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A new axiomatics for mesures II

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Abstract

Masures are generalizations of Bruhat-Tits buildings. They were introduced by Gaussent and Rousseau in order to study Kac-Moody groups over valued fields. We prove that the intersection of two apartments of a mesure is convex. Using this, we simplify the axiomatic definition of mesures given by Rousseau.

1 Introduction

Bruhat-Tits buildings are an important tool in the study of reductive groups over non-Archimedean local fields. They were introduced by Bruhat and Tits in [BT72] and [BT84]. Kac-Moody groups (à la Tits) are interesting infinite dimensional (if not reductive) generalizations of reductive groups. In order to study them over fields endowed with a discrete valuation, Gaussent and Rousseau introduced in [GR08] some spaces similar to Bruhat-Tits buildings, called mesures (also known as hovels), on which these groups act. Charignon and Rousseau generalized this construction in [Cha10], [Rou16] and [Rou17]: Charignon treated the almost split case and Rousseau suppressed restrictions on the base field and on the group. Thanks to these works, a mesure is now associated with each almost split Kac-Moody group over a valued field (with some additional assumptions on the field in the non-split case, see [Rou17]). Measures enable to obtain results on the representation theory of almost split Kac-Moody groups over non-Archimedean local fields. For example, Bardy-Panse, Gaussent and Rousseau used them to associate with each such group a spherical and an Iwahori-Hecke algebra (see [GR14] and [BPGR16], these algebras were already defined in the split affine case by Braverman, Kazhdan and Patnaik in [BK11] and [BKP16]).

Let G be a split reductive group over a valued field (\mathcal{K}, ω) , where $\omega : \mathcal{K} \rightarrow \mathbb{R} \cup \{\infty\}$ is a valuation. Let T be a maximal split torus of G . Let $\mathbb{A} = Y \otimes \mathbb{R}$, where Y is the cocharacter lattice of (G, T) . As a set, $\mathcal{I} = G \times \mathbb{A} / \sim$, where \sim is some equivalence relation on $G \times \mathbb{A}$. The definition of \sim is complicated and based on the notion of parahoric subgroups. However, many of the properties of \mathcal{I} can be recovered from the fact that it satisfies the crucial properties (I1) and (I2) below. Let Φ denote the root system of (G, T) and $\Lambda = \omega(\mathcal{K}) \setminus \{\infty\}$. For $\alpha \in \Phi$ and $k \in \Lambda$, let $H_{\alpha, k} = \{x \in \mathbb{A} \mid \alpha(x) = -k\}$. The elements of the form $H_{\alpha, k}$, for $\alpha \in \Phi$ and $k \in \Lambda$ are called **walls**. A half-space of \mathbb{A} delimited by some wall is called a half-apartment of \mathbb{A} . An **apartment** of \mathcal{I} (resp. a **half-apartment** of \mathcal{I}) is a set of the form $g \cdot \mathbb{A}$ (resp. $g \cdot D$), for some $g \in G$ (resp. and some half-apartment D of \mathbb{A}). We call a

set **enclosed** if it is a finite intersection of half-apartments. Then \mathcal{I} satisfies the following properties:

- (I1) for all $x, y \in \mathcal{I}$, there exists an apartment containing $\{x, y\}$,
- (I2) if A and A' are two apartments of \mathcal{I} , then $A \cap A'$ is enclosed in A and there exists $g \in G$ such that $g.A = A'$ and such that g fixes $A \cap A'$.

Note that (I1) is a building theoretic translation of the Cartan decomposition of G .

Let G be a split Kac-Moody group over a valued field (\mathcal{K}, ω) . Then similarly to the reductive case, the measure \mathcal{I} of G is defined as $G \times \mathbb{A} / \sim$, for some equivalence relation \sim on $G \times \mathbb{A}$ (see [GR08] and [Rou16]). As the Cartan decomposition does not hold in G (unless G is reductive), property (I1) is not necessarily satisfied by \mathcal{I} . It can be replaced by an axiom involving “chimneys”, which are certain objects at the infinity of \mathcal{I} (see (MA iii) in 2.2.2). This axiom corresponds to the Iwasawa and the Birkhoff decompositions in G . Gaussent and Rousseau proved weak versions of (I2) in [GR08] and [Rou11]. More precisely, let \mathcal{T} be the Tits cone of \mathbb{A} . If $x, y \in \mathcal{I}$, one writes $x \leq y$ if there exists $g \in G$ such that $g.x, g.y \in \mathbb{A}$ and $g.y - g.x \in \mathcal{T}$. It is proved in [GR08] that if A, A' are two apartments, if $x, y \in A \cap A'$ are such that $x \leq y$, then $[x, y]_A$ (the line segment in A joining x to y) is equal to $[x, y]_{A'}$ and there exists $g \in G$ such that $g.A = A'$ and such that g fixes $[x, y]_A$. This property is called preordered convexity and is fundamental to most of the applications of the theory of mesures so far. However, very few was known about convexity properties for pairs of points which are not preordered. In [Héb20], we proved that when \mathcal{I} is associated with an affine Kac-Moody group, then \mathcal{I} satisfies (I2). In general we proved that (I2) is satisfied for pairs of apartments A, A' such that $A \cap A'$ is “large enough” (more precisely, when $A \cap A'$ contains a generic ray, see [Héb20, Theorem 4.22]). We used this to simplify the definition of mesures. In this paper we prove that (I2) is satisfied without assumption on $A \cap A'$. More precisely, let \mathcal{I} be a measure in the sense of [Rou11, Définition 2.1], satisfying some mild technical assumption (see assumption 2.1). Then:

Theorem 1. (see Theorem 3.6) Let A and A' be two apartments of \mathcal{I} . Then $A \cap A'$ is enclosed and there exists an apartment isomorphism $\phi : A \rightarrow A'$ fixing $A \cap A'$.

Note that in the theorem above, the half-apartments of \mathbb{A} are delimited by real roots of (G, T) (and not by imaginary roots as it was the case in [GR08]).

We then use this theorem to give a simplified definition of mesures equivalent to [Rou11, Définition 2.1]. This also simplifies the definition given in [Héb20, Theorem 1.5] (see Corollary 3.7).

Framework Actually we do not limit our study to mesures associated with Kac-Moody groups: for us a measure is a set satisfying the axioms of [Rou11] and whose apartments are associated with a root generating system (and thus with a Kac-Moody matrix). We do not assume the existence of a group acting strongly transitively on it. Our results apply to the mesures associated with split Kac-Moody groups over valued fields constructed in [GR08] and [Rou16] and to mesures associated with almost split Kac-Moody groups over valued fields (satisfying some additional conditions, see [Rou17, 6.1]) in [Rou17]. Contrary to [Rou11], we assume that the family of simple coroots is free in \mathbb{A} (see assumption 2.1).

Comments on the proof of Theorem 1 In [Héb20], we proved that if A and A' are apartments containing a common generic ray, then $A \cap A'$ is enclosed by following the steps below:

- (1) We prove that if B, B' are any two apartments, then $B \cap B'$ can be written as a finite union of enclosed subsets of B .
- (2) We prove that $A \cap A'$ is convex. Using (1), we deduce that it is enclosed.

