



HAL
open science

LÉVY AREA WITHOUT APPROXIMATION

Isao Sauzedde

► **To cite this version:**

| Isao Sauzedde. LÉVY AREA WITHOUT APPROXIMATION. 2021. hal-03107197

HAL Id: hal-03107197

<https://hal.science/hal-03107197>

Preprint submitted on 12 Jan 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

LÉVY AREA WITHOUT APPROXIMATION

ISAO SAUZEDDE

ABSTRACT. We give asymptotic estimations on the area of the sets of points with large Brownian winding, and study the average winding between a planar Brownian motion and a Poisson point process of large intensity on the plane. This allows us to give a new definition of the Lévy area which does not rely on approximations of the Brownian path. It also does not depend on the metric structure on the plane.

CONTENTS

Introduction	1
1. First definitions and main results	3
1.1. Average winding of a curve	3
1.2. Strategy of the proof	5
2. Preliminary lemmas	7
2.1. Statements of the lemma	7
2.2. Proofs	8
3. Asymptotics in L^2	13
3.1. Asymptotic of the mean	13
3.2. Decomposition into small pieces	17
3.3. Asymptotic for the second moment	19
4. From L^2 to almost sure estimates	21
5. Computation of the position parameter	23
5.1. Some properties of Cauchy-like laws	24
5.2. Computation for the position parameter of the Brownian motion	27
5.3. Proof of Theorem 0.1	29
6. From L^2 estimates to L^p estimates	31
7. A similar result for Young integration	32
8. Further discussion	36
Acknowledgements	37
References	38

INTRODUCTION

For a smooth, simple, closed curve γ on the plane, Stokes' theorem allows one to express the integral $\int_{\gamma} x dy$ (up to sign) as the area delimited by the curve γ . If we lift the assumption that

Key words and phrases. Stokes's formula; Planar Brownian motion; Lévy's area.

the curve is simple, we have to take a multiplicity into account. The formula becomes

$$\int_{\gamma} x \, dy = \int_{\mathbb{R}^2} \theta_{\gamma}(z) \, dz, \quad (1)$$

where $\theta_{\gamma}(z)$ is the integer winding of γ around the point z (defined for z outside the range of γ).

This integral can then be evaluated by a Monte Carlo method. If \mathcal{P} is a Poisson point process with intensity $K \, dz$ with K large, then the normalized sum

$$\frac{1}{K} \sum_{z \in \mathcal{P}} \theta_{\gamma}(z) \quad (2)$$

is approximately equal to the integral $\int_{\gamma} x \, dy$.

When one substitutes the smooth curve γ with a Brownian motion, the integral on the right-hand side of (1) does not make sense anymore. Werner already remarked this fact in [14], where he defined a family of approximations for the right-hand side of (1) and proved a convergence in probability toward the left-hand side. In this paper, we use a different family of approximations and prove an almost sure convergence. We also link these approximations with (2).

The study of the winding function for the Brownian motion, started with the celebrated result of Spitzer about the large time asymptotics around a given point in the plane [12], is a long-standing subject. Yor gave in [15] an explicit form for the law of the winding of a Brownian loop around a fixed point (the result can also be found in [10]). In [11], Shi gave a detailed analysis of the distribution of the winding around a fixed point as a process in time.

Werner studied the behaviour as N tends to infinity of the area A_N of the set of points with winding N in the L^2 sense. In [13], he showed in particular that A_N behaves as $\frac{1}{2\pi N^2}$ (see Equation (6) below for a precise statement). He also derived the leading term of the asymptotic expansion of the area D_N of the set of points with winding at least N . We will push the analysis of D_N further, and obtain a bound on the difference between D_N and the leading term. Besides, we will show that both the leading term and the bound remain valid in L^p (for any p) and almost surely. This allows us to prove the following result, which is the main result of this paper. Let us recall that the Cauchy distribution with position parameter $p \in \mathbb{R}$ and scale parameter $\sigma > 0$ is the distribution with density f given by

$$f(x) = \frac{\sigma}{\pi} \frac{1}{(x-p)^2 + \sigma^2}.$$

Theorem 0.1. *Let $B = (X, Y) : [0, 1] \rightarrow \mathbb{R}^2$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and γ be the concatenation of B with a straight segment from B_1 to B_0 . Let also $\mathcal{P} = \mathcal{P}(K)$ be a Poisson process with intensity $K \, dz$ on a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$.*

Then, \mathbb{P} -almost surely, the normalized sum (2) converges in distribution, as $K \rightarrow \infty$, towards a Cauchy distribution with position parameter

$$\int_0^1 X \, dY - \frac{X_0 + X_1}{2} (Y_1 - Y_0)$$

where the integral is to be understood in the sense of Ito.

From Spitzer's result, one expects that the random variable given by (2) should converge to a Cauchy law as K goes to infinity. We show that this is the case indeed. What might be more

surprising is that the convergence actually holds almost surely. The fact that the asymptotic behaviour of D_N is deterministic at the leading order can be understood, very roughly, as follows. The value of the winding, when it is large, depends ‘mostly’ on a small piece of the path. We can therefore expect to be able to decompose D_N into a sum of independent and identically distributed contributions from different pieces of the path. The random fluctuations of each of those contributions around their mean cancel out, so that D_N should indeed be deterministic as N is large. The difficulty is to control the ‘mostly’, as well as the speed at which the random fluctuations cancel out.

We also show a similar theorem for slightly more regular curves.

Theorem 0.2. *Let $p, q \geq 1$ be reals such that $\delta = \frac{1}{p} + \frac{1}{q} - 1 > 0$. Let $\gamma = (x, y) : [0, 1] \rightarrow \mathbb{R}^2$ be a continuous closed curve such that x has finite p -variation and y has finite q -variation. Then, the range of γ has zero Lebesgue measure and $\theta_\gamma \in L^{\delta'}(\mathbb{R}^2, \mathbb{Z})$ for any $\delta' \in [1, \delta)$. Besides, the equality*

$$\int_0^1 x_t \, dy_t - \frac{x_0 + x_1}{2}(y_1 - y_0) = \int_{\mathbb{R}^2} \theta_\gamma(z) \, dz. \quad (3)$$

holds if the left-hand side is interpreted as a Young integral.

The paper is organized as follows. The first section is a summary of the main results. Sections 2 to 5 are devoted to the proof of Theorem 0.1. Section 2 contains technical bounds that will be used at different points in the paper. Section 3 extends the estimation of Werner on D_N by giving an L^2 bound on $D_N - \frac{1}{2\pi N}$. In Section 4, we obtain a maximal inequality that allows us to extract an almost sure bound from the result of the previous section. We show in Section 5 some general facts about families of Cauchy-like variables, which allows us to compute the position parameter that appears in Theorem 0.1. In Section 6, we extend the estimations previously obtained in L^2 to L^p . Section 7 consists mostly on the proof of Theorem 7.1. In the last section, we conclude with a few remarks about the dependence, or not, of the quantity $\int_{\mathbb{R}^2} \theta_\gamma(z) \, dz + \frac{x_0 + x_1}{2}(y_1 - y_0)$ with respect to the ambient Riemannian metric.

1. FIRST DEFINITIONS AND MAIN RESULTS

1.1. Average winding of a curve. We denote Borel sets with curly letters $(\mathcal{A}, \mathcal{D}, \dots)$ and use the same straight letters (A, D, \dots) for their Lebesgue measures. We also denote the Lebesgue measure by $|\cdot|$. We write $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ and $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$.

Let $s < t$ and $\gamma : [s, t] \rightarrow \mathbb{R}^2$ be a continuous planar curve. Let z be a point of \mathbb{R}^2 that does not lie on the range of γ nor on the segment joining γ_t to γ_s . We denote by $\theta_\gamma(z)$ the winding around z of the closed curve obtained by concatenating γ with the segment $[\gamma_t, \gamma_s]$. Provided the range of γ has zero Lebesgue measure, the function θ_γ is defined almost everywhere, measurable, and takes its values in \mathbb{Z} .

We define a finite measure μ_γ on \mathbb{Z}^* by setting, for all $n \in \mathbb{Z}^*$,

$$\mu_\gamma(\{n\}) = |\{z \in \mathbb{R}^2 : \theta_\gamma(z) = n\}|.$$

In words, μ_γ is the restriction to \mathbb{Z}^* of the image by the function θ_γ of the Lebesgue measure on \mathbb{R}^2 .

Provided $\mu_\gamma(\mathbb{Z}^*) \neq 0$, we also define ν_γ as the probability law obtained by normalization of μ_γ :

$$\nu_\gamma = \frac{\mu_\gamma}{\mu_\gamma(\mathbb{Z}^*)}.$$

Provided the tail of the measure μ_γ decreases fast enough, its first moment

$$\sum_{n \in \mathbb{Z}^*} n \mu_\gamma(n)$$

is well defined, and is the algebraic area enclosed by the curve γ .

For a smooth curve γ , this quantity can also be expressed as an integral. Indeed, denoting, for all $t \in [0, 1]$ by x_t and y_t the coordinates of γ_t , it is equal to the integral

$$\int_0^1 x_t y'_t dt - \frac{x_1 + x_0}{2} (y_1 - y_0).$$

Note that the second term in this expression (with the minus sign) is the integral of $x dy$ along the segment $[\gamma_1, \gamma_0]$. The equality of the two quantities is a consequence of Stokes' formula¹. We are interested in a less regular situation, in which μ_γ does not necessarily possess a first moment, but can still be assigned a quantity which will play the role of a substitute for the non-existing first moment.

Let us recall that the Cauchy distribution $C(p, \sigma)$ with position parameter $p \in \mathbb{R}$ and scale parameter $\sigma > 0$ is the following probability measure on \mathbb{R} :

$$C(p, \sigma) = \frac{\sigma dx}{\pi(\sigma^2 + (x - p)^2)}.$$

We also set $C(p, 0) = \delta_p$. We recall also that, for any p, σ , $C(p, \sigma)$ is a 1-stable law: if X and Y are independent random variables distributed according to $C(p, \sigma)$, then $\frac{X+Y}{2}$ is also distributed according to $C(p, \sigma)$.

A probability measure ν on \mathbb{R} is said to lie in the attraction domain of a Cauchy distribution if there exists sequences $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$ of reals such that for an i.i.d. sequence $(Z_n)_{n \geq 0}$ with common law ν ,

$$\frac{Z_1 + \dots + Z_n}{a_n} - b_n \xrightarrow[n \rightarrow +\infty]{(d)} C(p, \sigma) \quad (4)$$

for some $p \in \mathbb{R}$, $\sigma > 0$.²

It is known that (4) is equivalent to some condition about the asymptotics of the tail (see for example [4]). In particular, it is sufficient that the cumulative distribution function F_ν of ν satisfies the two tail conditions

$$1 - F_\nu(x) \underset{x \rightarrow +\infty}{\sim} \frac{\sigma}{\pi x} \quad \text{and} \quad F_\nu(x) \underset{x \rightarrow -\infty}{\sim} -\frac{\sigma}{\pi x}.$$

In this general situation, the position parameter p of the limiting Cauchy distribution has no particular meaning, as it can be changed arbitrarily by shifting the sequence $(b_n)_{n \geq 1}$.

¹For a smooth curve with non-vanishing derivative, compactness and the implicit function theorem allow one to split the interval $[0, 1]$ into finitely many sub-intervals on each of which one coordinate of the curve is a smooth function of the other. The formula holds on each sub-interval, hence on $[0, 1]$ by additivity.

²One cannot include the case $\sigma = 0$ without imposing some restrictions on the sequence $(a_n)_{n \geq 1}$.

We will make use of the following less common (and more restrictive) definition. We say that a probability measure ν on \mathbb{R} lies in the *strong attraction domain* (of a Cauchy distribution) with scale parameter $\sigma \geq 0$ if there exists $\delta > 0$ such that

$$F_\nu(x) \underset{x \rightarrow -\infty}{=} \frac{\sigma}{\pi|x|} + o\left(\frac{1}{|x|^{1+\delta}}\right) \quad 1 - F_\nu(x) \underset{x \rightarrow +\infty}{=} \frac{\sigma}{\pi x} + o\left(\frac{1}{x^{1+\delta}}\right). \quad (\text{C})$$

We use here the terminology of [8] (Definition 5.2). It is shown in [8, Lemma 5.1] that the condition (C) implies the existence of a Cauchy distribution ν' , a real $\delta > 0$, and a coupling (X, Y) with X distributed according to ν and Y according to ν' , such that $\mathbb{E}[|X - Y|^{1+\delta}]$ is finite. In particular, not only does ν lie in the attraction domain, but the convergence (4) holds with the choices $a_n = n$, $b_n = 0$ (see Theorem 1.2 in [8]³).

We then denote by p_ν the position parameter of the limiting Cauchy distribution (for these choices of a_n, b_n), and by σ_ν its scale parameter. The scale parameter σ_ν is also the value of the σ that appears on (C). We call p_ν the *position parameter* of ν , and $\sigma_\nu u$ its *scale parameter*. Any distribution with a finite moment of order strictly greater than 1 also satisfies (C) with $\sigma = 0$, and in that case p_ν is equal to the first moment of ν . However, the distributions that satisfies (C) with $\sigma \neq 0$ do not even have a moment of order 1. We will show that when ν lies on the strong attraction domain, p_ν is given by the explicit formulas

$$p_\nu = \sum_{N \geq 1} (\nu([N, +\infty)) - \nu((-\infty, -N])) = \sum_{N \geq 1} N(\nu(N) - \nu(-N)).$$

We can extend these definitions to finite measures on \mathbb{R} . If μ is a finite measure with mass Z , and the probability measure $\nu = \frac{\mu}{Z}$ satisfies condition (C), then we set $p_\mu = Zp_\nu$ (resp. $\sigma_\mu = Z\sigma_\nu$) and we call it the position parameter of μ (resp. the scale parameter of μ). We then say that μ lies on the strong attraction domain (of the Cauchy distribution).

