



**HAL**  
open science

# Nash equilibrium structure of Cox process Hotelling games

Venkat Anantharam, François Baccelli

► **To cite this version:**

Venkat Anantharam, François Baccelli. Nash equilibrium structure of Cox process Hotelling games. *Advances in Applied Probability*, 2022, 54 (2), pp.570-598. 10.1017/apr.2021.45 . hal-03107798

**HAL Id: hal-03107798**

**<https://hal.science/hal-03107798v1>**

Submitted on 13 Jan 2025

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Nash equilibrium structure of Cox process Hotelling games

Venkat Anantharam  
*EECS Department*  
*University of California, Berkeley*  
ananth@eecs.berkeley.edu

François Baccelli  
*Department of Mathematics and ECE Department*  
*University of Texas, Austin*  
and  
*INRIA/ENS, Paris, France*

francois.baccelli@austin.utexas.edu *or* francois.baccelli@inria.fr

August 18, 2020

## Abstract

We study an  $N$ -player game where a pure action of each player is to select a non-negative function on a Polish space supporting a finite diffuse measure, subject to a finite constraint on the integral of the function. This function is used to define the intensity of a Poisson point process on the Polish space. The processes are independent over the players, and the value to a player is the measure of the union of its open Voronoi cells in the superposition point process. Under randomized strategies, the process of points of a player is thus a Cox process, and the nature of competition between the players is akin to that in Hotelling competition games. We characterize when such a game admits Nash equilibria and prove that when a Nash equilibrium exists, it is unique and comprised of pure strategies that are proportional in the same proportions as the total intensities. We give examples of such games where Nash equilibria do not exist. A better understanding of

the criterion for the existence of Nash equilibria remains an intriguing open problem.

**Keywords:** Constant sum game; Cox process; Game theory; Hotelling competition; Nash equilibrium; Poisson process.

**Classification:** 60G55;91A06;91A60

# 1 Introduction

## 1.1 Informal problem formulation

We study *games of spatial competition of the Hotelling type*, where  $N$  players compete for space. Here space is modeled as a complete separable metric space (i.e. a *Polish space*)  $D$  supporting a *diffuse* (i.e. non-atomic) finite positive measure  $\eta$  on its Borel  $\sigma$ -field, which we denote by  $\mathcal{B}$ . Throughout the paper we think of  $D$  as being endowed with a fixed metric  $d$  generating its topology. The game in question is a one-shot game. Each player's action consists in selecting a nonnegative measure on  $D$ <sup>1</sup> which is absolutely continuous with respect to  $\eta$ , among the set of all measures with a fixed finite and positive total mass. This mass constraint depends on the player and represents the total available intensity that the player can deploy over the space. The measure chosen by each player results in a Poisson process of points on  $D$  with this measure as its intensity measure, these processes being independent. Thus, if a player uses a randomized strategy, the point process on  $D$  generated by each player is a *Cox point process*, namely a point process which is conditionally Poisson given its random intensity. The payoff of each player is the total  $\eta$  measure of the union of the *open Voronoi cells* of the points of the point process of this player, where the open Voronoi cells are evaluated with respect to the superposition of the point processes of all players (if there are no points, which can happen with positive probability, each player gets zero value; further, we need to make a technical assumption on the metric structure of  $D$  which ensures that the union of the open Voronoi cells is of full measure, conditioned on there being at least one point in the superposition point process). Our goal in this paper is to study the *Nash equilibria* of this game.

---

<sup>1</sup>Since we will always consider  $D$  as being metrized by the metric  $d$  and endowed with its Borel  $\sigma$ -algebra  $\mathcal{B}$ , we will not mention  $d$  and  $\mathcal{B}$  where these can be inferred from the context. Thus, for instance, by a nonnegative measure on  $D$  what we actually mean is a nonnegative measure on  $(D, \mathcal{B})$ . Similarly, when we talk about the open Voronoi cell of a point in a configuration of points from  $D$ , we implicitly mean that the Voronoi cells are based on the metric  $d$ .

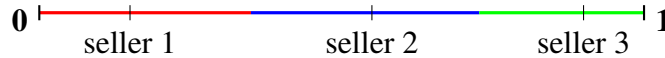


Figure 1: Hotelling competition on the unit interval with three sellers, with locations indicated. The individual open Voronoi cells are indicated in color. In the absence of price discrimination, consumers residing at a location in the open Voronoi cell of a seller will go to that seller.

In the 2-player case, one motivation for such a formulation comes from a model for defense against threats. The underlying space may be thought of as a model for the set of possible attack modalities, with the metric indicating how similar attacks are to each other. The defender and the attacker are respectively interested in defending against or deploying the different kinds of attacks, and evaluating value by the total  $\eta$  measure of the union of the open Voronoi cells of the points of each of them evaluates how well each of them is doing with regard to its individual objective of getting the upper hand over the other. The stochastic nature of the placement of the points of a player is meant to capture the idea that the deployment of effort only results in success in a stochastic way. The finite total intensity that each each player can deploy represents individual budget constraints. The study of Nash equilibria is then motivated by the goal of getting some insights into how the individual players (i.e. the attacker and defender) might play when faced with such an environment. Apart from its possible intrinsic interest, it turns out that it is no more difficult to study the  $N$ -player version of the game than the 2-player version, for reasons that we will soon see, so we have formulated the problem we study at this apparently broader level of generality.

## 1.2 Literature survey

The origin of the study of spatial competition models is generally attributed to a paper by Hotelling [17]. Hotelling’s model has some additional features, such as prices set by the sellers, which will not play a role in our formulation. Building on the *Cournot duopoly model* [7], [31, Sec. 27.5], as refined by Bertrand [2], [31, Sec. 27.9] and Edgeworth [12], the key innovation of [17] is to introduce spatial aspects to the modeling of the competition between sellers for buyers. Specifically, Hotelling considers the problem faced by two sellers as to where to position themselves along an interval of fixed length, which models a market along which consumers are uniformly distributed, and how to individually set prices for

the one identical good that they sell so that each seller maximizes its profit, given the strategy of the other seller. In the model of Hotelling each consumer incurs a transportation cost proportional to its distance from the seller from which it buys, which then determines which seller it prefers, given the location of the sellers.

Notice that for fixed locations of the two sellers and if price discrimination is not possible (i.e. prices are identical at the two sellers) consumers would go to the closer seller, so the problem of each seller becomes that of how to position itself, in reaction to position of the other seller, so as to maximize the length of its Voronoi cell in the *Voronoi decomposition* associated to the locations of the two sellers. See [17, Fig. 1, pg. 45]. The problem considered in this paper is of this purely Voronoi-cell-decomposition type. It is worth noting that the solution concept in [17] or, for that matter, in [7] is already of the type that one would today call a pure-strategy Nash equilibrium, though of course these works predate by several decades the work of von Neumann and Morgenstern [32] and Nash [24, 25], which formalized the notion of Nash equilibrium.

There is by now a vast literature on the Hotelling competition model, and many different variants have been developed and studied. Rather than attempt to survey this work, particularly since the precise problem formulation we consider does not appear in the prior literature in this area, we refer the reader to the surveys in [16, 14, 13, 27], and to the papers in the recent edited volume [22].

Another growing body of work that is related to the themes of this paper is that of Voronoi games, see the seminal papers [1] and [6]. Here also the players are competing to capture regions of space according to the Voronoi decomposition of the underlying space based on the choice of the locations of their points but, in contrast to the model we consider, the players are assumed to have control over exactly where they can place their points with the constraint, in some of the literature on Voronoi games, that these locations should lie in a given finite set of potential locations in the ambient space. Further, in contrast to the model we consider, much of this literature assumes that the players place their points one at a time, alternating between the players. There is also a particular interest in this literature in the study of Voronoi games on graphs, see e.g. [30, 11]. Note that this is incompatible with the non-atomic nature of the underlying measure  $\eta$  required for our formulation. Nevertheless, our formulation was partly inspired by a recent work in this area, on so-called *Voronoi choice games*, by Boppana et al. [4].

There are also some similarities between the game we analyze and the study of the so-called *Colonel Blotto games*, see [28], on variants of which also there is a rapidly growing literature. In the game we study, the ability of a player to control exactly where to place its points, as in the literature on Hotelling games

or Voronoi games, can be thought of as being replaced by a softer ability of being able to control the intensity or, in effect, the local mean number of points, in the same way as the formulation of the so-called *General Lotto game*, see [20], softens the ability of individual players to fix the number of soldiers to be placed on each battlefield of the Colonel Blotto game. However, the reward structure of the players in a Colonel Blotto game is completely different from that in the Hotelling games or Voronoi games.

### 1.3 Notational conventions

$\mathbb{N}$  denotes the set of natural numbers.  $\mathbb{R}_+$  denotes the set of non-negative real numbers. We use  $:=$  and  $=:$  for equality by definition. The indicator of a set or an event  $E$  is written as  $1_E$  or  $1(E)$ .

### 1.4 Structure of the paper

Section 2 sets up the structure of 2-player Cox processes Hotelling games. Section 3 introduces the  $N$ -player Cox process Hotelling games and develops several general properties of the value function in these games; these are used later to prove the main results. Section 4 is focused on the case where the underlying Polish space on which the game is played is compact and admits a transitive group of metric preserving automorphisms, with the base measure of the game being invariant under this group. Section 5 studies the general case of the  $N$ -player game and determines the structure of the Nash equilibria when they exist. A necessary and sufficient condition for the existence of Nash equilibria is provided, and also examples are given where no Nash equilibrium exists. Although Nash equilibrium, when it exists, is unique and comprised of pure strategies, it is established that Cox process Hotelling games are not ordinal potential games in general. A family of so-called restricted Cox process Hotelling games is defined to provide a vehicle to better understand the meaning of the criterion for the existence of Nash equilibria.

## 2 Cox process Hotelling games

### 2.1 Diffuse non-conflicting finite positive measures

Let  $D$  be a Polish space. We write  $\mathcal{B}$  for the Borel  $\sigma$ -field of  $D$ . A finite positive measure  $\eta$  on  $D$  is called *diffuse* if for every Borel set  $B \in \mathcal{B}$  with  $\eta(B) > 0$  there is a Borel subset  $C \subset B$  with  $0 < \eta(C) < \eta(B)$ . One can define a Poisson process on  $D$  based on such a diffuse positive measure  $\eta$  on  $D$ , see e.g. [8, Ch. 9]. A point process on  $D$  is a random counting measure on  $D$ . The Poisson process on  $D$  with intensity measure  $\eta$ , where  $\eta$  is any finite positive diffuse measure on  $D$ , is the point process obtained by first selecting the total number of points according to the Poisson distribution on  $\mathbb{N}$  with mean  $\eta(D)$ , and then sampling independently the location of the points, if any, according to the measure  $\eta(\cdot)/\eta(D)$  on  $\mathcal{B}$ .

Recall that  $d$  denotes the fixed metric on  $D$  generating its topology. The *open Voronoi cells* of any point process  $\Phi$  with respect to  $d$  can be defined in the usual way. Namely, for  $x$  in the support of the point process  $\Phi$ , the open Voronoi cell  $W_\Phi(x)$  is comprised of those  $z \in D$  such that  $d(x, z) < d(x', z)$  for all points  $x' \neq x$  in the support of  $\Phi$ . For our purposes, we need to impose on  $\eta$  the condition that the union of the open Voronoi cells of the Poisson process on  $D$  with intensity measure  $\eta$  has full measure  $\eta(D)$  with probability 1, conditioned on there being at least one point in this Poisson process. If  $\eta$  satisfies this condition, we call it *non-conflicting*. A sufficient condition for this to hold is given in Appendix A.

The importance of imposing a non-conflicting condition can be understood by considering the following example.

**Example 1.** Suppose  $D$  is the disjoint union of two unit intervals  $I$  and  $J$ , each of length 1. Assume that every point in  $I$  is at distance 2 from every point in  $J$ . It is straightforward to check that the resulting metric makes  $D$  a Polish space and that the measure  $\eta$  on the Borel  $\sigma$ -field of  $D$  which corresponds to Lebesgue measure on the two unit intervals comprising  $D$  is a finite positive diffuse measure. However,  $\eta$  is not non-conflicting. This is because, for instance, on the event of positive probability that the Poisson process on  $D$  with intensity measure  $\eta$  has two points in  $I$  and no points in  $J$ , every point of  $J$  will be distance 2 from each of the two points of the Poisson process in  $I$  and so does not belong to the open Voronoi cell of either point.  $\square$

Throughout the paper,  $\eta$  is a finite positive diffuse non-conflicting measure on  $D$  which will be referred to as the *base measure*. It is straightforward to check that if  $\nu$  is any finite positive measure on  $D$  that is absolutely continuous with

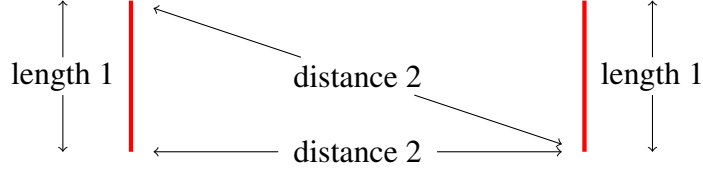


Figure 2: A metric space comprised of two intervals, each of length 1. Every point in the interval on the left is at distance 2 from every point in the interval on the right. The metric on each interval is the usual one.

respect to  $\eta$ , then  $\nu$  is also diffuse and non-conflicting. We write  $\mathcal{M}(D)$  for the set of nonnegative finite measures on  $D$ .

## 2.2 Radon-Nikodym derivatives

In the *Cox process Hotelling games* considered in this paper, we think of each individual player as choosing a positive measure on  $D$  of fixed total mass that is absolutely continuous with respect to the base measure  $\eta$ . The measure chosen by the player then serves as the intensity measure for a Poisson process of points on  $D$ , which we think of as the points belonging to that player. By the Radon-Nikodym theorem, we may identify the measure chosen by the player with its likelihood function with respect to the base measure.