As we already proved (1) in [Héb20], the main difficulty in the proof of Theorem 1 is to prove the convexity of $A \cap A'$ (without assumption on A and A'). In this paper, we prove it directly, without using step (2) of [Héb20]. Our proof of the convexity of $A \cap A'$ is based on the study of some kind of Hecke paths, that is, on the study of the images of line segments by retractions centered at a sector-germ. Our proof of the convexity is actually simpler than the proof of step (2) in [Héb20] (and more general).

Organization of the paper In section 2, we describe the general framework and recall the definition of mesures.

In section 3, we prove Theorem 1.

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2 General framework, Measure

In this section, we define our framework and recall the definition of mesures. We give the definition of [Héb20].

2.1 Standard apartment

2.1.1 Root generating system

Let A be a **Kac-Moody matrix** (also known as generalized Cartan matrix) i.e a square matrix $A = (a_{i,j})_{i,j \in I}$ with integers coefficients, indexed by a finite set I and satisfying:

1. $\forall i \in I, a_{i,i} = 2$
2. $\forall (i,j) \in I^2 | i \neq j, a_{i,j} \leq 0$
3. $\forall (i,j) \in I^2, a_{i,j} = 0 \Leftrightarrow a_{j,i} = 0$.

A **root generating system** of type A is a 5-tuple $\mathcal{S} = (A, X, Y, (\alpha_i)_{i \in I}, (\alpha_i^\vee)_{i \in I})$ made of a Kac-Moody matrix A indexed by I , of two dual free \mathbb{Z} -modules X (of **characters**) and Y (of **cocharacters**) of finite rank $\text{rk}(X)$, a family $(\alpha_i)_{i \in I}$ (of **simple roots**) in X and a family $(\alpha_i^\vee)_{i \in I}$ (of **simple coroots**) in Y . They have to satisfy the following compatibility condition: $a_{i,j} = \alpha_j(\alpha_i^\vee)$ for all $i, j \in I$.

Assumption 2.1. We assume that $(\alpha_i)_{i \in I}$ is free in \mathbb{A}^* and that $(\alpha_i^\vee)_{i \in I}$ is free in \mathbb{A} .

Let $\mathbb{A} = Y \otimes \mathbb{R}$. Every element of X induces a linear form on \mathbb{A} . We consider X as a subset of the dual \mathbb{A}^* of \mathbb{A} : the α_i , $i \in I$ are viewed as linear forms on \mathbb{A} . For $i \in I$, we define an involution r_i of \mathbb{A} by $r_i(v) = v - \alpha_i(v)\alpha_i^\vee$ for all $v \in \mathbb{A}$. Its space of fixed points is $\ker \alpha_i$. The subgroup of $\text{GL}(\mathbb{A})$ generated by the α_i for $i \in I$ is denoted by W^v and is called the **vectorial Weyl group** of \mathcal{S} . Then $(W^v, \{r_i | i \in I\})$ is a Coxeter system.

One defines an action of the group W^v on \mathbb{A}^* as follows: if $x \in \mathbb{A}$, $w \in W^v$ and $\alpha \in \mathbb{A}^*$, then $(w.\alpha)(x) = \alpha(w^{-1}.x)$. Let $\Phi = \{w.\alpha_i | (w, i) \in W^v \times I\}$ be the set of **(real) roots**. Then $\Phi \subset Q$, where $Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$. Let $Q^+ = \bigoplus_{i \in I} \mathbb{N}\alpha_i$, $\Phi^+ = Q^+ \cap \Phi$ and $\Phi^- = (-Q^+) \cap \Phi$. Then $\Phi = \Phi^+ \sqcup \Phi^-$.

We set $Q_{\mathbb{R}}^\vee = \bigoplus_{i \in I} \mathbb{R}\alpha_i^\vee$ and $Q_{\mathbb{R}^+}^\vee = \bigoplus_{i \in I} \mathbb{R}_+\alpha_i^\vee$. For $x, y \in \mathbb{A}$, we write $x \leq_{Q_{\mathbb{R}}^\vee} y$ if $y - x \in Q_{\mathbb{R}^+}^\vee$.

2.1.2 Vectorial faces and Tits cone

Define $C_f^v = \{v \in \mathbb{A} | \alpha_i(v) > 0, \forall i \in I\}$. We call it the **fundamental vectorial chamber**. For $J \subset I$, one sets $F^v(J) = \{v \in \mathbb{A} | \alpha_i(v) = 0 \forall i \in J, \alpha_i(v) > 0 \forall i \in I \setminus J\}$. Then the closure $\overline{C_f^v}$ of C_f^v is the union of the $F^v(J)$ for $J \subset I$. The **positive** (resp. **negative**) **vectorial faces** are the sets $w.F^v(J)$ (resp. $-w.F^v(J)$) for $w \in W^v$ and $J \subset I$. A **vectorial face** is either a positive vectorial face or a negative vectorial face. We call **positive chamber** (resp. **negative**) every cone of the form $w.C_f^v$ for some $w \in W^v$ (resp. $-w.C_f^v$). For all $x \in C_f^v$ and for all $w \in W^v$, $w.x = x$ implies that $w = 1$. In particular the action of w on the positive chambers is simply transitive. The **Tits cone** \mathcal{T} is defined by $\mathcal{T} = \bigcup_{w \in W^v} w.\overline{C_f^v}$. One defines a W^v -invariant preorder \leq on \mathbb{A} , the **Tits preorder** by:

$$\forall (x, y) \in \mathbb{A}^2, x \leq y \Leftrightarrow y - x \in \mathcal{T}.$$

2.1.3 Affine Weyl group of \mathbb{A}

We now define the Weyl group W of \mathbb{A} . If \mathcal{X} is an affine subspace of \mathbb{A} , one denotes by $\vec{\mathcal{X}}$ its direction. One equips \mathbb{A} with a family \mathcal{M} of affine hyperplanes called **real walls** such that:

1. For all $M \in \mathcal{M}$, there exists $\alpha_M \in \Phi$ such that $\vec{M} = \ker(\alpha_M)$.
2. For all $\alpha \in \Phi$, there exists an infinite number of hyperplanes $M \in \mathcal{M}$ such that $\alpha = \alpha_M$.
3. If $M \in \mathcal{M}$, we denote by r_M the reflection with respect to the hyperplane M whose associated linear map is r_{α_M} . We assume that the group W generated by the r_M for $M \in \mathcal{M}$ stabilizes \mathcal{M} .

The group W is the Weyl group of \mathbb{A} . We assume that 0 is special (i.e we assume that $\ker \alpha \in \mathcal{M}$ for every $\alpha \in \Phi$) and thus $W \supset W^v$.

For $\alpha \in \mathbb{A}^*$ and $k \in \mathbb{R}$, set $M(\alpha, k) = \{v \in \mathbb{A} | \alpha(v) + k = 0\}$. Then for all $M \in \mathcal{M}$, there exists $\alpha \in \Phi$ and $k_M \in \mathbb{R}$ such that $M = M(\alpha, k_M)$. For $\alpha \in \Phi$, set $\Lambda_\alpha = \{k_M | M \in \mathcal{M} \text{ and } \vec{M} = \ker(\alpha)\}$. Then $\Lambda_{w.\alpha} = \Lambda_\alpha$ for all $w \in W^v$ and $\alpha \in \Phi$.