We will prove the following statement, from which we will deduce Theorem 0.1 at the end of Section 5.

Theorem 1.1. *Let $B = (X, Y) : [0, 1] \rightarrow \mathbb{R}^2$ be a Brownian motion. With probability 1, the measure μ_B lies in the strong attraction domain of the Cauchy distribution, and the position parameter $p_B = p_{\mu_B}$ is related to the Lévy area of B by the formula*

$$p_B = \int_0^1 X \, dY - \frac{X_1 + X_0}{2}(Y_1 - Y_0). \quad (5)$$

1.2. Strategy of the proof. We will prove Theorem 1.1 by showing that the measure $\nu_B = \frac{\mu_B}{\mu_B(\mathbb{Z}^*)}$ satisfies almost surely the condition (C) with $\delta = \frac{1}{6}$. For this, our main object of interest will naturally be the tail of ν_B .

Let us choose, on a probability space $(\Omega, \mathcal{F}, (\mathbb{P}_z)_{z \in \mathbb{R}^2})$, a Brownian motion $(B_t)_{t \in [0, 1]}$. It is understood that under \mathbb{P}_z , the Brownian motion is started from z . We will often consider the Brownian motion started from 0, and we write $\mathbb{P} = \mathbb{P}_0$.

³There seems to be a minor mistake in the assumptions of this theorem. The condition “ $\mathbb{E}[X_i] = 0$ if $\alpha > 1$ ” should be replaced with the condition “if $\beta > 1$, there exists an α -stable random variable Y such that $\mathbb{E}[X_i - Y] = 0$ ” in order to deal correctly with the case $\alpha \leq 1 < \beta$. The last inequality on the proof (p. 841) is true only under this stronger condition.

For every integer $N \geq 1$, we define

$$\mathcal{D}_N = \{z \in \mathbb{R}^2 : \theta_B(z) \geq N\} \quad \text{and} \quad D_N = |\mathcal{D}_N| = \mu_B([N, +\infty)).$$

It is known from the work of Werner [13] that

$$N^2 \mu_B(N) \xrightarrow[N \rightarrow +\infty]{L^2} \frac{1}{2\pi}. \quad (6)$$

Thus, we expect D_N to be of the order of $\frac{1}{2\pi N}$.

After the preliminary estimations of Section 2, we will study both the expectation and the variance of ND_N , in Section 3. We will obtain the two following lemmas.

Lemma 1.2. *There exists $C \geq 0$ such that for all $N \geq 1$,*

$$N \left| \mathbb{E}[ND_N] - \frac{1}{2\pi} \right| \leq C.$$

Lemma 1.3. *For all $\delta \in (0, \frac{1}{3})$, there exists $C \geq 0$ such that for all $N \geq 1$,*

$$N^{2\delta} \text{Var}[ND_N] \leq C.$$

We will obtain these two estimates by quite different methods. The proof of the first one relies mostly on the study of some explicit, analytical expression, and we consider it as not very enlightening. On the other hand, the proof of the second estimate is based on making rigorous the idea that D_N can be decomposed into a sum of ‘local’ quantities. We hope that the ideas used there may be applied to solve different but similar problems. These two lemmas merge into the following proposition.

Proposition 1.4. *For all $\delta \in (0, \frac{1}{3})$, there exists $C \geq 0$ such that for all $N \geq 1$,*

$$\mathbb{E} \left[N^{2\delta} \left(ND_N - \frac{1}{2\pi} \right)^2 \right] \leq C. \quad (7)$$

From this result in L^2 , we will deduce with some extra work that the measure μ_B satisfies almost surely the condition (C): this is the subject of Section 4. Informally, the goal is to put a maximum under the expectation in (7). We do this at the cost of lowering the upper bound on δ from $\frac{1}{3}$ to $\frac{1}{6}$. This also gives a probabilistic control on the remainder of (C).

At this point, the first assertion of Theorem 1.1 will be proven, and there will remain to study the position parameter of the limiting Cauchy distribution. We will do this on Section 5, using a few general results on ‘Cauchy-like’ distributions. We will in particular make a repeated use of the gap between the dominant term in $|x|^{-1}$ and the first correction in $|x|^{-1-\delta}$ imposed on the definition of the strong Cauchy domains.

Theorem 1.1 will then be proved. Section 6 presents the extensions of some of the result in L^2 into results in L^p , for arbitrary large p . We expect to use these additional estimations on a forthcoming work, in which the approach given here is the cornerstone to study others stochastic integrals, including some non-trivial ones. The main conclusion of this section will be the following.

Theorem 1.5. *For all $p \in [2, \infty)$ and all $\delta < \frac{2}{3}$, there exists a constant C such that for all $N \in \mathbb{N}^*$,*

$$\mathbb{E}\left[N^\delta \left|ND_N - \frac{1}{2\pi}\right|^p\right] \leq C. \quad (8)$$

2. PRELIMINARY LEMMAS

We will split the Brownian trajectory into many small pieces, and study the winding of the whole trajectory as resulting from the individual contributions of each of these pieces. For this, we will need to understand something of the joint distribution of the winding of two of these small pieces. Up to scaling, this amounts to studying the joint winding of two Brownian trajectories started at different points.

Let us assume that our probability space carries a second process $(B'_t)_{t \in [0,1]}$ with values in \mathbb{R}^2 and a family $(\mathbb{P}_{z,z'})_{z,z' \in \mathbb{R}^2}$ of probability measures such that under $\mathbb{P}_{z,z'}$, the processes B and B' are independent Brownian motions, respectively started from z and z' .

We define, for all $N \geq 0$,

$$\mathcal{D}_N^{(2)} = \{z \in \mathbb{R}^2 : |\theta_B(z)| \geq N \text{ and } |\theta_{B'}(z)| \geq N\}.$$

Note the absolute values in this definition. As usual, we denote by $D_N^{(2)}$ the Lebesgue measure of $\mathcal{D}_N^{(2)}$.

We will state several results which say in various ways that $D_N^{(2)}$ is small.

2.1. Statements of the lemma. The first lemma deals with the expectation of $D_N^{(2)}$ in several situations regarding the relative positions of the starting points of the Brownian motions.

Lemma 2.1. *For all $\varepsilon > 0$, there exists $C \geq 0$ such that for all $N \geq 1$,*

$$\sup_{z,z' \in \mathbb{R}^2} \mathbb{E}_{z,z'}[D_N^{(2)}] \leq CN^{-2+\varepsilon}.$$

Besides, for all $\delta > 0$, there exists $C \geq 0$ such that for all $N \geq 1$,

$$\sup_{\|z-z'\| \geq N^\delta} \mathbb{E}_{z,z'}[D_N^{(2)}] \leq Ce^{-\frac{1}{17}N^{2\delta}}.$$

Finally, for all $\varepsilon > 0$, there exists $C \geq 0$ such that for all $N \geq 1$, for all $\sigma \geq 0$,

$$\mathbb{E}_{z,\mathcal{N}(z,\sigma^2)}[D_N^{(2)}] \leq C \frac{N^{-2+\varepsilon}}{1+\sigma^2}.$$

To be clear, we used the notation $\mathbb{E}_{z,\mathcal{N}(z,\sigma^2)}$ in the following sense:

$$\mathbb{E}_{z,\mathcal{N}(z,\sigma^2)}[\cdot] = \int_{\mathbb{R}^2} e^{-\frac{\|z'-z\|^2}{2\sigma^2}} \mathbb{E}_{z,z'}[\cdot] \frac{dz'}{2\pi\sigma^2}.$$

In the proof of this lemma, we will need the following estimate of the probability that the Brownian motion started at 0 winds at least N times around a point z .

Sublemma 2.2. For all $z \in \mathbb{R}^2$, define

$$f_N(z) = \mathbb{P}_0(|\theta_B(z)| \geq N).$$

For every $\delta, \beta > 0$, there exists a constant $C > 0$ such that for all $N \geq 1$, for all $z \in \mathbb{R}^2$,

$$f_N(z) \leq \begin{cases} 1 & \text{if } z \in B(0, N^{-\beta}) \\ C \log(N) N^{-1} & \text{if } z \in B(0, \frac{N^\delta}{2}) \setminus B(0, N^{-\beta}) \\ C e^{-\frac{1}{4}\|z\|^2} & \text{if } z \in \mathbb{C} \setminus B(0, \frac{N^\delta}{2}). \end{cases} \quad (9)$$

The second lemma is the analogue of Lemma 2.1 for the second moment of $D_N^{(2)}$.

Lemma 2.3. For all $\varepsilon > 0$, there exists $C \geq 0$ such that for all $N \geq 1$,

$$\sup_{z, z' \in \mathbb{R}^2} \mathbb{E}_{z, z'} \left[(D_N^{(2)})^2 \right] \leq C N^{-4+\varepsilon}.$$

Besides, for all $\delta > 0$, there exists $C \geq 0$ such that for all $N \geq 1$,

$$\sup_{\|z-z'\| \geq N^\delta} \mathbb{E}_{z, z'} \left[(D_N^{(2)})^2 \right] \leq C e^{-\frac{N^{2\delta}}{17}}.$$

Finally, for all $\varepsilon > 0$, there exists $C \geq 0$ such that for all $N \geq 1$, for all $\sigma \geq 0$,

$$\mathbb{E}_{z, \mathcal{N}(z, \sigma^2)} \left[(D_N^{(2)})^2 \right] \leq C \frac{N^{-4+\varepsilon}}{1 + \sigma^2}.$$

We will also need a preparatory result in the vein of Lemma 2.2.

Sublemma 2.4. For all $z_1, z_2 \in \mathbb{R}^2$, define

$$f_N^{(2)}(z_1, z_2) = \mathbb{P}_0(|\theta_B(z_1)| \geq N \text{ and } |\theta_B(z_2)| \geq N).$$

For every $\delta, \beta > 0$ define

$$T_{\beta, \delta} = \{(z_1, z_2) \in B(0, N^\delta)^2 : \min(\|z_1\|, \|z_2\|, \|z_1 - z_2\|) \leq N^{-\beta}\}.$$

There exists a constant $C > 0$ such that for all $N \geq 1$, for all $z_1, z_2 \in \mathbb{R}^2$,

$$f_N^{(2)}(z_1, z_2) \leq \begin{cases} 1 & \text{if } (z_1, z_2) \in T_{\beta, \delta} \\ C \log(N)^2 N^{-2} & \text{if } (z_1, z_2) \in B(0, N^\delta/2)^2 \setminus T_{\beta, \delta} \\ C e^{-\frac{1}{4} \max(\|z_1\|, \|z_2\|)^2} & \text{if } (z_1, z_2) \in (\mathbb{R}^2 \times \mathbb{R}^2) \setminus B(0, N^\delta/2)^2. \end{cases} \quad (10)$$

2.2. Proofs. We now begin the proofs of these four lemma. We will denote by $C, C', C^{(1)}, C^{(2)}, \dots$ for different constants that appear during the proofs. We sometime use additional subscripts to emphasize their dependency on some parameter.

Proof of Sublemma 2.2. : The first case is a triviality.

For the third case, set $B^* = \sup_{t \in [0, 1]} \|B_t\|$. Since θ_B is zero outside the ball of radius B^* ,

$$f_N(z) \leq \mathbb{P}(B^* \geq \|z\|) \leq 8\Phi(\|z\|/\sqrt{2})$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{t^2}{2}} dt$. The fact that $\Phi(x) \leq e^{-\frac{x^2}{2}}$ for $x \geq 1$ allows us to conclude this case.

The second case is a direct application of an inequality which can be found page 117 in Shi's article [11].

The inequality states that for t and x positive reals such that $t \log(x)$ is large enough,

$$\mathbb{P}_0 \left(\sup_{0 \leq u \leq t} \theta_{B_{|[0,u]}}(1, 0) \geq x \right) \leq \frac{8}{x} + \frac{2 \log(16t \log(x))}{x}. \quad (11)$$

Here, $t = \|z\|^{-2}$ and $x = N$. For $z \notin B(0, N^{-\beta})$, $\|z\|^{-2} \log(N) \geq N^{2\beta} \log(N)$, which becomes arbitrarily large when N is large. Therefore, we can apply the inequality, at least when N is larger than some N_0 which does not depend on z . Using also the scaling property of the Brownian motion, we end up with

$$\mathbb{P}_0 \left(\sup_{0 \leq u \leq 1} \theta_{B_{|[0,u]}}(z) \geq N \right) \leq \frac{8}{N} + \frac{2 \log(16\|z\|^{-2} \log(N))}{N}. \quad (12)$$

There is a constant C , which does not depend on $z \in \mathbb{R}^2 \setminus B(0, N^{-\beta})$, such that the right-hand side is less than $C \frac{\log(N)}{N}$. Up to modification of the constant C , the inequality

$$\mathbb{P}_0 \left(\sup_{0 \leq u \leq 1} \theta_{B_{|[0,u]}}(z) > N \right) \leq C \frac{\log(N)}{N} \quad (13)$$

stays true for $N < N_0$, which concludes the proof. \square

Proof of Lemma 2.1. First, we rewrite $\mathbb{E}_{z,z'}[D_N^{(2)}]$:

$$\begin{aligned} \mathbb{E}_{z,z'}[D_N^{(2)}] &= \mathbb{E}_{z,z'} \left[\int_{\mathbb{R}^2} \mathbb{1}_{|\theta_B(\xi)| \geq N} \mathbb{1}_{|\theta_{B'}(\xi)| \geq N} d\xi \right] \\ &= \int_{\mathbb{R}^2} \mathbb{P}_z(|\theta_B(\xi)| \geq N) \mathbb{P}_{z'}(|\theta_{B'}(\xi)| \geq N) d\xi \\ &= \int_{\mathbb{R}^2} f_N(\xi - z) f_N(\xi - z') d\xi. \end{aligned}$$

Now we distinguish between the three results we want to show.