With this viewpoint in mind, for any  $\rho > 0$ , let

$$\mathcal{C}(\rho) := \{f : D \rightarrow \mathbb{R}_+ \text{ s.t. } \int_D f(x)\eta(dx) = \rho\}. \quad (1)$$

$\mathcal{C}(\rho)$  can be thought of as a subset of  $L^1(\eta)$ , i.e. the Lebesgue space of  $\mathcal{B}$ -measurable functions on  $D$  that are absolutely integrable with respect to the base measure  $\eta$ , and also as a set of measures on  $D$  by identifying  $f \in \mathcal{C}(\rho)$  with the measure  $f\eta$ . With the latter viewpoint in mind, we think of  $\mathcal{C}(\rho)$  as endowed with the topology of weak convergence of measures [26], which it inherits as a subset of  $\mathcal{M}(D)$ . Note that, for all  $\rho > 0$  and all  $f \in \mathcal{C}(\rho)$ , the measure  $f\eta$  is diffuse and non-conflicting. Also note that, for all  $\rho > 0$ , the set  $\mathcal{C}(\rho)$  is a convex subset of  $\mathcal{M}(D)$ . However, since we have endowed  $\mathcal{C}(\rho)$  with the topology of weak convergence of measures on  $D$ , this set is not closed.

For any  $\rho > 0$ , the subset of  $\mathcal{M}(D)$  comprised of measures of total mass  $\rho$ , with the topology of weak convergence, can be metrized so as to make it a



complete separable metric space [26, Thm. 6.2 and Thm. 6.5]. As a metric space it is first countable, so it suffices to discuss convergence of sequences rather than of nets [19, Thm. 8]. To discuss randomized strategies of the individual players we need to be able to discuss probability distributions on subsets of the type  $\mathcal{C}(\rho)$  of  $\mathcal{M}(D)$ . This is made possible by the following result.

**Lemma 2.** For each  $\rho > 0$ , the set  $\mathcal{C}(\rho)$  is a Borel subset of  $\mathcal{M}(D)$  when  $\mathcal{M}(D)$  is endowed with the topology of weak convergence.

*Proof.* See [21, Thm. 3.5]. □

### 2.3 2-player games

In the 2-player version of the game, a pure action of Alice consists in choosing an intensity measure which is absolutely continuous with respect to the measure  $\eta$ . Namely Alice chooses a nonnegative measurable function  $f_A : D \rightarrow \mathbb{R}_+$  as the Radon Nikodym derivative with respect to  $\eta$  of the intensity measure of its Poisson point process. We denote the set of functions from which this choice must be made as  $\mathcal{C}_A$ , i.e.

$$\mathcal{C}_A := \mathcal{C}(\rho_A) = \{f_A : D \rightarrow \mathbb{R}_+ \text{ s.t. } \int_D f_A(x)\eta(dx) = \rho_A\}. \quad (2)$$

Thus  $\rho_A$  can be thought of as a constraint on the total intensity that Alice can deploy for her point process.

Similarly, Bob chooses a nonnegative measurable function  $f_B : D \rightarrow \mathbb{R}_+$  within the class of functions  $\mathcal{C}_B$ , where

$$\mathcal{C}_B := \mathcal{C}(\rho_B) = \{f_B : D \rightarrow \mathbb{R}_+ \text{ s.t. } \int_D f_B(x)\eta(dx) = \rho_B\}. \quad (3)$$

Here  $\rho_A$  and  $\rho_B$  are fixed positive constants. Note that  $\mathcal{C}_A$  and  $\mathcal{C}_B$  are convex subsets of  $\mathcal{M}(D)$ , but neither of them is closed in the topology of weak convergence on  $\mathcal{M}(D)$ .

We denote such a 2-player Cox process Hotelling game by  $(D, \eta, \rho_A, \rho_B)$ .

Assume that Alice plays  $f_A$  and Bob plays  $f_B$ . The resulting value for Alice, denoted by  $V_A(f_A, f_B)$ , is defined as follows. Let  $\Phi_A$  and  $\Phi_B$  be independent Poisson point processes on  $D$  with intensity measures  $f_A\eta$  and  $f_B\eta$ , respectively. We think of the points of  $\Phi_A$  as the points of Alice, since these result from the choice of  $f_A$  by Alice. Similarly we think of the points of  $\Phi_B$  as the points of

Bob. Let  $\Phi := \Phi_A + \Phi_B$ . For all  $x \in \Phi$ , let  $W_\Phi(x)$  denote the open Voronoi cell of  $x$  with respect to  $\Phi$ . Then,

$$V_A(f_A, f_B) := \mathbb{E}\left[\sum_{x \in \Phi_A} \eta(W_\Phi(x))\right],$$

where the expectation is with respect to the joint law of  $\Phi_A$  and  $\Phi_B$ , which are assumed to be independent. Here, by definition, a sum over an empty set is 0. The value for Bob resulting from this pair of actions, denoted  $V_B(f_A, f_B)$ , is determined similarly, i.e. it is

$$V_B(f_A, f_B) := \mathbb{E}\left[\sum_{x \in \Phi_B} \eta(W_\Phi(x))\right].$$

In words, the value of Alice is the mean value of the sum of the  $\eta$ -measures of the open  $\Phi$ -Voronoi cells centered at her points if she has points, and is 0 otherwise, and likewise for the value of Bob. Intuitively, one thinks of the points of Alice and those of Bob as competing to capture the ambient space, with a point in the ambient space  $D$  belonging to Alice if the closest point to it is one of Alice, and to Bob otherwise. Under our non-conflicting condition on the base measure  $\eta$ , there is no need to worry about how to break ties.

Note that

$$V_A(f_A, f_B) + V_B(f_A, f_B) = \eta(D)(1 - e^{-\rho}), \quad (4)$$

for all  $(f_A, f_B) \in \mathcal{C}_A \times \mathcal{C}_B$ , where  $\rho := \rho_A + \rho_B$ . This is because, by virtue of our assumption that  $\eta$  is non-conflicting, the sum of the values of Alice and Bob is  $\eta(D)$ , except on the event where  $\Phi_A$  and  $\Phi_B$  both do not have points, which is an event of probability  $1 - e^{-\rho}$ . This formula in Equation (4) is formally established in Lemma 5 below.

## 2.4 Nash equilibrium in 2-player games

In view of Lemma 2, any mixed strategy pair in the 2-player game  $(D, \eta, \rho_A, \rho_B)$  between Alice and Bob can be written as  $(f_A(M_A), f_B(M_B))$ , where  $M_A \in \mathcal{M}_A$  and  $M_B \in \mathcal{M}_B$  are independent random variables representing the randomizations used by Alice and Bob respectively in implementing randomized strategies, and  $f_A(m_A) \in \mathcal{C}_A$  (respectively  $f_B(m_B) \in \mathcal{C}_B$ ) is the choice of Alice (respectively Bob) in case the realization of her random variable  $M_A$  is  $m_A$  (respectively, the realization of his random variable  $M_B$  is  $m_B$ ). The value of Alice in such a mixed strategy is  $\mathbb{E}[V_A(f_A(M_A), f_B(M_B))]$ , while that of Bob is

$\mathbb{E}[V_B(f_A(M_A), f_B(M_B))]$ , the expectations being taken with respect to the joint distribution of  $M_A$  and  $M_B$ , which are independent. As a direct consequence of Equation (4), we have

$$\mathbb{E}[V_A(f_A(M_A), f_B(M_B))] + \mathbb{E}[V_B(f_A(M_A), f_B(M_B))] = \eta(D)(1 - e^{-\rho}), \quad (5)$$

where  $\rho := \rho_A + \rho_B$ .

**Definition 3.** The pair of independently randomized strategies  $(f_A(M_A), f_B(M_B)) \in \mathcal{C}_A \times \mathcal{C}_B$  is called a *Nash equilibrium* of the game between Alice and Bob if, for all  $g_A \in \mathcal{C}_A$  and  $g_B \in \mathcal{C}_B$ , we have

$$\mathbb{E}[V_A(f_A(M_A), f_B(M_B))] \geq \mathbb{E}[V_A(g_A, f_B(M_B))], \quad (6)$$

and

$$\mathbb{E}[V_B(f_A(M_A), f_B(M_B))] \geq \mathbb{E}[V_B(f_A(M_A), g_B)]. \quad (7)$$

In words, Alice sees no advantage in playing the strategy  $g_A$  instead of her randomized strategy  $f_A(M_A)$ , given that Bob is playing his randomized strategy  $f_B(M_B)$ , and similarly for Bob.  $\square$

## 3 Supporting lemmas

### 3.1 A formula for the value

Consider a 2-player Cox process Hotelling game  $(D, \eta, \rho_A, \rho_B)$  between Alice and Bob, and suppose that the players play the action pair  $(f_A, f_B)$ . Let  $\Phi_A$  and  $\Phi_B$  denote the Poisson processes of points of Alice and Bob respectively on  $D$ . Recall that these are independent point processes. Let  $\mathbb{P}_A^x$  (resp.  $\mathbb{P}_B^x$ ) denote the Palm probability [18, Chap. 6] with respect to  $\Phi_A$  (resp.  $\Phi_B$ ) at  $x$ . By Slivynak's theorem, [18, Lemma 6.14] under the Palm probability with respect to  $\Phi_A$  at  $x$ ,  $\Phi_A$  consists of a point at  $x$  and a Poisson process on  $D$  of intensity  $f_A \eta$ , and  $\Phi_B$  consists of an independent Poisson process on  $D$  of intensity  $f_B \eta$ . The Palm probability with respect to  $\Phi_B$  at  $x$  has a symmetrical description.

From Campbell's formula [18, Sec. 6.1] we have

$$V_A(f_A, f_B) = \int_D f_A(x) \mathbb{E}_A^x[\eta(W_{\Phi}(x))] \eta(dx).$$

Here the expectation is with respect to the law of  $\Phi$  under the Palm distribution  $\mathbb{P}_A^x$ . From the description of this Palm distribution, we have

$$\begin{aligned}\mathbb{E}_A^x[\eta(W_\Phi(x))] &= \mathbb{E}_A^x\left[\int_{y \in D} 1_{\{y \in W_\Phi(x)\}} \eta(dy)\right] \\ &\stackrel{(a)}{=} \int_{y \in D} \mathbb{P}_A^x(y \in W_\Phi(x)) \eta(dy) \\ &\stackrel{(b)}{=} \int_{y \in D} e^{-\int_{B(y \rightarrow x)} (f_A(u) + f_B(u)) \eta(du)} \eta(dy),\end{aligned}$$

where

$$B(y \rightarrow x) := \{z \in D \text{ s.t. } d(z, y) \leq d(x, y)\}.$$

The notation  $B(y \rightarrow x)$  is supposed to bring to mind a closed ball centered at  $y$  having  $x$  at its boundary.

In the chain of equations above, the step (a) comes from an application of Fubini's theorem and step (b) from Slivnyak's theorem. Hence

$$V_A(f_A, f_B) = \int_{x \in D} f_A(x) \int_{y \in D} e^{-\int_{B(y \rightarrow x)} (f_A(u) + f_B(u)) \eta(du)} \eta(dy) \eta(dx). \quad (8)$$

### 3.2 $N$ -player games

The point process analysis above and the resulting formula in Equation (8) in the 2-player case also leads to a formula in an  $N$ -player Cox process Hotelling game for the value  $V_i(f_1, \dots, f_N)$  seen by player  $i$  when the pure strategies deployed by the individual players are  $f_1, \dots, f_N$  respectively. For this, let  $f_j : D \rightarrow \mathbb{R}_+$  be a nonnegative measurable function on  $D$  belonging to

$$\mathcal{C}_j := \mathcal{C}(\rho_j) = \{f_j : D \rightarrow \mathbb{R}_+ \text{ s.t. } \int_D f_j(x) \eta(dx) = \rho_j\}, \quad (9)$$

where  $\rho_j > 0$  are fixed for  $1 \leq j \leq N$ . Note that each  $\mathcal{C}_j$  is a convex subset of  $\mathcal{M}(D)$ , but is not closed in the topology of weak convergence. Assume that the individual players play  $f_i \in \mathcal{C}_i$  for  $1 \leq i \leq N$ . Let  $\Phi_i$  be a Poisson point process on  $D$  with intensity measure  $f_i \eta$ , with these processes being mutually independent for  $1 \leq i \leq N$ , and let  $\Phi := \sum_{i=1}^N \Phi_i$ . We think of the points of  $\Phi_i$  as the points of player  $i$ , since these result from the choice of  $f_i$ , which was made by that player. Then the value of player  $i$ , denoted  $V_i(f_1, \dots, f_N)$ , is given by

$$V_i(f_1, \dots, f_N) := \mathbb{E}\left[\sum_{x \in \Phi_i} \eta(W_\Phi(x))\right].$$

The expectation is with respect to the joint law of  $(\Phi_j, 1 \leq j \leq n)$ , which are independent. Here, by definition, a sum over an empty set is 0.

We denote such an  $N$ -player Cox process Hotelling game by  $(D, \eta, \rho_1, \dots, \rho_N)$ .

We can write a formula for  $V_i(f_1, \dots, f_N)$ , based on equation (8), by thinking of the points of player  $i$  as competing for space in  $D$  with the union of the points of the other players. Thus we have:

$$V_i(f_1, \dots, f_N) = \int_{x \in D} f_i(x) \int_{y \in D} e^{-\int_{u \in B(y \rightarrow x)} f(u) \eta(du)} \eta(dy) \eta(dx), \quad (10)$$

where  $f := \sum_{i=1}^N f_i$ .

Note that we must have

$$\sum_{i=1}^N V_i(f_1, \dots, f_N) = \eta(D)(1 - e^{-\rho}), \quad (11)$$

where  $\rho := \sum_{j=1}^N \rho_j$ . This is because we have  $\int_{x \in D} f(x) \eta(dx) = \rho$ , so the law of the total number of points is Poisson with mean  $\rho$ , and so, by virtue of our assumption that  $\eta$  is non-conflicting, the total value of all the players is  $\eta(D)$  except on the event that there are no points in  $\Phi$ , in which case the total value is 0. The formula in Equation (11) is established in Lemma 5 below.

### 3.3 Nash equilibrium in $N$ -player games

In view of Lemma 2, any mixed strategy pair in an  $N$ -player Cox process Hotelling game  $(D, \eta, \rho_1, \dots, \rho_N)$  can be written as  $(f_1(M_1), \dots, f_N(M_N))$ , where  $(M_j \in \mathcal{M}_j, 1 \leq j \leq N)$  are independent random variables representing the randomizations used by the individual players in implementing their randomized strategies, and  $f_j(m_j) \in \mathcal{C}_j$  is the choice of action of player  $j$  in case the realization of her random variable  $M_j$  is  $m_j$ . The value of player  $i$  in such a mixed strategy is  $\mathbb{E}[V_i(f_1(M_1), \dots, f_N(M_N))]$ , the expectation being taken with respect to the joint distribution of  $(M_j, 1 \leq j \leq N)$ , which are independent. As a direct consequence of Equation (11), we have

$$\sum_{i=1}^N \mathbb{E}[V_i(f_1(M_1), \dots, f_N(M_N))] = \eta(D)(1 - e^{-\rho}), \quad (12)$$

where  $\rho := \sum_{i=1}^N \rho_i$ .