If $\alpha \in \Phi$, one denotes by $\tilde{\Lambda}_\alpha$ the subgroup of \mathbb{R} generated by Λ_α . By (3), $\Lambda_\alpha = \Lambda_\alpha + 2\tilde{\Lambda}_\alpha$ for all $\alpha \in \Phi$. In particular, $\Lambda_\alpha = -\Lambda_\alpha$ and when Λ_α is discrete, $\tilde{\Lambda}_\alpha = \Lambda_\alpha$ is isomorphic to \mathbb{Z} .

One sets $Q^\vee = \bigoplus_{\alpha \in \Phi} \tilde{\Lambda}_\alpha \alpha^\vee$. This is a subgroup of \mathbb{A} stable under the action of W^v . Then one has $W = W^v \ltimes Q^\vee$.

2.1.4 Filters

Definition 2.2. A filter on a set E is a nonempty set F of nonempty subsets of E such that, for all subsets S, S' of E , if $S, S' \in F$ then $S \cap S' \in F$ and, if $S' \subset S$, with $S' \in F$, then $S \in F$.

If F is a filter on a set E , and E' is a subset of E , one says that F contains E' if every element of F contains E' . If E' is nonempty, the set $F_{E'}$ of subsets of E containing E' is a filter. We will sometimes regard E' as a filter on E by identifying $F_{E'}$ and E' . If F is a filter on E , its closure \overline{F} (resp. its convex hull) is the filter of subsets of E containing the closure (resp. the convex envelope) of some element of F . A filter F is said to be contained in another filter F' : $F \subset F'$ (resp. in a subset Z in E : $F \subset Z$) if and only if any set in F' (resp. if Z) is in F .

If $x \in \mathbb{A}$ and Ω is a subset of \mathbb{A} containing x in its closure, then the **germ** of Ω in x is the filter $germ_x(\Omega)$ of subsets of \mathbb{A} containing a neighborhood of x in Ω .

A **sector** in \mathbb{A} is a set of the form $\mathfrak{s} = x + C^v$ with $C^v = \pm w.C_f^v$ for some $x \in \mathbb{A}$ and $w \in W^v$. A point u such that $\mathfrak{s} = u + C^v$ is called a **base point** of \mathfrak{s} and C^v is its **direction**. The intersection of two sectors of the same direction is a sector of the same direction.

The **sector-germ** of a sector $\mathfrak{s} = x + C^v$ is the filter \mathfrak{S} of subsets of \mathbb{A} containing an \mathbb{A} -translate of \mathfrak{s} . It only depends on the direction C^v . We denote by $+\infty$ (resp. $-\infty$) the sector-germ of C_f^v (resp. of $-C_f^v$).

A ray δ with base point x and containing $y \neq x$ (or the interval $]x, y] = [x, y] \setminus \{x\}$ or $[x, y]$ or the line containing x and y) is called **preordered** if $x \leq y$ or $y \leq x$ and **generic** if $y - x \in \pm \overset{\circ}{\mathcal{T}}$, the interior of $\pm \mathcal{T}$.

For $\alpha \in \Phi$, and $k \in \mathbb{R} \cup \{+\infty\}$, let $D(\alpha, k) = \{v \in \mathbb{A} \mid \alpha(v) + k \geq 0\}$ (and $D(\alpha, +\infty) = \mathbb{A}$) and $D^\circ(\alpha, k) = \{v \in \mathbb{A} \mid \alpha(v) + k > 0\}$ (for $\alpha \in \Phi$ and $k \in \mathbb{R} \cup \{+\infty\}$).

Let \mathcal{L} be the set of families $(\Lambda'_\alpha)_{\alpha \in \Phi}$ such that $\Lambda_\alpha \subset \Lambda'_\alpha \subset \mathbb{R}$ and $\Lambda'_\alpha = -\Lambda'_{-\alpha}$, for $\alpha \in \Phi$.

An **apartment** is a root generating system equipped with a Weyl group W (i.e with a set \mathcal{M} of real walls, see 2.1.3) and a family $\Lambda' \in \mathcal{L}$. Let $\underline{\mathbb{A}} = (\mathcal{S}, W, \Lambda')$ be an apartment and \mathbb{A} be the underlying affine space. A set of the form $M(\alpha, k)$, with $\alpha \in \Phi$ and $k \in \Lambda'_\alpha$ is called a **wall** of \mathbb{A} and a set of the form $D(\alpha, k)$, with $\alpha \in \Phi$ and $k \in \Lambda'_\alpha$ is called a **half-apartment** of \mathbb{A} . A subset X of \mathbb{A} is said to be **enclosed** if there exist $k \in \mathbb{N}$, $\beta_1, \dots, \beta_k \in \Phi$ and $(\lambda_1, \dots, \lambda_k) \in \prod_{i=1}^k \Lambda'_{\beta_i}$ such that $X = \bigcap_{i=1}^k D(\beta_i, \lambda_i)$.

2.2 Measure

In this section, we define measures.

2.2.1 Definitions of faces, chimneys and related notions

Let $\underline{\mathbb{A}} = (\mathcal{S}, W, \Lambda')$ be an apartment and \mathbb{A} be the underlying affine space.

Let $x \in \mathbb{A}$ and F^v be a vectorial face. The **local-face** $F^\ell(x, F^v) = germ_x(x + F^v)$ is the filter defined as the intersection of $x + F^v$ with the the filter of neighborhoods of x in

\mathbb{A} . The **face** $F(x, F^v)$ is the filter consisting of the subsets containing a finite intersection of half-spaces $D(\alpha, \lambda_\alpha)$ or $D^\circ(\alpha, \lambda_\alpha)$, with $\lambda_\alpha \in \Lambda'_\alpha \cup \{+\infty\}$ for all $\alpha \in \Phi$ (at most one $\lambda_\alpha \in \Lambda_\alpha$ for each $\alpha \in \Phi$). We say that F is **positive** (or **negative**) if F^v is.

Let F, F' be two faces. We say that F' **dominates** F if $F \subset \overline{F'}$. The dimension of a face F is the smallest dimension of an affine space generated by some $S \in F$. Such an affine space is unique and is called its **support**. A face is said to be **spherical** if the direction of its support meets the open Tits cone $\overset{\circ}{\mathcal{T}}$; then its pointwise stabilizer W_F in W^v is finite.

A **chamber** (or alcove) is a face of the form $F(x, C^v)$ where $x \in \mathbb{A}$ and C^v is a vectorial chamber of \mathbb{A} .

A **panel** is a face which is maximal (for the domination relation) among the faces contained in at least one wall.

Let $F = F(x, F^v)$ be a face ($x \in \mathbb{A}$ and F^v is a vectorial face). The **chimney** $\mathfrak{r}(F, F^v)$ is the filter consisting of the sets containing an enclosed set containing $F + F^v$. The face F is the basis of the chimney and the vectorial face F^v its direction. A chimney is **splayed** if F^v is spherical.