First case. Using the Cauchy–Schwarz inequality and the sublemma, we find

$$\begin{aligned} \int_{\mathbb{R}^2} f_N(\xi - z) f_N(\xi - z') d\xi &\leq \int_{\mathbb{R}^2} f_N(\xi)^2 d\xi \\ &\leq |B(0, N^{-\beta})| + C^2 \log(N)^2 N^{-2} |B(0, N^\delta/2)| + C^2 \int_{\mathbb{R}^2 \setminus B(0, N^\delta/2)} e^{-\frac{1}{4}\|\xi\|^2} d\xi \\ &\leq C' N^{-2\beta} + C' \log(N)^2 N^{-2+2\delta} + C' e^{-\frac{1}{16} N^{2\delta}}. \end{aligned}$$

We take $0 < \delta < \frac{\varepsilon}{4}$ and $\beta > 1 - \frac{\varepsilon}{2}$ to obtain the desired result.

Second case. We assume that $\|z - z'\| \geq N^\delta$. Since f_N is bounded by 1, we have:

$$\begin{aligned} \int_{\mathbb{R}^2} f_N(\xi - z) f_N(\xi - z') \, d\xi &\leq \int_{B(z, N^\delta/2)} f_N(\xi - z') \, d\xi + \int_{\mathbb{R}^2 \setminus B(z, N^\delta/2)} f_N(\xi - z) \, d\xi \\ &\leq C N^{2\delta} e^{-\frac{1}{16} N^{2\delta}} + C \int_{\mathbb{R}^2 \setminus B(0, N^\delta/2)} e^{-\frac{1}{16} \|\xi\|^2} \, d\xi \\ &\leq C' N^{2\delta} e^{-\frac{1}{16} N^{2\delta}} \\ &\leq C'' e^{-\frac{1}{17} N^{2\delta}}. \end{aligned}$$

Third case. We denote the 2-dimensional heat kernel by

$$p(t, x, y) = \frac{1}{2\pi t} \exp\left(-\frac{\|x - y\|^2}{2t}\right).$$

We use the two first cases with some $\varepsilon' < \varepsilon$, and we set $\delta = \frac{\varepsilon - \varepsilon'}{2}$.

$$\begin{aligned} \mathbb{E}_{z, \mathcal{N}(z, \sigma^2)} [D_N^{(2)}] &\leq \int_{B(z, N^\delta)} p(\sigma^2, z, \xi) \mathbb{E}_{z, \xi} [D_N^{(2)}] \, d\xi + \int_{\mathbb{R}^2 \setminus B(z, N^\delta)} p(\sigma^2, z, \xi) \mathbb{E}_{z, \xi} [D_N^{(2)}] \, d\xi \\ &\leq |B(z, N^\delta)| \|p(\sigma^2, z, \cdot)\|_\infty \sup_{z, z' \in \mathbb{R}^2} \mathbb{E}_{z, z'} [D_N^{(2)}] + \sup_{\|z - z'\| \geq N^\delta} \mathbb{E}_{z, z'} [D_N^{(2)}] \\ &\leq C \frac{N^{-2 + \varepsilon' + 2\delta}}{\sigma^2} + C e^{-\frac{1}{17} N^{2\delta}} \\ &\leq C' \frac{N^{-2 + \varepsilon}}{\sigma^2}. \end{aligned}$$

For $\sigma \geq 1$, we can directly replace σ^2 by $\sigma^2 + 1$ and C' by $2C'$. For $\sigma \leq 1$, the third statement directly reduces to the first one, so we can again replace σ^2 with $\sigma^2 + 1$. \square

Proof of Sublemma 2.4. Once again, the first case is trivial. For the third one, we remark that $f_N^{(2)}(z_1, z_2) \leq f_N(\max(\|z_1\|, \|z_2\|))$ and we apply the third case in Lemma 2.2.

The second case is a bit subtler than the one in lemma 2.2, because we had a turnkey inequality which is not sufficient to conclude here. The very rough idea is that if B must turn N times around z_1 and N times around z_2 , then there is a time S when it has turned $\frac{N}{2}$ times around one of them (say, z_1), but still has to turn $\frac{N}{2}$ times around the other one. In order to do the first part of that, it has to go close to z_1 , and then go close to z_2 . Both conditions have a small probability to be satisfied (this is the content of Lemma 2.2). Besides, they are formulated in a way such that one depends on B before S , whilst the other depends on B after S . This permits us to benefit from the Markov property.

We introduce the rays $d^1 = z_1 + \mathbb{R}_+(z_1 - z_2)$ and $d^2 = z_2 + \mathbb{R}_+(z_2 - z_1)$. In words, d^1 starts at z_1 and goes away from z_2 , whilst d^2 starts at z_2 and goes away from z_1 .

We also define the following stopping times, with the convention that the infimum of an empty set is equal to 1, and with $i \in \{1, 2\}$.

$$\begin{aligned} S^i &= \inf\{u \in [0, 1] : X_u \in d^i, |\theta_{B_{[0,u]}}(z_i)| = \lfloor N/2 \rfloor\} \\ S &= \min(S^1, S^2) \\ U &= \begin{cases} \inf\{u \geq S : |\theta_{B_{[S,u]}}(z_1)| = \lfloor N/2 \rfloor\} & \text{if } S = S^2 \\ \inf\{u \geq S : |\theta_{B_{[S,u]}}(z_2)| = \lfloor N/2 \rfloor\} & \text{if } S = S^1. \end{cases} \end{aligned}$$

In words, we wait for B to wind $N/2$ times around either z_1 or z_2 (whatever happens first), and then we wait for it to wind $N/2$ more times around the other point.

The event $\{|\theta_B(z_1)| \geq N \text{ and } |\theta_B(z_2)| \geq N\}$ is then included in the event $\{U < 1\}$ (up to the negligible event $B_1 \in d^1 \cup d^2$).

We write $\theta^*(s, z) = \max_{u \in [0, s]} (\theta_{B_{[0,u]}}(z))$. In particular, we have the following properties:

- The function $s \mapsto \theta^*(s, z)$ is increasing.
- For all $z \in \mathbb{R}^2$, under \mathbb{P}_0 , we have the equality in law $\theta^*(1, z) = \theta^*(\|z\|^{-2}, 1)$, where we write 1 for the vector $(1, 0) \in \mathbb{R}^2$.

Using the Markov property between the first and the second line, we have:

$$\begin{aligned} f_N^{(2)}(z_1, z_2) &\leq \mathbb{P}_0(S = S^1 \text{ and } U < 1) + \mathbb{P}_0(S = S^2 \text{ and } U < 1) \\ &\leq \mathbb{P}_0(\theta_B^*(1, z_1) \geq \lfloor N/2 \rfloor) \sup_{y \in d^1} \mathbb{P}_y(\theta^*(1, z_2) \geq \lfloor N/2 \rfloor) \\ &\quad + \mathbb{P}_0(\theta_B^*(1, z_2) \geq \lfloor N/2 \rfloor) \sup_{y \in d^2} \mathbb{P}_y(\theta^*(1, z_1) \geq \lfloor N/2 \rfloor) \\ &= \mathbb{P}_0(\theta^*(\|z_1\|^{-2}, 1) \geq \lfloor N/2 \rfloor) \mathbb{P}_0(\theta^*(\|z_2 - z_1\|^{-2}, 1) \geq \lfloor N/2 \rfloor) \\ &\quad + \mathbb{P}_0(\theta^*(\|z_2\|^{-2}, 1) \geq \lfloor N/2 \rfloor) \mathbb{P}_0(\theta^*(\|z_2 - z_1\|^{-2}, 1) \geq \lfloor N/2 \rfloor). \end{aligned}$$

We use again the inequality (11), which implies that there exists a constant $C > 0$ such that for all $z \in \mathbb{R}^2 \setminus B(0, N^{-\delta})$, for all $N \geq 1$,

$$\mathbb{P}_0(\theta^*(\|z\|^{-2}, 1) \geq \lfloor N/2 \rfloor) \leq C \frac{\log(N)}{N}.$$

Outside $T_{\beta, \delta}$, the condition is satisfied for $z = z_1$, for $z = z_2$, and for $z = z_2 - z_1$, so that for all $(z_1, z_2) \notin T_{\beta, \delta}$,

$$f_N(z_1, z_2) \leq C^2 \log(N)^2 N^{-2}.$$

This finishes the proof. □

Proof of Lemma 2.3. : By translation we can assume $z' = 0$. Then:

$$\begin{aligned}
\mathbb{E}_{z,0} \left[(D_N^{(2)})^2 \right] &= \mathbb{E}_{z,0} \left[|\{\xi \in \mathbb{C} : |\theta_B(\xi)| \geq N \text{ and } |\theta_{B'}(\xi)| \geq N\}|^2 \right] \\
&= \mathbb{E}_{z,0} \left[\int_{(\mathbb{R}^2)^2} \mathbb{1}_{|\theta_B(\xi)| \geq N} \mathbb{1}_{|\theta_{B'}(\xi)| \geq N} \mathbb{1}_{|\theta_B(\xi')| \geq N} \mathbb{1}_{|\theta_{B'}(\xi')| \geq N} d\xi d\xi' \right] \\
&= \int_{(\mathbb{R}^2)^2} \mathbb{P}_0 (|\theta_B(\xi - z)| \geq N, |\theta_{B'}(\xi' - z)| \geq N) \mathbb{P}_0 (|\theta_B(\xi)| \geq N, |\theta_{B'}(\xi')| \geq N) d\xi d\xi' \\
&= \int_{(\mathbb{R}^2)^2} f_N^{(2)}(\xi - z, \xi' - z) f_N^{(2)}(\xi, \xi') d\xi d\xi'.
\end{aligned}$$

Here again, we distinguish the three cases.

First case. Using the Cauchy–Schwarz inequality and the sublemma, we find

$$\begin{aligned}
&\int_{(\mathbb{R}^2)^2} f_N^{(2)}(\xi - z, \xi' - z) f_N^{(2)}(\xi, \xi') d\xi d\xi' \\
&\leq \int_{(\mathbb{R}^2)^2} \left(f_N^{(2)}(\xi, \xi') \right)^2 d\xi d\xi' \\
&\leq C^2 \int_{(\mathbb{R}^2)^2 \setminus B(0, N^\delta/2)^2} e^{-\frac{1}{4} \max(\|\xi\|, \|\xi'\|)^2} d\xi d\xi' + C^2 \text{Vol}(B(0, N^\delta/2)^2) \log(N)^4 N^{-4} + \text{Vol}(T_{\beta, \delta}) \\
&\leq C' e^{-\frac{1}{17} N^{2\delta}} + C' \log(N)^4 N^{-4+4\delta} + C' N^{2\delta-2\beta}.
\end{aligned}$$

We take $0 < \delta < \frac{\varepsilon}{8}$ and $\beta > 2 + \delta - \frac{\varepsilon}{2}$ to obtain the desired result.

Second case. We assume that $\|z\| \geq N^\delta$.

$$\begin{aligned}
&\int_{(\mathbb{R}^2)^2} f_N^{(2)}(\xi - z, \xi' - z) f_N^{(2)}(\xi, \xi') d\xi d\xi' \\
&\leq \int_{B(0, N^\delta/2) \times \mathbb{R}^2} f_N^{(2)}(\xi - z, \xi' - z) d\xi d\xi' + \int_{(\mathbb{R}^2 \setminus B(0, N^\delta/2)) \times \mathbb{R}^2} f_N^{(2)}(\xi, \xi') d\xi d\xi' \\
&\leq C N^{2\delta} \int_{\mathbb{C}} e^{-\frac{1}{4} \max(\frac{1}{4} N^{2\delta}, \|\xi'\|^2)} d\xi' + \int_{\mathbb{R}^2 \setminus B(0, N^\delta/2) \times \mathbb{R}^2} e^{-\frac{1}{4} \max(\|\xi\|, \|\xi'\|)^2} d\xi d\xi' \\
&\leq C' e^{-\frac{1}{17} N^{2\delta}}.
\end{aligned}$$

Third case. We use the two first cases with some $\varepsilon' < \varepsilon$, and we set $\delta = \frac{\varepsilon - \varepsilon'}{2}$. Then,

$$\begin{aligned}
\mathbb{E}_{0, \mathcal{N}(0, \sigma^2)} \left[(D_N^{(2)})^2 \right] &\leq \int_{B(0, N^\delta)} p(\sigma^2, 0, \xi) \mathbb{E}_{0, \xi} \left[(D_N^{(2)})^2 \right] d\xi + \int_{\mathbb{R}^2 \setminus B(0, N^\delta)} p(\sigma^2, 0, \xi) \mathbb{E}_{0, \xi} \left[(D_N^{(2)})^2 \right] d\xi \\
&\leq |B(z, N^\delta)| \|p(\sigma^2, 0, \cdot)\|_\infty \sup_{z \in \mathbb{R}^2} \mathbb{E}_{0, z} \left[(D_N^{(2)})^2 \right] + \sup_{\|z\| \geq N^\delta} \mathbb{E}_{0, z} \left[(D_N^{(2)})^2 \right] \\
&\leq C \frac{N^{-4+\varepsilon'+2\delta}}{\sigma^2} + C e^{-\frac{1}{17} N^{2\delta}} \\
&\leq C' \frac{N^{-4+\varepsilon}}{\sigma^2}.
\end{aligned}$$

For $\sigma \geq 1$, we can directly replace σ^2 by $\sigma^2 + 1$ and C' by $2C'$. For $\sigma \leq 1$, the third statement directly reduces to the first one, so we can again replace σ^2 with $\sigma^2 + 1$. \square

3. ASYMPTOTICS IN L^2

3.1. Asymptotic of the mean. Recall that D_N denotes the area of the set of points z for which the winding $\theta_B(z)$ is at least N . Our temporary goal is to obtain a nice bound on the quantity $\mathbb{E}[(ND_N - \frac{1}{2\pi})^2]$, which is already known to converge to 0 as $N \rightarrow +\infty$. We first show the following asymptotic, which is a reformulation of Lemma 1.2.

