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Axicon optical forces and other kinds of transverse optical forces exerted by off-axis Bessel beams in the Rayleigh regime in the framework of generalized Lorenz-Mie theory.

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Abstract

In two recent papers, it has been demonstrated that, beside usual scattering and optical forces, new optical forces and terms are exhibited when a Rayleigh particle is illuminated by an off-axis Bessel beam. Namely, (i) axicon optical forces are associated with scattering optical forces and (ii) additional axicon terms are associated with gradient optical forces. These extra-axicon forces and terms are zero when the axicon angle is zero and/or when an on-axis configuration is considered rather than an off-axis configuration. This study was devoted to longitudinal forces. The present paper is devoted to transverse forces and demonstrates the existence of newly additional axicon forces associated to both the usual scattering and gradient forces. Again, these new extra-axicon forces are zero when the axicon angle is zero and/or when an on-axis configuration is considered rather than an off-axis configuration.

Keywords: optical forces; gradient and scattering optical forces; axicon forces; Bessel beams; generalized Lorenz-Mie theory; Rayleigh regime.

1 Introduction.

After Arthur Ashkin's pioneering work in optical levitation and manipulation, e.g. a collection of preprints in [1], a great deal of work has been devoted to the study of optical forces exerted by laser beams on particles, e.g. [2], [3], [4], [5], [6], [7], [8], [9], [10] for reviews. In two recent papers [11], [12], the attention was focused on the case of longitudinal optical forces exerted on Rayleigh particles illuminated by off-axis Bessel beams. It has then unexpectedly been found that scattering optical forces are accompanied by extra-axicon optical forces (which may be called axicon scattering forces), and that gradient optical forces (which are found to be zero) are accompanied by extra-axicon terms (which may be called axicon gradient terms) which are not zero. Furthermore, these extra-forces and terms are zero when the axicon angle is zero and/or when an on-axis configuration is considered instead of an off-axis configuration.

Although we are taking the risk of a repetition, it is important to insist on the meaning of new expressions introduced in the present paper and in previous papers [11], [12]. The new terminology introduces "axicon forces" and "axicon terms". Axicon forces are forces which are deduced from the expressions for pressure radiation cross-sections (or forces) but which have no counterpart in expressions dealing with scattering expressions, e.g. such as expressed by the Poynting vector, which may be called axicon scattering forces as examined in [11], or which have no counterpart in expressions dealing with the usual gradients of the gradient optical forces theory, which may be called axicon gradient forces, and which are examined in the present paper. Conversely, axicon terms are extra-terms which do not arise from the expressions for pressure radiation cross-sections (or forces) but which arise from gradient expressions. Axicon terms associated with gradient expressions have been examined in a previous paper [12].

While previous studies were devoted to longitudinal forces, the present paper is devoted to a similar study devoted to transverse forces. We shall demonstrate the existence of axicon optical forces (which may be called axicon gradient forces) associated with gradient optical forces (which are found to be zero) and of additional axicon forces (which may be called axicon scattering forces) which are associated with the scattering optical forces. Again, these new extra-axicon forces are zero when the axicon angle is zero and/or when an on-axis configuration is considered rather than an off-axis configuration. This study is carried out in the framework of the generalized Lorenz-Mie which was the framework used as well previously for the study of longitudinal optical forces.

The paper is organized as follows. Section 2 provides a background to be used in the sequel, concerning (i) optical forces in the framework of GLMT, (ii) Poynting vector in the framework of GLMT, (iii) basic mathematical formulae for further use and (iv) the description of Bessel beams. Section 3 deals with optical forces exerted by Bessel beams on Rayleigh particles. In Section 4, the optical forces derived in the previous section are analyzed in terms of gradient and scattering forces, revealing the existence of the aforementioned axicon

forces. Section 5 provides a discussion. Section 6 provides a complementary discussion serving as well as a conclusion.

2 Physical and mathematical background.

The GLMT *stricto sensu* describes the interaction between an arbitrary (structured) shaped beam and a homogeneous spherical particle defined by its diameter and its complex refractive index, e.g. [13], [14], [15], and references therein. Also, see reviews [16], [17], in particular for other GLMTs devoted to the interaction between laser beams and other kinds of scatterers. In the present paper, we consider a Rayleigh particle located at the origin O_P of a Cartesian coordinate system O_Pxyz illuminated by an off-axis Bessel beam propagating along the z -direction.

2.1 Optical forces.

In the GLMT framework, we express the optical force components F_i by using pressure radiation cross-section components $C_{pr,i}$ ($i = x, y, z$) with the relationship $F_i = C_{pr,i}/c$ ([18], p.14). We may conveniently use a metonymique abuse of language in which we shall use "force" standing for "cross section". In the present paper, we deal with the transverse cross-sections (along x - and y -directions) reading as, e.g. Eqs.(3.181) and (3.185) in [15], with an "obvious" typo in Eq. (3.178) in which the last term should involve $g_{n,TE}^p g_{m,TM}^{p+1*}$ instead of $g_{n,TE}^p g_{m,TE}^{p+1*}$:

$$\begin{aligned}
C_{pr,x} &= \frac{\lambda^2}{2\pi} \sum_{p=1}^{\infty} \sum_{n=p}^{\infty} \sum_{m=p-1 \neq 0}^{\infty} \frac{(n+p)!}{(n-p)!} \\
&\times [\text{Re}(S_{mn}^{p-1} + S_{nm}^{-p} - 2U_{mn}^{p-1} - 2U_{nm}^{-p}) (\frac{\delta_{m,n+1}}{m^2} - \frac{\delta_{n,m+1}}{n^2}) \\
&+ \frac{2n+1}{n^2(n+1)^2} \delta_{nm} \text{Re}(T_{mn}^{p-1} - T_{nm}^{-p} - 2V_{mn}^{p-1} + 2V_{nm}^{-p})]
\end{aligned} \tag{1}$$

in which:

$$S_{nm}^p = (a_n + a_m^*) g_{n,TM}^p g_{m,TM}^{p+1*} + (b_n + b_m^*) g_{n,TE}^p g_{m,TE}^{p+1*} \tag{2}$$

$$T_{nm}^p = -i(a_n + b_m^*) g_{n,TM}^p g_{m,TE}^{p+1*} + i(b_n + a_m^*) g_{n,TE}^p g_{m,TM}^{p+1*} \tag{3}$$

$$U_{nm}^p = a_n a_m^* g_{n,TE}^p g_{m,TM}^{p+1*} + b_n b_m^* g_{n,TE}^p g_{m,TE}^{p+1*} \quad (4)$$

$$V_{nm}^p = i b_n a_m^* g_{n,TE}^p g_{m,TM}^{p+1*} - i a_n b_m^* g_{n,TM}^p g_{m,TE}^{p+1*} \quad (5)$$

with notations being the ones of [13]. In particular, we used the normalization condition $E_0 H_0^* / 2 = 1$. The time-dependence of the wave is $\exp(+i\omega t)$ which is the usual choice in GLMT. Furthermore, specifically, a_n and b_n are the usual Mie coefficients of the usual Lorenz-Mie theory, $g_{n,TM}^m$ and $g_{n,TE}^m$, with TM and TE standing for "Transverse Magnetic" and "Transverse Electric" respectively, are the beam shape coefficients (BSCs) encoding the description of the beam, either in terms of scalar potentials, e.g. [15] or in terms of vector spherical wave functions [19], see [20], sections 2.5 and 2.6, for a review concerning the evaluations of BSCs. Also, the star denotes a complex conjugation and δ_{nm} is the Kronecker symbol. The y -component $C_{pr,y}$ is obtained from the x -component by changing Re to Im.