A **shortening** of a chimney $\mathfrak{r}(F, F^v)$, with $F = F(x, F_0^v)$ is a chimney of the form $\mathfrak{r}(F(x + \xi, F_0^v), F^v)$ for some $\xi \in \overline{F^v}$. The **germ** of a chimney \mathfrak{r} is the filter of subsets of \mathbb{A} containing a shortening of \mathfrak{r} (this definition of shortening is slightly different from the one of [Rou11] 1.12 but follows [Rou17] 3.6) and we obtain the same germs with these two definitions).

2.2.2 Measure

An **apartment of type \mathbb{A}** is a set \underline{A} with a nonempty set $\text{Isom}(\mathbb{A}, A)$ of bijections (called **Weyl-isomorphisms**) such that if $f_0 \in \text{Isom}(\mathbb{A}, A)$ then $f \in \text{Isom}(\mathbb{A}, A)$ if and only if there exists $w \in W$ satisfying $f = f_0 \circ w$. We will say **isomorphism** instead of Weyl-isomorphism in the sequel. An isomorphism between two apartments $\phi : A \rightarrow A'$ is a bijection such that ($f \in \text{Isom}(\mathbb{A}, A)$ if, and only if, $\phi \circ f \in \text{Isom}(\mathbb{A}, A')$). We extend all the notions that are preserved by W to each apartment. Thus sectors, enclosures, faces and chimneys are well defined in any apartment of type \mathbb{A} . If A and B are two apartments, and $\phi : A \rightarrow B$ is an apartment isomorphism fixing some set X , we write $\phi : A \xrightarrow{X} B$.

Definition 2.3. *A measure of type $\underline{\mathbb{A}} = (\mathcal{S}, W, \Lambda')$ is a set \mathcal{I} endowed with a covering \mathcal{A} of subsets called **apartments** such that:*

(MA i) *Any $A \in \mathcal{A}$ is equipped with the structure of an apartment of type \mathbb{A} .*

(MA ii) : *if two apartments A, A' contain a generic ray, then $A \cap A'$ is enclosed and there exists an apartment isomorphism $\phi : A \xrightarrow{A \cap A'} A'$.*

(MA iii): *if \mathfrak{R} is the germ of a splayed chimney and if F is a face or a germ of a chimney, then there exists an apartment containing \mathfrak{R} and F .*

In this definition, we say that an apartment contains a germ of a filter if it contains at least one element of this germ. We say that a map fixes a germ if it fixes at least one element of this germ.

The main examples of measures are the measures associated with (almost) split Kac-Moody group over valued fields constructed in [Rou16] and [Rou17].

By [Héb20, Theorem 5.1], definition 2.3 is equivalent to [Rou11, Définition 2.1] (at least under Assumption 2.1).

2.2.3 Example: measure associated with a split Kac-Moody group over a valued field

Let A be a Kac-Moody matrix and \mathcal{S} be a root generating system of type A . We consider the group functor \mathbf{G} associated with the root generating system \mathcal{S} in [Tit87] and in [Rém02, Chapitre 8]. This functor is a functor from the category of rings to the category of groups satisfying axioms (KMG1) to (KMG 9) of [Tit87]. When R is a field, $\mathbf{G}(R)$ is uniquely determined by these axioms by Theorem 1' of [Tit87]. This functor contains a toric functor \mathbf{T} , from the category of rings to the category of commutative groups (denoted \mathcal{T} in [Rém02]) and two functors \mathbf{U}^+ and \mathbf{U}^- from the category of rings to the category of groups.

Let \mathcal{K} be a field equipped with a non-trivial valuation $\omega : \mathcal{K} \rightarrow \mathbb{R} \cup \{+\infty\}$, \mathcal{O} be its ring of integers and $G = \mathbf{G}(\mathcal{K})$ (and $U^+ = \mathbf{U}^+(\mathcal{K})$, ...). For all $\epsilon \in \{-, +\}$, and all $\alpha \in \Phi_\epsilon$, we have an isomorphism x_α from \mathcal{K} to a group U_α . For all $k \in \mathbb{R}$, one defines a subgroup $U_{\alpha,k} := x_\alpha(\{u \in \mathcal{K} \mid \omega(u) \geq k\})$. Let \mathcal{I} be the measure associated with G constructed in [Rou16]. Then for all $\alpha \in \Phi$, $\Lambda_\alpha = \Lambda'_\alpha = \omega(\mathcal{K}^*)$. If moreover $\omega(\mathcal{K}^*)$ is discrete, one has (up to renormalization) $\Lambda_\alpha = \mathbb{Z}$ for all $\alpha \in \Phi$. Moreover, we have:

- the fixator of \mathbb{A} in G is $H = \mathbf{T}(\mathcal{O})$ (by remark 3.2 of [GR08])
- for all $\alpha \in \Phi$ and $k \in \mathbb{Z}$, the fixator of $D(\alpha, k)$ in G is $H.U_{\alpha,k}$ (by 4.2 7) of [GR08])
- for all $\epsilon \in \{-, +\}$, $H.U^\epsilon$ is the fixator of $\epsilon\infty$ (by 4.2 4) of [GR08])
- when moreover the residue field of (\mathcal{K}, ω) contains \mathbb{C} , the fixator of $\{0\}$ in G is $K_s = \mathbf{G}(\mathcal{O})$ (by example 3.14 of [GR08]).

If moreover, \mathcal{K} is local, with residue cardinal q , each panel is contained in $1 + q$ chambers.

The group G is reductive if and only if W^v is finite. In this case, \mathcal{I} is the usual Bruhat-Tits building of G and one has $\mathcal{T} = \mathbb{A}$.

2.2.4 Tits preorder on \mathcal{I}

As the Tits preorder \leq on \mathbb{A} is invariant under the action of W^v , one can equip each apartment A with \leq_A . Let A be an apartment of \mathcal{I} and $x, y \in A$ be such that $x \leq_A y$. Then by [Rou11, Proposition 5.4], if B is an apartment containing x and y , $[x, y]_A = [x, y]_B$ and there exists an apartment isomorphism $\psi : A \xrightarrow{[x,y]} B$. In particular, $x \leq_B y$. This defines a relation \leq on \mathcal{I} . By Théorème 5.9 of [Rou11], \leq is a preorder on \mathcal{I} . It is invariant by apartment isomorphisms: if A, B are apartments, $\phi : A \rightarrow B$ is an apartment isomorphism and $x, y \in A$ are such that $x \leq y$, then $\phi(x) \leq \phi(y)$. We call it the **Tits preorder on \mathcal{I}** .

2.2.5 Retractions centered at sector-germs

Let \mathfrak{s} be a sector-germ of \mathcal{I} and A be an apartment containing it. Let $x \in \mathcal{I}$. By (MA iii), there exists an apartment A_x of \mathcal{I} containing x and \mathfrak{s} . By (MA ii), there exists an apartment

isomorphism $\phi : A_x \rightarrow A$ fixing \mathfrak{s} . By [Rou11, 2.6], $\phi(x)$ does not depend on the choices we made and thus we can set $\rho_{A,\mathfrak{s}}(x) = \phi(x)$.

The map $\rho_{A,\mathfrak{s}} : \mathcal{I} \rightarrow A$ is the **retraction onto A centered at \mathfrak{s}** .

We denote by $\rho_{+\infty} : \mathcal{I} \rightarrow \mathbb{A}$ (resp. $\rho_{-\infty}$) the retraction onto \mathbb{A} centered at $+\infty$ (resp. $-\infty$).