Lemma 3.1. *As N tends to infinity,*

$$\mathbb{E}[D_N] = \frac{1}{2\pi N} + O(N^{-2}). \quad (14)$$

Proof. We use the complex coordinate z on the real plane \mathbb{R}^2 , and we denote by $\tilde{\theta}(z)$ the value at time 1 of the continuous determination of the angle of B around z , initialized to be 0 at time 0. It will be convenient here to study $\tilde{\theta}$ instead of the integer-valued winding number θ that we use in the rest of the paper. The reason is that, for a given z , explicit formulas are known about the law of $\tilde{\theta}(z)$. For any point $z \in \mathbb{R}^2$ for which $\theta_B(z)$ is well-defined, the quantities $\theta_B(z)$ and $\tilde{\theta}(z)$ are related by the bound

$$|\theta_B(z) - \frac{1}{2\pi}\tilde{\theta}(z)| \leq \frac{1}{2}. \quad (15)$$

It is also convenient to eliminate from our analysis the points on the plane which are very close from the starting point of our trajectory. Indeed, such points have a higher probability to have a large winding number. It is thus more convenient to simply bound this probability by 1 than to try to control this high probability. We thus introduce, for $N \geq 1$, the set

$$\tilde{\mathcal{D}}_N = \{z \in \mathbb{C} \setminus B(0, e^{-N}) : \tilde{\theta}(z) \geq 2\pi N\},$$

of which we denote the Lebesgue measure by \tilde{D}_N .

From (15), we deduce the following inclusions:

$$\tilde{\mathcal{D}}_{N+2} \subseteq \mathcal{D}_{N+1} \subseteq \tilde{\mathcal{D}}_N \cup B(0, e^{-N}) \subseteq \mathcal{D}_{N-1} \cup B(0, e^{-N}).$$

In particular,

$$\mathbb{E}[D_N] = \frac{1}{2\pi N} + O(N^{-2}) \iff \mathbb{E}[\tilde{D}_N] = \frac{1}{2\pi N} + O(N^{-2}).$$

We prove the right-hand side. First, we give an integral representation of the quantity $\mathbb{E}[\tilde{D}_N]$. We denote by I_0 the modified Bessel function of the first kind with parameter 0. The single thing about this function that we will need is the inequality $I_0(x) \geq 1$ for $x \geq 0$. Then, for $\theta > \pi$ and $\rho, r \geq 0$, Mansuy and Yor showed in [10] (Theorem 5.2) the following equality:

$$\mathbb{P}_r(\tilde{\theta}(0) \geq \theta \mid |B_1| = \rho) = \frac{1}{2\pi^2 I_0(r\rho)} \int_{\theta-\pi}^{\theta+\pi} \int_0^\infty e^{-r\rho \cosh(t)} \frac{x}{x^2 + t^2} dt dx. \quad (16)$$

Here of course, the conditioning corresponds to the disintegration with respect to the (continuous) density of B_1 . By integrating back with respect to ρ (with the appropriate density), we obtain:

$$\mathbb{P}_r(\tilde{\theta}(0) \geq \theta) = \frac{1}{2\pi^2} \int_0^\infty \frac{\rho}{I_0(r\rho)} \int_0^{2\pi} p_1(r, \rho e^{i\phi}) d\phi \int_{\theta-\pi}^{\theta+\pi} \int_0^\infty e^{-r\rho \cosh(t)} \frac{x}{x^2 + t^2} dt dx d\rho.$$

Using the invariance of the Brownian motion with respect to translation, we have

$$\begin{aligned}
\mathbb{E}_0[\tilde{D}_N] &= \int_{\mathbb{C}^2 \setminus B(0, e^{-N})} \mathbb{P}_0(\tilde{\theta}(z) \geq 2\pi N) \, dz \\
&= 2\pi \int_{e^{-N}}^{\infty} r \mathbb{P}_r(\tilde{\theta}(0) \geq 2\pi N) \, dr \\
&= \frac{1}{\pi} \int_{e^{-N}}^{\infty} r \int_0^{\infty} \frac{\rho}{I_0(r\rho)} \int_0^{2\pi} p_1(r, \rho e^{i\phi}) \, d\phi \int_{(2N-1)\pi}^{(2N+1)\pi} \int_0^{\infty} e^{-r\rho \cosh(t)} \frac{x}{x^2 + t^2} \, dt \, dx \, d\rho \, dr. \quad (17)
\end{aligned}$$

Roughly speaking, on the asymptotic regime $N \rightarrow +\infty$, we have $x \simeq 2\pi N \rightarrow +\infty$ and we expect that $\frac{x}{x^2+t^2} \simeq \frac{1}{x}$. We also expect that the bound e^{-N} can freely be replaced with 0. The multiple integral (17) then decouples into

$$A \int_{(2N-1)\pi}^{(2N+1)\pi} \frac{dx}{x} \quad (18)$$

where

$$A = \frac{1}{\pi} \int_0^{\infty} r \int_0^{\infty} \frac{\rho}{I_0(r\rho)} \int_0^{2\pi} p_1(r, \rho e^{i\phi}) \, d\phi \int_0^{\infty} e^{-r\rho \cosh(t)} \, dt \, d\rho \, dr.$$

The reader might be puzzled by the fact we first introduced a kind of ‘‘cutoff’’ e^{-N} in the definition of \tilde{D}_N to then remove it with computations. The thing is we remove the cutoff *after* we replace $\frac{x}{x^2+t^2}$ with $\frac{1}{x}$. The rest of this proof consists on a lengthy but elementary computation to show that the difference between (17) and (18) is a $O(N^{-2})$.

We denote by I_N the right-hand side of (17) but with $\frac{1}{x^2+t^2}$ replaced by $\frac{1}{x^2}$. Set $\delta_N = I_N - \mathbb{E}_0[\tilde{D}_N]$. That is,

$$\delta_N = \frac{1}{\pi} \int_{e^{-N}}^{\infty} r \int_0^{\infty} \frac{\rho}{I_0(r\rho)} \int_0^{2\pi} p_1(r, \rho e^{i\phi}) \, d\phi \int_{(2N-1)\pi}^{(2N+1)\pi} \int_0^{\infty} e^{-r\rho \cosh(t)} \frac{t^2}{x(x^2 + t^2)} \, dt \, dx \, d\rho \, dr.$$

Observe that $\delta_N \geq 0$. We decompose δ_N as $\delta_N^1 + \delta_N^2$ by splitting the first integral, with respect to r , at $r = 1$:

$$\delta_N^1 = \frac{1}{\pi} \int_{e^{-N}}^1 \dots \, dr \quad \text{and} \quad \delta_N^2 = \frac{1}{\pi} \int_1^{\infty} \dots \, dr.$$

To estimate δ_N^1 , we use the bounds

$$p_1(x, y) \leq p_1(0, 0); \quad \frac{1}{x^2 + t^2} \leq \frac{1}{x^2}; \quad \int_{(2N-1)\pi}^{(2N+1)\pi} \frac{dx}{x^3} \leq \frac{2\pi}{((2N-1)\pi)^3}; \quad I_0(r\rho) \geq 1.$$

Integrating then with respect to ρ , t and finally r , we obtain

$$\begin{aligned}
\delta_N^1 &\leq \frac{2\pi}{\pi} p_1(0, 0) \frac{2\pi}{((2N-1)\pi)^3} \int_{e^{-N}}^1 r \int_0^{\infty} \rho \int_0^{\infty} e^{-r\rho \cosh(t)} t^2 \, dt \, d\rho \, dr \\
&= O(N^{-3}) \int_{e^{-N}}^1 r \int_0^{\infty} \frac{t^2}{(r \cosh(t))^2} \, dt \, dr \\
&= O(N^{-2}).
\end{aligned}$$

To estimate δ_N^2 , we also use the bounds $e^{-r\rho\cosh(t)} \leq e^{-\rho\cosh(t)}$ (for $r \geq 1$) and $\int_{\mathbb{C}} p_t(x, y) dy = 1$. We then obtain

$$\delta_N^2 \leq \frac{2}{(2\pi(2N-1))^3} \int_0^\infty \rho \int_0^\infty e^{-\rho\cosh(t)} t^2 dt d\rho.$$

Computing first the integral on ρ , we obtain

$$\delta_N^2 = O(N^{-3}) \int_0^\infty \frac{t^2}{\cosh(t)^2} dt.$$

The remaining integral is clearly finite. We conclude that $\delta_N = O(N^{-2})$, that is, $\mathbb{E}_0[\tilde{D}_N] = I_N + O(N^{-2})$. We now wish to eliminate the cutoff, that is to replace I_N with $I_N + J_N$ where

$$J_N = \int_0^{e^{-N}} \dots dr.$$

Remark that $I_N + J_N$ is, as we hoped, exactly the right-hand side of (18). Inverting the integrals on the definition of J , we can write it

$$J_N = \int_0^\infty \dots d\rho.$$

and we then split the integral with respect to ρ at $\rho = 1$. We thus define

$$\begin{aligned} J_N^1 &= \int_0^{e^{-N}} \int_0^1 \frac{r\rho}{I_0(r\rho)} \int_0^{2\pi} p_1(r, \rho e^{i\phi}) d\phi \int_0^\infty e^{-r\rho\cosh(t)} dt d\rho dr \\ J_N^2 &= \int_0^{e^{-N}} \int_1^\infty \frac{r\rho}{I_0(r\rho)} \int_0^{2\pi} p_1(r, \rho e^{i\phi}) d\phi \int_0^\infty e^{-r\rho\cosh(t)} dt d\rho dr, \end{aligned}$$

and we have

$$J_N = (J_N^1 + J_N^2) \log \frac{2N+1}{2N-1}.$$

We will use the following estimation: there exists some finite C such that for any $\rho \in (0, \frac{1}{\cosh(1)})$,

$$f(\rho) = \int_0^{+\infty} \frac{1 - (1 + \rho \cosh(t))e^{-\rho \cosh(t)}}{\rho^2 \cosh(t)^2} dt \leq C + 2 \ln(\rho^{-1}). \quad (19)$$

We use now this inequality and we postpone its computation to the end of the proof. Using the facts that p_1 is maximal at $(0, 0)$, and that $I_0 \geq 1$, we have

$$\begin{aligned}
J_N^1 &\leq 2\pi p_1(0, 0) \int_0^{e^{-N}} \int_0^1 r \rho \int_0^\infty e^{-r\rho \cosh(t)} dt d\rho dr \\
&\leq \int_0^{e^{-N}} \int_0^r u \int_0^\infty e^{-u \cosh(t)} dt \frac{du}{r} dr \quad (u = r\rho) \\
&= \int_0^{e^{-N}} \int_0^\infty \frac{1 - (1 + r \cosh(t))e^{-r \cosh(t)}}{r \cosh(t)^2} dt dr \quad (\text{computing the integral on } u). \\
&= \int_0^{e^{-N}} r f(r) dr \\
&\leq \int_0^{e^{-N}} r(C + 2 \log(r^{-1})) dr \\
&= O(Ne^{-2N}).
\end{aligned}$$

For J_N^2 , since $\rho \geq 1$ inside the integral, we can bound $e^{-r\rho \cosh(t)}$ by $e^{-r \cosh(t)}$. Then, we have

$$\begin{aligned}
J_N^2 &\leq \int_0^{e^{-N}} r \left[\int_0^{2\pi} \int_0^\infty p_1(r, \rho e^{i\phi}) \rho d\rho d\phi \right] \int_0^\infty e^{-r \cosh(t)} dt dr \\
&= \int_0^{e^{-N}} \int_0^\infty r e^{-r \cosh(t)} dt dr \\
&= \int_0^\infty \frac{1 - (1 + e^{-N} \cosh(t))e^{e^{-N} \cosh(t)}}{\cosh(t)^2} dt \quad (\text{computing the integral on } r) \\
&= e^{-2N} f(e^{-N}) \\
&= O(Ne^{-2N}).
\end{aligned}$$

From this, we deduce in particular that $J_N = O(N^{-2})$ and finally that

$$\mathbb{E}[D_N] = A \int_{(2N-1)\pi}^{(2N+1)\pi} \frac{dx}{x} + O(N^{-2}) = \frac{A}{N} + O(N^{-2}).$$

Wiener's estimate on D_N gives $\mathbb{E}[D_N] = \frac{1}{2\pi N} + o(\frac{1}{N})$. It follows that $A = \frac{1}{2\pi}$ and that

$$\mathbb{E}[D_N] = \frac{1}{2\pi N} + O(N^{-2}).$$

To conclude the proof, we only need to show the inequality (19). It is easily proven that, for any $\rho, t \geq 0$,

$$0 \leq 1 - (1 + \rho \cosh(t))e^{-\rho \cosh(t)} \leq \rho^2 \cosh(t)^2.$$

For $t \geq 1$, we also have $2 \sinh(t) \geq \cosh(t)$. Thus, we have

$$\begin{aligned}
f(\rho) &= \int_0^1 \frac{1 - (1 + \rho \cosh(t))e^{-\rho \cosh(t)}}{\rho^2 \cosh(t)^2} dt + \int_1^{+\infty} \frac{1 - (1 + \rho \cosh(t))e^{-\rho \cosh(t)}}{\rho^2 \cosh(t)^2} dt \\
&\leq 1 + \int_1^{+\infty} \frac{1 - (1 + \rho \cosh(t))e^{-\rho \cosh(t)}}{\rho^2 \cosh(t)^2} \frac{2\rho \sinh(t)}{\rho \cosh(t)} dt \\
&= 1 + 2 \int_{\rho \cosh(1)}^{+\infty} \frac{1 - (1 + u)e^{-u}}{u^3} du \quad (u = \rho \cosh(t)) \\
&= 1 + 2 \int_{\rho \cosh(1)}^1 \frac{1 - (1 + u)e^{-u}}{u^3} du + 2 \int_1^{+\infty} \frac{1 - (1 + u)e^{-u}}{u^3} du.
\end{aligned}$$

Because of the exponential decay, the last integral is finite. Using $e^{-u} \geq 1 - u$, we then obtain, for some finite C ,

$$\begin{aligned}
f(\rho) &\leq C + 2 \int_{\rho \cosh(1)}^1 \frac{1 - (1 + u)(1 - u)}{u^3} du \\
&= C + 2 \int_{\rho \cosh(1)}^1 \frac{1}{u} du \\
&= C + 2 \ln(\rho^{-1} \cosh(1)) \\
&= C' + 2 \ln(\rho^{-1}).
\end{aligned}$$

This is the announced inequality. \square

Remark 3.2. *If we replace the Brownian motion with a Brownian loop, the integrals on ϕ and ρ disappear (it is the case $\rho = 0$), and the Bessel function reduces to 1. In that case, Garban and Ferreras obtained in [7, Theorem 5.2] the exact value*

$$\mathbb{E}_{x,x,1}[A_N] = \frac{1}{2\pi N^2}.$$

Their computation also uses the explicit expression (16) given by Mansuy and Yor. They then compute the integrals by performing a residue computation.