**Definition 4.** The vector of independently randomized strategies

$$(f_1(M_1), \dots, f_N(M_N)) \in \mathcal{C}_1 \times \dots \times \mathcal{C}_N$$

is called a *Nash equilibrium* of the  $N$ -player game if, for all  $g_j \in \mathcal{C}_j$ ,  $1 \leq j \leq N$ , we have, for all  $1 \leq i \leq N$ ,

$$\mathbb{E}[V_i(f_1(M_1), \dots, f_N(M_N))] \geq \mathbb{E}[V_i(g_i, (f_j(M_j), j \neq i))]. \quad (13)$$

In words, each player  $i$  perceives no advantage in playing the strategy  $g_i$  instead of its randomized strategy  $f_i(M_i)$ , given that the other players, i.e. the players  $j \neq i$ , are playing the individually randomized strategies  $(f_j(M_j), j \neq i)$ .  $\square$

### 3.4 A conservation law

In order to establish the conservation law in Equation (11) and its special case in Equation (4), it suffices to demonstrate that for all  $\rho > 0$  and all  $f \in \mathcal{C}(\rho)$ , we have equation (14) below. This is because the desired conservation laws would then follow immediately from equation (10) and its special case in Equation (8), respectively, by summing these formulas over the individual players. We establish (14) formally in the following lemma.

**Lemma 5.** For any  $\rho > 0$  and  $f \in \mathcal{C}(\rho)$ , we have:

$$\int_{x \in D} f(x) \int_{y \in D} e^{-\int_{B(y \rightarrow x)} f(u)\eta(du)} \eta(dy)\eta(dx) = \eta(D) (1 - e^{-\rho}). \quad (14)$$

*Proof.* We give two proofs of this formula. The first one is probabilistic. Let  $\Phi$  denote a Poisson process on  $D$  with intensity measure  $f\eta$ . By Campbell's formula and Slivnyak's theorem [18], the integral on the left hand side of Equation (14) is just

$$\mathbb{E}\left[\sum_{x \in \Phi} \eta(W_\Phi(x))\right],$$

where  $W_\Phi(x)$  denotes the open Voronoi cell of  $x \in \Phi$  with respect to the Poisson process  $\Phi$ . If  $\Phi(D) = 0$ , which happens with probability  $e^{-\rho}$ , the sum is 0. On the complementary event the sum is  $\rho(D)$ , since, by virtue of the assumption that  $\eta$  is non-conflicting, the collection of sets  $(W_\Phi(x), x \in \Phi)$  form a partition of  $D$  up to a set of  $\eta$ -measure 0.

The second proof is analytical. We have, for each  $y \in D$ ,

$$\int_{x \in D} f(x) e^{-\int_{u \in B(y \rightarrow x)} f(u) \eta(du)} \eta(dx) = 1 - e^{-\rho},$$

since, when  $\eta$  is non-conflicting, the integral on the left hand side of the preceding equation is just the probability that a nonhomogeneous Poisson process with intensity function  $f(x)$  with respect to  $\eta$  has at least one point in  $D$  (to see this, think of moving out from  $y$  by balls of increasing radius till one covers all of  $D$ ). Thus we have

$$\int_{y \in D} \int_{x \in D} f(x) e^{-\int_{u \in B(y \rightarrow x)} f(u) \eta(du)} \eta(dy) \eta(dx) = \eta(D)(1 - e^{-\rho}).$$

But the left hand side of the preceding equation equals the left hand side of Equation (14), by an application of Fubini's theorem. □

### 3.5 Constant strategies

Consider a 2-player Cox process Hotelling game  $(D, \eta, \rho_A, \rho_B)$ . Write  $\bar{\rho}_A$  for the strategy of Alice where she chooses  $f_A$  to be the constant  $\frac{\rho_A}{\eta(D)}$ , and similarly write  $\bar{\rho}_B$  for the strategy of Bob where he chooses  $f_B$  to be the constant  $\frac{\rho_B}{\eta(D)}$ . Note that  $\bar{\rho}_A \in \mathcal{C}_A$  and  $\bar{\rho}_B \in \mathcal{C}_B$ .

The following useful lemma gives an explicit formula for the value obtained by Alice and that obtained by Bob in the 2-player game  $(D, \eta, \rho_A, \rho_B)$  when the players play the constant strategies  $\bar{\rho}_A$  and  $\bar{\rho}_B$  respectively. Note that, while the strategies are constant, the resulting intensities are  $\frac{\rho_A}{\eta(D)}\eta$  and  $\frac{\rho_B}{\eta(D)}\eta$  for Alice and Bob respectively.

**Lemma 6.** Consider a 2-player Cox process Hotelling game  $(D, \eta, \rho_A, \rho_B)$ . Then  $V_A(\bar{\rho}_A, \bar{\rho}_B) = \frac{\rho_A}{\rho} \eta(D)(1 - e^{-\rho})$  and  $V_B(\bar{\rho}_A, \bar{\rho}_B) = \frac{\rho_B}{\rho} \eta(D)(1 - e^{-\rho})$ , where  $\rho := \rho_A + \rho_B$ .

*Proof.* We have

$$\begin{aligned}
V_A(\bar{\rho}_A, \bar{\rho}_B) &\stackrel{(a)}{=} \int_{x \in D} \frac{\rho_A}{\eta(D)} \int_{y \in D} e^{-\frac{\rho}{\eta(D)} \eta(B(y \rightarrow x))} \eta(dy) \eta(dx) \\
&= \frac{\rho_A}{\rho} \int_{x \in D} \int_{y \in D} \frac{\rho}{\eta(D)} e^{-\frac{\rho}{\eta(D)} \eta(B(y \rightarrow x))} \eta(dy) \eta(dx) \\
&\stackrel{(b)}{=} \frac{\rho_A}{\rho} \int_{y \in D} \int_{x \in D} \frac{\rho}{\eta(D)} e^{-\frac{\rho}{\eta(D)} \eta(B(y \rightarrow x))} \eta(dx) \eta(dy) \\
&\stackrel{(c)}{=} \frac{\rho_A}{\rho} \eta(D) (1 - e^{-\rho}).
\end{aligned}$$

Here step (a) is from equation (8), step (b) is an application of Fubini's theorem, and step (c) comes from Lemma 5. The formula for  $V_B(\bar{\rho}_A, \bar{\rho}_B)$  results from interchanging the roles of Alice and Bob in this calculation.  $\square$

In an  $N$ -player Cox process Hotelling game  $(D, \eta, \rho_1, \dots, \rho_M)$ , write  $\bar{\rho}_i$  for the strategy of player  $i$ ,  $1 \leq i \leq N$ , where she chooses  $f_i$  to be the constant  $\frac{\rho_i}{\eta(D)}$ . We then have the following analog of Lemma 6.

**Lemma 7.** Consider an  $N$ -player Cox process Hotelling game  $(D, \eta, \rho_1, \dots, \rho_N)$ . Then, for all  $1 \leq i \leq N$ , we have

$$V_i(\bar{\rho}_1, \dots, \bar{\rho}_N) = \frac{\rho_i}{\rho} \eta(D) (1 - e^{-\rho}),$$

where  $\rho := \sum_{i=1}^N \rho_i$ .

*Proof.* The proof is similar to that of Lemma 6, when one starts with Equation (10) instead of Equation (8) and makes the obvious modifications.  $\square$

### 3.6 Concavity of the value

To close this section, we record a convexity property that will play a key role in establishing the main claims of this paper.

**Lemma 8.** Consider a 2-player Cox process Hotelling game  $(D, \eta, \rho_A, \rho_B)$ . Fix  $f_B \in \mathcal{C}_B$  and  $x \in D$ . Then the mapping

$$f_A \mapsto \int_{y \in D} e^{-\int_{B(y \rightarrow x)} (f_A(u) + f_B(u)) \eta(du)} \eta(dy)$$



is strictly convex on  $\mathcal{C}_A$ , which, we recall, is a convex set.

As a consequence,  $f_A \mapsto V_B(f_A, f_B)$ , for fixed  $f_B \in \mathcal{C}_B$ , is strictly convex on  $\mathcal{C}_A$ , and hence  $f_A \mapsto V_A(f_A, f_B)$ , for fixed  $f_B \in \mathcal{C}_B$ , is strictly concave on  $\mathcal{C}_A$ .

*Proof.* Let  $f_A, f'_A \in \mathcal{C}_A$  and  $\theta \in [0, 1]$ . Then we have

$$\begin{aligned} & \int_{y \in D} e^{-\int_{B(y \rightarrow x)} (\theta f_A(u) + (1-\theta) f'_A(u) + f_B(u)) \eta(du)} \eta(dy) \\ & \leq \theta \int_{y \in D} e^{-\int_{B(y \rightarrow x)} (f_A(u) + f_B(u)) \eta(du)} \eta(dy) \\ & \quad + (1-\theta) \int_{y \in D} e^{-\int_{B(y \rightarrow x)} (f'_A(u) + f_B(u)) \eta(du)} \eta(dy), \end{aligned}$$

from the convexity of the exponential function, and this inequality is strict if  $\theta \notin \{0, 1\}$  and  $f_A \neq f'_A$ . This proves the first claim of the lemma.

From Equation (8), we have

$$V_B(f_A, f_B) = \int_{x \in D} f_B(x) \int_{y \in D} e^{-\int_{B(y \rightarrow x)} (f_A(u) + f_B(u)) \eta(du)} \eta(dy) \eta(dx). \quad (15)$$

Since, for fixed  $f_B \in \mathcal{C}_B$ , the inner integral is strictly convex on  $\mathcal{C}_A$  for each  $x \in D$ , the overall integral is also strictly convex on  $\mathcal{C}_A$ , which establishes the second claim of the lemma.

Finally, from the conservation rule in Equation (4), we have  $V_A(f_A, f_B) = \eta(D)(1 - e^{-\rho}) - V_B(f_A, f_B)$ , where  $\rho := \rho_A + \rho_B$ . From the second claim of the lemma, the third claim now follows immediately.  $\square$

The main claim of Lemma 8 is the third one about the strict concavity of the value function. This also holds for  $N$ -player Cox process Hotelling games. We state this claim formally in the following lemma.

**Lemma 9.** Consider an  $N$ -player Cox process Hotelling game  $(D, \eta, \rho_1, \dots, \rho_N)$ . Recall that the sets of pure actions,  $\mathcal{C}_i$  for player  $i$ ,  $1 \leq i \leq N$ , as defined in Equation (9), are convex subsets of  $\mathcal{M}(D)$ .

For each  $1 \leq i \leq N$ , the map  $f_i \mapsto V_i(f_1, \dots, f_N)$ , for fixed  $f_j \in \mathcal{C}_j$ ,  $j \neq i$ , is strictly concave on  $\mathcal{C}_i$ .

*Proof.* This is a direct consequence of the third claim of Lemma 8, once one observes that value of player  $i$  when it plays  $f_i$  in response to the actions  $f_j$  of the players  $j \neq i$  in the  $N$ -player game is that same as its value when it plays  $f_i$  in response to the pure action  $\sum_{j \neq i} f_j$  of the opposing player in a 2-player game where the intensity constraint of the opposing player is  $\sum_{j \neq i} \rho_j$ .  $\square$

## 4 Invariance under a transitive group action

In this section, we consider the case where  $D$  is compact, admits a transitive group of metric-preserving automorphisms, and  $\eta$  is invariant under this group of automorphisms. This scenario covers several interesting concrete cases, such as the metric tori derived from lattice fundamental regions in  $\mathbb{R}^d$ , with the metric of  $\mathbb{R}^d$  and Lebesgue measure; spheres of a fixed radius with the associated uniform measure, which is invariant under the rigid rotations; etc.

### 4.1 Exploiting concavity of the value

The following lemma, which it suffices to state in the 2-player case, is the key technical result driving the game-theoretic results in this section.

**Lemma 10.** Consider a 2-player Cox process Hotelling game  $(D, \eta, \rho_A, \rho_B)$  where  $D$  is compact, admits a transitive group of metric-preserving automorphisms, and  $\eta$  is invariant under this group of automorphisms. Then, for any strategy  $f_A \in \mathcal{C}_A$  of Alice, we have

$$V_A(f_A, \bar{\rho}_B) \leq V_A(\bar{\rho}_A, \bar{\rho}_B) = \frac{\rho_A}{\rho} \eta(D)(1 - e^{-\rho}),$$

where  $\rho := \rho_A + \rho_B$ . Further, we have the strict inequality

$$V_A(f_A, \bar{\rho}_B) < V_A(\bar{\rho}_A, \bar{\rho}_B),$$

except in the case  $f_A = \bar{\rho}_A$ .

*Proof.* From Lemma 6, we have  $V_A(\bar{\rho}_A, \bar{\rho}_B) = \frac{\rho_A}{\rho} \eta(D)(1 - e^{-\rho})$ , which is one of the claims of this lemma. From Lemma 8 for the choice  $f_B = \bar{\rho}_B$ , we conclude that  $V_A(f_A, \bar{\rho}_B)$  is strictly concave on  $\mathcal{C}_A$ . We now use this to conclude that  $V_A(f_A, \bar{\rho}_B)$  is uniquely maximized over  $\mathcal{C}_A$  by the choice  $f_A = \bar{\rho}_A$ . Indeed, if  $f_A \in \mathcal{C}_A$ ,  $f_A \neq \bar{\rho}_A$ , then we can find a translate  $f'_A$  of  $f_A$  such that  $f_A \neq f'_A$ . We have  $V_A(f_A, \bar{\rho}_B) = V_A(f'_A, \bar{\rho}_B)$ , because  $f_A$  and  $f'_A$  are translates of each other. However, since  $f_A \neq f'_A$ , we have

$$V_A\left(\frac{1}{2}(f_A + f'_A), \bar{\rho}_B\right) > \frac{1}{2}V_A(f_A, \bar{\rho}_B) + \frac{1}{2}V_A(f'_A, \bar{\rho}_B) = V_A(f_A, \bar{\rho}_B).$$

Hence,  $f_A$  cannot be the maximizer of  $V_A(f_A, \bar{\rho}_B)$  over  $\mathcal{C}_A$  unless it is translation invariant, i.e. unless it equals  $\bar{\rho}_A$ . This concludes the proof of the lemma.  $\square$

## 4.2 Nash equilibrium structure for 2-player games

We are now in a position to determine the Nash equilibrium structure of 2-player Cox process Hotelling games  $(D, \eta, \rho_A, \rho_B)$  in the context of this section.