3 Intersection of two apartments in a measure

Let A, B be two apartments of \mathcal{I} . We prove below that $A \cap B$ is enclosed and that there exists $\phi : A \xrightarrow{A \cap B} B$ (see Theorem 3.6). Let us sketch our proof. We assume that $\mathbb{A} = B$. By results of [Héb20], the main difficulty is to prove that $A \cap \mathbb{A}$ is convex. We first assume that $A \cap \mathbb{A}$ has nonempty interior. Then using [Héb21, Proposition 9] or [Héb18, Proposition 4.2.8], we write $A = \bigcup_{i=1}^k P_i$, where $k \in \mathbb{N}$, the P_i are enclosed and if $i \in \llbracket 1, k \rrbracket$, there exists an apartment A_i containing $P_i, -\infty$ and an apartment isomorphism $A \xrightarrow{P_i} A_i$ (note that in [Héb21], the measure \mathcal{I} is moreover assumed to be semi-discrete, which means that $\Lambda'_\alpha = \mathbb{Z}$ for all $\alpha \in \Phi$, but this assumption is easily dropped for the mentioned results, see [Héb18]). Let \mathcal{H} be a finite set of walls delimiting the P_i . If $a, b \in A$ we denote by $\tau_{a,b} : [0, 1] \rightarrow A$ the affine parametrization of $[a, b]_A$ such that $\tau_{a,b}(0) = a$ and $\tau_{a,b}(1) = b$. Let $a, b \in A$ be such that at each time t such that $\tau(t)$ is in some wall of \mathcal{H} , then this wall is unique (as we shall see, almost every pair $(a, b) \in A^2$ satisfies this property). We prove that $\pi = \rho_{-\infty} \circ \tau_{a,b}$ is a piecewise linear path whose left-hand and right-hand derivatives satisfy some growth property with respect to $\leq_{Q_{\mathbb{R}}^{\vee}}$ (see Lemma 3.4). We deduce that if $b \in A \cap \mathbb{A}$, then for almost all $a \in \mathbb{A} \cap A$, $\pi'_+(0) \leq_{Q_{\mathbb{R}}^{\vee}} b - a$. Applying the analogous inequality to $\rho_{+\infty} \circ \tau$, we deduce that $\rho_{-\infty} \circ \tau = \rho_{+\infty} \circ \tau$. We then deduce that $\tau \subset \mathbb{A}$ and we conclude by using a density argument.

We equip every apartment with the topology defined by its structure of a finite dimensional real-affine space. If E is a subset of an apartment A , we denote by $\text{Int}(E)$ or by $\overset{\circ}{E}$ its interior, depending on the context. This does not depend on the choice of an apartment containing E by [Héb20, Proposition 3.26]. Let A be an apartment. For $a, b \in A$, $a \neq b$, we denote by $\mathcal{L}(a, b) = \mathcal{L}_A(a, b)$ the line containing a and b in A . Let $x, y \in \mathcal{I}$ be such that $x \leq y$ and A be an apartment containing x and y . We write $x \overset{\circ}{<} y$ if there exists an apartment isomorphism $\phi : A \rightarrow \mathbb{A}$ such that $\phi(y) - \phi(x) \in \overset{\circ}{\mathcal{T}}$. This does not depend on the choices of A and ϕ .

Lemma 3.1. *Let $x \in \mathcal{I}$ and A, B be two apartments containing x . We assume that there exists a neighborhood V of x in A and a half-apartment D of A such that $A \cap B \supset D \cap V$. Then:*

1. *Either $x \in \text{Int}(A \cap B)$ or there exists a neighborhood \tilde{V} of x in A such that $A \cap B \cap \tilde{V} = D \cap \tilde{V}$.*
2. *If $x \notin \text{Int}(A \cap B)$, there exists an apartment B' such that $B \cap B'$ is a half-apartment and such that $x \in \text{Int}(A \cap B')$.*

Proof. Let M be the wall of A delimiting D and P be a positive panel of M based at x . Then $P \subset A \cap B$. Let C be the chamber of A dominating P and not contained in D . We can assume, reducing V if necessary, that V is convex and open. By [Héb20, Proposition 3.26], there exists $\phi : A \xrightarrow{D \cap V} B$. Let $D_B = \phi(D)$. We have

$$\phi(D \cap V) = D \cap V \subset \phi(D) \cap V = D_B \cap V. \quad (1)$$

Let B' be an apartment containing D_B and C (the existence of such an apartment is provided by [Rou11, Proposition 2.9 1]). Then $B' \cap A$ contains $D_B \cap V$. Let Q be the sector of A based at x and containing C . Reducing V if necessary, we may assume that $V \cap Q \subset B'$. Let $y \in V \cap Q$. Then $y \overset{\circ}{>} x$. Let $V_1 \subset V$ be a neighborhood of x in A such that $y \overset{\circ}{>} V_1$. Let $z \in \mathcal{L}(x, y) \cap \text{Int}(D \cap V)$. Let $V_2 \subset V_1$ be a neighborhood of z in A such that $V_2 \subset D \cap V$. Then $V_3 := \bigcup_{v \in V_2} [y, v]_A$ contains x in its interior. By [Rou11, Proposition 5.4],

$$V_3 \subset A \cap B'. \quad (2)$$

Suppose $B \supset C$. By applying the result above with $B' = B$ we deduce that $x \in \text{Int}(A \cap B)$. Consequently $A \cap B$ contains C if and only if $x \in \text{Int}(A \cap B)$. Suppose now $x \notin \text{Int}(A \cap B)$. Then $B \not\supseteq C$. Then $B' \cap B$ contains a half-apartment and thus it contains a generic ray. Thus we have:

- $B \cap B'$ is enclosed in B (by (MA ii)),
- $B \cap B' \not\supseteq C$,
- $B \cap B' \supset D_B$.

Therefore $B \cap B' = D_B$.

Let $\tilde{V} \subset V_3$ be a convex neighborhood of x in A . Let us prove the equality

$$D \cap \tilde{V} = D_B \cap \tilde{V}. \quad (3)$$

By [Héb20, Proposition 3.26], there exists an apartment isomorphism $f : B' \xrightarrow{\tilde{V}} A$. By (MA ii), there exists $\psi : B' \xrightarrow{D_B} B$. By (1), $\phi^{-1} \circ \psi : B' \rightarrow A$ fixes pointwise $D \cap V \supset D \cap \tilde{V}$. As $D \cap \tilde{V}$ has nonempty interior, we have $f = \phi^{-1} \circ \psi$. Let now $y' \in D_B \cap \tilde{V}$. Then

$$f(y') = y' = \phi^{-1}(\psi(y')) = \phi^{-1}(y')$$

and thus ϕ^{-1} fixes $D_B \cap \tilde{V}$ pointwise. Since $\tilde{V} \subset V$, ϕ also fixes pointwise $D \cap \tilde{V}$. Similarly to (1), we deduce that $D_B \cap \tilde{V} \subset D \cap \tilde{V}$ and $D_B \cap \tilde{V} \supset D \cap \tilde{V}$, which proves (3).

