-order multipole contributions. Such high-order contributions arise from the structure of Eq.(1) in [1] which expresses the longitudinal radiation pressure cross-section $C_{pr,z}$. BSCs then occur in a coupled way in terms reading as $g_{n,TM}^m g_{n+1,TM}^{m*}$, $g_{n,TE}^m g_{n+1,TE}^{m*}$ and $g_{n,TM}^m g_{n,TE}^{m*}$ ($m = -n, \dots, +n$). In the Rayleigh limit of the GLMT, we only retain ($n = 1$)- partial waves so that we have to deal with BSCs coupling terms of the form $g_{1,TM}^m g_{2,TM}^{m*}$, $g_{1,TE}^m g_{2,TE}^{m*}$ and $g_{1,TM}^m g_{1,TE}^{m*}$ ($m = -1, 0, +1$). The standard scattering force $C_{pr,z}^{s11}$ is expressed by Eq.(67) of [1] containing couplings of the form $g_{1,TM}^{-1} g_{1,TE}^{-1*}$ and $g_{1,TM}^1 g_{1,TE}^{1*}$ which therefore do not involve any higher-order multipole contributions. Conversely, the new axicon forces $C_{pr,z}^{s12}$ and $C_{pr,z}^{s0}$ are expressed by Eqs.(49) and (50) of [1], which are expressed in terms of quantities denoted G^{12} and G^0 given by Eqs.(45) and (46) of [1]. We then observe that G^{12} involves couplings of the form $g_{1,TM}^{-1} g_{2,TM}^{-1*}$ and $g_{1,TM}^1 g_{2,TM}^{1*}$ while G^0 involves $g_{1,TM}^0 g_{2,TM}^{0*}$, that is to say involves coupling with higher-order multipole contributions, namely between ($n = 1$)– and ($n = 2$)–multipoles. In other words, the Rayleigh limit of the GLMT does not exactly identify with the Rayleigh dipole theory of optical forces.

6.2 Relationship with the localization principle of van de Hulst.

A plane wave of infinite extent may be thought of as being made up of separate rays of light each independently pursuing its own path. According to the van de Hulst principle of localization, e.g. p.208 of [16], a partial wave of order n in the electromagnetic description of the plane wave corresponds to a ray passing at a distance from the origin equal to $(n + 1/2)(\lambda/2\pi)$. Here, the partial wave of order n refers to a term in the classical Lorenz-Mie theory which contains Bessel functions and spherical harmonics, with the integer n ranging from 1 to

∞ . This van de Hulst principle of localization has been successfully extended from the case of plane waves to the case of structured beams and has been at the origin of the localized approximation to the evaluation of BSCs, e.g. [33], [34], see [35] for a review.

Let us first consider an on-axis location of the particle or, equivalently, in the present context, a coordinate system centered on the beam, and let us call z_{on} its axis which is as well the direction of propagation of the beam (and which is parallel to the z -axis considered in the off-axis configuration studied in the present paper). The GLMT expands the on-axis wave in terms of partial waves characterized by two subscripts n_{on} (the order of the partial wave) and m_{on} (the azimuthal mode). The van de Hulst principle then associates the partial wave n_{on} with a ray passing at a distance $(n_{on} + 1/2)(\lambda/2\pi)$ from the axis z_{on} . Note however that the value of this distance is questionable because its derivation, using the method of stationary phase – not available from van de Hulst’s book – requires that $n \gg 1$ (James Lock, private communication), but this feature does not change the gist of the argument discussed in this subsection. In the Rayleigh limit, only the $n_{on} = 1$ partial wave interacts with the particle because only the Mie coefficient a_1 is to be retained. This corresponds to a ray passing at a distance equal to $d_{on} = 3\lambda/(4\pi) \approx \lambda/4$ from the z_{on} -axis.