Remark 3.3. *With a much simpler computation, we also obtain, for every $z \neq 0$ the estimate $\mathbb{P}_0(\theta_B(z) \geq N) = \frac{C_z}{N} + O(N^{-2})$. Our estimate (14) does not follow from this simpler estimation, since the remainder is not uniform near $z = 0$.*

3.2. Decomposition into small pieces. Let $\gamma : [0, 1] \rightarrow \mathbb{R}^2$ be continuous, and such that its range has vanishing Lebesgue measure. We will introduce a decomposition that allows us to relate the large winding set for γ with the large winding sets of different pieces of γ . Two inclusions are obtained here, and will be used again many times during the paper. Though we show them in a general framework, we first explain briefly how we will use them.

We fix three positive *large* integers N, M and T , such that $T(M + 1) < N$. Typically, T and M will be (the integer part of) some fractional power of N . The integer T will be the number of pieces we cut the curve into: we will write γ as the concatenation $\gamma^1 \cdots \gamma^T$, where γ^i is the restriction of γ to the interval $[\frac{i-1}{T}, \frac{i}{T}]$. Using the self-similarity of the Brownian motion,

the inclusions will induce inequalities in distribution satisfied by the large winding set of the Brownian motion. The integer M is used as a barrier between two different situations. Basically, we want it to be large enough that one can asymptotically neglect the set of points around which two different pieces both wind at least M times, but small enough that the relation $T(M+1) < N$ holds. We invite the reader to always keep in mind the idea that when a Brownian path winds a lot around some point, only a small piece of the path is responsible for almost all of these windings.

We now start a rigorous reasoning. Let us introduce some notations. Let N, M, T be three integers such that $T(M+1) < N$. Set $0 = t_0 < \dots < t_T = 1$. Set also γ^i the restriction of γ to $[t_{i-1}, t_i]$. We denote by γ^{pl} the piecewise linear curve with interpolation times t_0, \dots, t_T . That is, for $i \in \{1, \dots, T\}$ and $u \in [0, 1]$,

$$\gamma^{pl}(t_{i-1} + u(t_i - t_{i-1})) = \gamma(t_{i-1}) + u(\gamma(t_i) - \gamma(t_{i-1})).$$

We denote the function θ_{γ^i} by θ_i . The following equality of measurable functions holds almost everywhere:

$$\theta_\gamma = \theta_{\gamma^{pl}} + \sum_{i=1}^T \theta_i.$$

It actually holds pointwise at any point $z \in \mathbb{R}^2$ which does not lie on the range of γ , nor on the range of γ^{pl} , nor on the segment between γ_0 and γ_1 . We fix such a z , and we assume that it satisfies $\theta_\gamma(z) \geq N$.

It is easy to see that $|\theta_{\gamma^{pl}}|$ is bounded by $\frac{T}{2}$, hence by T . This implies $\sum_{i=1}^T \theta_i(z) \geq N - T$. Discussing whether the second highest value of $\{\theta_i(z) : i \in \{1, \dots, T\}\}$ is more or less than M , we have two cases. Either there exist two integers $i, j \in \{1, \dots, T\}, i \neq j$ such that

$$\theta_i(z) \geq M \quad \text{and} \quad \theta_j(z) \geq M,$$

or there exists an integer $i \in \{1, \dots, T\}$ such that $\theta_i(z) \geq N - T - (T-1)M$.

This gives us a condition implied by $\theta_\gamma(z) \geq N$. Similarly, we show that $\theta_\gamma(z) \geq N$ is implied by the condition that there exists one integer $i \in \{1, \dots, T\}$ such that $\theta_i(z) \geq N + T + (T-1)M$, and that there is no pair of integers $i, j \in \{1, \dots, T\}, i \neq j$ such that

$$|\theta_i(z)| \geq M \quad \text{and} \quad |\theta_j(z)| \geq M.$$

We introduce the following notation, for all positive integers N and M (with implicit dependency on T and on the decomposition into T pieces):

$$\mathcal{D}_N(\gamma) = \{z \in \mathbb{R}^2 : \theta_\gamma(z) \geq N\}$$

$$\mathcal{D}_N^i(\gamma) = \{z \in \mathbb{R}^2 : \theta_i(z) \geq N\}$$

$$\mathcal{D}_M^{i,j}(\gamma) = \{z \in \mathbb{R}^2 : |\theta_i(z)| \geq M \text{ and } |\theta_j(z)| \geq M\}.$$

The same notation without γ means that we take $\gamma = B$, a standard Brownian motion from $[0, 1]$ to \mathbb{R}^2 , with $t_i = \frac{i}{T}$ for $i \in \{0, \dots, T\}$.

Then, our previous reasoning translates into the following inclusions:

$$\bigcup_{i=1}^T \mathcal{D}_{N+T+M(T-1)}^i(\gamma) \setminus \bigcup_{1 \leq i < j \leq T} \mathcal{D}_M^{i,j}(\gamma) \subseteq \mathcal{D}_N(\gamma) \subseteq \bigcup_{i=1}^T \mathcal{D}_{N-T-M(T-1)}^i(\gamma) \cup \bigcup_{1 \leq i < j \leq T} \mathcal{D}_M^{i,j}(\gamma). \quad (20)$$

Using the inclusion $\mathcal{D}_{N+T+M(T-1)}^i(\gamma) \cap \mathcal{D}_{N+T+M(T-1)}^j(\gamma) \subseteq \mathcal{D}_M^{i,j}(\gamma)$ and the Bonferroni's inequalities, we have corresponding inequalities at the level of measures:

$$\sum_{i=1}^T D_{N+T+M(T-1)}^i(\gamma) - \sum_{1 \leq i < j \leq T} D_M^{i,j}(\gamma) \leq D_N(\gamma) \leq \sum_{i=1}^T D_{N-T-M(T-1)}^i(\gamma) + \sum_{1 \leq i < j \leq T} D_M^{i,j}(\gamma). \quad (21)$$

Let us remark that in this last step, we could have use any measure μ instead of the Lebesgue measure, under the sole conditions that $\mu(\text{Range}(\gamma)) = \mu(\text{Range}(\gamma^{pl})) = 0$.

3.3. Asymptotic for the second moment. We now prove Lemma 1.3 about the second moment of D_N . Let us first state it again.

Lemma 3.4. *For all $\delta \in (0, \frac{1}{3})$, there exists $C \geq 0$ such that for all $N \geq 1$,*

$$N^{2\delta} \text{Var}[ND_N] \leq C.$$

Proof. We write $d_N = \mathbb{E}[D_N]$, and x_+ (resp. x_-) for the positive (resp. negative) part of a real number x . We also set $N^+ = N + T + M(T-1)$, and $N^- = N - T - M(T-1)$. The value of M and T will be given later on by $M = \lfloor N^m \rfloor$ and $T = \lfloor N^t \rfloor$ for some positive exponents m, t , so that N^+ and N^- only depends on N . Besides, the exponents are assumed to satisfy $m + t < 1$ so that

$$N^+ \underset{N \rightarrow +\infty}{\sim} N \underset{N \rightarrow +\infty}{\sim} N^-.$$

We know from (21) that

$$D_N \leq \sum_{i=1}^T D_{N^+}^i + \sum_{1 \leq i < j \leq T} D_M^{i,j}$$

so that

$$N(D_N - d_N) \leq N \sum_{i=1}^T (D_{N^+}^i - \frac{d_{N^+}}{T}) + N(d_{N^+} - d_N) + N \sum_{1 \leq i < j \leq T} D_M^{i,j}.$$

Taking positive parts, squares, and expectations, and using the identity

$$(a + b + c)^2 \leq 3(a^2 + b^2 + c^2),$$

we obtain

$$\begin{aligned} \mathbb{E}_x \left[(N(D_N - d_N)_+)^2 \right] &\leq 3 \sum_{i=1}^T \text{Var}[ND_{N^+}^i] + 3N^2(d_N - d_{N^+})^2 + 3N^2 \mathbb{E} \left[\left(\sum_{1 \leq i < j \leq T} D_M^{i,j} \right)^2 \right] \\ &\leq 3T^{-1} \text{Var}[ND_{N^+}] + 3N^2(d_N - d_{N^+})^2 + 3N^2 \mathbb{E} \left[\left(\sum_{1 \leq i < j \leq T} D_M^{i,j} \right)^2 \right]. \end{aligned}$$

For the second inequality, we used the fact that the variables $D_{N^+}^i$ are i.i.d., and distributed as $T^{-1}D_{N^+}$. This in turn follows directly from the Markov property, and scale and translation invariance of the Brownian motion. The apparently insignificant T^{-1} factor that appears here is actually the core of the proof: the sum of the fluctuations of the $D_{N^+}^i$ is of lesser order than the sum of the absolute values of these fluctuations. This is why D_N itself has very small fluctuations.

We now fix some $\varepsilon > 0$. Using the preliminary lemma 2.3, with the scaling $B \leftrightarrow T^{-1}B$, we have the following inequalities, for some constant C, C' :

$$\begin{aligned} \mathbb{E}_x \left[\left(\sum_{1 \leq i < j \leq T} D_M^{i,j} \right)^2 \right] &\leq T^2 \sum_{1 \leq i < j \leq T} \mathbb{E}_x [(D_M^{i,j})^2] \\ &\leq T^2 \sum_{1 \leq i < j \leq T} \frac{T^{-2} C M^{-4+\varepsilon}}{j-i} \\ &\leq C' M^{-4+\varepsilon} T \log(T). \end{aligned} \quad (22)$$

To control nicely $d_N - d_{N^-}$, we summon the convergence shown by W. Werner:

$$N^2 |\{z \in \mathbb{R}^2 : \theta_B(z) = N\}| \xrightarrow{L^2} \frac{1}{2\pi}. \quad (23)$$

In particular, this implies that the family $n^2(d_{n+1} - d_n)$ is bounded, so that

$$|d_N - d_{N^-}| \leq \sum_{n=N^-}^N \frac{C}{n^2} \leq \frac{C'(N - N^-)}{NN^-} \underset{N \rightarrow +\infty}{\sim} C' M T N^{-2}.$$

We do the same computation for the negative part, and we obtain the existence of some C such that

$$\text{Var}[ND_N] \leq C(T^{-1}(\text{Var}[ND_{N^-}] + \text{Var}[ND_{N^+}]) + \frac{M^2 T^2}{N^2} + N^2 M^{-4+\varepsilon} T \log(T)). \quad (24)$$

Using now the explicit expressions of M and T , we obtain, for $m, t \in (0, 1)$ such that $m + t < 1$, and $\varepsilon > 0$ arbitrary:

$$\text{Var}[ND_N] \leq C(N^{-t}(\text{Var}[ND_{N^-}] + \text{Var}[ND_{N^+}]) + N^{-2+2m+2t+\varepsilon} + N^{2-4m+t+\varepsilon}). \quad (25)$$

If we have some asymptotic bound on $\text{Var}[D_N]$, we can now put it on the right-hand side of the equation and hope that it will lead to a *better* asymptotic bound: if the additional terms are small enough, we should gain a factor CN^{-t} . We will iterate this process, with different values for t and m at each iteration. We recursively define a sequence α by $\alpha_0 = 0$, $\alpha_{k+1} = \frac{5\alpha_k + 2}{8}$. We show the following:

Claim 3.5. *For all $k \in \mathbb{N}$, for all $\varepsilon > 0$, there exists C such that for all N , $\text{Var}[ND_N] \leq CN^{-\alpha_k + \varepsilon}$.*

For $k = 0$, this follows directly from (6).

Now assume this to be true for some k , and use (25) with $t = \frac{2-3\alpha_k}{8} > 0$, $m = \frac{4-t}{6} - \varepsilon > 0$:

$$\begin{aligned} \text{Var}[ND_N] &\leq C' N^{-\frac{2-3\alpha_k}{8} - \alpha_k + \varepsilon} + C' N^{2 + \frac{2-3\alpha_k}{8} - 4(\frac{4-t}{6} - \varepsilon) + \varepsilon} \log(N) \\ &\leq C'' N^{-\alpha_{k+1} + 6\varepsilon} \end{aligned}$$

which conclude the recursion. Since $\alpha_k \xrightarrow[k \rightarrow \infty]{} \frac{2}{3}$, we deduce Lemma 1.3. \square

4. FROM L^2 TO ALMOST SURE ESTIMATES

Our goal in this section is to go from the asymptotic estimation of D_N in L^2 to an asymptotic estimation of D_N in the almost sure sense. We achieve this by inserting a supremum under the expectation. Then, the Bienaymé–Tchebychev inequality allows us to deduce an almost sure bound. We first show the following general maximal inequality, with assumptions suited to our purpose. As it is formulated, this lemma also allows one to work in L^p instead of L^2 .