**Theorem 11.** Consider a 2-player Cox process Hotelling game  $(D, \eta, \rho_A, \rho_B)$  between two players, Alice and Bob, where  $D$  is compact, admits a transitive group of metric-preserving automorphisms, and  $\eta$  is invariant under this group of automorphisms. Then  $(\bar{\rho}_A, \bar{\rho}_B)$  is the unique Nash equilibrium for the game.

*Proof.* We need to show that if  $(f_A(M_A), f_B(M_B))$  is a Nash equilibrium for the game, then  $f_A(M_A) = \bar{\rho}_A$  and  $f_B(M_B) = \bar{\rho}_B$  with probability 1.

Suppose first that  $f_B(M_B) = \bar{\rho}_B$  with probability 1. If  $\mathbb{P}(f_A(M_A) \neq \bar{\rho}_A) > 0$  then, by Lemma 10, we have

$$\mathbb{E}[V_A(f_A(M_A), f_B(M_B))] = \mathbb{E}[V_A(f_A(M_A), \bar{\rho}_B)] < V_A(\bar{\rho}_A, \bar{\rho}_B).$$

On the other hand, since  $f_A(M_A)$  is a best response by Alice to the strategy  $f_B(M_B)$  of Bob, we have

$$\mathbb{E}[V_A(f_A(M_A), f_B(M_B))] \geq \mathbb{E}[V_A(\bar{\rho}_A, f_B(M_B))] = V_A(\bar{\rho}_A, \bar{\rho}_B).$$

This contradiction establishes that if  $f_B(M_B) = \bar{\rho}_B$  with probability 1 then we must have  $f_A(M_A) = \bar{\rho}_A$  with probability 1. A similar argument works to show that if  $f_A(M_A) = \bar{\rho}_A$  with probability 1 then we must have  $f_B(M_B) = \bar{\rho}_B$  with probability 1.

Thus, it remains to handle the case where we have both  $\mathbb{P}(f_A(M_A) \neq \bar{\rho}_A) > 0$  and  $\mathbb{P}(f_B(M_B) \neq \bar{\rho}_B) > 0$ . In this case, since  $f_A(M_A)$  is a best response by Alice to the strategy  $f_B(M_B)$  of Bob, we have

$$\mathbb{E}[V_A(f_A(M_A), f_B(M_B))] \geq \mathbb{E}[V_A(\bar{\rho}_A, f_B(M_B))], \quad (16)$$

and, since  $f_B(M_B)$  is a best response by Bob to the strategy  $f_A(M_A)$  of Alice, we have

$$\mathbb{E}[V_B(f_A(M_A), f_B(M_B))] \geq \mathbb{E}[V_B(f_A(M_A), \bar{\rho}_B)]. \quad (17)$$

Now, since  $\mathbb{P}(f_B(M_B) \neq \bar{\rho}_B) > 0$ , by Lemma 10 we have

$$\mathbb{E}[V_B(\bar{\rho}_A, f_B(M_B))] < V_B(\bar{\rho}_A, \bar{\rho}_B),$$

so that, by Equation (5), we have

$$\mathbb{E}[V_A(\bar{\rho}_A, f_B(M_B))] > V_A(\bar{\rho}_A, \bar{\rho}_B).$$

Combining this with Equation (16), we get

$$\mathbb{E}[V_A(f_A(M_A), f_B(M_B))] > V_A(\bar{\rho}_A, \bar{\rho}_B).$$

Similar reasoning, based on Equation (17), gives

$$\mathbb{E}[V_B(f_A(M_A), f_B(M_B))] > V_B(\bar{\rho}_A, \bar{\rho}_B).$$

However, putting these inequalities together contradicts the conservation law in equation (5). This completes the proof of the theorem.  $\square$

### 4.3 Nash equilibrium structure for $N$ -player games

Theorem 11 is actually a special case of a uniqueness theorem for Nash equilibria in the general  $N$ -player case. The proof in the  $N$ -player case also depends on a peculiar feature of Cox process Hotelling games, which is that a player faced with the strategies of the other players, i.e. their individual choices of likelihood functions with respect to the underlying measure  $\eta$  which result in their individual intensities, receives the same value as she would in a 2-player game where she is faced with a single player playing a likelihood with respect to the underlying measure  $\eta$  that results in an intensity equal to the sum of the intensities corresponding to the strategies of the other players. With this in mind, we turn now to the  $N$ -player case.

**Theorem 12.** Consider an  $N$ -player Cox process Hotelling game  $(D, \eta, \rho_1, \dots, \rho_N)$  where  $D$  is compact, admits a transitive group of metric-preserving automorphisms, and  $\eta$  is invariant under this group of automorphisms. Let  $\bar{\rho}_j \in \mathcal{C}_j$  denote the constant function  $\frac{\rho_j}{\eta(D)}$ , which results in the constant intensity  $\frac{\rho_j}{\eta(D)}\eta$  for player  $j$ .

Then  $(\bar{\rho}_1, \dots, \bar{\rho}_N)$  is the unique Nash equilibrium for this game.

*Proof.* We need to show that if  $(f_1(M_1), \dots, f_N(M_N))$  is a Nash equilibrium, then  $\mathbb{P}(f_j(M_j) = \bar{\rho}_j) = 1$  for all  $1 \leq j \leq N$ . To do this, suppose first, after reindexing if needed, that we have  $\mathbb{P}(f_1(M_1) \neq \bar{\rho}_1) > 0$  and  $\mathbb{P}(\sum_{j=2}^N f_j(M_j) = \sum_{j=2}^N \bar{\rho}_j) = 1$ . Then, because  $f_1(M_1)$  is a best reaction of player 1 to the individually randomized strategies  $(f_j(M_j), 2 \leq j \leq N)$  of the other players, we have

$$\mathbb{E}[V_1(f_1(M_1), f_2(M_2), \dots, f_N(M_N))] \geq \mathbb{E}[V_1(\bar{\rho}_1, f_2(M_2), \dots, f_N(M_N))]. \quad (18)$$

On the other hand, by Lemma 10, we have

$$\mathbb{E}[V_1(f_1(M_1), f_2(M_2), \dots, f_N(M_N))] < \mathbb{E}[V_1(\bar{\rho}_1, f_2(M_2), \dots, f_N(M_N))]. \quad (19)$$

To see this, observe that  $V_1(f_1(M_1), f_2(M_2), \dots, f_N(M_N))$  is the same as the value of player 1 in the 2-player game in which she plays the randomized strategy  $f_1(M_1)$  against a single opponent playing the strategy  $\sum_{j=2}^N f_j(M_j)$ , which we have assumed equals the constant  $\sum_{j=2}^N \bar{\rho}_j$  with probability 1, and also  $V_1(\bar{\rho}_1, f_2(M_2), \dots, f_N(M_N))$  is the same as the value of player 1 in the 2-player game in which she plays the constant strategy  $\bar{\rho}_1$  against a single opponent playing the strategy  $\sum_{j=2}^N f_j(M_j)$ , which we have assumed equals the constant  $\sum_{j=2}^N \bar{\rho}_j$  with probability 1, and so the scenario of Lemma 10 applies to allow us to compare these two values. Note that equations (18) and (19) contradict each other. Thus, we can conclude that if  $(f_1(M_1), \dots, f_N(M_N))$  is a Nash equilibrium then for every player  $1 \leq i \leq N$  for which  $\mathbb{P}(f_i(M_i) \neq \bar{\rho}_i) > 0$ , we must also have  $\mathbb{P}(\sum_{j \neq i} f_j(M_j) \neq \sum_{j \neq i} \bar{\rho}_j) > 0$ .

Suppose now, after reindexing if necessary, that  $\mathbb{P}(f_1(M_1) \neq \bar{\rho}_1) > 0$ . We have established that we must also have  $\mathbb{P}(\sum_{j=2}^N f_j(M_j) \neq \sum_{j=2}^N \bar{\rho}_j) > 0$ . Since  $f_1(M_1)$  is a best reaction of player 1 to the individually randomized strategies  $(f_j(M_j), 2 \leq j \leq N)$  of the other players, the inequality in Equation (18) holds. Because  $\mathbb{P}(\sum_{j=2}^N f_j(M_j) \neq \sum_{j=2}^N \bar{\rho}_j) > 0$ , from Lemma 10 also have

$$\sum_{j=2}^N \mathbb{E}[V_j(\bar{\rho}_1, f_2(M_2), \dots, f_N(M_N))] < \sum_{j=2}^N V_j(\bar{\rho}_1, \bar{\rho}_2, \dots, \bar{\rho}_N). \quad (20)$$

To see this, note that the sum of the values of the other players  $2 \leq j \leq N$ , when player 1 plays the constant strategy  $\bar{\rho}_1$ , is the same as the value of a single player playing the strategy  $\sum_{j=2}^N f_j(M_j)$  against player 1 playing the constant strategy  $\bar{\rho}_1$  in a 2-player game between player 1 and this single player, where the overall intensity constraint of player 1 continues to be  $\rho_1$  and that of this single player is  $\sum_{j=2}^N \rho_j$ . Since  $\mathbb{P}(\sum_{j=2}^N f_j(M_j) \neq \sum_{j=2}^N \bar{\rho}_j) > 0$ , Lemma 10 allows us to conclude that this value is strictly less the value this single player would get by playing the constant strategy  $\sum_{j=2}^N \bar{\rho}_j$  against player 1, who is playing the constant strategy  $\bar{\rho}_1$ . But this is equal to the sum of the values of individual players  $2 \leq j \leq N$  in the given  $N$ -player game when they individually play the constant strategies  $(\bar{\rho}_j, 2 \leq j \leq N)$  respectively, and player 1 is playing the constant strategy  $\bar{\rho}_1$ .

Now, in view of the conservation law in Equation (12), we can conclude from Equation (20) that

$$\mathbb{E}[V_1(\rho_1, f_2(M_2), \dots, f_N(M_N))] > V_1(\bar{\rho}_1, \bar{\rho}_2, \dots, \bar{\rho}_N).$$

Thus, so far, what we have concluded is that if  $(f_1(M_1), \dots, f_N(M_N))$  is a Nash equilibrium, then, every  $1 \leq i \leq N$  such that  $\mathbb{P}(f_i(M_i) \neq \bar{\rho}_i) > 0$ , we must have

$$\mathbb{E}[V_i(\bar{\rho}_i, (f_j(M_j), j \neq i))] > V_i(\bar{\rho}_1, \bar{\rho}_2, \dots, \bar{\rho}_N). \quad (21)$$

Finally, suppose that  $\mathbb{P}(f_i(M_i) = \bar{\rho}_i) = 1$  for some  $1 \leq i \leq N$ . Then we must

$$\mathbb{E}[V_i(\bar{\rho}_i, (f_j(M_j), j \neq i))] \geq V_i(\bar{\rho}_1, \bar{\rho}_2, \dots, \bar{\rho}_N). \quad (22)$$

To see this note that the  $\sum_{k \neq i} \mathbb{E}[V_k(\bar{\rho}_i, (f_j(M_j), j \neq i))]$  is the same as the value of a single player who plays the strategy  $\sum_{j \neq i} f_j(M_j)$  in the 2-player game against player  $i$  playing the constant strategy  $\bar{\rho}_i$  and, by Lemma 10 this is no bigger than the value this single player would get if she played the constant strategy  $\sum_{j \neq i} \bar{\rho}_j$ , but this value is the same as  $\sum_{j \neq i} V_j(\bar{\rho}_1, \bar{\rho}_2, \dots, \bar{\rho}_N)$ . To conclude Equation (22) from this logic, apply the conservation rule in Equation (12).

The inequalities in equations (21) and (22) together result in a contradiction of Equation (12) unless we have  $\mathbb{P}(f_i(M_i) = \bar{\rho}_i) = 1$  for all  $1 \leq i \leq N$ . This concludes the proof of the theorem. □

## 5 General results

In this section we discuss the structure of Nash equilibria in a general  $N$ -player Cox process Hotelling game  $(D, \eta, \rho_1, \dots, \rho_N)$  without the group-theoretic assumptions of Section 4.

### 5.1 Structure of the Nash equilibria, assuming one exists

Leaving aside for the moment the question of existence of Nash equilibria, the strict concavity of the value function of a player for fixed choices of the pure actions of the other players, which was established in Lemmas 8 and 9, ensures that any Nash equilibrium that exists must be pure. We discuss this first for the 2-player case.

**Theorem 13.** Consider a 2-player Cox process Hotelling game  $(D, \eta, \rho_A, \rho_B)$  between the players Alice and Bob. Suppose  $(f_A(M_A), f_B(M_B)) \in \mathcal{C}_A \times \mathcal{C}_B$  is a Nash equilibrium of the game, as defined in Definition 3. Then there exist  $g_A \in \mathcal{C}_A$  and  $g_B \in \mathcal{C}_B$  such that

$$\mathbb{P}((f_A(M_A), f_B(M_B)) = (g_A, g_B)) = 1.$$

*Proof.* We write

$$\begin{aligned}
\mathbb{E}[V_A(f_A(M_A), f_B(M_B))] &= \mathbb{E}[\mathbb{E}[V_A(f_A(M_A), f_B(M_B)) | M_B]] \\
&\stackrel{(a)}{\leq} \mathbb{E}[\mathbb{E}[V_A(\mathbb{E}[f_A(M_A) | M_B], f_B(M_B)) | M_B]] \\
&\stackrel{(b)}{=} \mathbb{E}[\mathbb{E}[V_A(\mathbb{E}[f_A(M_A)], f_B(M_B)) | M_B]] \\
&= \mathbb{E}[V_A(\mathbb{E}[f_A(M_A)], f_B(M_B))].
\end{aligned}$$

Here step (a) comes from the concavity property of the value function established in Lemma 8 and step (b) comes from independence of  $M_A$  and  $M_B$ . Since  $(f_A(M_A), f_B(M_B))$  is a Nash equilibrium pair, we see from Definition 3 that the inequality in the chain of equations above must be an equality. But then, by the strict concavity property of the value function established in Lemma 8, it follows that  $\mathbb{P}(f_A(M_A) = \mathbb{E}[f_A(M_A)]) = 1$ . A similar argument interchanging the roles of Alice and Bob completes the proof, with  $g_A$  being  $\mathbb{E}[f_A(M_A)]$  and  $g_B$  being  $\mathbb{E}[f_B(M_B)]$  in the notation of the statement of the lemma.  $\square$

The analog of Theorem 13 also holds in the  $N$ -player case.