The off-axis coordinate system is the one used in the present paper, with Cartesian coordinates (x, y, z) and partial waves characterized by the order n_{off} and the azimuthal mode m_{off} (more simply denoted as n and m in the present paper). The off-axis description of the beam is more complicated than the on-axis description of the beam, but such a complication is required in order to express the light scattered by the particle in terms of spherical coordinates (r, θ, φ) centered on the particle which is located at the origin of the Cartesian coordinates of the off-axis beam configuration. We now consider the van de Hulst principle of localization in the off-axis coordinate system. Similarly as above, in the Rayleigh regime, only the $n_{off} = 1$ partial wave interacts with the particle, corresponding to a ray passing at a distance $d_{off} = \lambda/4$ from the z_{off} -axis which may be at a distance $d_{on/off}$ which is large with respect to the z_{on} -axis.

The fact that only $n_{on} = 1$ and $n_{off} = 1$ are to be retained in the on-axis and off-axis configurations respectively does not depend on the respective values of BSCs associated with these particular partial waves, but results from the fact that only the Mie scattering coefficient a_1 is to be retained in the calculation. But the fact that optical forces generated by rays passing at d_{on} from z_{on} in the on-axis configuration or by rays passing at $d_{on/off}$ from the same axis in the off-axis configuration is linked to the specific values taken by the BSCs in these different configurations.

Furthermore, in both cases, a large enough particle would interact with all partial waves, whether it is in an on-axis or in an off-axis configuration, and therefore the gradient force would be zero insofar as there is no longitudinal gradient of $\mathbf{E} \cdot \mathbf{E}^*$ associated in the whole beam, in agreement with the standard theory of optical forces expressed in terms of scattering and gradient forces. Conversely, a Rayleigh particle interacting only with the $n = 1$ partial waves

does not interact with the whole beam but only with a part of the whole beam. The axicon terms revealed in the present paper then indeed show that the longitudinal gradient of $\widetilde{\mathbf{E} \cdot \mathbf{E}^*}$ is not zero, from which we could expect a longitudinal gradient force which would be non zero as well. The fact that, nevertheless we have $C_{pr,z}^g = 0$, see Eq.(42) in [1], that is to say that the longitudinal gradient force is still zero may then be surprising. This is to be viewed as a coincidence or as a significant result requiring an explanation still to be revealed. In particular, the evaluation of $C_{pr,z}^g$ accounts for higher-order multipoles with $n = 2$ which does not occur in the gradient of $\widetilde{\mathbf{E} \cdot \mathbf{E}^*}$ and which are not included in the use of the van de Hulst principle. Whatever the future of this story, our calculation exhibits a case when there is a zero longitudinal gradient force although the longitudinal gradients "felt" by the particle are not zero, but are indeed expressed by what we have called axicon terms.

6.3 Further works.

Further works are required to complement what is already known and understood. The next step should be a study of transverse optical forces similar to the previous work [1] and to the present work, both of them devoted to longitudinal optical forces [36]. Next, the on-axis configuration of Bessel beams should be studied (i) to confirm in a simplified framework that axicon forces and terms indeed cancel in such a configuration and (ii) to provide a comparison between the cases of dark and non dark beams which exhibit structural different features [19], [37]. We may anticipate that specific departures from the standard theory could occur for dark beams, although the anticipation could be deceived. Also, the case of off-axis Gaussian beams which can be analytically handled using a localized beam model [38] should be investigated in order to check whether the newly axicon forces and terms are specific of axicon beams, or whether they occur in more general situations. Our current anticipation, which could again be deceived, is that indeed the newly studied terms occur in other situations than the one of axicon beams.