2.2 Poynting vector.

The transverse components of the Poynting vector are found to read as [21], [22]:

$$S_x = \frac{1}{2} E_0 H_0^* \operatorname{Re} \sum_{n=1}^{\infty} \sum_{m=-n}^{+n} \sum_{p=1}^{\infty} \sum_{q=-p}^{+p} c_n^{pw} c_p^{pw*} e^{i(m-q)\varphi} \quad (6)$$

$$\times \left[\frac{k \sin \varphi}{r} (\psi_n'' + \psi_n) A_{nmpq} + \frac{ik \cos \theta \cos \varphi}{r} (\psi_n'' + \psi_n) B_{nmpq} + \frac{i \sin \theta \cos \varphi}{r^2} C_{nmpq} \right]$$

$$S_y = \frac{1}{2} E_0 H_0^* \operatorname{Re} \sum_{n=1}^{\infty} \sum_{m=-n}^{+n} \sum_{p=1}^{\infty} \sum_{q=-p}^{+p} c_n^{pw} c_p^{pw*} e^{i(m-q)\varphi} \quad (7)$$

$$\times \left[-\frac{k \cos \varphi}{r} (\psi_n'' + \psi_n) A_{nmpq} + \frac{ik \cos \theta \sin \varphi}{r} (\psi_n'' + \psi_n) B_{nmpq} + \frac{i \sin \theta \sin \varphi}{r^2} C_{nmpq} \right]$$

in which :

$$A_{nmpq} = [\alpha_{nmpq} \psi_p' \tau_p^{|q|} + q \beta_{nmpq} \psi_p \pi_p^{|q|}] P_n^{|m|} \quad (8)$$

$$B_{nmpq} = [-q\alpha_{nmpq}\psi'_p\pi_p^{|q|} - \beta_{nmpq}\psi_p\tau_p^{|q|}]P_n^{|m|} \quad (9)$$

$$\begin{aligned} C_{nmpq} = & \alpha_{nmpq}(m\psi'_n\psi'_p\pi_n^{|m|}\tau_p^{|q|} + q\psi_n\psi_p\tau_n^{|m|}\pi_p^{|q|}) \\ & + \beta_{nmpq}(mq\psi'_n\psi_p\pi_n^{|m|}\pi_p^{|q|} - \psi_n\psi'_p\tau_n^{|m|}\tau_p^{|q|}) \end{aligned} \quad (10)$$

in which :

$$\alpha_{nmpq} = g_{p, TM}^{q*}g_{n, TE}^m - g_{n, TM}^m g_{p, TE}^{q*} \quad (11)$$

$$\beta_{nmpq} = g_{n, TM}^m g_{p, TM}^{q*} + g_{p, TE}^{q*} g_{n, TE}^m \quad (12)$$

in which the coefficients c_n^{pw} which are plane wave coefficients of the usual Lorenz-Mie theory in the Bromwich formulation read as [23]:

$$c_n^{pw} = \frac{1}{ik}(-i)^n \frac{2n+1}{n(n+1)} \quad (13)$$

in which k is the wavenumber, $P_n^{|m|} = P_n^{|m|}(\cos \theta)$ are associated Legendre functions, $\tau_n^{|m|} = \tau_n^{|m|}(\cos \theta)$ and $\pi_n^{|m|} = \pi_n^{|m|}(\cos \theta)$ are generalized Legendre functions, and $\psi_n = \psi_n(kr)$ are Ricatti-Bessel functions, while a prime denotes a derivative with respect to the argument (and a double prime a second derivative). In Eqs.6-10, and in other equations in the sequel, arguments are omitted for convenience

2.3 Mathematical background.

Associated Legendre functions are defined according to Hobson's notation [24]:

$$P_n^{|m|}(\cos \theta) = (-1)^{|m|}(\sin \theta)^{|m|} \frac{d^{|m|} P_n(\cos \theta)}{(d \cos \theta)^{|m|}} \quad (14)$$

in which $P_n(\cos \theta)$ are the usual Legendre polynomials. For the description of Bessel beams, we shall also need associated Legendre functions with a negative superscript. They can be obtained from [25]:

$$P_n^{-m}(\cos \theta) = (-1)^m \frac{(n-m)!}{(n+m)!} P_n^m(\cos \theta) \quad (15)$$

Generalized Legendre functions may then be evaluated according to:

$$\tau_n^m(\cos \theta) = \frac{dP_n^m(\cos \theta)}{d\theta} \quad (16)$$

$$\pi_n^m(\cos \theta) = \frac{P_n^m(\cos \theta)}{\sin \theta} \quad (17)$$

Ricatti-Bessel functions $\psi_n(kr)$ may be expressed in terms of spherical Bessel functions $j_n(kr)$ according to [26]:

$$\psi_n(x) = x j_n(x) \quad (18)$$

Furthermore, we may establish, see again [26]:

$$\psi_n'(x) = (n+1)j_n(x) - x j_{n+1}(x) \quad (19)$$

and we have [26]:

$$\lim_{x \rightarrow 0} j_n(x) = \frac{x^n}{(2n+1)!!} \quad (20)$$

We also have:

$$\left(\frac{d^2}{dr^2} + k^2\right) r j_n(kr) = \frac{n(n+1)}{r} j_n(kr) \quad (21)$$

2.4 Bessel beams.

Bessel beams have been introduced by Durnin and co-workers [27], [28]. They possess the appealing property of being self-healing and non diffracting, and, more important in the context of the present paper, they furthermore possess a propagation invariance property, namely the intensity of the beam is constant along the direction of propagation. There exist an infinity of kinds of Bessel beams, in particular depending on the value given to an arbitrary function $g(\alpha_0)$, in which α_0 is the axicon angle (or half-cone angle) of the beam, with different linear and circular polarizations [29], [30], [31], [32]. All of them generically exhibit the following structure:

$$K_i = k_i(x, y)e^{-ik_z z}, \quad i = x, y, z, \quad K = E \text{ or } H \quad (22)$$

in which $k_z = k \cos \alpha$ is the longitudinal wavenumber. Among all of them, we shall pay a particular attention to circularly symmetric Bessel beams of arbitrary order whose BSCs in an off-axis configuration read as [32]:

$$g_{n, TM}^m = -g(\alpha_0)e^{ik_z z_0} [i^{l-m+1} e^{i(l-m+1)\phi_0} A_n^m + i^{l-m-1} e^{i(l-m-1)\phi_0} B_n^m] \quad (23)$$

$$g_{n, TE}^m = ig(\alpha_0)e^{ik_z z_0} [i^{l-m+1} e^{i(l-m+1)\phi_0} A_n^m - i^{l-m-1} e^{i(l-m-1)\phi_0} B_n^m] \quad (24)$$

in which:

$$A_n^m = (-1)^{(m-|m|)/2} \frac{(n-m)!}{(n+|m|)!} J_{l-m+1}(\sigma_0) [\tau_n^m(\cos \alpha_0) + m\pi_n^m(\cos \alpha_0)] \quad (25)$$

$$B_n^m = (-1)^{(m-|m|)/2} \frac{(n-m)!}{(n+|m|)!} J_{l-m-1}(\sigma_0) [\tau_n^m(\cos \alpha_0) - m\pi_n^m(\cos \alpha_0)] \quad (26)$$

in which $\sigma_0 = k_t \rho_0$, $k_t = k \sin \alpha_0$ is the transverse wavenumber, $\rho_0 = (x_0^2 + y_0^2)^{1/2}$, $\phi_0 = \tan^{-1}(y_0/x_0)$, (x_0, y_0, z_0) denotes the location of the beam origin with respect to the origin of the particle coordinate system at which the scatterer

is located, and therefore specifies the off-axis location of the beam, and $J_k(\cdot)$ is the k -order Bessel function of the first kind. When $g(\alpha_0) = (1 + \cos \alpha_0)/4$, the beam reduces to a Davis circularly symmetric Bessel beam as discussed in [29], [33]. When $g(\alpha_0) = 1/2$, the beam reduces to another kind of Bessel beam discussed in [34], [35], [36]. In the sequel, it will be convenient to introduce the notations $C = \cos \alpha_0$ and $S = \sin \alpha_0$.