**Theorem 14.** Consider an  $N$ -player Cox process Hotelling game  $(D, \eta, \rho_1, \dots, \rho_N)$ . Suppose  $(f_1(M_1), \dots, f_N(M_N)) \in \mathcal{C}_1 \times \dots \times \mathcal{C}_N$  is a Nash equilibrium of the game, as defined in Definition 4. Then there exist  $g_i \in \mathcal{C}_i$ ,  $1 \leq i \leq N$ , such that

$$\mathbb{P}((f_1(M_1), \dots, f_N(M_N)) = (g_1, \dots, g_N)) = 1.$$

*Proof.* The proof is similar to that of Theorem 13, with the obvious modifications. The key observation, if one wants to base the proof of Lemma 8, is that the value of player  $i$  when she plays the randomized strategy  $f_i(M_i)$  in response to the randomized strategies  $f_j(M_j)$ ,  $j \neq i$  of the other players in the  $N$ -player game is the same as her value when she plays the randomized strategy  $f_i(M_i)$  in response to the randomized strategy  $\sum_{j \neq i} f_j(M_j)$  of the opposing player in a 2-player game where the opposing player has the intensity constraint  $\sum_{j \neq i} \rho_j$ . This observation then leads to the conclusion that  $\mathbb{P}(f_i(M_i) = \mathbb{E}[f_i(M_i)]) = 1$ , by following the lines of the proof of Theorem 13, and since this holds for all  $1 \leq i \leq N$ , this completes the proof.

Alternately, one can write out the obvious analog of the sequence of equations in the proof of Theorem 13 by conditioning on  $(M_j, j \neq i)$ , for each  $1 \leq i \leq N$ , and base the proof on the strict concavity property for the  $N$ -player game proved in Lemma 9.  $\square$

In fact the strict concavity property of the value function of a player that was established in Lemmas 8 and 9, together with the constant sum nature of the game, ensures that, if a Nash equilibrium exists, then it is not only pure, but is unique. We state this first in the 2-player case.

**Theorem 15.** Consider a 2-player Cox process Hotelling game  $(D, \eta, \rho_A, \rho_B)$  between the players Alice and Bob. Suppose  $(f_A, f_B) \in \mathcal{C}_A \times \mathcal{C}_B$  and  $(g_A, g_B) \in \mathcal{C}_A \times \mathcal{C}_B$  are pure Nash equilibria of the game. Then  $f_A = g_A$  and  $f_B = g_B$ .

*Proof.* We will first show that

$$V_A(g_A, g_B) = V_A(f_A, f_B). \quad (23)$$

The procedure to do this is standard in the theory of constant sum games, but is reproduced here for convenience. To verify Equation (23), note that  $V_A(g_A, f_B) \leq V_A(f_A, f_B)$ , because  $(f_A, f_B)$  is a Nash equilibrium, see Equation (6). But then, by the constant sum nature of the game, see Equation (5), we have  $V_B(g_A, f_B) \geq V_B(f_A, f_B)$ . However, since  $(g_A, g_B)$  is a Nash equilibrium, we have  $V_B(g_A, g_B) \geq V_B(g_A, f_B)$ , so we conclude that  $V_B(g_A, g_B) \geq V_B(f_A, f_B)$ . Interchanging the roles of  $(f_A, f_B)$  and  $(g_A, g_B)$  then gives  $V_B(f_A, f_B) \geq V_B(g_A, g_B)$ , which establishes Equation (23).

We next show that

$$V_A(g_A, f_B) = V_A(f_A, f_B). \quad (24)$$

We have  $V_B(g_A, g_B) \geq V_B(g_A, f_B)$  because  $(g_A, g_B)$  is a Nash equilibrium. Hence, by the constant sum nature of the game, see Equation (5), we have  $V_A(g_A, g_B) \leq V_A(g_A, f_B)$ . In view of Equation (23), this gives  $V_A(f_A, f_B) \leq V_A(g_A, f_B)$ , but since  $(f_A, f_B)$  is a Nash equilibrium, this can only hold with equality, i.e. Equation (24) holds.

Now, since  $(f_A, f_B)$  is a Nash equilibrium we know that  $f_A$  is a best response of Alice to the pure strategy  $f_B$  of Bob. Thus, Equation (24) tells us that  $g_A$  is also a best response of Alice in response to  $f_B$ . The strict concavity property of the value function of Alice proved in Lemma 8 shows that this is only possible if  $g_A = f_A$ . Interchanging the roles of Alice and Bob, we conclude that we must also have  $g_B = f_B$ . This concludes the proof of the lemma.  $\square$

Assume Nash equilibria exist. Beyond there being a unique pure Nash equilibrium, the special structure of the game allows us to say more about the form of this unique Nash equilibrium. We state the result first in the 2-player case.



**Theorem 16.** Consider a 2-player Cox process Hotelling game  $(D, \eta, \rho_A, \rho_B)$  between the players Alice and Bob. Suppose  $(f_A, f_B) \in \mathcal{C}_A \times \mathcal{C}_B$  is a pure Nash equilibrium of the game. Then  $f_A = \rho_A f$  and  $f_B = \rho_B f$ , where  $f := f_A + f_B$ .

*Proof.* Suppose Alice reacts to  $f_B$  by playing  $\frac{\rho_A}{\rho_B} f_B$ . Let  $\rho := \rho_A + \rho_B$ . We have

$$\begin{aligned} V_A\left(\frac{\rho_A}{\rho_B} f_B, f_B\right) &\stackrel{(a)}{=} \frac{\rho_A}{\rho_B} \int_{x \in D} f_B(x) \int_{y \in D} e^{-\int_{B(y \rightarrow x)} \frac{\rho}{\rho_B} f_B(u) \eta(du)} \eta(dy) \eta(dx) \\ &= \frac{\rho_A}{\rho} \int_{x \in D} \frac{\rho}{\rho_B} f_B(x) \int_{y \in D} e^{-\int_{B(y \rightarrow x)} \frac{\rho}{\rho_B} f_B(u) \eta(du)} \eta(dy) \eta(dx) \\ &\stackrel{(b)}{=} \frac{\rho_A}{\rho} \eta(D) (1 - e^{-\rho}), \end{aligned}$$

where step (a) is from Equation (8) and step (b) is from Equation (14). Since  $(f_A, f_B)$  is a Nash equilibrium, it follows that

$$V_A(f_A, f_B) \geq \frac{\rho_A}{\rho} \eta(D) (1 - e^{-\rho}).$$

Interchanging the roles of Alice and Bob gives

$$V_B(f_A, f_B) \geq \frac{\rho_B}{\rho} \eta(D) (1 - e^{-\rho}).$$

In view of the constant sum property in Equation (5), it then follows that we have equality in both these inequalities, i.e.

$$V_A(f_A, f_B) = \frac{\rho_A}{\rho} \eta(D) (1 - e^{-\rho}),$$

and

$$V_B(f_A, f_B) = \frac{\rho_B}{\rho} \eta(D) (1 - e^{-\rho}).$$

But then we have  $V_A(f_A, f_B) = V_A\left(\frac{\rho_A}{\rho_B} f_B, f_B\right)$ , so the strict concavity property of the value of Alice proved in Lemma 8 together with the assumption that  $(f_A, f_B)$  is a Nash equilibrium implies that  $f_A = \frac{\rho_A}{\rho_B} f_B$ , which suffices to establish the claim (one can also go through this argument again interchanging the roles of Alice and Bob, but it is not necessary).  $\square$

The analog of Theorem 15 and the stronger statement in Theorem 16 also hold in the  $N$ -player case. We state these claims together.

**Theorem 17.** Consider an  $N$ -player Cox process Hotelling game  $(D, \eta, \rho_1, \dots, \rho_N)$ . Suppose  $(f_1, \dots, f_N) \in \mathcal{C}_1 \times \dots \times \mathcal{C}_N$  and  $(g_1, \dots, g_N) \in \mathcal{C}_1 \times \dots \times \mathcal{C}_N$  are pure Nash equilibria of the game. Then  $f_i = g_i$  for  $1 \leq i \leq N$ . Indeed, if  $(f_1, \dots, f_N) \in \mathcal{C}_1 \times \dots \times \mathcal{C}_N$  is a pure Nash equilibrium of the game. Then  $f_i = \rho_i f$  for  $1 \leq i \leq N$ , where where  $f := \sum_{i=1}^N f_i$ .

*Proof.* We prove the stronger statement. The proof is similar to that of Theorem 16. Suppose player  $i$  reacts to the strategy profile  $(f_j, j \neq i)$  by playing  $\frac{\rho_i}{\sum_{j \neq i} \rho_j} \sum_{j \neq i} f_j$ . Let  $\rho := \sum_{i=1}^N \rho_i$ . We have

$$\begin{aligned} & V_i\left(\frac{\rho_i}{\sum_{j \neq i} \rho_j} \sum_{j \neq i} f_j, (f_j, j \neq i)\right) \\ & \stackrel{(a)}{=} \frac{\rho_i}{\sum_{j \neq i} \rho_j} \int_{x \in D} \sum_{j \neq i} f_j(x) \int_{y \in D} e^{-\int_{u \in B(y \rightarrow x)} \frac{\rho}{\sum_{j \neq i} \rho_j} \sum_{j \neq i} f_j(u) \eta(du)} \eta(dy) \eta(dx) \\ & = \frac{\rho_i}{\rho} \int_{x \in D} \frac{\rho}{\sum_{j \neq i} \rho_j} \sum_{j \neq i} f_j(x) \int_{y \in D} e^{-\int_{u \in B(y \rightarrow x)} \frac{\rho}{\sum_{j \neq i} \rho_j} \sum_{j \neq i} f_j(u) \eta(du)} \eta(dy) \eta(dx) \\ & \stackrel{(b)}{=} \frac{\rho_i}{\rho} \eta(D)(1 - e^{-\rho}), \end{aligned}$$

where step (a) is from Equation (10) and step (b) is from Equation (14). Since  $(f_1, \dots, f_N)$  is a Nash equilibrium, it follows that

$$V_i(f_1, \dots, f_N) \geq \frac{\rho_i}{\rho} \eta(D)(1 - e^{-\rho}).$$

Since this holds for all  $1 \leq i \leq N$ , it follows from Equation (11) that this inequality must hold with equality for all  $1 \leq i \leq N$ , i.e. that we have

$$V_i(f_1, \dots, f_N) = \frac{\rho_i}{\rho} \eta(D)(1 - e^{-\rho}),$$

for all  $1 \leq i \leq N$ . But then, for each  $1 \leq i \leq N$  we have

$$V_i\left(f_i = \frac{\rho_i}{\sum_{j \neq i} \rho_j} \sum_{j \neq i} f_j, (f_j, j \neq i)\right) = V_i(f_1, \dots, f_N),$$

so from the strict concavity property of the value function of player  $i$  proved in Lemma 9 and the fact that  $f_i$  is a best response of player  $i$  to the strategy profile  $(f_j, j \neq i)$  of the other players, we must have  $f_i = \frac{\rho_i}{\sum_{j \neq i} \rho_j} \sum_{j \neq i} f_j$ , which is the same as  $f_i = \frac{\rho_i}{\rho} f$ , where  $f := \sum_{i=1}^N f_i$ . This completes the proof of the theorem.  $\square$

The properties of Nash equilibria of Cox process Hotelling games established so far, assuming a Nash equilibrium exists, can be gathered into the following statement.

**Theorem 18.** Consider an  $N$ -player Cox process Hotelling game  $(D, \eta, \rho_1, \dots, \rho_N)$ . Let  $\rho := \sum_{i=1}^N \rho_i$ . The game admits a Nash equilibrium if and only if there is a function  $f \in \mathcal{C}(\rho)$  such that

$$\begin{aligned} & \int_{x \in D} f(x) \int_{y \in D} e^{-\int_{u \in B(y \rightarrow x)} f(u) \eta(du)} \eta(dy) \eta(dx) \\ & \geq \int_{x \in D} g(x) \int_{y \in D} e^{-\int_{u \in B(y \rightarrow x)} f(u) \eta(du)} \eta(dy) \eta(dx), \end{aligned} \quad (25)$$

for all  $g \in \mathcal{C}(\rho)$ . If such a function exists, the game has a unique Nash equilibrium, given by the pure strategy profile  $(\frac{\rho_1}{\rho} f, \dots, \frac{\rho_N}{\rho} f)$ .

*Proof.* Suppose first that the game admits a Nash equilibrium. By Theorem 17, we know that this Nash equilibrium is unique and is a pure Nash equilibrium of the form  $(\frac{\rho_1}{\rho} f, \dots, \frac{\rho_N}{\rho} f)$ , for some  $f \in \mathcal{C}(\rho)$ . Since  $\frac{\rho_i}{\rho} f$  is a best response of player  $i$  to the strategy profile  $(f_j = \frac{\rho_j}{\rho} f, j \neq i)$  of the other players, by the definition of Nash equilibrium, see Equation (13), we must have

$$\begin{aligned} & \int_{x \in D} \frac{\rho_i}{\rho} f \int_{y \in D} e^{-\int_{u \in B(y \rightarrow x)} f(u) \eta(du)} \eta(dy) \eta(dx) \\ & \geq \int_{x \in D} g_i(x) \int_{y \in D} e^{-\int_{u \in B(y \rightarrow x)} f(u) \eta(du)} \eta(dy) \eta(dx), \end{aligned}$$

for all  $g_i \in \mathcal{C}(\rho_i)$ , which is the same as the condition in Equation (25).

Conversely, suppose the condition in Equation (25) holds for some function  $f \in \mathcal{C}(\rho)$ . Then, by the definition of Nash equilibrium in Equation (13), we see that the strategy profile  $(\frac{\rho_1}{\rho} f, \dots, \frac{\rho_N}{\rho} f)$  is a Nash equilibrium of the game. The existence of such a Nash equilibrium then guarantees, by Theorem 17, that it is the unique Nash equilibrium of the game.  $\square$

## 5.2 Relationship with ordinal potential games

Recall that an  $n$ -player game, with player  $i$  having action set  $\mathcal{Y}_i$ , is called an *ordinal potential game* [23] if there is a function  $P : \prod_{i=1}^n \mathcal{Y}_i \rightarrow \mathbb{R}$  such that, for

all  $1 \leq i \leq n$ ,  $y_i, z_i \in Y_i$ , and  $(y_j, j \neq i) \in \prod_{j \neq i} Y_j$ , we have  $V_i(y_i, (y_j, j \neq i)) > V_i(z_i, (y_j, j \neq i))$  iff  $P(y_i, (y_j, j \neq i)) > P(z_i, (y_j, j \neq i))$ .