Lemma 4.1. *Let $(D_N)_{N \in \mathbb{N}}$ be a random sequence which is almost surely decreasing and takes non-negative values. Assume that there exists $m \geq 0$, $\delta \in (0, 1)$ and $p > 1$ such that, for all N large enough,*

$$\mathbb{E}[|ND_N - m|^p] \leq N^{-\delta}.$$

Then, for $\delta' < \frac{p-1}{p}\delta$,

$$\mathbb{E}\left[\sup_{N \geq N_0} N^{\delta'} |ND_N - m|^p\right] \xrightarrow{N_0 \rightarrow \infty} 0.$$

Proof. Let $\omega \in \left(\frac{1}{\delta-\delta'}, \frac{p-1}{\delta'}\right)$. This set is non empty precisely when $\delta' < \frac{p-1}{p}\delta$. Let also $\tilde{\mathbb{N}} = \{\lfloor K^\omega \rfloor : K \in \mathbb{N}^*\}$. The main idea of the proof is to replace \mathbb{N} with $\tilde{\mathbb{N}}$ in the supremum. Since this set is ‘sparser’, we can then bound the supremum with a sum, and still get something finite. Of course, we then have to replace back $\tilde{\mathbb{N}}$ with \mathbb{N} . This is done by showing that ND_N varies slowly.

For $M \in \tilde{\mathbb{N}}$, let $s(M)$ be the successor of M in $\tilde{\mathbb{N}}$ (that is, the smallest element of $\tilde{\mathbb{N}}$ which is strictly larger than M). Then, for $N \in \mathbb{N}$, let N_- and N_+ be the two unique elements of $\tilde{\mathbb{N}}$ such that $N_- \leq N < N_+ = s(N_-)$.

Then, $N^{\frac{\delta'}{p}}(ND_N - m)$ is less than $N_+^{\frac{\delta'}{p}}(N_+D_{N_-} - m)$. We decompose this quantity into

$$N_-^{\frac{\delta'}{p}}(N_-D_{N_-} - m) + (N_+^{\frac{\delta'}{p}} - N_-^{\frac{\delta'}{p}})(N_-D_{N_-} - m) + N_+^{\frac{\delta'}{p}}(N_+ - N_-)D_{N_-}.$$

For $N_0 \in \tilde{\mathbb{N}}$, we obtain

$$\begin{aligned} \mathbb{E}\left[\max_{\substack{N \in \mathbb{N} \\ N \geq N_0}} \mathbb{1}_{ND_N - m \geq 0} N^{\delta'} (ND_N - m)^p\right] &\leq C_p \left(\mathbb{E}\left[\max_{\substack{M \in \tilde{\mathbb{N}} \\ M \geq N_0}} M^{\delta'} (MD_M - m)^p\right] \right. \\ &\quad \left. + \mathbb{E}\left[\max_{\substack{M \in \tilde{\mathbb{N}} \\ M \geq N_0}} (s(M))^{\frac{\delta'}{p}} - M^{\frac{\delta'}{p}})^p (MD_M - m)^p\right] + \mathbb{E}\left[\max_{\substack{M \in \tilde{\mathbb{N}} \\ M \geq N_0}} (s(M))^{\delta'} (s(M) - M)^p D_M^p\right] \right). \end{aligned}$$

The first term on the right-hand side is the one that we wanted in the first place: the same thing as our initial maximum, but with $\tilde{\mathbb{N}}$ instead of \mathbb{N} .

To bound the two other terms, let us remark that for any $\alpha \neq 0$, $s(M) \sim M$ and $s(M)^\alpha - M^\alpha \sim C_{\alpha,\omega} M^{\alpha - \frac{1}{\omega}}$ for some constant $C_{\alpha,\omega}$. The previous expression can then be reduced to

$$\begin{aligned} \mathbb{E} \left[\max_{\substack{N \in \tilde{\mathbb{N}} \\ N \geq N_0}} \mathbb{1}_{ND_N \geq m} N^{\delta'} (ND_N - m)^p \right] &\leq C' \left(\mathbb{E} \left[\max_{\substack{M \in \tilde{\mathbb{N}} \\ M \geq N_0}} M^{\delta'} (MD_M - m)^p \right] \right. \\ &\quad \left. + \mathbb{E} \left[\max_{\substack{M \in \tilde{\mathbb{N}} \\ M \geq N_0}} (M^{\delta' - \frac{p}{\omega}} (MD_M - m)^p) \right] + \mathbb{E} \left[\max_{\substack{M \in \tilde{\mathbb{N}} \\ M \geq N_0}} M^{\delta' + p - \frac{p}{\omega}} D_M^p \right] \right) \\ &\leq C'' \left(\mathbb{E} \left[\max_{\substack{M \in \tilde{\mathbb{N}} \\ M \geq N_0}} M^{\delta'} (MD_M - m)^p \right] + \mathbb{E} \left[\max_{\substack{M \in \tilde{\mathbb{N}} \\ M \geq N_0}} M^{\delta' + p - \frac{p}{\omega}} D_M^p \right] \right). \end{aligned} \quad (26)$$

Let us denote $K_0 = \lfloor N_0^{\frac{1}{\omega}} \rfloor$. Then,

$$\begin{aligned} \mathbb{E} \left[\max_{\substack{N \in \tilde{\mathbb{N}} \\ N \geq N_0}} N^{\delta'} |ND_N - m|^p \right] &\leq \sum_{\substack{N \in \tilde{\mathbb{N}} \\ N \geq N_0}} \mathbb{E} [N^{\delta'} |ND_N - m|^p] \\ &\leq \sum_{\substack{N \in \tilde{\mathbb{N}} \\ N \geq N_0}} N^{\delta' - \delta} \\ &\leq \sum_{\substack{K \in \mathbb{N} \\ K \geq K_0}} K^{\omega(\delta' - \delta)} \\ &\leq (N_0^{\frac{1}{\omega}})^{\omega(\delta' - \delta) - 1} (1 + o(1)) \quad (\text{since } \omega(\delta' - \delta) < -1) \\ &\leq N_0^{(\delta' - \delta) - \frac{1}{\omega}} (1 + o(1)). \end{aligned} \quad (27)$$

Replacing \mathbb{N} with $\tilde{\mathbb{N}}$ is necessary for the inequality from the second to the third line: the additional power ω makes the sum converge.

To control the last error term, we also need the following estimation:

$$\begin{aligned} \mathbb{E} \left[\max_{\substack{N \in \tilde{\mathbb{N}} \\ M \geq N_0}} N^{\delta' + p - \frac{p}{\omega}} D_N^p \right] &\leq \sum_{\substack{N \in \tilde{\mathbb{N}} \\ N \geq N_0}} N^{\delta' + p - \frac{p}{\omega}} \mathbb{E} [D_N^p] \\ &\leq C \sum_{\substack{N \in \tilde{\mathbb{N}} \\ N \geq N_0}} N^{\delta' - \frac{p}{\omega}} \\ &\leq C \sum_{\substack{K \in \mathbb{N} \\ K \geq K_0}} K^{\omega\delta' - p} \\ &\leq C' K_0^{\omega\delta' - p + 1}, \quad \text{since } \omega\delta' - p < -1. \end{aligned} \quad (28)$$

Putting (26), (27) and (28) together, we obtain

$$\mathbb{E}\left[\max_{\substack{N \in \mathbb{N} \\ N \geq N_0}} \mathbb{1}_{ND_N - m \geq 0} N^{\delta'} (ND_N - m)^p\right] \xrightarrow{N_0 \rightarrow +\infty} 0.$$

We show similarly that

$$\mathbb{E}\left[\max_{\substack{N \in \mathbb{N} \\ N \geq N_0}} \mathbb{1}_{ND_N - m \leq 0} N^{\delta'} (ND_N - m)^p\right] \xrightarrow{N_0 \rightarrow +\infty} 0,$$

which concludes the proof of the lemma. \square

Proposition 1.4 gives us the preliminary estimate needed to apply this lemma with $p = 2$ and $\delta \in (0, \frac{2}{3})$. We obtain the following bound.

Corollary 4.2. *Let $\delta \in (0, \frac{1}{6})$. Then, there exists a finite constant C such that*

$$\mathbb{E}\left[\max_{N \in \mathbb{N}^*} N^{2\delta} (ND_N - \frac{1}{2\pi})^2\right] \leq C.$$

By application of the Bienaymé–Tchebychev inequality, we immediately deduce the following.

Corollary 4.3. *Let $\delta \in (0, \frac{1}{6})$. Then, there exists a finite constant C_δ such that for all $\xi > 0$*

$$\mathbb{P}\left(\max_{N \in \mathbb{N}^*} N^\delta |ND_N - \frac{1}{2\pi}| \geq \frac{C_\delta}{\sqrt{\xi}}\right) \leq \xi.$$

The condition (C) is, in particular, almost surely satisfied by the winding measure μ_B . In other words, μ_B lies almost surely in the strong attraction domain of a Cauchy distribution. What remains to be done in order to prove Theorem 1.1 is the computation of the position parameter p_B of the limiting Cauchy distribution.

5. COMPUTATION OF THE POSITION PARAMETER

For the planar Brownian motion $B = (X, Y)$, define

$$\mathcal{A}_B = \int_0^1 X_t dY_t - \frac{X_1 + X_0}{2}(Y_1 - Y_0),$$

the Lévy area of B .

Our goal, in this section, is to show the equality between this Lévy area and the position parameter of the measure μ_B :

Lemma 5.1. *Let B be a planar Brownian motion. Then, almost surely,*

$$p_B = \mathcal{A}_B.$$

To prove this, we will first look at piecewise linear approximations of B . We will chose the dyadic approximations, since it is known that the integrals of $x dy$ along those approximations converge in the almost sure sense toward the stochastic integral $\int_0^1 X_t dY_t$. We compare ‘integral’ with ‘position parameter’ at the level of the approximations. We then show that there is no discontinuity of the sequence of position parameters when we pass to the limit. The situation here is the exact opposite of the one for the previous result: the non-vanishing of the scale parameter is only due to the small pieces between the Brownian path and its piecewise-linear

approximation. The position parameter, on the opposite, is very well approximated by the piecewise-linear approximation.

We will write Δ for the set of laws μ which lie on the strong attraction domain of a Cauchy law (that is, those which satisfy Condition (C) after normalization). We also denote by Δ the set of curves γ such that $\nu_\gamma \in \Delta$. The ambiguity will always be resolved by the context. For a given probability space $(\Omega, \mathcal{F}, \mathbf{P})$, we also set $\Delta(\Omega)$ the set of random variables on Ω whose distribution lies in Δ .

Before we proceed, we should warn the reader about the following facts, which might seem counter-intuitive: if Ω is large enough, the set $\Delta(\Omega)$ is not a linear space. Even worse is the fact that for a general additive subset S of $\Delta(\Omega)$, the map $p : S \rightarrow \mathbb{R}$ which maps a random variable to the position parameter of its law, is *not* additive in general. A counter-example to this was given by Chen and Shepp [3], where S is actually generated by two Cauchy random variables.

In Section 5.1, we introduce a formula to compute position parameters, and a way to bypass this global lack of additivity. The next section (Section 5.2), is dedicated to the computation of the position parameter for the Brownian motion.

5.1. Some properties of Cauchy-like laws. We will need the two following lemma, whose proofs, given below, consists in simple computations. In what follows, $(\Omega, \mathcal{F}, \mathbf{P})$ is a fixed probability space.

Lemma 5.2. *Consider $X \in \Delta(\Omega)$. Let p be its position parameter. For two real numbers a, k with $k > 0$, let also $(a)_k$ denote the quantity $\max(\min(a, k), -k)$. Then, we have the following equalities:*

$$p = \lim_{N \rightarrow \infty} N \mathbf{E}[\sin(X/N)] = \lim_{k \rightarrow \infty} \mathbf{E}[X \mathbb{1}_{|X| \leq k}] = \lim_{k \rightarrow \infty} \mathbf{E}[(X)_k].$$

This lemma will allow us to express the position parameter p_B in terms of the sequence D_N . We will also need the second following lemma, which roughly speaking states that the position parameters do add up as soon as the corresponding variables are not too strongly correlated in their tail behaviour.

Lemma 5.3. *Let $n \in \mathbb{N}$ and $X_1, \dots, X_n \in \Delta(\Omega)$ with position parameters p_1, \dots, p_n . Assume that there exists $\delta > 0$ such that, for all $i, j \in \{1, \dots, n\}$, $i \neq j$,*

$$\mathbf{P}(|X_i| \geq x, |X_j| \geq x) = o(x^{-(1+\delta)}) \quad \text{as } x \rightarrow +\infty.$$

Then $\sum_{i=1}^n X_i \in \Delta(\Omega)$ and its position parameter p is equal to $\sum_{i=1}^n p_i$.