3 Optical forces exerted by Bessel beams on Rayleigh particles.

The expressions for the optical forces contain an infinite number of Mie coefficients a_n and b_n . For Rayleigh particles, the Mie coefficients reduce to [37], and [18], pp. 143-144:

$$a_1 = \frac{2i}{3} \frac{m^2 - 1}{m^2 + 2} \alpha^3 + O(i\alpha^5) + \frac{4}{9} \left(\frac{m^2 - 1}{m^2 + 2} \right)^2 \alpha^6 \quad (27)$$

$$b_1 = O(i\alpha^5) \quad (28)$$

in which m is the refractive index (here taken to be real) with respect to the surrounding medium and α the size parameter $\pi d/\lambda$. The other coefficients a_n and b_n ($n > 1$) involves still higher powers of α . Real parts are then proportional to α^6 and imaginary parts are proportional to α^3 while higher powers are discarded in the Rayleigh regime and we therefore retain only:

$$\text{Im}(a_1) = \frac{2}{3} \frac{m^2 - 1}{m^2 + 2} \alpha^3 \quad (29)$$

$$\text{Re}(a_1) = \frac{4}{9} \left(\frac{m^2 - 1}{m^2 + 2} \right)^2 \alpha^6 \quad (30)$$

Let us remark that m is used as both the real refractive index of the particle and as a superscript in the GLMT framework (there is however no confusion possible). We then start from Eq.1, and retain only the ($n = 1$)-partial waves and the Mie coefficient a_1 . We shall use a tilde to denote quantities, named reduced quantities, evaluated under such restrictions. We then have to evaluate $\widetilde{C}_{pr,x}$. To achieve the calculations, we recommend to split $C_{pr,x}$ into three terms according to $C_{pr,x} = C_{pr,x}^\alpha + C_{pr,x}^\beta + C_{pr,x}^\gamma$ in which:

$$C_{pr,x}^\alpha = \frac{\lambda^2}{2\pi} \sum_{p=1}^{\infty} \sum_{n=p}^{\infty} \sum_{m=p-1 \neq 0}^{\infty} \frac{(n+p)!}{(n-p)!} \operatorname{Re}(S_{mn}^{p-1} + S_{nm}^{-p} - 2U_{mn}^{p-1} - 2U_{nm}^{-p}) \frac{\delta_{m,n+1}}{m^2} \quad (31)$$

$$C_{pr,x}^\beta = \frac{-\lambda^2}{2\pi} \sum_{p=1}^{\infty} \sum_{n=p}^{\infty} \sum_{m=p-1 \neq 0}^{\infty} \frac{(n+p)!}{(n-p)!} \operatorname{Re}(S_{mn}^{p-1} + S_{nm}^{-p} - 2U_{mn}^{p-1} - 2U_{nm}^{-p}) \frac{\delta_{n,m+1}}{n^2} \quad (32)$$

$$C_{pr,x}^\gamma = \frac{\lambda^2}{2\pi} \sum_{p=1}^{\infty} \sum_{n=p}^{\infty} \sum_{m=p-1 \neq 0}^{\infty} \frac{(n+p)!}{(n-p)!} \frac{2n+1}{n^2(n+1)^2} \delta_{nm} \operatorname{Re}(T_{mn}^{p-1} - T_{nm}^{-p} - 2V_{mn}^{p-1} + 2V_{nm}^{-p}) \quad (33)$$

The reduced partial pressure radiation cross sections along the x -direction are then found to be:

$$\widetilde{C}_{pr,x}^\alpha = \frac{\lambda^2}{4\pi} \operatorname{Re}[a_1 g_{2,TM}^{0*} (g_{1,TM}^1 + g_{1,TM}^{-1})] \quad (34)$$

$$\begin{aligned} \widetilde{C}_{pr,x}^\beta = & \frac{-3\lambda^2}{4\pi} \operatorname{Re}[a_1 (g_{1,TM}^0 g_{2,TM}^{1*} + g_{1,TM}^0 g_{2,TM}^{-1*} \\ & + 4g_{1,TM}^1 g_{2,TM}^{2*} + 4g_{1,TM}^{-1} g_{2,TM}^{-2*})] \end{aligned} \quad (35)$$

$$\begin{aligned} \widetilde{C}_{pr,x}^\gamma = & \frac{3\lambda^2}{4\pi} \operatorname{Re}\{ia_1 [g_{1,TM}^0 (g_{1,TE}^{-1*} - g_{1,TE}^{1*}) \\ & + g_{1,TE}^{0*} (g_{1,TM}^{-1} - g_{1,TM}^1)]\} \end{aligned} \quad (36)$$

Assembling everything, we may then obtain:

$$\widetilde{C}_{pr,x} = \frac{\lambda^2}{4\pi} \operatorname{Re}(a_1 H) \quad (37)$$

in which:

$$\begin{aligned}
H = & g_{2, TM}^{0*}(g_{1, TM}^1 + g_{1, TM}^{-1}) - 3g_{1, TM}^0(g_{2, TM}^{1*} + g_{2, TM}^{-1*}) \\
& -12(g_{1, TM}^1 g_{2, TM}^{2*} + g_{1, TM}^{-1} g_{2, TM}^{-2*}) \\
& +3i[g_{1, TM}^0(g_{1, TE}^{-1*} - g_{1, TE}^{1*}) + g_{1, TE}^{0*}(g_{1, TM}^{-1} - g_{1, TM}^1)]
\end{aligned} \tag{38}$$

Similarly, the reduced y -component reads as:

$$\widetilde{C}_{pr,y} = \frac{\lambda^2}{4\pi} \text{Im}(a_1 H) \tag{39}$$

Using Eqs.29-30, and:

$$\text{Re}(a_1 H) = \text{Re}(a_1) \text{Re}(H) - \text{Im}(a_1) \text{Im}(H) \tag{40}$$

we may rewrite $\widetilde{C}_{pr,x}$ as the sum of two terms, one $\widetilde{C}_{pr,x}^{\text{Re}}$ proportional to α^6 and the other one $\widetilde{C}_{pr,x}^{\text{Im}}$ proportional to α^3 , according to:

$$\widetilde{C}_{pr,x}^{\text{Re}} = \frac{\lambda^2}{9\pi} \left(\frac{m^2 - 1}{m^2 + 2}\right)^2 \alpha^6 \text{Re}(H) \tag{41}$$

$$\widetilde{C}_{pr,x}^{\text{Im}} = \frac{-\lambda^2}{6\pi} \frac{m^2 - 1}{m^2 + 2} \alpha^3 \text{Im}(H) \tag{42}$$

Similarly, $\widetilde{C}_{pr,y}$ of Eq.39 may be rewritten as the sum of two terms reading as:

$$\widetilde{C}_{pr,y}^{\text{Re}} = \frac{\lambda^2}{6\pi} \frac{m^2 - 1}{m^2 + 2} \alpha^3 \text{Re}(H) \tag{43}$$

$$\widetilde{C}_{pr,y}^{\text{Im}} = \frac{\lambda^2}{9\pi} \left(\frac{m^2 - 1}{m^2 + 2}\right)^2 \alpha^6 \text{Im}(H) \tag{44}$$

We recall that, according to the literature, e.g. [37], [38], [39], [40], [41], and many other papers including my two recent papers on longitudinal forces [11], [12], the α^6 -term and the α^3 -term should correspond to the scattering and the gradient forces respectively. This expectation will not be perfectly fulfilled.

4 Axicon, scattering and gradient forces.

4.1 Reduced Poynting vector components.

We now evaluate the reduced Poynting vector components, at the location of the particle (which is located at the origin of the coordinates), starting from Eqs.6-7. The location of the particle is going to be implemented using two steps (i) considering a location on the axis $\theta = 0$ and (ii) considering the origin of the coordinates on the axis $z = r = 0$. The x -component of the reduced Poynting vector on the axis reads as:

$$[\widetilde{S}_x]_{\theta=0} = \frac{9}{4} \frac{\psi_1'' + \psi_1}{kr} \operatorname{Re} \sum_{m=-1}^{+1} \sum_{q=-1}^{+1} e^{i(m-q)\varphi} [\sin \varphi A_{1m1q} + i \cos \varphi B_{1m1q}] \quad (45)$$

in which we have used $c_1^{pw} c_1^{pw*} = 9/(4k^2)$. This may be expanded as:

$$[\widetilde{S}_x]_{\theta=0} = \frac{9}{8} \frac{\psi_1'' + \psi_1}{kr} \operatorname{Re} i \sum_{j=-3}^{+3} a^j e^{ij\varphi} \quad (46)$$

in which:

$$a^{-3} = A_{1-111} + B_{1-111} \quad (47)$$

$$a^{-2} = A_{1-110} + B_{1-110} + A_{1011} + B_{1011} \quad (48)$$

$$a^{-1} = A_{1-11-1} + B_{1-11-1} - A_{1-111} + B_{1-111} + A_{1010} + B_{1010} + A_{1111} + B_{1111} \quad (49)$$

$$a^0 = -A_{1-110} + B_{1-110} + A_{101-1} + B_{101-1} - A_{1011} + B_{1011} + A_{1110} + B_{1110} \quad (50)$$

$$a^1 = -A_{1-11-1} + B_{1-11-1} - A_{1010} + B_{1010} + A_{111-1} + B_{111-1} - A_{1111} + B_{1111} \quad (51)$$

$$a^2 = -A_{101-1} + B_{101-1} - A_{1110} + B_{1110} \quad (52)$$

$$a^3 = -A_{111-1} + B_{111-1} \quad (53)$$

Coefficients A_{nmpq} and B_{nmpq} involved in Eqs.47-53 are afterward evaluated using Eqs.8-9 and expressing afterward the involved generalized Legendre functions and associated Legendre functions using Eqs.14-17 leading to:

$$a^{\pm 3} = a^{\pm 2} = a^{\pm 1} = 0 \quad (54)$$

and to:

$$a^0 = 2[\psi'_1(\alpha_{1011} - \alpha_{101-1}) + \psi_1(\beta_{1011} + \beta_{101-1})] \quad (55)$$

The reduced x -component of the Poynting vector on the axis then becomes:

$$[\widetilde{S}_x]_{\theta=0} = \frac{9}{4} \frac{\psi_1'' + \psi_1}{kr} [\psi_1' \operatorname{Re} i(\alpha_{1011} - \alpha_{101-1}) + \psi_1 \operatorname{Re} i(\beta_{1011} + \beta_{101-1})] \quad (56)$$

in which the coefficients α_{nmpq} and β_{nmpq} do not depend on the radial coordinate (see Eqs.11-12). When evaluating the component S_x of the Poynting vector, the terms $(\psi_1'' + \psi_1)$ on the one hand and $\psi_1'/(kr)$, $\psi_1/(kr)$ on the other hand came from different components of the electric and magnetic fields as we can see by using $S_x = \operatorname{Re}(E_y H_z^* - E_z H_y^*)/2$ and referring to Eqs.(3.39)-(3-50) in [15]. We may therefore consider separately the limits of these terms when $z = r = 0$. Using Eq.18 and referring to [26], we then readily establish:

$$\left[\frac{\psi_1(kr)}{kr} \right]_{r=0} = [j_1(kr)]_{kr=0} = 0 \quad (57)$$

Next, we use Eq.19, the fact that $[j_2(kr)]_{kr=0} = 0$, and Eq.(11.144) from [26] to establish:

$$\left[\frac{\psi_1'(kr)}{kr} \right]_{kr=0} = \frac{2}{3} \quad (58)$$

Next, we use Eqs.18, 21, and again Eq.(11.144) from [26] to establish:

$$[\psi_1''(kr) + \psi_1(kr)]_{kr=0} = \frac{2}{3} \quad (59)$$

Eq.56 then simply becomes:

$$[\widetilde{S}_x]_{\theta=r=0} = \text{Re } i(\alpha_{1011} - \alpha_{101-1}) \quad (60)$$

Using Eq.11, $[\widetilde{S}_x]_{\theta=r=0}$ may be eventually expressed in terms of BSCs as:

$$[\widetilde{S}_x]_{\theta=r=0} = \text{Re } i[g_{1,TM}^0(g_{1,TE}^{-1*} - g_{1,TE}^{1*}) + g_{1,TE}^0(g_{1,TM}^{1*} - g_{1,TM}^{-1*})] \quad (61)$$

Anticipating, it is more convenient to rewrite this result as $[\widetilde{S}_x]_{\theta=r=0} = [\widetilde{S}_x]_{\theta=r=0}^1 + [\widetilde{S}_x]_{\theta=r=0}^2$ in which:

$$[\widetilde{S}_x]_{\theta=r=0}^1 = \text{Re } i[g_{1,TM}^0(g_{1,TE}^{-1*} - g_{1,TE}^{1*})] \quad (62)$$

$$\begin{aligned} [\widetilde{S}_x]_{\theta=r=0}^2 &= \text{Re } i[g_{1,TE}^0(g_{1,TM}^{1*} - g_{1,TM}^{-1*})] \\ &= \text{Re } i[g_{1,TE}^{0*}(g_{1,TM}^{-1} - g_{1,TM}^1)] \end{aligned} \quad (63)$$

in which we have used $\text{Re}(iz) = -\text{Re}(iz^*)$.

To evaluate explicitly these quantities, we proceed using three steps as follows: (i) express the BSCs in terms of A_n^m 's and B_n^m 's using Eqs.23-24, (ii) express these coefficients in terms of Bessel functions and generalized Legendre functions using Eqs.25-26 and (iii) express the generalized Legendre functions in terms of $S = \sin \alpha_0$ and $C = \cos \alpha_0$ using Eqs.16-17 and 14-15 (in the sequel this will be called the three-step procedure). We then obtain:

$$\begin{aligned} [\widetilde{S}_x]_{\theta=r=0}^1 &= |g(\alpha_0)|^2 \frac{S}{2} \{ \sin(3\phi_0)(C-1)(-J_{l-1}J_{l+2} - J_{l-2}J_{l+1}) \\ &\quad + \sin \phi_0[(C-1)(J_{l-2}J_{l-1} + J_{l+1}J_{l+2}) - 2(C+1)J_l(J_{l-1} + J_{l+1})] \} \end{aligned} \quad (64)$$

$$\begin{aligned} [\widetilde{S}_x]_{\theta=r=0}^2 &= |g(\alpha_0)|^2 \frac{S}{2} (C-1) [\sin(3\phi_0)(J_{l-1}J_{l+2} + J_{l-2}J_{l+1}) \\ &\quad + \sin\phi_0(J_{l-2}J_{l-1} + J_{l+1}J_{l+2})] \end{aligned} \quad (65)$$

and:

$$\begin{aligned} [\widetilde{S}_x]_{\theta=r=0} &= [\widetilde{S}_x]_{\theta=r=0}^1 + [\widetilde{S}_x]_{\theta=r=0}^2 = |g(\alpha_0)|^2 S \sin\phi_0 \\ &\quad \times [(C-1)(J_{l-2}J_{l-1} + J_{l+1}J_{l+2}) - (C+1)J_l(J_{l-1} + J_{l+1})] \end{aligned} \quad (66)$$

Similarly, after repetition of the calculations for the y -component, we obtain:

$$[\widetilde{S}_y]_{\theta=r=0} = \text{Re}(\alpha_{1011} + \alpha_{101-1}) \quad (67)$$

Using Eq.11, $[\widetilde{S}_y]_{\theta=r=0}$ may be eventually expressed in terms of BSCs as:

$$[\widetilde{S}_y]_{\theta=r=0} = \text{Re}[g_{1,TE}^0(g_{1,TM}^{1*} + g_{1,TM}^{-1*}) - g_{1,TM}^0(g_{1,TE}^{-1*} + g_{1,TE}^{1*})] \quad (68)$$

Anticipating, it is more convenient to rewrite this result as $[\widetilde{S}_y]_{\theta=r=0} = [\widetilde{S}_y]_{\theta=r=0}^1 + [\widetilde{S}_y]_{\theta=r=0}^2$ in which:

$$[\widetilde{S}_y]_{\theta=r=0}^1 = \text{Re}(g_{1,TE}^0 g_{1,TM}^{1*} - g_{1,TM}^0 g_{1,TE}^{-1*}) \quad (69)$$

$$[\widetilde{S}_y]_{\theta=r=0}^2 = \text{Re}(g_{1,TE}^0 g_{1,TM}^{-1*} - g_{1,TM}^0 g_{1,TE}^{1*}) \quad (70)$$