We have established that when a Cox process Hotelling game admits a Nash equilibrium it admits a unique pure Nash equilibrium. Since ordinal potential games are a well-known class of games that admit pure strategy Nash equilibria, this leads naturally to the question of whether Cox process Hotelling games are ordinal potential games. However, it is possible to argue that in general this is not true.

To see this, consider the 2-player Cox process Hotelling game between Alice and Bob on  $D$ , taken to be a circle of radius 1 centered at the origin in  $\mathbb{R}^2$ , the base measure  $\eta$  being the Lebesgue measure on  $D$ . Thus  $\eta(D) = 2\pi$ . Suppose that  $\rho_A = \rho_B = \frac{\rho}{2}$ , where  $\rho$  should be thought of as being sufficiently large in a sense that we will make precise shortly. Let  $\epsilon > 0$  be sufficiently small (to be precise, we require that  $\epsilon < \frac{2\pi}{9}$ ).

We consider four pure strategies, i.e elements of  $\mathcal{C}(\frac{\rho}{2})$ , denoted by  $\sigma^R, \beta^R, \sigma^L$  and  $\beta^L$  respectively, defined as follows:

$\sigma^R$  is constant over the arc of the circle of length  $\epsilon$  centered at  $(1, 0)$  and is zero elsewhere;

Consider the arc of the circle of length  $\frac{\epsilon}{2}$  centered at  $(\frac{\sqrt{3}}{2}, \frac{1}{2})$  and the arc of the circle of length  $\frac{\epsilon}{2}$  centered at  $(\frac{\sqrt{3}}{2}, -\frac{1}{2})$ .  $\beta^R$  is constant over the union of these two arcs and is zero elsewhere;

$\sigma^L$  is constant over the arc of the circle of length  $\epsilon$  centered at  $(-1, 0)$  and is zero elsewhere. It can be considered to be the “left” version of  $\sigma^R$ , which is its “right” version;

$\beta^L$  is the “left” version of  $\beta^R$ , which is its “right” version. Namely,  $\beta^L$  is uniform over the union of the two arcs of the circle of length  $\frac{\epsilon}{2}$  centered at  $(-\frac{\sqrt{3}}{2}, \frac{1}{2})$  and  $(-\frac{\sqrt{3}}{2}, -\frac{1}{2})$  respectively, and is zero elsewhere.

To be consistent with the convention in the rest of the document, we will use a subscript to indicate the identity of the player playing the strategy. Thus, for instance, the strategy pair  $(\sigma_A^L, \sigma_B^R)$  indicates that Alice is playing the strategy  $\sigma^L$  and Bob is playing the strategy  $\sigma^R$ .

It is straightforward to check that

$$\lim_{\rho \rightarrow \infty} V_A(\sigma_A^R, \beta_B^R) = \frac{\pi}{6} + \frac{\epsilon}{4},$$

and

$$\lim_{\rho \rightarrow \infty} V_A(\sigma_A^L, \beta_B^R) = \frac{5\pi}{6} + \frac{\epsilon}{4}.$$

We will use these facts and their obvious consequences in the following argument.

Suppose there were an ordinal potential function  $P : \mathcal{C}(\frac{\rho}{2}) \times \mathcal{C}(\frac{\rho}{2}) \rightarrow \mathbb{R}$  for this 2-player Cox process Hotelling game.

For sufficiently large  $\rho$ , we have  $V_A(\sigma_A^R, \beta_B^R) < V_A(\sigma_A^L, \beta_B^R)$ , and so  $P(\sigma^R, \beta^R) < P(\sigma^L, \beta^R)$ .

For sufficiently large  $\rho$  we also have  $V_B(\sigma_A^L, \beta_B^R) < V_B(\sigma_A^L, \beta_B^L)$ , and so  $P(\sigma^L, \beta^R) < P(\sigma^L, \beta^L)$ .

But for sufficiently large  $\rho$  we also have  $V_A(\sigma_A^L, \beta_B^L) < V_A(\sigma_A^R, \beta_B^L)$ , and so  $P(\sigma^L, \beta^L) < P(\sigma^R, \beta^L)$ .

Finally, for sufficiently large  $\rho$  we also have  $V_B(\sigma_A^R, \beta_B^L) < V_B(\sigma_A^R, \beta_B^R)$ , and so  $P(\sigma^R, \beta^L) < P(\sigma^R, \beta^R)$ .

Putting these together leads to a contradiction. Hence this 2-player Cox process Hotelling game is not an ordinal potential game.

### 5.3 Nash equilibria may not exist

Consider a 2-player Cox process Hotelling game  $(D, \eta, \rho_A, \rho_B)$ . The results of Theorems 13, 15 and 16 are consistent with that of Theorem 11 in the case where  $D$  is compact, admits a transitive group of metric-preserving automorphisms, and  $\eta$  is an invariant measure under the action of this group. This might lead one to expect that the constant intensity pair  $(\bar{\rho}_A, \bar{\rho}_B)$  is a Nash equilibrium for a general 2-player Cox process Hotelling game  $(D, \eta, \rho_A, \rho_B)$ . The following simple example shows that this is not the case.

**Example 19.** Let  $D$  be the interval  $[-\frac{1}{2}, \frac{1}{2}]$  of the real line, with  $\eta$  being the Lebesgue measure restricted to  $D$ . Then, for every 2-player Cox process Hotelling game  $(D, \eta, \rho_A, \rho_B)$ , the constant intensity pair  $(\bar{\rho}_A, \bar{\rho}_B)$  is not a Nash equilibrium of the game.

To see this, first note that  $\eta(D) = 1$ , so  $\bar{\rho}_A = \rho_A$  and  $\bar{\rho}_B = \rho_B$ . Recall that  $\rho := \rho_A + \rho_B$ . From Theorem 18, it suffices to find  $g \in \mathcal{C}(\rho)$ , i.e.  $g : [-\frac{1}{2}, \frac{1}{2}] \rightarrow \mathbb{R}_+$  with  $\int_{u=-\frac{1}{2}}^{\frac{1}{2}} g(u) du = \rho$ , such that

$$\int_{x=-\frac{1}{2}}^{\frac{1}{2}} g(x) \int_{y=-\frac{1}{2}}^{\frac{1}{2}} e^{-\rho\eta(B(y \rightarrow x))} dy dx > \int_{x=-\frac{1}{2}}^{\frac{1}{2}} \rho \int_{y=-\frac{1}{2}}^{\frac{1}{2}} e^{-\rho\eta(B(y \rightarrow x))} dy dx, \quad (26)$$

where we have written  $dx$  and  $dy$  in the integrals instead of  $\eta(dx)$  and  $\eta(dy)$  respectively, because  $\eta$  is the Lebesgue measure. By Equation (14) the integral on the right hand side of Equation (26) is  $1 - e^{-\rho}$ . For the integral on the left

hand side of Equation (26), let us first replace  $g(x)dx$  by  $\rho\delta_0(dx)$  where  $\delta_0$  is the measure on  $[-\frac{1}{2}, \frac{1}{2}]$  giving mass 1 to the point at the origin, i.e. let us consider the integral

$$\begin{aligned}
& \rho \int_{x=-\frac{1}{2}}^{\frac{1}{2}} \int_{y=-\frac{1}{2}}^{\frac{1}{2}} e^{-\rho\eta(B(y\rightarrow x))} dy \delta_0(dx) \\
&= \rho \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-\rho\eta(B(y\rightarrow 0))} dy \\
&= 2\rho \int_0^{\frac{1}{4}} e^{-2\rho y} dy + 2\rho \int_{\frac{1}{4}}^{\frac{1}{2}} e^{-\frac{\rho}{2}} dy \\
&= 1 - e^{-\frac{\rho}{2}} + \frac{\rho}{2} e^{-\frac{\rho}{2}}.
\end{aligned}$$

For all  $\rho > 0$  this integral is strictly bigger than  $1 - e^{-\rho}$ . It follows that we can find  $g \in \mathcal{C}(\rho)$  to get the strict inequality in Equation (26), as desired.  $\square$

In the 2-player game considered in Example 19, one can in fact conclude that there is no Nash equilibrium when the sum of the intensities of the two players is sufficiently small. We state this formally.

**Theorem 20.** Let  $D$  be the interval  $[-\frac{1}{2}, \frac{1}{2}]$  of the real line, with  $\eta$  being the Lebesgue measure restricted to  $D$ . Then the 2-player Cox process Hotelling game  $(D, \eta, \rho_A, \rho_B)$  does not admit a Nash equilibrium when  $\rho := \rho_A + \rho_B < \log_e 4$ .

*Proof.* Theorem 18 tells us that, to prove that a Nash equilibrium does not exist for this 2-player game, it suffices to show that for every  $f \in \mathcal{C}(\rho)$  there is some  $g \in \mathcal{C}(\rho)$  such that

$$\int_{x=-\frac{1}{2}}^{\frac{1}{2}} g(x) \int_{y=-\frac{1}{2}}^{\frac{1}{2}} e^{-\int_{u \in B(y \rightarrow x)} f(u) du} dy dx > \int_{x=-\frac{1}{2}}^{\frac{1}{2}} f(x) \int_{y=-\frac{1}{2}}^{\frac{1}{2}} e^{-\int_{u \in B(y \rightarrow x)} f(u) du} dy dx. \tag{27}$$

Let  $f \in \mathcal{C}(\rho)$ . For  $x \in [-\frac{1}{2}, \frac{1}{2}]$ , define  $\psi_x : [-\frac{1}{2}, \frac{1}{2}] \rightarrow \mathbb{R}_+$  via:

$$\psi_x(y) := e^{-\int_{u \in B(y \rightarrow x)} f(u) du}.$$

Suppose  $\int_{y=-\frac{1}{2}}^{\frac{1}{2}} \psi_x(y) dy$  is not constant in  $x$ . Let  $x^* \in \arg \max_x \int_{y=-\frac{1}{2}}^{\frac{1}{2}} \psi_x(y) dy$ ,

which exists because  $\int_{y=-\frac{1}{2}}^{\frac{1}{2}} \psi_x(y)dy$  is continuous in  $x$  over  $[-\frac{1}{2}, \frac{1}{2}]$ . Since

$$\int_{y=-\frac{1}{2}}^{\frac{1}{2}} \psi_x(y)dy$$

is not constant in  $x$  over  $x \in [-\frac{1}{2}, \frac{1}{2}]$ , and since  $\int_{x=-\frac{1}{2}}^{\frac{1}{2}} f(x)dx = \rho$ , this implies that

$$\rho \int_{y=-\frac{1}{2}}^{\frac{1}{2}} e^{-\int_{u \in B(y \rightarrow x^*)} f(u)du} dy > \int_{x=-\frac{1}{2}}^{\frac{1}{2}} f(x) \int_{y=-\frac{1}{2}}^{\frac{1}{2}} e^{-\int_{u \in B(y \rightarrow x)} f(u)du} dy dx,$$

from which we can conclude the existence of  $g \in \mathcal{C}(\rho)$  satisfying the strict inequality in Equation (27).

From Theorem 18, we also know that if a Nash equilibrium exists it must be pure and of the form  $(\frac{\rho_A}{\rho} f, \frac{\rho_B}{\rho} f)$  for some  $f \in \mathcal{C}(\rho)$ . We claim that it must further be the case that  $f(u) = f(-u)$  for all  $u \in [-\frac{1}{2}, \frac{1}{2}]$ , i.e. that  $f$  is an *even* function. This is because, by symmetry, if  $(\frac{\rho_A}{\rho} f, \frac{\rho_B}{\rho} f)$  is a Nash equilibrium then so is  $(\frac{\rho_A}{\rho} \tilde{f}, \frac{\rho_B}{\rho} \tilde{f})$ , where  $\tilde{f}(u) := f(-u)$  (so we also have  $\tilde{f} \in \mathcal{C}(\rho)$ ), and then, because the Nash equilibrium is unique, it must be the case that  $\tilde{f} = f$ .

Thus it suffices to show that when  $\rho < \log_e 4$  it is impossible to find an even function  $f \in \mathcal{C}(\rho)$  such that  $\int_{y=-\frac{1}{2}}^{\frac{1}{2}} \psi_x(y)dy$  is constant in  $x$ . We will do this by establishing that for every even function  $f \in \mathcal{C}(\rho)$  we have

$$\int_{y=-\frac{1}{2}}^{\frac{1}{2}} \psi_{-\frac{1}{2}}(y)dy < \int_{y=-\frac{1}{2}}^{\frac{1}{2}} \psi_0(y)dy. \quad (28)$$

Observe that  $\psi_0(y)$  is an even function of  $y \in [-\frac{1}{2}, \frac{1}{2}]$ . Further, we have  $\psi_0(y) \geq e^{-\frac{\rho}{2}}$  for all  $y \in [-\frac{1}{2}, \frac{1}{2}]$ , and we have  $\psi_0(y) = e^{-\frac{\rho}{2}}$  for  $-\frac{1}{2} \leq y \leq -\frac{1}{4}$ .

Observe also that  $\psi_{-\frac{1}{2}}(y)$  is nonincreasing over  $y \in [-\frac{1}{2}, \frac{1}{2}]$ , with  $\psi_{-\frac{1}{2}}(-\frac{1}{2}) = 1$ ,  $\psi_{-\frac{1}{2}}(-\frac{1}{4}) = e^{-\frac{\rho}{2}}$ , and  $\psi_{-\frac{1}{2}}(y) = e^{-\rho}$  for  $0 \leq y \leq \frac{1}{2}$ .