Proof of Lemma 5.2. The first equality is a known result (see, for example [4], Part XVII, Theorem 3 p. 580, and conclusive remark p. 581), and relies on the study of the characteristic function of X . Let μ be the law of X . Let F denote its cumulative distribution function, and

set $F^-(x) = F(-x)$. Let $p_N = N\mathbf{E}[\sin(X/N)]$. Then,

$$\begin{aligned} p_N &= \lim_{k \rightarrow +\infty} \int_{-k}^k N \sin(x/N) dF(x) \\ &= \lim_{k \rightarrow +\infty} \int_0^k N \sin(x/N) d(1 + F - F^-)(x) \\ &= \lim_{k \rightarrow +\infty} \left(N \sin(k/N)(1 + F - F^-)(k) - \int_0^k \cos(x/N)(1 + F - F^-)(x) dx \right). \end{aligned}$$

From the fact that μ lies in the strong attraction domain, we deduce that, for some $\varepsilon > 0$,

$$1 + F(x) - F(-x) = o(x^{-1-\varepsilon}). \quad (29)$$

It follows that

$$|N \sin(k/N)(1 + F - F^-)(k)| \leq k|(1 + F - F^-)(k)| = o(1),$$

so that

$$p_N = - \int_0^\infty \mathbb{1}_{x \leq N} \cos(x/N)(1 + F(x) - F^-(x)) dx.$$

The integrand is dominated by the integrable function $1 + F - F^-$, and from pointwise convergence it follows that

$$p = \lim_{N \rightarrow \infty} p_N = - \int_0^\infty (1 + F(x) - F(-x)) dx.$$

Besides,

$$\begin{aligned} \mathbf{E}[X \mathbb{1}_{|X| \leq k}] &= \int_{-k}^k x dF(x) \\ &= \int_0^k x d(1 + F - F^-)(x) \\ &= k(1 - F(k) - F(-k)) - \int_0^k (1 + F(x) - F(-x)) dx \\ &\xrightarrow[k \rightarrow +\infty]{} - \int_0^\infty (1 + F(x) - F(-x)) dx \quad (\text{using (29) once again}). \end{aligned}$$

This implies the second equality.

For the third equality, it suffices to remark that

$$\mathbf{E}[(X)_k] - \mathbf{E}[X \mathbb{1}_{|X| \leq k}] = k(\mathbb{P}(X \geq k) - \mathbb{P}(X \leq -k)) \xrightarrow[k \rightarrow +\infty]{} 0.$$

The proof is complete. \square

Proof of Lemma 5.3. We first assume $n = 2$. We set a_1, a_2 and γ such that

$$\mathbf{P}(X_i \geq x) \underset{x \rightarrow +\infty}{=} \frac{a_i}{x} + o(x^{-1-\gamma}).$$

We also fix $\varepsilon : 0 < \varepsilon < 1 - \frac{1}{1+\delta}$, and assume $x^\varepsilon > 3$. We first show that $X_1 + X_2$ lies on $\Delta(\Omega)$:

$$\begin{aligned} \mathbf{P}(X_1 + X_2 \geq x) &\geq \mathbf{P}(X_1 + X_2 \geq x \text{ and } |X_2| \leq x^{1-\varepsilon}) \\ &\quad + \mathbf{P}(X_1 + X_2 \geq x \text{ and } |X_1| \leq x^{1-\varepsilon}) \\ &\geq \mathbf{P}(X_1 \geq x + x^{1-\varepsilon}) - \mathbf{P}(X_1 \geq x + x^{1-\varepsilon}, |X_2| \geq x^{1-\varepsilon}) \\ &\quad + \mathbf{P}(X_2 \geq x + x^{1-\varepsilon}) - \mathbf{P}(X_2 \geq x + x^{1-\varepsilon}, |X_1| \geq x^{1-\varepsilon}) \\ &\geq \frac{a_1 + a_2}{x} + O(x^{-1-\varepsilon}) + O(x^{-1-\gamma}) + O(x^{-(1-\varepsilon)(1+\delta)}). \end{aligned}$$

Besides,

$$\begin{aligned} \mathbf{P}(X_1 + X_2 \geq x) &\leq \mathbf{P}(X_1 \geq x - x^{1-\varepsilon}) + \mathbf{P}(X_2 \geq x - x^{1-\varepsilon}) + \mathbf{P}(X_1 \geq x^{1-\varepsilon} \text{ and } X_2 \geq x^{1-\varepsilon}) \\ &\leq \frac{a_1 + a_2}{x} + O(x^{-1-\varepsilon}) + O(x^{-1-\gamma}) + O(x^{-(1-\varepsilon)(1+\delta)}). \end{aligned}$$

The estimation near $-\infty$ is identical, and it follows that $X_1 + X_2$ lies on $\Delta(\Omega)$. To show that $p = p_1 + p_2$, we use Lemma 5.2. We write $k^\pm = k \pm k^{1-\varepsilon}$.

Then,

$$\begin{aligned} &\{X_1 \geq 0, |X_1 + X_2| \leq k^-\} \setminus \{X_1 \geq 0, |X_2| \geq k^{1-\varepsilon}, |X_1 + X_2| \leq k^-\} \\ &\subseteq \{X_1 \in [0, k]\} \\ &\subseteq \{X_1 \geq 0, |X_1 + X_2| \leq k^+\} \cup \{X_1 \in [0, k^{1-\varepsilon}], |X_2| \geq k\} \cup \{X_1 \in [k^{1-\varepsilon}, k], |X_2| \geq k^{1-\varepsilon}\}, \end{aligned}$$

so that

$$\begin{aligned} \mathbf{E}[X_1 \mathbb{1}_{X_1 \in [0, k]}] &\leq \mathbf{E}[X_1 \mathbb{1}_{X_1 \in [0, k], |X_1 + X_2| \leq k^+}] \\ &\quad + k^{1-\varepsilon} \mathbf{P}(\{|X_2| \geq k\}) + k \mathbf{P}(|X_1| \geq k^{1-\varepsilon} \text{ and } |X_2| \geq k^{1-\varepsilon}) \\ &\leq \mathbf{E}[X_1 \mathbb{1}_{X_1 \in [0, k], |X_1 + X_2| \leq k^+}] + k^{-\varepsilon} + k^{1-(1-\varepsilon)(1+\delta)}, \end{aligned} \tag{30}$$

and

$$\mathbf{E}[X_1 \mathbb{1}_{X_1 \in [0, k]}] \geq \mathbf{E}[X_1 \mathbb{1}_{X_1 \geq 0, |X_1 + X_2| \leq k^-}] - \mathbf{E}[X_1 \mathbb{1}_{|X_2| \geq k^{1-\varepsilon}, |X_1 + X_2| \leq k^-}].$$

To bound the last term, we introduce some ε' such that $\varepsilon < \varepsilon' < 1 - \frac{1}{1+\delta}$, and we separate the events $\{X_1 \leq k^{1-\varepsilon'}\}$, $\{X_1 > k^{1-\varepsilon'}\}$. We obtain

$$\mathbf{E}[X_1 \mathbb{1}_{|X_2| \geq k^{1-\varepsilon}, |X_1 + X_2| \leq k^-}] \leq k^{1-\varepsilon'} \mathbf{P}(|X_2| \geq k^{1-\varepsilon}) + k \mathbf{P}(|X_1| \geq k^{1-\varepsilon'} \text{ and } |X_2| \geq k^{1-\varepsilon}),$$

which is less than $k^{\varepsilon-\varepsilon'} + k^{1-(1-\varepsilon')(1+\delta)}$. Thus,

$$\mathbf{E}[X_1 \mathbb{1}_{X_1 \in [0, k]}] \geq \mathbf{E}[X_1 \mathbb{1}_{X_1 \geq 0, |X_1 + X_2| \leq k^-}] - k^{\varepsilon-\varepsilon'} - k^{1-(1-\varepsilon')(1+\delta)}. \tag{31}$$

Finally, writing F for the cumulative distribution function of X_1 , we have

$$\begin{aligned}
\mathbf{E}[X_1 \mathbb{1}_{X_1 \geq 0, |X_1 + X_2| \in [k^-, k^+]}] &\leq k^{1-\varepsilon} \mathbf{P}(|X_2| \geq k - 2k^{1-\varepsilon}) \\
&\quad + (k + 2k^{1-\varepsilon}) \mathbf{P}(X_1 \in [k^{1-\varepsilon}, k + 2k^{1-\varepsilon}], X_2 \geq k^{1-\varepsilon}) \\
&\quad + (k + 2k^{1-\varepsilon}) \mathbf{P}(X_1 \in [k - 2k^{1-\varepsilon}, k + 2k^{1-\varepsilon}]) \\
&\leq C(k^{-\varepsilon} + k^{1-(1-\varepsilon)(1+\delta)} + k(F(k + 2k^{1-\varepsilon}) - F(k - 2k^{1-\varepsilon}))) \\
&\leq C'(k^{-\varepsilon} + k^{1-(1-\varepsilon)(1+\delta)} + k(k^{\varepsilon-2} + k^{-1-\gamma})). \tag{32}
\end{aligned}$$

With (30), (31) and (32), we obtain

$$\mathbf{E}[X_1 \mathbb{1}_{X_1 \in [0, k]}] = \mathbf{E}[X_1 \mathbb{1}_{X_1 \in [0, k], |X_1 + X_2| \leq k^+}] + O(x^{-\xi})$$

where $\xi = \min(\varepsilon - \varepsilon', \gamma, 1 - \varepsilon, (1 - \varepsilon')(1 + \delta) - 1) > 0$. We do the same thing with $(-X_1, X_2)$, (X_2, X_1) , and $(-X_2, X_1)$ instead of (X_1, X_2) , and we obtain

$$\mathbf{E}[X_1 \mathbb{1}_{|X_1| \leq k}] + \mathbf{E}[X_2 \mathbb{1}_{|X_2| \leq k}] - \mathbf{E}[(X_1 + X_2) \mathbb{1}_{|X_1 + X_2| \leq k^+}] = O(x^{-\xi}) = o(1).$$

Taking the limit $k \rightarrow +\infty$, we obtain $p_1 + p_2 - p = 0$.

The proof is now complete in the case $n = 2$, and the inequality

$$\mathbf{P}\left(|X_n| \geq x \text{ and } \left| \sum_{i=1}^{n-1} X_i \right| \geq x\right) \leq \mathbf{P}\left(|X_n| \geq \frac{x}{n} \text{ and } \exists i \in \{1, \dots, n-1\} : |X_i| \geq \frac{x}{n}\right).$$

allows us to extend, by induction, the result to an arbitrary number of random variables. \square

Remark 5.4. For the case of the area measures of two independent Brownian motions, the condition of Lemma 5.3 follows from the preliminary lemma 2.3.

5.2. Computation for the position parameter of the Brownian motion. We now have the tools to show Lemma 5.1.

Proof of Lemma 5.1. For a positive integer N , we set

$$D_N^- = |\{z \in \mathbb{R}^2 : \theta_B(z) \leq -N\}|.$$

It is clear, by symmetry of the Brownian motion, that D_N^- is equal in distribution to D_N , and thus satisfies the same estimates. Using Lemma 5.2 (which extends directly to the case of measures with finite mass), we have

$$\mathbb{E}[|p_B|] = \mathbb{E}\left[\left| \sum_{N=1}^{\infty} (D_N - D_N^-) \right|\right].$$

The reader will remark that the dominant term in the asymptotic expansion of D_N cancels with the one of D_N^- , so that it is the second order term which is relevant here. We now use the L^2 estimation on Proposition 1.4, which tell us that, for $\delta < \frac{1}{3}$, for some constant C ,

$$\mathbb{E}[|p_B|] \leq \sum_{N=1}^{\infty} CN^{-1-\delta} < +\infty,$$

so that p_B has finite expectation.

Let us denote by $B^{pl,n}$ the dyadic piecewise linear approximation of B with 2^n steps: for $i \in \{0, 1, \dots, 2^n - 1\}$ and $u \in [0, 1)$,

$$B_{\frac{i+u}{2^n}}^{pl,n} = B_{\frac{i}{2^n}} + u(B_{\frac{i+1}{2^n}} - B_{\frac{i}{2^n}}).$$

We also let $B(i, n)$ be the restriction of B to the interval $[\frac{i-1}{2^n}, \frac{i}{2^n}]$, so that

$$\theta_B = \theta_{B^{pl,n}} + \sum_{i=1}^{2^n} \theta_{B(i,n)}.$$

Let us assume that the equality holds at the level of position parameters, that is

$$p_B = p_{B^{pl,n}} + \sum_{i=1}^{2^n} p_{B(i,n)}. \quad (\star)$$

Since the function $\theta_{B^{pl,n}}$ is bounded, it is easy to see that $p_{B^{pl,n}} = \int_{\mathbb{R}^2} \theta_{B^{pl,n}}$, and that this is equal to $\mathcal{A}_{B^{pl,n}}$. It is widely known, from the early introduction of the Lévy area, that $\mathcal{A}_{B^{pl,n}}$ converges toward \mathcal{A}_B , in the almost sure sense, as $n \rightarrow \infty$. Thus, under the assumption (\star) , the conclusion would follow from

$$\sum_{i=1}^{2^n} p_{B(i,n)} \xrightarrow[n \rightarrow \infty]{p.s.} 0. \quad (33)$$

Since we already know that

$$\sum_{i=1}^{2^n} p_{B(i,n)} \xrightarrow[n \rightarrow \infty]{p.s.} p_B - \mathcal{A}_B,$$

it is actually sufficient to show that the convergence (33) holds in distribution. Remark that the curves $(B(i, n) - B_{\frac{i-1}{2^n}})_{i \in \{1, \dots, 2^n\}}$ are i.i.d. Brownian motions, so that their position parameters are i.i.d. variables. Their position parameters $p_{B(i,n)}$ are equal in distribution to $\frac{p_B}{2^n}$, because of the scaling property of Brownian motion. Since p_B has finite expectation, the weak law of large numbers applies and ensures that $\sum_{i=1}^{2^n} p_{B(i,n)}$ converges in distribution towards the expectation of p_B . By symmetry of the Brownian motion, this expectation is zero, which implies (33). Remark that the strong law of large numbers does not apply directly, because we have a triangular array instead of a sequence of random variables.

There is only (\star) left to show.

Remark first that for any two curves γ, γ' , if γ lies in the strong attraction domain of the Cauchy law and $\mu_{\gamma'}$ admits a first moment, then $\gamma \cdot \gamma'$ lies in the strong attraction domain of the Cauchy law, and $p_{\gamma \cdot \gamma'} = p_\gamma + p_{\gamma'}$. This follows directly from Slutsky's Lemma.