By (2), we have $\tilde{V} \subset A \cap B'$. Therefore:

$$\begin{aligned} A \cap B \cap \tilde{V} &= A \cap B' \cap B \cap \tilde{V} \\ &= A \cap D_B \cap \tilde{V} \\ &= A \cap D \cap \tilde{V} \text{ (by (3))} \\ &= D \cap \tilde{V}, \end{aligned}$$

which completes the proof of the lemma. □

An element $r \in W^v$ is called a **reflection** if $r = wr_iw^{-1}$, for some $w \in W^v$ and $i \in I$.

Let B be an apartment containing $-\infty$. We denote by $\rho_{-\infty, B}$ the retraction onto B centered at $-\infty$. If $x, y \in B$ we denote $x \leq_{Q_{\mathbb{R}, B}^v} y$ if $\rho_{-\infty}(x) \leq_{Q_{\mathbb{R}}^v} \rho_{-\infty}(y)$.

For $t \in \mathbb{R}$, we denote by $]t^-, t]$ the filter on \mathbb{R} consisting of the sets containing a set of the form $[t - \epsilon, t]$, for some $\epsilon > 0$.

Lemma 3.2. *Let A be an apartment and $x \in A$. Let B be an apartment containing $-\infty$. We assume that there exists a half-apartment D of A and a neighborhood V of x in A such that $V \cap A \cap B = V \cap D$. Let $\tau : \mathbb{R} \rightarrow A$ be a non-constant affine map such that $\tau(0) = x$ and such that $\tau(]0^-, 0]) \subset D$. Let $\pi = \rho_{-\infty, B} \circ \tau$. Then $\pi'_+(0) \geq_{Q_{\mathbb{R}}^v} \pi'_-(0)$. If moreover $\pi'_+(0) \neq \pi'_-(0)$, then there exists a reflection \vec{r} of W^v such that $\pi'_+(0) = \vec{r}.\pi'_-(0)$.*

Proof. We assume that $\pi'_+(0) \neq \pi'_-(0)$, since otherwise there is nothing to prove. Then $x \notin \text{Int}(A \cap B)$. By Lemma 3.1, there exists an apartment B' such that $B' \cap B$ is a half-apartment and such that $B' \cap A$ contains a neighborhood of x . Maybe reducing V , we may assume that V is open and convex, that it is contained in $A \cap B'$ and that $A \cap B \cap V = D \cap V$. By [Héb20, Proposition 3.26], there exists $\phi : A \rightarrow B'$ fixing V . Therefore there is no loss of generality in assuming that $A = B'$ and thus that $A \cap B$ contains a half-apartment. Then $A \cap B = D$ (by (MA ii)). We identify B and \mathbb{A} .

Let $\psi : A \xrightarrow{A \cap \mathbb{A}} \mathbb{A}$. As $\pi'_-(0) \neq \pi'_+(0)$, A does not contain $-\infty$. Let D_A be the half-apartment of A opposite to D and $D_{\mathbb{A}}$ be the half-apartment of \mathbb{A} opposite to D . By [Rou11, Proposition 2.9 2)], $\tilde{A} := D_A \cup D_{\mathbb{A}}$ is an apartment. Let $\tilde{\psi} : \tilde{A} \xrightarrow{\tilde{A} \cap \mathbb{A}} \mathbb{A}$. Then as $\tilde{A} \supset -\infty$, we have $\rho_{-\infty}(y) = \tilde{\psi}(y)$ for every $y \in D_A$. Let r be the reflection of the affine Weyl group W with respect to the wall of D . By [Héb21, Lemma 6 2)], $\psi(\tau(t)) = r.\tilde{\psi}(\tau(t))$ for every $t \in \mathbb{R}_+$. Therefore $\pi'_-(0) = \vec{r}.\pi'_+(0)$, where \vec{r} is the linear map associated with r . Let $\alpha \in \Phi_+$ be such that $\vec{r} = r_\alpha$. Let $\vec{D}_{\mathbb{A}} = \alpha^{-1}(\mathbb{R}_-)$. Then $\vec{D}_{\mathbb{A}}$ is parallel to $D_{\mathbb{A}}$. Then as $\tau(]0^-, 0]) \subset D$ we have $\pi'_-(0) \in \vec{D}_{\mathbb{A}}$ and thus $\alpha(\pi'_-(0)) < 0$. Consequently $\pi'_+(0) = \pi'_-(0) - \alpha(\pi'_-(0))\alpha^v >_{Q_{\mathbb{R}}^v} \pi'_-(0)$. \square

Let A be an apartment and \mathfrak{q} be a sector-germ of \mathcal{I} . Then by [Héb21, Proposition 9] or [Héb18, Proposition 4.2.8], there exist $k \in \mathbb{N}$ and enclosed subsets P_1, \dots, P_k such that:

- $A = \bigcup_{i=1}^k P_i$,
- for every $i \in \llbracket 1, k \rrbracket$, there exists an apartment A_i containing P_i and \mathfrak{q} ,
- for every $i \in \llbracket 1, k \rrbracket$, there exists an apartment isomorphism $\phi_i : A \xrightarrow{P_i} A_i$,
- $\mathring{P}_i \neq \emptyset$, for $i \in \llbracket 1, k \rrbracket$.

The last condition follows from [Héb20, Lemma 3.10] applied with $U = X = A$.

For $i \in \llbracket 1, k \rrbracket$, we write $P_i = \bigcap_{j=1}^{k_i} D_{i,j}$, where $k_i \in \mathbb{N}$ and $D_{i,j}$ is a half-apartment of A . We denote by $M_{i,j}$ the wall of $D_{i,j}$. Then we set $\mathcal{H}(A, \mathfrak{q}) = \{M_{i,j} \mid i \in \llbracket 1, k \rrbracket, j \in \llbracket 1, k_i \rrbracket\}$. For $b \in A$, we then set

$$E(A, \mathfrak{q}, b) = A \setminus \bigcup_{M, M' \in \mathcal{H}(A, \mathfrak{q}), M \neq M'} \bigcup_{x \in M \cap M'} \mathcal{L}(x, b).$$

Note that $\mathcal{H}(A, \mathfrak{q})$ and $E(A, \mathfrak{q}, b)$ depend on the choices we made (we can for example artificially increase the number of P_i). However for our purpose, this dependency will not be important and we will not be specified: for any choice, $E(A, \mathfrak{q}, b)$ is obtained from A by removing finitely many hyperplanes (which are in general not walls). It is in particular dense in A . By definition, we have the following lemma:

Lemma 3.3. *Let A be an apartment, \mathfrak{q} be a sector-germ of \mathcal{I} , $b \in A$ and $a \in E(A, \mathfrak{q}, b)$. Let $\tau : [0, 1] \rightarrow [a, b]_A$ be an affine parametrization of $[a, b]_A$. Then for every $t \in [0, 1]$, we have either $\tau(t) \notin \bigcup_{M \in \mathcal{H}(A, \mathfrak{q})} M$ or there exists a unique $M \in \mathcal{H}(A, \mathfrak{q})$ such that $\tau(t) \in M$.*