Using the three-step procedure, these expressions become:

$$\begin{aligned} [\widetilde{S}_y]_{\theta=r=0}^1 &= |g(\alpha_0)|^2 \frac{S}{2} \{ \cos(3\phi_0)(C-1)(J_{l-1}J_{l+2} - J_{l-2}J_{l+1}) \\ &\quad + \cos\phi_0[2(C+1)J_{l-1}J_l - (C-1)(J_{l-2}J_{l-1} + J_{l+1}J_{l+2})] \} \end{aligned} \quad (71)$$

$$\begin{aligned} [\widetilde{S}_y]_{\theta=r=0}^2 &= |g(\alpha_0)|^2 \frac{S}{2} \{ \cos(3\phi_0)(C-1)(J_{l-2}J_{l+1} - J_{l-1}J_{l+2}) \\ &\quad + \cos\phi_0[2(C+1)J_lJ_{l+1} - (C-1)(J_{l-2}J_{l-1} + J_{l+1}J_{l+2})] \} \end{aligned} \quad (72)$$

and, as a whole:

$$\begin{aligned} [\widetilde{S}_y]_{\theta=r=0} &= [\widetilde{S}_y]_{\theta=r=0}^1 + [\widetilde{S}_y]_{\theta=r=0}^2 = |g(\alpha_0)|^2 S \cos\phi_0 \\ &\quad \times [(C+1)J_l(J_{l-1} + J_{l+1}) - (C-1)(J_{l-2}J_{l-1} + J_{l+1}J_{l+2})] \end{aligned} \quad (73)$$

We therefore have an appealing relation reading as:

$$\frac{[\widetilde{S}_x]_{\theta=r=0}}{[\widetilde{S}_y]_{\theta=r=0}} = -\tan\phi_0 \quad (74)$$

4.2 Scattering and axicon scattering forces.

We now recall that the reduced x -component of the α^6 -cross section $\widetilde{C}_{pr,x}^{\text{Re}}$ is given by Eq.41 with H given by Eq.38. It is now convenient to express $\widetilde{C}_{pr,x}^{\text{Re}}$ as a summation of three terms according to:

$$\widetilde{C}_{pr,x}^{\text{Re}} = (\widetilde{C}_{pr,x}^{\text{Re}})^{as} + (\widetilde{C}_{pr,x}^{\text{Re}})^{s1} + (\widetilde{C}_{pr,x}^{\text{Re}})^{s2} \quad (75)$$

in which:

$$\begin{aligned} (\widetilde{C}_{pr,x}^{\text{Re}})^{as} &= \frac{\lambda^2}{9\pi} \left(\frac{m^2-1}{m^2+2}\right)^2 \alpha^6 \\ &\quad \times \text{Re}[g_{2,TM}^{0*}(g_{1,TM}^1 + g_{1,TM}^{-1}) - 3g_{1,TM}^0(g_{2,TM}^{1*} + g_{2,TM}^{-1*}) \\ &\quad - 12(g_{1,TM}^1 g_{2,TM}^{2*} + g_{1,TM}^{-1} g_{2,TM}^{-2*})] \end{aligned} \quad (76)$$

$$(\widetilde{C}_{pr,x}^{\text{Re}})^{s1} = \frac{\lambda^2}{3\pi} \left(\frac{m^2-1}{m^2+2}\right)^2 \alpha^6 \text{Re} i [g_{1,TM}^0 (g_{1,TE}^{-1*} - g_{1,TE}^{1*})] \quad (77)$$

$$(\widetilde{C}_{pr,x}^{\text{Re}})^{s2} = \frac{\lambda^2}{3\pi} \left(\frac{m^2-1}{m^2+2}\right)^2 \alpha^6 \text{Re } i[g_{1,TE}^{0*}(g_{1,TM}^{-1} - g_{1,TM}^1)] \quad (78)$$

We then observe that:

$$(\widetilde{C}_{pr,x}^{\text{Re}})^{s1} / [\widetilde{S}_x]_{\theta=r=0}^1 = (\widetilde{C}_{pr,x}^{\text{Re}})^{s2} / [\widetilde{S}_x]_{\theta=r=0}^2 = \frac{\lambda^2}{3\pi} \left(\frac{m^2-1}{m^2+2}\right)^2 \alpha^6 \quad (79)$$

Therefore $(\widetilde{C}_{pr,x}^{\text{Re}})^{s1}$ and $(\widetilde{C}_{pr,x}^{\text{Re}})^{s2}$ may be interpreted as describing scattering forces. Let us also note that $(\widetilde{C}_{pr,x}^{\text{Re}})^s = (\widetilde{C}_{pr,x}^{\text{Re}})^{s1} + (\widetilde{C}_{pr,x}^{\text{Re}})^{s2}$ is proportional to $[\widetilde{S}_x]_{\theta=r=0} = [\widetilde{S}_x]_{\theta=r=0}^1 + [\widetilde{S}_x]_{\theta=r=0}^2$ with the same coefficient of proportionality and may then be interpreted as a total scattering force. This will not exactly be expressed in the same way for the y -component justifying that $[\widetilde{S}_x]_{\theta=r=0}$ has been expressed as the sum of two terms in order to provide a fair comparison between the results of the x - and y -components. The scattering forces (partial or total) may then be explicitly expressed by using Eq.66. It is observed that the Poynting components $[\widetilde{S}_x]_{\theta=r=0}$, $[\widetilde{S}_x]_{\theta=r=0}^1$, $[\widetilde{S}_x]_{\theta=r=0}^2$ and the corresponding scattering forces $(\widetilde{C}_{pr,x}^{\text{Re}})^s$, $(\widetilde{C}_{pr,x}^{\text{Re}})^{s1}$, and $(\widetilde{C}_{pr,x}^{\text{Re}})^{s2}$ are zero (i) when the axicon angle is zero because we then have $S = 0$ and (ii) when we consider an on-axis situation σ_0 as the result of the fact that $J_l(0) = \delta_{l0}$. Furthermore, the fact that all the quantities discussed just above are zero when the axicon angle is zero is in agreement with the fact that the Angular Spectrum Decomposition (ASD), e.g. [20], subsection 2.6 for a review in a GLMT framework, is made from non tilted plane waves

The term $(\widetilde{C}_{pr,x}^{\text{Re}})^{as}$ is therefore to be interpreted as an extra-force (actually an extra-force cross-section), namely an axicon force associated with the scattering forces, which does not occur in the usual traditional partition of optical forces in terms of scattering and scattering forces. Using the three-step procedure, this axicon force is found to read as:

$$(\widetilde{C}_{pr,x}^{\text{Re}})^{as} = \frac{\lambda^2}{9\pi} \left(\frac{m^2-1}{m^2+2}\right)^2 \alpha^6 \text{Re}(H_1 - 3H_2 - 12H_3) \quad (80)$$

in which:

$$\begin{aligned} \text{Re}(H_1) &= 3CS |g(\alpha_0)|^2 \left\{ \frac{1}{2}(C-1)(J_{l-2}J_{l+1} + J_{l-1}J_{l+2}) \sin(3\phi_0) \right. \\ &\quad \left. - [(C+1)J_l(J_{l+1} + J_{l-1}) + \frac{1}{2}(C-1)(J_{l-2}J_{l-1} + J_{l+1}J_{l+2})] \sin \phi_0 \right\} \end{aligned} \quad (81)$$

$$\begin{aligned}
\text{Re}(H_2) &= S |g(\alpha_0)|^2 \left\{ \frac{1}{2} (2C^2 - C - 1) (J_{l-2} J_{l+1} + J_{l-1} J_{l+2}) \sin(3\phi_0) \right. \\
&\quad \left. - [(2C^2 + C - 1) J_l (J_{l+1} + J_{l-1}) + \frac{1}{2} (2C^2 - C - 1) (J_{l-2} J_{l-1} + J_{l+1} J_{l+2})] \sin \phi_0 \right\} \quad (82)
\end{aligned}$$

$$\begin{aligned}
\text{Re}(H_3) &= \frac{S}{8} |g(\alpha_0)|^2 \left\{ -(C+1)(C-1) J_l (J_{l-3} + J_{l+3}) \sin(3\phi_0) \right. \\
&\quad + [(C+1)^2 J_l (J_{l+1} + J_{l-1}) + (C-1)^2 (J_{l-3} J_{l-2} + J_{l+2} J_{l+3}) \\
&\quad \left. + (C+1)(C-1) (J_{l-2} J_{l-1} + J_{l+1} J_{l+2}) \right] \sin \phi_0 \} \quad (83)
\end{aligned}$$

Eq.80 has been presented preserving a natural summation of three terms in order to avoid awkward expressions and to help the reader who would like to check details, and possibly identify computational mistakes if any. What is important however is to observe once again that these terms are zero (i) when the axicon angle is zero because we then have $S = 0$ and (ii) when we consider an on-axis situation.