7 Conclusion.

Since the work of Ashkin, see [39] for a compilation, we are used to think of optical forces in terms of gradient and scattering forces. However, in a recent paper studying longitudinal optical forces in the case of off-axis Bessel beams in the Rayleigh regime, a new unexpected kind of optical forces, associated with the scattering force procedure, named axicon forces have been uncovered. In the present paper, we demonstrated the existence of axicon terms associated with gradient forces. These axicon terms are zero if the axicon angle is zero

and/or if we are facing an on-axis configuration. It is expected that such axicon forces and terms would occur whenever axicon angles are involved in the beam description. This would encompass Bessel beams whatever the polarization, all kinds of beams obtained from Bessel beam superpositions, e.g. frozen waves, Mathieu and Lommel beams [40], [41], [42], [43], [44], [45], [46], [47], [48][49], [50], or Laguerre-Gauss beams focused by a lens [51], [52], [53], [54], [55].

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References

- [1] G. Gouesbet. Gradient, scattering and other kinds of longitudinal optical forces exerted by off-axis Bessel beams in the Rayleigh regime in the framework of generalized Lorenz-Mie theory. *Journal of Quantitative and Spectroscopy Transfer*, 246:Article 106913, 2020.
- [2] E.M. Furst. Interactions, structure, and microscopic response : complex fluid rheology using laser tweezers. *Soft Materials*, 1,2:167–185, 2003.
- [3] V.A. Soifer, V.V. Kotlyar, and S.N. Khonina. Optical microparticle manipulation : Advances and new possibilities created by diffractive optics. *Physics of Particles and Nuclei*, 35,6:733–766, 2004.
- [4] K.C. Neuman and S.M. Block. Optical trapping. *Review of Scientific Instruments*, 75,9:2787–2809, 2004.
- [5] T.A. Nieminen, G. Knoener, N.R. Heckenberg, and H. Rubinsztein-Dunlop. Physics of optical tweezers. *Laser Manipulation of Cells and Tissues*, 82:207–236, 2007.
- [6] M. Dienerowitz, M. Mazilu, and K. Dholakia. Optical manipulation of nanoparticles: a review. *Journal of nanophotonics*, 2:Paper 021875, 32 pages, 2008.
- [7] A. Jonas and P. Zemanek. Light at work: The use of optical forces for particle manipulation, sorting and analysis. *Electrophoresis*, 29:4813–4851, 2008.
- [8] T.A. Nieminen, N. du Preez-Wilkinson, A.B. Stilgoe, V.L.Y. Loke, A.A.M. Bui, and H. Rubinsztein-Dunlop. Optical tweezers: Theory and modelling. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 146:59–80, 2014.

- [9] A.-I. Bunea and J. Glückstad. Strategies for optical trapping in biological samples: Aiming at microrobotic surgeons. *Laser photonics Rev.*, Paper 1800227:17 pages, 2019.
- [10] G. Gouesbet. Generalized Lorenz-Mie theories and mechanical effects of laser light, on the occasion of Arthur Ashkin’s receipt of the 2018 Nobel prize in physics for his pioneering work in optical levitation and manipulation: A review. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 225:258–277, 2019.
- [11] G. Gouesbet, B. Maheu, and G. Gréhan. Light scattering from a sphere arbitrarily located in a Gaussian beam, using a Bromwich formulation. *Journal of the Optical Society of America A*, 5,9:1427–1443, 1988.
- [12] G. Gouesbet, G. Gréhan, and B. Maheu. *Combustion measurements, edited by N. Chigier*, chapter : Generalized Lorenz-Mie theory and applications to optical sizing, pages 339–384. Hemisphere Publishing Corporation, New-York, USA, 1991.
- [13] G. Gouesbet and G. Gréhan. *Generalized Lorenz-Mie theories, second edition*. Springer International Publishing AG, 2017.
- [14] G. Gouesbet. Latest achievements in generalized Lorenz-Mie theories: A commented reference database. *Annalen der Physik*, 526, 11-12:461–489, 2014.
- [15] G. Gouesbet. T-matrix methods for electromagnetic structured beams: A commented reference database for the period 2014-2018. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 230:247–281, 2019.
- [16] H.C. van de Hulst. *Light scattering by small particles*. Wiley, New York, 1957.
- [17] G. Gouesbet. T-matrix formulation and generalized Lorenz-Mie theories in spherical coordinates. *Optics Communications*, 283, 4:517–521, 2010.
- [18] G. Gouesbet. Partial wave expansions and properties of axisymmetric light beams. *Applied Optics*, 35,9:1543–1555, 1996.
- [19] G. Gouesbet. Poynting theorem in terms of beam shape coefficients and applications to axisymmetric, dark and non-dark, vortex and non-vortex beams. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 201:184–196, 2017.
- [20] G. Gouesbet and G. Gréhan. Sur la généralisation de la théorie de Lorenz-Mie. *Journal of Optics*, 13,2:97–103, 1982.
- [21] L. Robin. *Fonctions sphériques de Legendre et fonctions sphéroïdales. Volumes 1, 2, 3*. Gauthier-Villars, Paris, 1957.