From these two sets of observations, we have  $\psi_{-\frac{1}{2}}(y) \geq \psi_0(y)$  for  $y \in [-\frac{1}{2}, -\frac{1}{4}]$  and  $\psi_{-\frac{1}{2}}(y) \leq \psi_0(y)$  for  $y \in [-\frac{1}{4}, \frac{1}{2}]$ . Further, we have

$$\int_{y=-\frac{1}{2}}^{-\frac{1}{4}} \left( \psi_{-\frac{1}{2}}(y) - \psi_0(y) \right) dy \leq \frac{1}{4}(1 - e^{-\frac{\rho}{2}}),$$

and

$$\int_{y=0}^{\frac{1}{2}} (\psi_0(y) - \psi_{-\frac{1}{2}}(y)) dy \geq \frac{1}{2}(e^{-\frac{\rho}{2}} - e^{-\rho}),$$

while we also have

$$\int_{y=-\frac{1}{4}}^0 (\psi_0(y) - \psi_{-\frac{1}{2}}(y)) dy \geq 0.$$

From this we conclude that

$$\int_{y=-\frac{1}{2}}^{\frac{1}{2}} (\psi_0(y) - \psi_{-\frac{1}{2}}(y)) dy \geq \frac{1}{2}(e^{-\frac{\rho}{2}} - e^{-\rho}) - \frac{1}{4}(1 - e^{-\frac{\rho}{2}}) > 0,$$

if  $0 < \rho < \log_e 4$ , which establishes the strict inequality in Equation (28) and completes the proof. A more careful analysis will increase the range of  $\rho$  for which one can prove that the game does not admit a Nash equilibrium.  $\square$

## 5.4 Restricted Cox process Hotelling games

A more insightful characterization than the one in Theorem 18 of when Nash equilibria exist in Cox process Hotelling games remains an interesting open problem. The main technical difficulty in proving the existence of Nash equilibria in such games is that the action spaces of the individual players, which are of the form  $\mathcal{C}(\rho_i)$ , where  $\rho_i > 0$  is the intensity budget of player  $i$ , are not compact when endowed with the topology of weak convergence. Indeed, we have seen in Section 5.3 that Nash equilibria may not exist in some cases.

To better understand the question of when Nash equilibria exist in such games, we therefore propose to study a family of *restricted Cox Process Hotelling games*, where the action space of each individual player is now a compact set. The restrictions can be imposed in such a way that a unique Nash equilibria will be guaranteed to exist in each such restricted game, it will be pure and, if the restrictions imposed on the individual players are proportional in a sense made precise below, this unique pure Nash equilibrium will be of proportional form. Further, by varying the restriction, we can vary the compact action space of each player in such a way that the union over all such choices is the full space of allowed actions for that player, namely  $\mathcal{C}(\rho_i)$  for player  $i$  having a total intensity budget of  $\rho_i$ . If a Nash equilibrium did exist for the original Cox process Hotelling game, then



this guarantees it would be discovered as the Nash equilibrium for some profile of restrictions on the action spaces of the individual players, i.e. in one of the restricted Cox process Hotelling games that we consider. Pursuing this direction, which we leave as a topic for future research, may give more insight into what the characterization in Theorem 18 is actually saying.

To carry out this program, we first demonstrate, for each  $\rho > 0$ , a family of compact subsets of  $\mathcal{C}(\rho)$  whose union is  $\mathcal{C}(\rho)$ . Recall that  $\mathcal{C}(\rho)$  is a subset of  $L^1(\eta)$ , where  $f \in \mathcal{C}(\rho)$  is identified with the measure  $f\eta$  on  $D$ , and  $\mathcal{C}(\rho)$  endowed with the topology of weak convergence of measures which it inherits as a subset of  $\mathcal{M}(D)$ . We will now consider  $L^1(\eta)$  with its weak topology defined by considering it to be a Banach space with Banach dual  $L^\infty(\eta)$ , see [29] or [9, pg. 44]. To avoid confusion, recall that the topology of weak convergence on  $\mathcal{C}(\rho)$  is called the narrow topology in the theory of Banach spaces, see e.g. [3, Vol. 1, pg. 176], and is weaker than the weak topology.

From the theorem of Dunford and Pettis, [9, Thm. 3], [10] the closure in the weak topology of a subset of  $L^1(\eta)$  is compact in the weak topology iff it is uniformly integrable. Here we recall, see e.g. [3, Vol. 1, Defn. 4.5.1.], [9, pg. 41] that a subset  $S \subseteq L^1(\eta)$  is called uniformly integrable if

$$\lim_{c \rightarrow \infty} \sup_{f \in S} \int_D |f(x)| 1(|f(x)| \geq c) \eta(dx) = 0.$$

Further, by the theorem of de la Vallée Poussin [5], [9, Thm. 2] for  $S \subseteq L^1(\eta)$  to be uniformly integrable it is necessary and sufficient that there be a nondecreasing convex function  $\Theta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , with  $\Theta(0) = 0$  and  $\lim_{x \rightarrow \infty} \frac{\Theta(x)}{x} = \infty$ , such that

$$\sup_{f \in S} \int_D \Theta(|f(x)|) \eta(dx) < \infty.$$

For  $\rho > 0$ ,  $\Theta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  a nondecreasing convex function with  $\Theta(0) = 0$  and  $\lim_{x \rightarrow \infty} \frac{\Theta(x)}{x} = \infty$ , and  $0 < K < \infty$ , we propose to consider the subset of  $\mathcal{C}(\rho)$  defined by

$$\mathcal{C}(\rho, \Theta, K) := \left\{ f : D \rightarrow \mathbb{R}_+ \text{ s.t. } \int_D f(x) \eta(dx) = \rho, \int_D \Theta(f(x)) \eta(dx) \leq K \right\}. \quad (29)$$

It can be checked that  $\mathcal{C}(\rho, \Theta, K)$  is a closed subset of  $L^1(\eta)$  in the weak topology. By the theorem of de la Vallée Poussin,  $\mathcal{C}(\rho, \Theta, K)$  is uniformly integrable, so by the Dunford-Pettis theorem it is a compact subset of  $L^1(\eta)$  in the

weak topology. Since the weak topology on  $L^1(\eta)$  is stronger than the topology of weak convergence (i.e. the narrow topology) on  $L^1(\eta)$ ,  $\mathcal{C}(\rho, \Theta, K)$  is compact in the topology of weak convergence on  $L^1(\eta)$ .

For every  $f \in L^1(\eta)$  it can be checked that there is some nondecreasing convex function  $\Theta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  with  $\Theta(0) = 0$  and  $\lim_{x \rightarrow \infty} \frac{\Theta(x)}{x} = \infty$ , such that  $\int_D \Theta(|f(x)|) \eta(dx) < \infty$ . It follows that the union of  $\mathcal{C}(\rho, \Theta, K)$  over all choices of  $\Theta$  and  $K$  equals  $\mathcal{C}(\rho)$ . We thus have a family of compact subsets of  $\mathcal{C}(\rho)$  whose union is  $\mathcal{C}(\rho)$ .

From the convexity of  $\Theta$  it is also straightforward to show that each  $\mathcal{C}(\rho, \Theta, K)$  is a convex subset of  $L^1(\eta)$ . As a closed subset of  $\mathcal{M}(D)$  in the topology of weak convergence,  $\mathcal{C}(\rho, \Theta, K)$  is a Borel subset of  $\mathcal{M}(D)$ , so we are able to discuss probability measures on  $\mathcal{C}(\rho, \Theta, K)$ .

By an  $N$ -player *restricted Cox process Hotelling game* we mean a game with player  $i$ , for  $1 \leq i \leq N$ , having the intensity budget  $\rho_i > 0$  and the space of pure actions some  $\mathcal{C}(\rho_i, \Theta_i, K_i)$ , which, as we have seen, is a compact subset of  $\mathcal{C}(\rho_i)$  in the topology of weak convergence. Here, for each  $1 \leq i \leq N$ ,  $\Theta_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a nondecreasing convex function with  $\Theta_i(0) = 0$  and  $\lim_{x \rightarrow \infty} \frac{\Theta_i(x)}{x} = \infty$ , and  $0 < K_i < \infty$ . Suppose the individual players play the pure actions  $f_i \in \mathcal{C}(\rho_i, \Theta_i, K_i)$  for  $1 \leq i \leq N$ . Let  $\Phi_i$  be a Poisson point process on  $D$  with intensity measure  $f_i \eta$ , with these processes being mutually independent for  $1 \leq i \leq N$ , and let  $\Phi := \sum_{i=1}^N \Phi_i$ . We think of the points of  $\Phi_i$  as the points of player  $i$ , since these result from the choice of  $f_i$ , which was made by that player. Then, as before, the value of player  $i$ , denoted  $V_i(f_1, \dots, f_N)$ , is given by

$$V_i(f_1, \dots, f_N) := \mathbb{E}\left[\sum_{x \in \Phi_i} \eta(W_\Phi(x))\right].$$

The expectation is with respect to the joint law of  $(\Phi_j, 1 \leq j \leq n)$ , which are independent. Here, by definition, a sum over an empty set is 0.

Consider an  $N$ -player restricted Cox process Hotelling game where player  $i$  has the space of actions  $\mathcal{C}(\rho_i, \Theta_i, K_i)$ . Any mixed strategy  $N$ -tuple in this game can be written as  $(f_1(M_1), \dots, f_N(M_N))$ , where  $(M_i \in \mathcal{M}_j, 1 \leq i \leq N)$  are independent random variables representing the randomizations used by the individual players in implementing their randomized strategies, and  $f_i(m_i) \in \mathcal{C}(\rho_i, \Theta_i, K_i)$  is the choice of action of player  $i$  in case the realization of her random variable  $M_i$  is  $m_i$ . The value of player  $i$  in such a mixed strategy is  $\mathbb{E}[V_i(f_1(M_1), \dots, f_N(M_N))]$ , the expectation being taken with respect to the joint distribution of  $(M_j, 1 \leq j \leq N)$ , which are independent. The vector of indepen-

dently randomized strategies

$$(f_1(M_1), \dots, f_N(M_N)) \in \mathcal{C}(\rho_1, \Theta_1, K_1) \times \dots \times \mathcal{C}(\rho_N, \Theta_N, K_N)$$

is called a *Nash equilibrium* of the game if, for all  $g_j \in \mathcal{C}(\rho_j, \Theta_j, K_j)$ ,  $1 \leq j \leq N$ , we have, for all  $1 \leq i \leq N$ ,

$$\mathbb{E}[V_i(f_1(M_1), \dots, f_N(M_N))] \geq \mathbb{E}[V_i(g_i, (f_j(M_j), j \neq i))]. \quad (30)$$

Since each  $\mathcal{C}(\rho_i, \Theta_i, K_i)$  is compact and each  $V_i : \prod_{j=1}^N \mathcal{C}(\rho_j, \Theta_j, K_j) \rightarrow \mathbb{R}_+$  is continuous, the existence of a mixed strategy Nash equilibrium for every restricted  $N$ -player Cox process Hotelling game is guaranteed [15].

We now formally state and prove that every restricted Cox process Hotelling game has unique Nash equilibrium, and that this is comprised of pure strategies. The following Theorem is a combination of the analog of Theorem 14 and the  $N$ -player version of Theorem 15 for restricted Cox process Hotelling games.

**Theorem 21.** Consider an  $N$ -player restricted Cox process Hotelling game on the Polish space  $D$  with base measure  $\eta$  where player  $i$ , for  $1 \leq i \leq N$ , has the intensity budget  $\rho_i > 0$  and the space of pure actions  $\mathcal{C}(\rho_i, \Theta_i, K_i)$ , where  $\Theta_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a nondecreasing convex function with  $\Theta_i(0) = 0$  and  $\lim_{x \rightarrow \infty} \frac{\Theta_i(x)}{x} = \infty$ , and  $0 < K_i < \infty$ . Suppose  $(f_1(M_1), \dots, f_N(M_N))$  is a Nash equilibrium of the game, which we know exists. Then there exist  $g_i \in \mathcal{C}_i$ ,  $1 \leq i \leq N$ , such that

$$\mathbb{P}((f_1(M_1), \dots, f_N(M_N)) = (g_1, \dots, g_N)) = 1,$$

i.e. the Nash equilibrium is a pure strategy Nash equilibrium. Furthermore, if  $(g_1, \dots, g_N)$  and  $(h_1, \dots, h_N)$  are two pure strategy Nash equilibria for the game, then  $g_i = h_i$  for all  $1 \leq i \leq N$ , i.e. the Nash equilibrium is unique.

*Proof.* The proof is similar to those of the  $N$ -player versions of Theorem 13, and Theorem 15, with the obvious modifications. All that is being used in those proofs is the convexity of the set of allowed actions of each player and the strict concavity of the value function of each player in its own action when the actions of its opponents are fixed. These properties continue to hold in the restricted Cox process Hotelling games that we are now considering.  $\square$

When the restrictions on the individual players in an  $N$ -player restricted Cox process Hotelling game are proportional to their allowed intensities we can characterize the Nash equilibrium of the game, which we know by Theorem 21 is

unique and comprised of pure strategies, in a manner analogous to what was done in Theorem 17. To define what we mean by proportional restrictions, given  $\alpha > 0$  and a nondecreasing convex function  $\Theta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  with  $\Theta(0) = 0$  and  $\lim_{x \rightarrow \infty} \frac{\Theta(x)}{x} = \infty$ , we define the function  $\Theta^{(\alpha)} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  by

$$\Theta^{(\alpha)}(x) := \Theta\left(\frac{x}{\alpha}\right), \quad x \in \mathbb{R}_+.$$

Note that  $\Theta^{(\alpha)} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a nondecreasing convex function with  $\Theta^{(\alpha)}(0) = 0$  and  $\lim_{x \rightarrow \infty} \frac{\Theta^{(\alpha)}(x)}{x} = \infty$ . We can then make the following simple observation.

**Lemma 22.** Given  $\rho > 0$ , a nondecreasing convex function  $\Theta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  with  $\Theta(0) = 0$  and  $\lim_{x \rightarrow \infty} \frac{\Theta(x)}{x} = \infty$ , and  $0 < K < \infty$ , we have  $f \in \mathcal{C}(\rho, \Theta^{(\rho)}, K)$  iff  $\frac{f}{\rho} \in \mathcal{C}(1, \Theta, K)$ .

*Proof.* To check if  $\frac{f}{\rho} \in \mathcal{C}(1, \Theta, K)$  we need to check if  $\int_D \frac{f(x)}{\rho} \eta(dx) = 1$  and if  $\int_D \Theta\left(\frac{f(x)}{\rho}\right) \eta(dx) \leq K$ . Equivalently, we need to check whether  $\int_D f(x) \eta(dx) = \rho$  and  $\int_D \Theta^{(\rho)}(f(x)) \eta(dx) \leq K$ , i.e. whether  $f \in \mathcal{C}(\rho, \Theta^{(\rho)}, K)$ .  $\square$

The following result characterizes the Nash equilibria of  $N$ -player restricted Cox process Hotelling games when the restrictions on the individual players are in proportion to their allowed intensities.