Scattering along the direction y is obtained similarly. We then recall that the reduced y -component of the α^6 -cross section $\widetilde{C}_{pr,y}^{\text{Im}}$ is given by Eq.44 with H again given by Eq.38. This is again conveniently expressed as a summation of three terms according to:

$$\widetilde{C}_{pr,y}^{\text{Im}} = (\widetilde{C}_{pr,y}^{\text{Im}})^{as} + (\widetilde{C}_{pr,y}^{\text{Im}})^{s1} + (\widetilde{C}_{pr,y}^{\text{Im}})^{s2} \quad (84)$$

in which:

$$\begin{aligned}
(\widetilde{C}_{pr,y}^{\text{Im}})^{as} &= \frac{\lambda^2}{9\pi} \left(\frac{m^2 - 1}{m^2 + 2} \right)^2 \alpha^6 \\
&\quad \text{Im} [g_{2,TM}^{0*} (g_{1,TM}^1 + g_{1,TM}^{-1}) - 3g_{1,TM}^0 (g_{2,TM}^{1*} + g_{2,TM}^{-1*}) \\
&\quad - 12(g_{1,TM}^1 g_{2,TM}^{2*} + g_{1,TM}^{-1} g_{2,TM}^{-2*})] \quad (85)
\end{aligned}$$

$$\begin{aligned}
(\widetilde{C}_{pr,y}^{\text{Im}})^{s1} &= \frac{\lambda^2}{3\pi} \left(\frac{m^2 - 1}{m^2 + 2} \right)^2 \alpha^6 \text{Im} i (g_{1,TM}^0 g_{1,TE}^{-1*} - g_{1,TE}^{0*} g_{1,TM}^1) \\
&= \frac{\lambda^2}{3\pi} \left(\frac{m^2 - 1}{m^2 + 2} \right)^2 \alpha^6 \text{Re} (g_{1,TM}^0 g_{1,TE}^{-1*} - g_{1,TE}^0 g_{1,TM}^{1*}) \quad (86)
\end{aligned}$$

$$\begin{aligned}
(\widetilde{C}_{pr,y}^{\text{Im}})^{s2} &= \frac{\lambda^2}{3\pi} \left(\frac{m^2 - 1}{m^2 + 2} \right)^2 \alpha^6 \text{Im} i (g_{1,TE}^{0*} g_{1,TM}^{-1} - g_{1,TM}^0 g_{1,TE}^{1*}) \quad (87) \\
&= \frac{\lambda^2}{3\pi} \left(\frac{m^2 - 1}{m^2 + 2} \right)^2 \alpha^6 \text{Re} (g_{1,TE}^0 g_{1,TM}^{-1*} - g_{1,TM}^0 g_{1,TE}^{1*})
\end{aligned}$$

in which we have used $\text{Im}(iz) = \text{Im}(iz^*) = \text{Re}(z)$. We then observe that:

$$(\widetilde{C}_{pr,y}^{\text{Im}})^{s2} / [\widetilde{S}_y]_{\theta=r=0}^2 = -(\widetilde{C}_{pr,y}^{\text{Im}})^{s1} / [\widetilde{S}_y]_{\theta=r=0}^1 = \frac{\lambda^2}{3\pi} \left(\frac{m^2 - 1}{m^2 + 2} \right)^2 \alpha^6 \quad (88)$$

Therefore $(\widetilde{C}_{pr,y}^{\text{Im}})^{s1}$ and $(\widetilde{C}_{pr,y}^{\text{Im}})^{s2}$ may be interpreted as describing scattering forces. However, in contrast with the x -component situation where we used a summation $(\widetilde{C}_{pr,x}^{\text{Re}})^s = (\widetilde{C}_{pr,x}^{\text{Re}})^{s1} + (\widetilde{C}_{pr,x}^{\text{Re}})^{s2}$, we now compulsorily have to distinguish between $(\widetilde{C}_{pr,y}^{\text{Im}})^{s1}$ and $(\widetilde{C}_{pr,y}^{\text{Im}})^{s2}$ due to the minus sign of Eq.88. Once again, it is observed that the total Poynting component $[\widetilde{S}_y]_{\theta=r=0}$ of Eq.68, the partial Poynting components $[\widetilde{S}_y]_{\theta=r=0}^1$, $[\widetilde{S}_y]_{\theta=r=0}^2$ of Eqs.69 and 70, and the corresponding scattering forces, $(\widetilde{C}_{pr,y}^{\text{Re}})^{s1}$, and $(\widetilde{C}_{pr,y}^{\text{Re}})^{s2}$ are zero (i) when the axicon angle is zero because we then have $S = 0$ and (ii) when we consider an on-axis configuration. Once again, this was to be expected if we refer to the fact that the ASD of a Bessel beam with an axicon angle equal to zero is made of non tilted plane waves.

The term $(\widetilde{C}_{pr,y}^{\text{Im}})^{as}$ is therefore to be interpreted as an extra-force (actually an extra-force cross-section), namely an axicon force associated with the scattering forces, which again does not occur in the usual traditional partition of optical forces in terms of scattering and scattering forces. Using the three-step procedure, this axicon force is found to read as:

$$(\widetilde{C}_{pr,y}^{\text{Im}})^{as} = \frac{\lambda^2}{9\pi} \left(\frac{m^2 - 1}{m^2 + 2} \right)^2 \alpha^6 \text{Im}(H_1 - 3H_2 - 13H_3) \quad (89)$$

in which:

$$\begin{aligned}
\text{Im}(H_1) &= 3CS |g(\alpha_0)|^2 \left\{ \frac{1}{2} (C - 1) (J_{l-2} J_{l+1} - J_{l-1} J_{l+2}) \cos(3\phi_0) \right. \quad (90) \\
&\quad \left. + [(C + 1) J_l (J_{l-1} - J_{l+1}) + \frac{1}{2} (C - 1) (J_{l+1} J_{l+2} - J_{l-1} J_{l+2})] \cos \phi_0 \right\}
\end{aligned}$$

$$\begin{aligned}
\text{Im}(H_2) &= S |g(\alpha_0)|^2 \left\{ \frac{1}{2} (2C^2 - C - 1) (J_{l-1} J_{l+2} - J_{l-2} J_{l+1}) \cos(3\phi_0) \right. \\
&\quad \left. + [(2C^2 + C - 1) J_l (J_{l+1} - J_{l-1}) + \frac{1}{2} (2C^2 - C - 1) (J_{l-2} J_{l-1} - J_{l+1} J_{l+2})] \cos \phi_0 \right\} \quad (91)
\end{aligned}$$

$$\begin{aligned}
\text{Im}(H_3) &= \frac{S}{8} |g(\alpha_0)|^2 \left\{ (C + 1)(C - 1) J_l (J_{l-3} - J_{l+3}) \cos(3\phi_0) \right. \\
&\quad \left. + [(C + 1)^2 J_l (J_{l+1} - J_{l-1}) + (C - 1)^2 (J_{l+2} J_{l+3} - J_{l-3} J_{l-2}) \right. \\
&\quad \left. + (C + 1)(C - 1) (J_{l-2} J_{l-1} - J_{l+1} J_{l+2}) \right\} \cos \phi_0 \quad (92)
\end{aligned}$$

in which we again preferred to preserve a natural summation of three terms. Once again these terms are zero (i) when the axicon angle is zero because we then have $S = 0$ and (ii) when we consider an on-axis situation.