- [22] G.B. Arfken and H.J. Weber. *Mathematical methods for physicists, sixth edition*. Elsevier Academic Press, Amsterdam, 2005.
- [23] J. Durnin, Jr. J.J. Miceli, and J.H. Eberly. Diffraction-free beams. *Physical Review Letters*, 58, 15:1499–1501, 1987.
- [24] J. Durnin. Exact solutions for nondiffracting beams. I. The scalar theory. *Journal of the Optical Society of America A*, 4, 4:651–654, 1987.
- [25] J.A. Lock. Angular spectrum and localized model of Davis-type beam. *Journal of the Optical Society of America A*, 30, 3:489–500, 2013.
- [26] J.J. Wang, T. Wriedt, J.A. Lock, and L. Mädler. General description of circularly symmetric Bessel beams of arbitrary order. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 184:218–232, 2016.
- [27] J.J. Wang, T. Wriedt, J.A. Lock, and Y.C. Jiao. General description of transverse mode Bessel beams and construction of basis Bessel fields. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 195:8–17, 2017.
- [28] J.J. Wang, T. Wriedt, L.Mädler, Y.P. Han, and P. Hartmann. Multipole expansion of circularly Bessel beams of arbitrary order for scattering calculations. *Optics Communications*, 387:102–109, 2017.
- [29] S.R. Mishra. A vector wave analysis of a Bessel beam. *Optics Communications*, 85:159–161, 1991.
- [30] T. Cizmar, V. Kollarova, Z. Bouchal, and P. Zemanek. Sub-micron particle organization by self-imaging of non-diffracting beams. *New Journal of Physics*, 8 (3), 43, 2006.
- [31] J.M. Taylor and G.D. Love. Multipole expansion of Bessel and Gaussian beams for Mie scattering calculations. *Journal of the Optical Society of America A*, 26, 2:278–282, 2009.
- [32] J. Chen, J. Ng, P. Wang, and Z. Lin. Analytical partial wave expansion of vector Bessel beam and its application to optical binding. *Optics Letters*, 35, 10:1674–1676, 2010.
- [33] G. Gréhan, B. Maheu, and G. Gouesbet. Scattering of laser beams by Mie scatter centers: numerical results using a localized approximation. *Applied Optics*, 25,19:3539–3548, 1986.
- [34] B. Maheu, G. Gréhan, and G. Gouesbet. Generalized Lorenz-Mie theory: first exact values and comparisons with the localized approximation. *Applied Optics*, 26,1:23–25, 1987.
- [35] G. Gouesbet, J.A. Lock, and G. Gréhan. Generalized Lorenz-Mie theories and description of electromagnetic arbitrary shaped beams: localized approximations and localized beam models, a review. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 112:1–27, 2011.