**Theorem 23.** Consider an  $N$ -player restricted Cox process Hotelling game on the Polish space  $D$  with base measure  $\eta$  where player  $i$ , for  $1 \leq i \leq N$ , has the intensity budget  $\rho_i > 0$  and the space of pure actions  $\mathcal{C}(\rho_i, \Theta^{(\rho_i)}, K)$ , where  $\Theta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a nondecreasing convex function with  $\Theta(0) = 0$  and  $\lim_{x \rightarrow \infty} \frac{\Theta(x)}{x} = \infty$ , and  $0 < K < \infty$ . Then the game has a unique Nash equilibrium, which is of the form  $(f_1, \dots, f_N)$ , where  $f_i = \frac{\rho_i}{\rho} f$  for some  $f \in \mathcal{C}(\rho, \Theta^{(\rho)}, K)$ , where  $\rho := \sum_{i=1}^N \rho_i$ .

*Proof.* The proof is similar to that of Theorem 17, with the obvious modifications. The only thing that needs to be observed is that for any choice of  $f_i \in \mathcal{C}(\rho_i, \Theta^{(\rho_i)}, K)$  for  $1 \leq i \leq N$ , we also have:

- (i)  $\frac{\rho_i}{\sum_{j \neq i} \rho_j} \sum_{j \neq i} f_j \in \mathcal{C}(\rho_i, \Theta^{(\rho_i)}, K)$  for all  $1 \leq i \leq N$ ;
- (ii)  $\sum_{i=1}^N f_i =: f \in \mathcal{C}(\rho, \Theta^{(\rho)}, K)$ , and;
- (iii)  $\frac{\rho_i}{\rho} f \in \mathcal{C}(\rho_i, \Theta^{(\rho_i)}, K)$  for all  $1 \leq i \leq N$ .

To show (i), by Lemma 22 what we need to show, for all  $1 \leq i \leq N$ , is that  $\frac{1}{\sum_{j \neq i} \rho_j} \sum_{j \neq i} f_j \in \mathcal{C}(1, \Theta, K)$ . By Lemma 22 again, we have  $\frac{f_j}{\rho_j} \in \mathcal{C}(1, \Theta, K)$  for all  $j \neq i$ . The desired claim follows from the convexity of  $\mathcal{C}(1, \Theta, K)$ .

To show (ii), by Lemma 22 what we need to show is that  $\frac{f}{\rho} \in \mathcal{C}(1, \Theta, K)$ . By Lemma 22 we have  $\frac{f_i}{\rho_i} \in \mathcal{C}(1, \Theta, K)$  for all  $1 \leq i \leq N$ . The desired claim follows from the convexity of  $\mathcal{C}(1, \Theta, K)$ .

As for (iii), it is an immediate consequence of Lemma 22 since we have established the claim in (ii).  $\square$

Finally, with a view to using restricted Cox process Hotelling games as a vehicle for better understanding the characterization in Theorem 18, we can prove the following analog of that result for restricted Cox process Hotelling games.

**Theorem 24.** Consider an  $N$ -player restricted Cox process Hotelling game on the Polish space  $D$  with base measure  $\eta$  where player  $i$ , for  $1 \leq i \leq N$ , has the intensity budget  $\rho_i > 0$  and the space of pure actions  $\mathcal{C}(\rho_i, \Theta^{(\rho_i)}, K)$ , where  $\Theta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a nondecreasing convex function with  $\Theta(0) = 0$  and  $\lim_{x \rightarrow \infty} \frac{\Theta(x)}{x} = \infty$ , and  $0 < K < \infty$ . The game admits a unique Nash equilibrium, comprised of pure strategies, defined in terms of a function  $f \in \mathcal{C}(\rho, \Theta^{(\rho)}, K)$ , where  $\rho := \sum_{i=1}^N \rho_i$ , such that

$$\begin{aligned} \int_{x \in D} f(x) \int_{y \in D} e^{-\int_{u \in B(y \rightarrow x)} f(u) \eta(du)} \eta(dy) \eta(dx) \\ \geq \int_{x \in D} g(x) \int_{y \in D} e^{-\int_{u \in B(y \rightarrow x)} f(u) \eta(du)} \eta(dy) \eta(dx), \end{aligned} \quad (31)$$

for all  $g \in \mathcal{C}(\rho, \Theta^{(\rho)}, K)$ . The corresponding unique Nash equilibrium of the game is given by the pure strategy profile  $(\frac{\rho_1}{\rho} f, \dots, \frac{\rho_N}{\rho} f)$ .

*Proof.* The proof is similar to that of Theorem 18, with the obvious modifications. The key point we need to observe is that, for all  $1 \leq i \leq N$ , we have  $g \in \mathcal{C}(\rho, \Theta^{(\rho)}, K)$  iff  $\frac{\rho_i}{\rho} g \in \mathcal{C}(\rho_i, \Theta^{(\rho_i)}, K)$ . Also, as we have observed earlier, in view of the compactness of the action spaces of the individual players and the continuity of the payoff of an individual player in its action when the actions of its opponents are fixed, the existence of a mixed strategy Nash equilibrium is guaranteed [15].  $\square$

**Remark 25.** For the  $N$ -player restricted Cox process Hotelling games with proportional restrictions of the kind considered in Theorem 24 that theorem tells us

that for each choice of  $\rho_i > 0$  for  $1 \leq i \leq N$ , nondecreasing convex function  $\Theta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  satisfying  $\Theta(0) = 0$  and  $\lim_{x \rightarrow \infty} \frac{\Theta(x)}{x} = \infty$ , and  $0 < K < \infty$ , there is a unique function  $f \in \mathcal{C}(\rho, \Theta^{(\rho)}, K)$  which satisfies the inequality in equation (31) for all  $g \in \mathcal{C}(\rho, \Theta^{(\rho)}, K)$ , where  $\rho := \sum_{i=1}^N \rho_i$ . We also know that if, for the given choices of  $\rho_i > 0$  for  $1 \leq i \leq N$ , the original  $N$ -player Cox player Hotelling game considered in Theorem 18 admits a Nash equilibrium then there will be a function  $f \in \mathcal{C}(\rho)$  satisfying the inequality in Equation (25) for all  $g \in \mathcal{C}(\rho)$ , and, most importantly, that this  $f$  will be manifest itself as the one verifying Equation (31) for some choices of nondecreasing convex function  $\Theta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  satisfying  $\Theta(0) = 0$  and  $\lim_{x \rightarrow \infty} \frac{\Theta(x)}{x} = \infty$ , and of  $0 < K < \infty$ . It is in this sense that the discussion of  $N$ -player restricted Cox process Hotelling games gives a vehicle, in principle, to better understand the meaning of the criterion in Equation (25) for the existence of Nash equilibria in  $N$ -player Cox process Hotelling games.

## Acknowledgements

VA acknowledges support from the NSF grants CNS-1527846, CCF-1618145, CCF-1901004, the NSF Science & Technology Center grant CCF-0939370 (Science of Information), and the William and Flora Hewlett Foundation supported Center for Long Term Cybersecurity at Berkeley. FB was supported by the ERC NEMO grant, under the European Union's Horizon 2020 research and innovation programme, grant agreement number 788851 to INRIA. The authors thank Prof. Yadati Narahari for posing the question of whether Cox process Hotelling games might be potential games.

## References

- [1] H.-K. Ahn, S.-W. Cheng, O. Cheong, M. Golin and R. van Oostrum. "Competitive facility location: the Voronoi game." *Theoretical Computer Science*, Vol. 310, 2004, pp. 457 -467.
- [2] J. Bertrand. "Review of 'Théorie mathématique de la richesse sociale' and 'Recherches sur les principes mathématiques de la théorie des richesses'," *Journal des Savants*, 1883, pp. 499 -508.
- [3] V. I. Bogachev. *Measure Theory*. Springer, 2000.

- [4] M. Boppana, R. Hod, M. Mitzenmacher and T. Morgan. “Voronoi choice games.” *arXiv:1604.07084v1*, 24 April 2016.
- [5] D. Chafai “De la Vallée Poussin on uniform integrability.” Blog entry at: <http://djalil.chafai.net/blog/2014/03/09/de-la-vallee-poussin-on-uniform-integrability/>, 2014.
- [6] O. Cheong, S. Har-Peled, N. Linial and J. Matoušek. “The one-round Voronoi game.” *Discrete and Computational Geometry*, Vol. 31, 2004, pp. 125 -138.
- [7] A. Cournot. *Recherches sur les Principes Mathématiques de la Théorie des Richesses*. Hachette, 1838.
- [8] D. J. Daley and D. Vere-Jones. *An Introduction to the Theory of Point Processes, Vol. II: General Theory and Structure*. Second Edition. Probability and its Applications, Springer, 2008.
- [9] J. Diestel. “Uniform integrability: an introduction.” In *Rendiconti dell’Istituto di Matematica dell’Università di Trieste*, Vol. 23, 1991, pp. 41 -80.
- [10] N. Dunford and B. J. Pettis. “Linear operators on summable functions.” *Transactions of the American Mathematical Society*, Vol. 47, 1940, pp. 323-392.
- [11] C. Dürr and N. K. Thang. “Nash equilibria in Voronoi games on graphs.” In *Lecture Notes in Computer Science*, Vol. 4698, L. Arge, M. Hoffman and E. Welzl (Editors), ESA-2007, Springer-Verlag, 2007, pp. 17 -28.
- [12] F. Y. Edgeworth. “The pure theory of monopoly.” In *Papers Relating to Political Economy*, Vol. 1, Macmillan and Co., Ltd., 1925, pp. 111 -142.
- [13] H. A. Eiselt, G. Laporte and J.-F. Thisse. “Competitive location models: A framework and bibliography.” *Transportation Science*, Vol. 27, No. 1, Feb. 1993, pp. 44 -54.
- [14] J. J. Gabszewicz and J.-F. Thisse. “Location.” Chapter 9 in *Handbook of Game Theory*, Vol. 1, Edited by R. J. Aumann and S. Hart, Elsevier Science Publishers. B .V. 1992, pp. 281 -304.

- [15] I. L. Glicksberg. "A further generalization of the Kakutani fixed point theorem, with application to Nash equilibrium points." *Proceedings of the American Mathematical Society*, Vol. 3, No. 1, 1952, pp. 170-174.
- [16] D. Graitson. "Spatial competition à la Hotelling: A selective survey." *The Journal of Industrial Economics*, Vol. 31, No. 1/2, Sep. -Dec. 1982, pp. 11-25.
- [17] H. Hotelling. "Stability in competition." *The Economic Journal*, Vol. 39, No. 153, Mar 1929, pp. 41-57.
- [18] O. Kallenberg. *Random measures, theory and applications*. Springer, 2017.
- [19] J. L. Kelley. *General Topology*. Graduate Texts in Mathematics, Vol. 27, Springer, 1991.
- [20] D. Kovenock and B. Roberson. "Generalizations of the General Lotto and Colonel Blotto games." *Economic Science Institute*, Working Paper No. 15-07, 2015.
- [21] K. Lange. "Borel sets of probability measures." *Pacific Journal of Mathematics*, Vol. 48, No. 1, 1973, pp. 141-161.
- [22] L. Mallozi, E. D'Amato and P. M. Pardalzo (Editors). *Spatial Interaction Models - Facility Location Using Game Theory*. Springer Optimization and its Applications, Vol. 118, Springer, 2017.
- [23] D. Monderer and L. S. Shapley. "Potential games." *Games and Economic Behavior*, Vol. 14, 1996, pp. 124-143.
- [24] J. F. Nash, Jr. "Equilibrium points in  $N$ -person games." *Proceedings of the National Academy of Sciences*, Vol. 36, 1950, pp. 48-49.
- [25] J. F. Nash, Jr. "Non-cooperative games." *Annals of Mathematics*, Vol. 54, No. 2, Sep. 1951, pp. 286-295.
- [26] K. R. Parthasarathy. *Probability Measures on Metric Spaces*. Academic Press, 1967.
- [27] C. S. ReVelle and H. A. Eiselt. "Location analysis: A synthesis and survey." *European Journal of Operations Research*, Vol. 165, 2005, pp. 1-19.



- [28] B. Roberson. “The Colonel Blotto game.” *Economic Theory*, 29: 1, 2006, pp. 1 -24.
- [29] W. Rudin. *Functional Analysis*. McGraw-Hill Inc., 1973.
- [30] S. Teramoto, E. D. Demaine and R. Uehara. “The Voronoi game on graphs and its complexity.” *Journal of Graph Algorithms and Applications*, Vol. 15, No. 4, 2011, pp. 485 -501.
- [31] H. R. Varian. *Intermediate Microeconomics - A Modern Approach*. Seventh Edition, W. W. Norton & Co., 2006.
- [32] J. von Neumann and O. Morgenstern. *Theory of Games and Economic Behavior*. Princeton University Press, 1953.

## Appendix

### A A sufficient condition for $f\eta$ to be non-conflicting

We give here a sufficient condition for the  $\eta$ -measure of the boundary of the Voronoi tessellation of a Poisson point process  $\Phi$  of density  $f$  w.r.t.  $\eta$  on  $D$  to be zero a.s. The setting is that of Subsection 2.1, with  $\int_D f(x)\eta(dx) = \rho < \infty$ , so that  $\Phi$  has a finite number of points a.s.

A sufficient condition for the desired property to hold is that

$$\eta\{z \in D \text{ s.t. } \exists X \neq Y \in \Phi \text{ with } d(z, X) = d(z, Y)\} = 0, \quad \text{a.s.},$$

which holds if

$$\mathbb{E}[\eta\{z \in D \text{ s.t. } \exists X \neq Y \in \Phi \text{ with } d(z, X) = d(z, Y)\}] = 0.$$

The latter can be written as

$$\sum_{n \geq 2} \frac{\rho^n}{n!} e^{-\rho} n(n-1) \int_{z \in D} \int_{x \in D} \int_{y \in D} 1_{d(z,x)=d(z,y)} \eta(dz) f(x) \eta(dx) f(y) \eta(dy) = 0,$$

so that a sufficient condition for the desired property to hold is that

$$\int_{z \in D} \int_{x \in D} \int_{y \in D} 1_{d(z,x)=d(z,y)} \eta(dz) f(x) \eta(dx) f(y) \eta(dy) = 0. \quad (32)$$