4.3 Gradient and axicon gradient forces.

We still have to clarify the meaning of the cross sections $\widetilde{C}_{pr,x}^{\text{Im}}$ and $\widetilde{C}_{pr,y}^{\text{Re}}$ given by Eqs.42 and 43 which would be expected to represent gradient forces (this expectation will not be fulfilled). Gradient forces along x and y should be proportional to the gradients of the (reduced) electric intensity $\widetilde{I}_E = \mathbf{E} \cdot \mathbf{E}^*$ with respect to the x - and y -directions respectively. The electric intensity has been evaluated in [12]. It reads as:

$$\widetilde{I}_E = \sum_{k=-2}^{+2} \widetilde{I}_E^k \quad (93)$$

with:

$$\begin{aligned}
\widetilde{I}_E^{-2} = \widetilde{I}_E^{2*} &= \frac{9 |E_0|^2}{4 k^2} e^{-2i\varphi} \sin^2 \theta \left\{ [k^2 (\psi_1'' + \psi_1)^2 - \frac{\psi_1' \psi_1'}{r^2}] g_{1, TM}^{-1} g_{1, TM}^{1*} + \frac{\psi_1 \psi_1}{r^2} g_{1, TE}^{-1} g_{1, TE}^{1*} \right\} \\
&\quad (94)
\end{aligned}$$

$$\begin{aligned}
\widetilde{I}_E^{-1} &= \widetilde{I}_E^{1*} = \frac{9 |E_0|^2}{4 k^2} e^{-i\varphi} \sin \theta \left\{ \cos \theta \left[\frac{\psi_1' \psi_1'}{r^2} - k^2 (\psi_1'' + \psi_1)^2 \right] (g_{1, TM}^{-1} g_{1, TM}^{0*} + g_{1, TM}^0 g_{1, TM}^{1*}) \right. \\
&\quad \left. + \frac{\psi_1' \psi_1}{r^2} (g_{1, TM}^0 g_{1, TE}^{1*} - g_{1, TE}^{-1} g_{1, TM}^{0*} - g_{1, TM}^{-1} g_{1, TE}^{0*} + g_{1, TE}^0 g_{1, TM}^{1*}) \right. \\
&\quad \left. + \cos \theta \frac{\psi_1 \psi_1}{r^2} (g_{1, TE}^{-1} g_{1, TE}^{0*} + g_{1, TE}^0 g_{1, TE}^{1*}) \right\} \quad (95)
\end{aligned}$$

$$\begin{aligned}
\widetilde{I}_E^0 &= \frac{9}{4} \frac{|E_0|^2}{k^2} e^{0\varphi} \{ [k^2 \sin^2 \theta (\psi_1'' + \psi_1)^2 + \cos^2 \theta \frac{\psi_1' \psi_1'}{r^2}] (g_{1, TM}^{-1} g_{1, TM}^{-1*} + g_{1, TM}^1 g_{1, TM}^{1*}) \\
&\quad + [k^2 \cos^2 \theta (\psi_1'' + \psi_1)^2 + \sin^2 \theta \frac{\psi_1' \psi_1'}{r^2}] g_{1, TM}^0 g_{1, TM}^{0*} \\
&\quad + 2 \cos \theta \frac{\psi_1' \psi_1'}{r^2} (g_{1, TM}^1 g_{1, TE}^{1*} - g_{1, TM}^{-1} g_{1, TE}^{-1*} + g_{1, TE}^1 g_{1, TM}^{1*} - g_{1, TE}^{-1} g_{1, TM}^{-1*}) \\
&\quad + (\cos^2 \theta + 1) \frac{\psi_1 \psi_1}{r^2} (g_{1, TE}^1 g_{1, TE}^{1*} + g_{1, TE}^{-1} g_{1, TE}^{-1*}) \\
&\quad + \sin^2 \theta \frac{\psi_1 \psi_1}{r^2} g_{1, TE}^0 g_{1, TE}^{0*} + \frac{\psi_1' \psi_1'}{r^2} (g_{1, TM}^{-1} g_{1, TM}^{-1*} + g_{1, TM}^1 g_{1, TM}^{1*}) \}
\end{aligned} \tag{96}$$

We then have to evaluate the derivatives of various x - and y -dependent quantities involved in these expressions, taken at the location of the particle. In doing so, we do not take into account the φ -dependent terms occurring in $\widetilde{I}_E^{\pm 2}$ and in $\widetilde{I}_E^{\pm 1}$. The reason why is that the terms to be derived are of the form $\sin \theta f(r, \theta) g(\varphi)$ whose derivative may be written as $[\partial \sin \theta f(r, \theta) / \partial x] g(\varphi) + \sin \theta f(r, \theta) \partial g(\varphi) / \partial x$. Having to take the derivative at the location of the particle, the second term of the summation $\sin \theta f(r, \theta) \partial g(\varphi) / \partial x$ is always zero because the particle is located at $\theta = 0$, so that we are left with the derivative of the first term. Another point of view might be to say that the derivative with respect to x is taken along the x -axis, i.e. for a fixed value of $\varphi = 0^\circ$ (and similarly for the derivative with respect to y taken along the y -axis, i.e. for $\varphi = 90^\circ$). We obtain:

$$\left\{ \frac{\partial}{\partial x} \sin^2 \theta [k^2 (\psi_1'' + \psi_1)^2 - \frac{\psi_1' \psi_1'}{r^2}] \right\}_{x=y=0} = \left\{ \frac{\partial}{\partial y} \sin^2 \theta [k^2 (\psi_1'' + \psi_1)^2 - \frac{\psi_1' \psi_1'}{r^2}] \right\}_{x=y=0} = 0 \tag{97}$$

$$\left(\frac{\partial}{\partial x} \sin^2 \theta \frac{\psi_1 \psi_1}{r^2} \right)_{x=y=0} = \left(\frac{\partial}{\partial y} \sin^2 \theta \frac{\psi_1 \psi_1}{r^2} \right)_{x=y=0} = 0 \tag{98}$$

$$\left\{ \frac{\partial}{\partial x} \sin \theta \cos \theta \left[\frac{\psi_1' \psi_1'}{r^2} - k^2 (\psi_1'' + \psi_1)^2 \right] \right\}_{x=y=0} = \left\{ \frac{\partial}{\partial y} \sin \theta \cos \theta \left[\frac{\psi_1' \psi_1'}{r^2} - k^2 (\psi_1'' + \psi_1)^2 \right] \right\}_{x=y=0} = 0 \tag{99}$$

$$\left(\frac{\partial}{\partial x} \sin \theta \frac{\psi_1' \psi_1'}{r^2} \right)_{x=y=0} = \left(\frac{\partial}{\partial y} \sin \theta \frac{\psi_1' \psi_1'}{r^2} \right)_{x=y=0} = 0 \tag{100}$$

$$\left(\frac{\partial}{\partial x} \sin \theta \cos \theta \frac{\psi_1 \psi_1}{r^2}\right)_{x=y=0} = \left(\frac{\partial}{\partial y} \sin \theta \cos \theta \frac{\psi_1 \psi_1}{r^2}\right)_{x=y=0} = 0 \quad (101)$$

$$\left\{\frac{\partial}{\partial x} [k^2 \sin^2 \theta (\psi_1'' + \psi_1)^2 + \cos^2 \theta \frac{\psi_1' \psi_1'}{r^2}]\right\}_{x=y=0} = \left\{\frac{\partial}{\partial y} [k^2 \sin^2 \theta (\psi_1'' + \psi_1)^2 + \cos^2 \theta \frac{\psi_1' \psi_1'}{r^2}]\right\}_{x=y=0} = 0 \quad (102)$$

$$\left\{\frac{\partial}{\partial x} [k^2 \cos^2 \theta (\psi_1'' + \psi_1)^2 + \sin^2 \theta \frac{\psi_1' \psi_1'}{r^2}]\right\}_{x=y=0} = \left\{\frac{\partial}{\partial y} [k^2 \cos^2 \theta (\psi_1'' + \psi_1)^2 + \sin^2 \theta \frac{\psi_1' \psi_1'}{r^2}]\right\}_{x=y=0} = 0 \quad (103)$$

$$\left(\frac{\partial}{\partial x} \cos \theta \frac{\psi_1' \psi_1'}{r^2}\right)_{x=y=0} = \left(\frac{\partial}{\partial y} \cos \theta \frac{\psi_1' \psi_1'}{r^2}\right)_{x=y=0} = 0 \quad (104)$$

$$\left[\frac{\partial}{\partial x} (\cos^2 \theta + 1) \frac{\psi_1 \psi_1}{r^2}\right]_{x=y=0} = \left[\frac{\partial}{\partial y} (\cos^2 \theta + 1) \frac{\psi_1 \psi_1}{r^2}\right]_{x=y=0} = 0 \quad (105)$$

$$\left(\frac{\partial}{\partial x} \sin^2 \theta \frac{\psi_1 \psi_1}{r^2}\right)_{x=y=0} = \left(\frac{\partial}{\partial y} \sin^2 \theta \frac{\psi_1 \psi_1}{r^2}\right)_{x=y=0} = 0 \quad (106)$$

$$\left(\frac{\partial}{\partial x} \frac{\psi_1' \psi_1'}{r^2}\right)_{x=y=0} = \left(\frac{\partial}{\partial y} \frac{\psi_1' \psi_1'}{r^2}\right)_{x=y=0} = 0 \quad (107)$$