- [36] G. Gouesbet. Axicon optical forces and other kinds of transverse optical forces exerted by off-axis Bessel beams in the Rayleigh regime in the framework of generalized Lorenz-Mie theory. *In preparation*.
- [37] G. Gouesbet and J.A. Lock. A darkness theorem for the beam shape coefficients and its relationship to higher-order non vortex Bessel beams. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 201:229–235, 2017.
- [38] G. Gouesbet and J.A. Lock. Rigorous justification of the localized approximation to the beam shape coefficients in generalized Lorenz-Mie theory. II. Off-axis beams. *Journal of the Optical Society of America A*, 11,9:2516–2525, 1994.
- [39] A. Ashkin. *Optical trapping and manipulations of neutral particles using lasers: A reprint volume with commentaries*. World Scientific, Singapore, 2006.
- [40] M. Zamboni-Rached. Stationary optical wave fields with arbitrary longitudinal shape by superposing equal frequency Bessel beams: Frozen waves. *Optics Express*, 12, 17:4001–4006, 2004.
- [41] L.A. Ambrosio and M. Zamboni-Rached. Analytical approach of ordinary frozen waves for optical trapping and micromanipulation. *Applied Optics*, 54, 10:2584–2593, 2015.
- [42] L.A. Ambrosio and M. Zamboni-Rached. Optical forces experienced by arbitrary-sized spherical scatterers from superpositions of equal-frequency Bessel beams. *Journal of the Optical Society of America B*, 32, 5:B37–B46, 2015.
- [43] L.A. Ambrosio. Circularly symmetric frozen waves: Vector approach for light scattering calculations. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 204:112–119, 2018.
- [44] L.A. Ambrosio, M. Zamboni-Rached, and G. Gouesbet. Discrete vector frozen waves in generalized Lorenz-Mie theory: linear, azimuthal and radial polarization. *Applied Optics*, 57, 12:3293–3300, 2018.
- [45] L.A. Ambrosio, L.F.M. Votto, G. Gouesbet, and J.J. Wang. Assessing the validity of the localized approximation for discrete superposition of Bessel beams. *Journal of the Optical Society of America B*, 35, 11:2690–2698, 2018.
- [46] L.A. Ambrosio, M. Zamboni Rached, and G. Gouesbet. Zeroth-order continuous vector frozen waves for light scattering: exact multipole expansion in the generalized Lorenz-Mie theory. *Journal of the Optical Society of America B*, 36, 1:81–89, 2019.

- [47] A. Chafiq, Z. Hricha, and A. Belafhal. Paraxial approximation of Mathieu beams through an apertured ABCD optical system. *Optics Communications*, 253:223–230, 2005.
- [48] A. Chafiq and A. Belafhal. Radiation pressure cross section exerted on homogeneous dielectric spherical particle by zeroth order Mathieu beams. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 179:170–176, 2016.
- [49] A. Chafiq, L.A. Ambrosio, G. Gouesbet, and A. Belafhal. On the validity of the integral localized approximation for on-axis zeroth-order Mathieu beams. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 204:27–34, 2018.
- [50] A. Chafiq and A. Belafhal. Scattering of Lommel beams by homogeneous spherical particle in generalized Lorenz-Mie theory. *Optical and Quantum Electronics*, 2/2018, 2018.
- [51] A.S. van de Nes, S.F. Pereira, and J.J.M. Braat. On the conservation of fundamental optical quantities in non-paraxial imaging systems. *Journal of Modern Optics*, 53, 5-6:677–687, 2006.
- [52] A.S. van de Nes and P. Török. Rigorous analysis of spheres in Gauss-Laguerre beams. *Optics Express*, 15,20:13360–13374, 2007.
- [53] P. Török and P.R.T. Munro. The use of Gauss-Laguerre vector beams in STED microscopy. *Optics Express*, 12, 15:3605–3617, 2004.
- [54] L.A. Ambrosio and G. Gouesbet. On localized approximations for Laguerre-Gauss beams focused by a lens. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 218:100–114, 2018.
- [55] G. Gouesbet, L.A. Ambrosio, and L.F.M. Votto. Finite series expressions to evaluate the beam shape coefficients of a Laguerre-Gauss beam focused by a lens in an on-axis configuration. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 242:Paper 106759, 17 pages, 2019.