A **piecewise linear continuous path of \mathbb{A}** is a continuous path $\pi : [0, 1] \rightarrow \mathbb{A}$ such that there exist $n \in \mathbb{N}$ and $0 < t_1 < \dots < t_n < 1$ such that $\pi|_{[t_i, t_{i+1}]}$ is a line segment, for every $i \in \llbracket 1, n-1 \rrbracket$. In [GR08, 6.1], Gaussent and Rousseau study the image π by $\rho_{-\infty}$ of a segment $\tau : [0, 1] \rightarrow \mathcal{I}$ such that $\tau(0) \leq \tau(1)$. They prove that this image is a Hecke path (see [GR08, Definition 5.2 and Theorem 6.2]). Roughly speaking, this means that π is a piecewise linear continuous path such that for all t such that $\pi'_-(t) \neq \pi'_+(t)$, $\pi'_+(t)$ “is farther from $-\infty$ ” than $\pi'_-(t)$. More precisely this means that there exists a sequence $\xi_1 = \pi'_-(t)$, $\xi_2, \dots, \xi_n = \pi'_+(t)$ such that for all $i \in \llbracket 1, n-1 \rrbracket$, $\xi_{i+1} = w_i \cdot \xi_i$, where w_i is a reflection of W^v with respect to some wall M_i , such that if D_i is the half-apartment of \mathbb{A} delimited by M_i and containing $-\infty$, then $\xi_i \in D_i$. In the lemma below, we study the image π by $\rho_{-\infty}$ of a segment $\tau : [0, 1] \rightarrow \mathcal{I}$ (satisfying some technical properties). Here we do not assume that $\tau(0) \leq \tau(1)$. We prove that π is a piecewise linear continuous path such that for all t such that $\pi'_-(t) \neq \pi'_+(t)$, $\pi'_+(t)$ “is farther from $-\infty$ ” than $\pi'_-(t)$. Thanks to our technical assumption on τ , we have $n = 2$ at each folding. We deduce an inequality similar to that of [BPGR16, 1.8 Lemma].

Lemma 3.4. *Let A be an apartment. Let $a, b \in A$. We assume that: $a \in E(A, -\infty, b)$.*

Let $\tau : [0, 1] \rightarrow [a, b]_A$ be the affine parametrization such that $\tau(0) = a$ and $\tau(1) = b$. Let $\pi = \rho_{-\infty} \circ \tau$. Then π is a piecewise linear continuous path such that:

1. *for all $t \in [0, 1[$ (resp. $t \in]0, 1]$), $\pi'_+(t) \in W^v \cdot \pi'_+(0)$ (resp. $\pi'_-(t) \in W^v \cdot \pi'_-(0)$),*
2. *for every $t, t' \in [0, 1]$ such that $0 < t < t' < 1$, we have:*

$$\pi'_+(0) \leq_{Q_{\mathbb{R}}^v} \pi'_-(t) \leq_{Q_{\mathbb{R}}^v} \pi'_+(t) \leq_{Q_{\mathbb{R}}^v} \pi'_-(t') \leq_{Q_{\mathbb{R}}^v} \pi'_+(t') \leq_{Q_{\mathbb{R}}^v} \pi'_-(1).$$

We have $\rho_{-\infty}(b) - \rho_{-\infty}(a) \geq_{Q_{\mathbb{R}}^v} \pi'_+(0)$ and if π is not a line segment, then this inequality is strict. In particular, if $a, b \in \mathbb{A}$, then $\pi'_+(0) \leq_{Q_{\mathbb{R}}^v} b - a$.

Proof. As $\rho_{-\infty}$ is an affine morphism on every P_i , $i \in \llbracket 1, k \rrbracket$, π is a piecewise linear continuous path. Let $n \in \mathbb{N}$ and $0 < t_1 < \dots < t_n < 1$ be such that $\pi|_{[t_i, t_{i+1}]}$ is a line segment, for every $i \in \llbracket 1, n-1 \rrbracket$.

We now prove that for all $i \in \llbracket 1, n \rrbracket$, $\pi'_-(t_i) \leq_{Q_{\mathbb{R}}^v} \pi'_+(t_i)$. Let $i \in \llbracket 1, n \rrbracket$. If $\pi'_-(t_i) = \pi'_+(t_i)$, there is nothing to prove. We assume that $\pi'_-(t_i) \neq \pi'_+(t_i)$. Let P be an enclosed subset of A with non-empty interior such that:

1. P is delimited by walls of $\mathcal{H} := \mathcal{H}(A, -\infty)$,

2. $\tau([t_i^-, t_i]) \subset P$,
3. there exist an apartment B containing $-\infty$ and P and an apartment isomorphism $\phi : B \xrightarrow{P} A$.

As $\pi'_-(t_i) \neq \pi'_+(t_i)$, there exists $M \in \mathcal{H}$ such that $\tau(t_i) \in M$ and by Lemma 3.3, this wall is unique. Let V be a convex neighborhood of $\tau(t_i)$ in A such that $V \cap \bigcup_{M' \in \mathcal{H} \setminus \{M\}} M' = \emptyset$. Let D be the half-apartment of A delimited by M and containing P . Then $V \cap A \cap B \supset V \cap P$. Let us prove that $V \cap P = V \cap D$. As D and P are delimited by M , we have $P \subset D$. Suppose $V \cap D \neq V \cap P$. Let $y \in (V \cap D) \setminus P$. Let $x \in \dot{P} \cap V$. Then there exists $z \in [x, y]$ such that z is in the boundary $\text{Fr}(P)$ of P . Then $z \in V$ and by assumption on V , we have $z \in M$. As $x \in \dot{D}$, we have $y = z \in M$. But then $[x, y] \subset P$: a contradiction. Therefore $V \cap P = V \cap D$. Then by Lemma 3.1, maybe replacing V by a smaller neighborhood of $\tau(t_i)$, we can assume that $V \cap A \cap B = V \cap D$. Set $\pi_B = \rho_{-\infty, B} \circ \tau$. By Lemma 3.2, we have $\pi'_{B,-}(t_i) \leq_{Q_{\mathbb{R}, B}^{\vee}} \pi'_{B,+}(t_i)$ and $\pi'_-(t_i) \in W^v \cdot \pi'_+(t_i)$. Therefore $\pi'_-(t_i) \leq_{Q_{\mathbb{R}}^{\vee}} \pi'_+(t_i)$. By induction we deduce that π satisfies (1) and (2). By integrating π' between 0 and 1, we deduce the lemma. \square

Lemma 3.5. *Let $A, B \in \mathcal{A}$ be two apartments such that $\text{Int}(A \cap B) \neq \emptyset$. Then $A \cap B$ is convex.*