This implies $\partial \widetilde{I}_E / \partial x = \partial \widetilde{I}_E / \partial y = 0$ and therefore that the gradient forces along the x - and y -directions should be expected to be zero. Following the strategy used in [11], [12], this conclusion may be reinforced by considering as well the gradients of the energy density $\varepsilon_0 \widetilde{\mathbf{E}} \cdot \widetilde{\mathbf{E}}^* + \mu_0 \widetilde{\mathbf{H}} \cdot \widetilde{\mathbf{H}}^*$ and of the transverse components of the Poynting vector. The gradients $\partial \widetilde{I}_H / \partial x$ and $\partial \widetilde{I}_H / \partial y$ of the magnetic intensity $\widetilde{I}_H = \widetilde{\mathbf{H}} \cdot \widetilde{\mathbf{H}}^*$ are found to be zero as well because the magnetic intensity \widetilde{I}_H involves the same x - and y -dependent terms than the electric intensity \widetilde{I}_E . Finally, as far the gradients of the transverse components of the Poynting vector are concerned, let us start from Eq.46 for \widetilde{S}_x , without however taking the limit $\theta = 0$, and list all the x - and y -dependent terms involved in it. Since \widetilde{S}_y is deduced from \widetilde{S}_x by changing Re to Im, this list is the same for both \widetilde{S}_x and \widetilde{S}_y . It is then found that this list is made from a number of

terms similar to the ones displayed in Eqs.97-107 but sufficiently longer to omit it here. Let us simply mention that the x - and y -derivatives of all these terms taken at $x = y = 0$ are found to be zero, so that we have $\partial\widetilde{S}_x/\partial x = \partial\widetilde{S}_y/\partial y$ confirming the fact that gradient forces are zero. In doing so, we do not have to worry with φ -derivatives insofar as only the $e^{0\varphi}$ -term, i.e. a_0 is to be retained.

Therefore, the quantities $\widetilde{C}_{pr,x}^{\text{Im}}$ and $\widetilde{C}_{pr,y}^{\text{Re}}$ are not gradient forces but complementary axicon forces associated with gradient forces. They involve $\text{Re}(H)$ and $\text{Im}(H)$ which have already been calculated. The important point to mention is that, once again, these axicon forces are zero when the axicon angle is zero and/or when an on-axis configuration is considered.

5 Discussion.

5.1 Relationship with the standard theory of scattering and optical forces.

A previous paper, devoted to longitudinal optical forces [11], introduced new axicon forces, associated with scattering forces, which have no counterpart in the usual standard theory of optical forces expressed in terms of scattering and of gradient forces. It has been shown that these axicon forces were generated by the occurrence of higher-order multipole contributions which are involved in the Rayleigh limit of the GLMT [12]. In the present paper, devoted to transverse optical forces, we again uncover similar axicon forces. Axicon forces associated with scattering forces are given in Eq.76 expressing $(\widetilde{C}_{pr,x}^{\text{Re}})^{as}$ and in Eq.85 expressing $(\widetilde{C}_{pr,y}^{\text{Im}})^{as}$. We observe once again the occurrence of higher-order multipole contributions which are conveyed by BSCs reading as $g_{2,TM}^0$, $g_{2,TM}^{\pm 1}$ and $g_{2,TM}^{\pm 2}$. Such BSCs (and therefore such higher-order multipole contributions) occur as well in Eqs.42 and 43 for the axicon forces $\widetilde{C}_{pr,x}^{\text{Im}}$ proportional to $\alpha^3 \text{Im}(H)$ and $\widetilde{C}_{pr,y}^{\text{Re}}$ proportional to $\alpha^3 \text{Re}(H)$ respectively, associated with the gradient forces (which are zero since the gradients are zero). There is however an unpleasant detail, namely that these axicon forces also contain first-order multipoles $g_{1,TM}^0$, $g_{1,TE}^0$, $g_{1,TM}^{\pm 1}$ and $g_{1,TE}^{\pm 1}$ that we would have preferred to be associated with genuine gradient forces.

To trace the origin of the higher-multipole contributions discussed above, we may return to Eq.(1) in [11] which expresses the longitudinal radiation pressure cross-section $C_{pr,z}$ in the framework of the GLMT. BSCs then occur in a coupled way in terms reading as $g_{n,TM}^m g_{n+1,TM}^{m*}$, $g_{n,TE}^m g_{n+1,TE}^{m*}$ and $g_{n,TM}^m g_{n,TE}^{m*}$ ($m = -n, \dots, +n$). In the Rayleigh limit of the GLMT, we only retain ($n = 1$)- partial waves so that we have to deal with BSCs coupling terms of the form $g_{1,TM}^m g_{2,TM}^{m*}$, $g_{1,TE}^m g_{2,TE}^{m*}$ and $g_{1,TM}^m g_{1,TE}^{m*}$ ($m = -1, 0, +1$), therefore implying the somewhat hidden occurrence of ($n = 2$)-partial waves. In other words, the Rayleigh limit

of the GLMT does not exactly identify with the Rayleigh dipole theory of optical forces.

5.2 Further works.

Further works are required to complement what is already known and understood. It is then planned to study the on-axis configuration of Bessel beams in order (i) to confirm in a simplified framework that axicon forces indeed cancel in such a configuration and (ii) to provide a comparison between the cases of dark and non dark beams which exhibit different structural features [22], [42]. We may anticipate that specific departures from the standard theory could occur for dark beams, although the anticipation could be deceived. Also, the case of off-axis Gaussian beams which can be analytically handled using a localized beam model [43] should be investigated in order to check whether the newly axicon forces are specific of axicon beams, or whether they occur in more general situations. Our current anticipation, which could again be deceived, is that indeed the newly studied terms occur in other situations than the one of axicon beams.

6 Conclusion.

The present paper complements two previous papers [11], [12] which demonstrated the unexpected existence of new axicon forces and terms associated with some Bessel beams along the direction of propagation. The present paper demonstrates that such terms exist as well if transverse directions, rather than longitudinal direction, are considered. They extend the traditional classification of optical forces carried out in terms of scattering and gradient forces.

When dealing with longitudinal forces, these results were unexpected. However, it is to be noted that computations involved in [11] have been independently fully checked by a reviewer. Also, computations involved in [12] have been independently checked by a co-author. We cannot pretend that the tedious computations involved in the present paper are completely immune to (minor) computational mistakes (nightmares for theoreticians) but the conclusions reached in this work are certainly immune to them, if any. This warning is motivated by the fact that the requirement of introducing partitions for the scattering forces has been felt to be unpleasant, although it has been resistant to checkings. Another unpleasant feature is the occurrence of first-order multipoles included in axicon forces associated with zero gradients, although they were expected to describe genuine gradient forces.

We specifically studied the case of circularly symmetric Bessel beams of arbitrary order but, conclusions we obtained for such beams very likely extend

to all kinds of beams exhibiting axicon angles comprising Bessel beams whatever the polarization, all kinds of beams obtained from Bessel beam superpositions, e.g. frozen waves, Mathieu and Lommel beams [44], [45], [46], [47], [48], [49], [50], [51], [52][53], [54], or Laguerre-Gauss beams focused by a lens [55], [56], [57], [58], [59].

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