Proof. Using isomorphisms of apartments, we may assume that $B = \mathbb{A}$. Let $a \in \text{Int}(A \cap \mathbb{A})$ and $b \in A \cap \mathbb{A}$. We begin by proving that $[a, b]_A \subset \mathbb{A}$. We choose a sequence $(a_n) \in \text{Int}(A \cap \mathbb{A})^{\mathbb{N}}$ such that $a_n \rightarrow a$ and such that for all $n \in \mathbb{N}$, $a_n \in E(A, -\infty, b) \cap E(A, +\infty, b)$ (this is possible since $E(A, -\infty, b)$ and $E(A, +\infty, b)$ are obtained from A by removing finitely many hyperplanes). Let $n \in \mathbb{N}$. Let $\tau : [0, 1] \mapsto [a_n, b]_A$ be the affine parametrization of $[a_n, b]$ such that $\tau(0) = a_n$ and $\tau(1) = b$. Let $\pi = \rho_{-\infty} \circ \tau$. Then by Lemma 3.4, $\pi'_+(0) \leq_{Q_{\mathbb{R}}^{\vee}} b - a_n$. Let $\tilde{\pi} = \rho_{+\infty} \circ \tau : [0, 1] \rightarrow \mathbb{A}$. Then similarly to Lemma 3.4, we have $\tilde{\pi}'_+(0) \geq_{Q_{\mathbb{R}}^{\vee}} b - a_n$. Moreover $\tau([0, \epsilon]) \subset \mathbb{A}$ for $\epsilon > 0$ small enough. Therefore $\rho_{+\infty} \circ \tau|_{[0, \epsilon]} = \rho_{-\infty} \circ \tau|_{[0, \epsilon]} = \tau|_{[0, \epsilon]}$. Consequently

$$\pi'_+(0) = \tilde{\pi}'_+(0) \leq_{Q_{\mathbb{R}}^{\vee}} b - a_n \leq_{Q_{\mathbb{R}}^{\vee}} \tilde{\pi}'_+(0) = \pi'_+(0).$$

As $(\alpha_i^{\vee})_{i \in I}$ is free, we deduce that $\pi'_+(0) = \tilde{\pi}'_+(0) = b - a_n$. The path π is thus a line segment, since otherwise we would have $b - a_n = \pi(1) - \pi(0) >_{Q_{\mathbb{R}}^{\vee}} b - a_n$ by Lemma 3.4. By symmetry, $\tilde{\pi}$ is the line segment joining a_n to b . Therefore $\pi = \tilde{\pi}$ and thus $\rho_{-\infty} \circ \tau(t) = \rho_{+\infty} \circ \tau(t)$ for $t \in [0, 1]$. By [Héb20, Proposition 3.7], $\tau(t) \in \mathbb{A}$ for every $t \in [0, 1]$. Therefore $[a_n, b] \subset \mathbb{A}$. Moreover

$$\overline{\bigcup_{n \in \mathbb{N}} [a_n, b]_A} \supset [a, b]_A.$$

By [Héb20, Proposition 3.9], $A \cap \mathbb{A}$ is closed and hence $[a, b]_A \subset \mathbb{A}$.

Let V be an open neighborhood of a contained in $A \cap \mathbb{A}$. Then by what we proved, $\bigcup_{v \in V} [v, b]_A \subset A \cap \mathbb{A}$. Moreover if $n \in \mathbb{N}$, then $(1 - \frac{1}{n})b + \frac{1}{n}V$ is a neighborhood of $(1 - \frac{1}{n})b + \frac{1}{n}a$ contained in $\bigcup_{v \in V} [v, b]_A \subset A \cap \mathbb{A}$. Therefore $b \in \text{Int}(A \cap \mathbb{A})$. Therefore $\text{Int}(A \cap \mathbb{A})$ is dense in $A \cap \mathbb{A}$. We also proved that $\text{Int}(A \cap \mathbb{A})$ is convex and as $A \cap \mathbb{A}$ is closed, the lemma follows. \square

Theorem 3.6. *Let A and B be two apartments. Then $A \cap B$ is enclosed and there exists an apartment isomorphism $\phi : A \rightarrow B$ fixing $A \cap B$.*

Proof. We assume that $A \cap B$ is non-empty. Let $a, b \in A \cap B$. Let C_a (resp. C_b) be a chamber of A (resp. of B) based at a (resp. b). By [Héb20, Proposition 5.17 (ii)], there exists an apartment \tilde{A} containing C_a and C_b . By Lemma 3.5, $A \cap \tilde{A}$ and $\tilde{A} \cap B$ are convex. By [Héb20, Proposition 3.26], there exists $\phi_A : A \xrightarrow{A \cap \tilde{A}} \tilde{A}$ and $\phi_B : \tilde{A} \xrightarrow{\tilde{A} \cap B} B$. Set $f_{a,b} = \phi_B \circ \phi_A$. Then $[a, b]_A = [a, b]_{\tilde{A}} = [a, b]_B$ and $f_{a,b}$ fixes $[a, b]_A$. Therefore $A \cap B$ is convex.

Let H be the support of $A \cap B$, that is, H is the smallest affine subspace of A containing $A \cap B$. If $E \subset H$, we denote by $\text{Int}_H(E)$ its interior in H . By [Héb20, Proposition 3.14], there exist $k \in \mathbb{N}$, enclosed subsets P_1, \dots, P_k of A such that $A \cap B = \bigcup_{i=1}^k P_i$ and such that for all $i \in \llbracket 1, k \rrbracket$, there exists an apartment isomorphism $\phi_i : A \xrightarrow{P_i} B$. As $A \cap B$ and the P_i are convex, there exists $i \in \llbracket 1, k \rrbracket$ such that $\text{Int}_H(P_i) \neq \emptyset$.

Let us prove that ϕ_i fixes $A \cap B$. Let $a \in \text{Int}_H(P_i)$. Let $b \in A \cap B$. Let $f = \phi_i^{-1} \circ f_{a,b} : A \rightarrow A$. Then f fixes a neighborhood of a in $[a, b]$ and as it is an affine isomorphism, it fixes $\mathcal{L}(a, b)$. In particular $f(b) = b$ and hence $\phi_i(b) = f_{a,b}(b) = b$, which proves that ϕ_i fixes $A \cap B$. By [Héb20, Proposition 3.22], $A \cap B$ is enclosed, which completes the proof of the theorem. \square

Corollary 3.7. *Let \mathcal{I} be a set endowed with a covering \mathcal{A} of subsets called **apartments**. We assume that $(\alpha_i)_{i \in \mathcal{I}}$ is free in \mathbb{A}^* and that $(\alpha_i^\vee)_{i \in \mathcal{I}}$ is free in \mathbb{A} . Then \mathcal{I} is a measure in the sense of [Rou11, Définition 2.1] if and only if it satisfies the following axioms:*

(MA I)=(MA i): *Any $A \in \mathcal{A}$ is equipped with the structure of an apartment of type \mathbb{A} .*

(MA II) : *Let A, A' be two apartments. Then $A \cap A'$ is enclosed and there exists an apartment isomorphism $\phi : A \xrightarrow{A \cap A'} A'$.*

(MA III)=(MA iii): *if \mathfrak{R} is the germ of a splayed chimney and if F is a face or a germ of a chimney, then there exists an apartment containing \mathfrak{R} and F .*

Remark 3.8. *By [Héb20, Proposition 4.25 and Remark 4.26], if \mathcal{I} is thick (which means that each panel of \mathcal{I} is dominated by at least three chambers) and if there exists a group acting strongly transitively on \mathcal{I} (see [Héb18, 3.1.5 and Corollary 4.4.40] for the definition of such an action), then for every enclosed subset P of \mathbb{A} such that $\overset{\circ}{P} \neq \emptyset$, there exists an apartment A such that $A \cap \mathbb{A} = P$. This is in particular true when \mathcal{I} is the measure associated with a split Kac-Moody group over a valued field.*

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