Since $\theta_{B^{pl,n}}$ is a bounded function, $\mu_{\gamma'}$ admits a first moment (recall that n is fixed here). We let $(B^{pl,n})^{-1}$ be the curve $B^{pl,n}$ with reversed orientation, and $B \cdot (B^{pl,n})^{-1}$ be the concatenation of B and $(B^{pl,n})^{-1}$, so that

$$p_B = p_{B \cdot (B^{pl,n})^{-1}} + p_{B^{pl,n}}.$$

Remark that the following equality holds almost everywhere

$$\theta_{B \cdot (B^{pl,n})} = \sum_{i=1}^{2^n} \theta_{B(i,n)}.$$

We now want to apply Lemma 5.3.

Let (X_1, \dots, X_{2^n}) be a family of \mathbb{Z} -valued random variables such that

$$\mathbf{P}((X_1, \dots, X_{2^n}) = (0, \dots, 0)) = 0$$

and for any $(k_1, \dots, k_{2^n}) \in \mathbb{Z}^{2^n} \setminus \{0\}$,

$$\mathbf{P}((X_1, \dots, X_{2^n}) = (k_1, \dots, k_{2^n})) = \frac{1}{Z} |\{z \in \mathbb{R}^2 : \forall i \in \{1, \dots, 2^n\}, \theta_{B(i,n)}(z) = k_i\}|$$

where the normalizing constant Z is such that \mathbf{P} is a probability measure.

For $i, j \in \{1, \dots, 2^n\}$, $i \neq j$, set

$$\begin{aligned} D_{n,N}^{i,j} &= |\{z \in \mathbb{R}^2 : |\theta_{B(i,n)}(z)| \geq N \text{ and } |\theta_{B(j,n)}(z)| \geq N\}| \\ &= Z \mathbf{P}(|X_i| \geq N \text{ and } |X_j| \geq N). \end{aligned}$$

Then, for $\delta, \varepsilon > 0$ such that $2\delta + \varepsilon < 2$,

$$\begin{aligned} \mathbb{P}(\exists N \geq N_0 : D_{n,N}^{i,j} \geq N^{-1-\delta}) &\leq \sum_{k=\lfloor \log_2(N_0) \rfloor}^{\infty} \mathbb{P}(D_{n,2^k}^{i,j} \geq 2^{(k+1)(-1-\delta)}) \\ &\leq \sum_{k=\lfloor \log_2(N_0) \rfloor}^{\infty} 2^{2(k+1)(1+\delta)} \mathbb{E} \left[\left(D_{n,2^k}^{i,j} \right)^2 \right] \\ &\leq \sum_{k=\lfloor \log_2(N_0) \rfloor}^{\infty} C 2^{-2n} 2^{2k(1+\delta)} 2^{k(-4+\varepsilon)} \quad (\text{using Lemma 2.3}) \\ &\xrightarrow{N_0 \rightarrow +\infty} 0. \end{aligned} \tag{34}$$

This implies that, for all n, i, j , \mathbb{P} -almost surely, the hypothesis of Lemma 5.3 is satisfied for (X_1, \dots, X_{2^n}) (under \mathbf{P}). Thus, \mathbb{P} -almost surely, (\star) holds. This ends the proof of Lemma 5.1, and thus of our main theorem 1.1. \square

5.3. Proof of Theorem 0.1. The goal of this section is to show that Theorem 1.1 does imply Theorem 0.1, as we announced earlier. We first show the following lemma.

Lemma 5.5. *Let $(X_i)_{i \in \mathbb{N}}$ be a family of i.i.d. random variables. For any $N \in \mathbb{N}$, let $P(N)$ be a Poisson random variable with parameter N and independent from the family $(X_i)_{i \in \mathbb{N}}$. Assume that the random variables*

$$Z_N = \frac{1}{N} \sum_{i=1}^N X_i$$

converge in distribution as $N \rightarrow +\infty$. Then,

$$\tilde{Z}_N = \frac{1}{N} \sum_{i=1}^{P(N)} X_i$$

also converge in distribution as $N \rightarrow +\infty$, and the limiting distributions are the same.

Proof. Let ϕ (resp. $\phi_N, \tilde{\phi}_N$) be the characteristic function of X_i (resp. Z_N, \tilde{Z}_N). Set also ϕ_∞ the characteristic function of the limit distribution of the Z_N .

Let $\theta \in \mathbb{R}$ and set $u_N = \phi(\frac{\theta}{N})$. As $N \rightarrow +\infty$, $u_N \rightarrow \phi(0) = 1$. Hence $u_N - 1 \underset{N \rightarrow \infty}{\sim} \text{Log}(u_N)$ where Log is a determination of the logarithm continuous at 1 and with $\text{Log}(1) = 0$. Then

$$N(u_N - 1) \underset{N \rightarrow \infty}{\sim} N \text{Log}(u_N).$$

In $\mathbb{C}/2i\pi\mathbb{Z}$, for N large enough, $u_N \neq 0$ and then $N \text{Log}(u_N) = \text{Log}(u_N^N) = \text{Log}(\phi_N(\theta))$. The assumption of the lemma ensures that $\phi_N(\theta) \underset{N \rightarrow \infty}{\rightarrow} \phi_\infty(\theta)$. Hence, in $\mathbb{C}/2i\pi\mathbb{Z}$,

$$N(u_N - 1) \underset{N \rightarrow \infty}{\rightarrow} \text{Log}(\phi_\infty(\theta)).$$

It follows that $\exp(N(u_N - 1)) \underset{N \rightarrow +\infty}{\rightarrow} \phi_\infty(\theta)$. The random variable \tilde{Z}_N is a compound Poisson variable, and $\tilde{\phi}_N(\theta)$ is equal to $\exp(N(u_N - 1))$, so $\tilde{\phi}_N(\theta) \underset{N \rightarrow +\infty}{\rightarrow} \phi_\infty(\theta)$. Since this is true for any $\theta \in \mathbb{R}$, the conclusion of the lemma follows from the Lévy's continuity theorem. \square

We now start the proof of Theorem 0.1.

Proof. Let Ω_0 be the full probability event of Theorem 1.1. It is a subset of the probability space Ω in which the Brownian motion is defined. We set $\omega \in \Omega_0$.

Set $R = \sup_{t \in [0,1]} \|B_t(\omega)\|$ and ν_R the probability law defined in \mathbb{Z} (including 0) by

$$\nu_R(N) = \frac{|\{z \in B(0, R) \setminus \text{Range}(B(\omega)) : \theta_{B(\omega)}(z) = N\}|}{|B(0, R)|}.$$

The probability law is related to the probability law $\nu_{B(\omega)}$ by the relation

$$\nu_R(N) = \frac{Z}{|B(0, R)|} \nu_{B(\omega)} + \frac{|B(0, R)| - Z}{|B(0, R)|} \delta_0 \quad (35)$$

with Z the mass of $\mu_{B(\omega)}$.

We denote by N_K the cardinal of $\mathcal{P}(K) \cap B(0, R)$, which is a Poisson random variable with parameter $|B(0, R)|K$. Set $(X_i)_{i \in \mathbb{N}}$ a family of i.i.d. random variables distributed as ν_R , and independent from N_K . Then,

$$\sum_{z \in \mathcal{P}(K)} \theta_{B(\omega)}(z) = \sum_{z \in \mathcal{P}(K) \cap B(0, R)} \theta_{B(\omega)}(z) \stackrel{(d)}{=} \sum_{i=1}^{N_K} X_i \quad (36)$$

Using (35), we can write $X_i = B_i Y_i$ where the B_i are Bernoulli's random variables with parameter $\frac{Z}{|B(0, R)|}$, the Y_i are distributed as $\nu_{B(\omega)}$ and B_i is independent from Y_i . Since the X_i, N_K are globally independent, we can further assume that the B_i, Y_i, N_K are also globally independent. Set $M_K = |\{i \in \{1, \dots, N_K\} : B_i = 1\}|$, which is easily seen to be a Poisson random variable with parameter ZK . Then,

$$\sum_{i=1}^{N_K} X_i \stackrel{(d)}{=} \sum_{i=1}^{M_K} Y_i. \quad (37)$$

Theorem 1.1 implies that $\sum_{i=1}^{Z^K} Y_i$ converges in distribution toward a Cauchy distribution with position parameter $p_{B(\omega)}$. Lemma 5.5 then implies that $\sum_{i=1}^{M^K} Y_i$ also converges in distribution toward a Cauchy distribution with position parameter $p_{B(\omega)}$. Together with (36) and (37), we obtain Theorem 0.1. \square

6. FROM L^2 ESTIMATES TO L^p ESTIMATES

We now extend the second order bound that we obtained from L^2 to L^p , for any $p \in (2, +\infty)$. The goal of this section is show Theorem 1.5. We first show a ‘large deviation’ bound.

Lemma 6.1. *For any $k > 0$ and $\varepsilon > 0$, there is a constant C such that for all $N \in \mathbb{N}^*$,*

$$\mathbb{P}(D_N \geq N^{-1+\varepsilon}) \leq CN^{-k}. \quad (38)$$

Proof. The proof follows the same pattern as many that we presented already: we start with an estimation at some rank, we cut the long trajectory into smaller ones, we rescale, and we apply the known estimation to obtain the one at next rank.

Set $p_{N,\varepsilon,C} = \mathbb{P}(D_N \geq CN^{-1+\varepsilon})$. Using the Markov inequality and the L^2 convergence, we first obtain, for all $\varepsilon > 0$, and for all $C > 0$, that there is some C' such that for all $N \in \mathbb{N}^*$,

$$p_{N,\varepsilon,C} \leq C'N^{-2\varepsilon}.$$

Suppose that we already know, for all $\varepsilon > 0$ for some $a > 1$, for all $C > 0$, for some C' , that

$$\forall \varepsilon > 0, p_{N,\varepsilon,C} \leq C'N^{-a\varepsilon}.$$

Then, set $t = \frac{a\varepsilon}{1+a}$ and $T = \lfloor N^t \rfloor$. This is always less than N^ε , so that when N is large enough, $T > 0$ and $N - T > \frac{N}{2}$.

We use here a dichotomy which is simpler than before. Recall that when we pass from a trajectory from $[0, 1]$ to its pieces of length T^{-1} , we need to replace N with $N - T$ because of the winding of the piecewise-linear approximation of X . For D_N to be larger than some value D , either there exists some i such that D_{N-T}^i is larger than $\frac{D}{2}$, or there is at least two indices $i \neq j$ such that D_{N-T}^i and D_{N-T}^j are both larger than $\frac{D}{2T}$. We thus have the following inequalities:

$$\begin{aligned} p_{N,\varepsilon,C} &= \mathbb{P}(D_N \geq CN^{-1+\varepsilon}) \\ &\leq \mathbb{P}(\exists i \in \{1, \dots, T\} : D_{N-T}^i \geq \frac{C}{2}N^{-1+\varepsilon}) \\ &\quad + \mathbb{P}(\exists i \neq j \in \{1, \dots, T\} : D_{N-T}^i \geq \frac{C}{2}\frac{N^{-1+\varepsilon}}{T} \text{ and } D_{N-T}^j \geq \frac{C}{2}\frac{N^{-1+\varepsilon}}{T}) \\ &\leq Tp_{N-T,\varepsilon+t,\frac{C}{2}} + T^2 p_{N-T,\varepsilon,\frac{C}{2}}^2 \\ &\leq C'N^{t-a(\varepsilon+\delta)} + C'N^{2t-2a\varepsilon} \\ &= 2C'N^{-\frac{2a^2\varepsilon}{1+a}}. \end{aligned}$$

Remark that $\frac{2a^2}{1+a} > a$ as soon as $a > 1$, so that we improved our estimation, but it would not have worked if we had started with $a = 1$. In particular, this method cannot be used as a quick way to improve the L^1 convergence of ND_N into an L^2 convergence.

Setting now $a_0 = 2$, and $a_{n+1} = \frac{2a_n^2}{1+a_n}$, we recursively obtain that for all $n \in \mathbb{N}$, for all $\varepsilon > 0$, for all $C > 0$, that there is some C' such that for all $N \in \mathbb{N}^*$,

$$p_{N,\varepsilon,C} \leq C' N^{-a_n \varepsilon}.$$

The function $f : x \mapsto \frac{2x^2}{1+x} - x$ is strictly positive on $[2, +\infty)$. It follows that $a_n \xrightarrow[n \rightarrow \infty]{} +\infty$, which concludes the proof. \square

Remark 6.2. *We actually expect $\mathbb{P}(D_N \geq N^{-1+\varepsilon})$ to decay at least power-exponentially in N .*

We can now show Theorem 1.5.

Proof. First, remark that for any $N \in \mathbb{N}^*$, $D_N \leq D_1 \leq \pi \|B\|_{\infty,[0,1]}^2$. The right-hand term admits moments of all order. Set $C_p = \mathbb{E}[D_1^p]$.

Choose $\varepsilon \in \left(0, \frac{1}{3(p-2)}\right)$ and $q > p$. Then, using a disjunction and Hölder inequality, we have

$$\mathbb{E}\left[\left|ND_N - \frac{1}{2\pi}\right|^p\right] \leq \mathbb{E}\left[\left|ND_N - \frac{1}{2\pi}\right|^p \mathbb{1}_{D_N \leq N^{-1+\varepsilon}}\right] + \mathbb{E}\left[\left|ND_N - \frac{1}{2\pi}\right|^q\right] \mathbb{P}(D_N \geq N^{-1+\varepsilon}). \quad (39)$$

The first term can be bounded by

$$\mathbb{E}\left[\left(ND_N - \frac{1}{2\pi}\right)^2 (N^{\varepsilon(p-2)} + (2\pi)^{2-p})\right].$$

According to 1.4, for any $\delta < \frac{1}{3}$, for some C , this is less than $CN^{-\delta+\varepsilon(p-2)}$.

According to the previous lemma (Lemma 6.1), for all $q > 0$, there exists some C' such that the second term can be bounded by $(N^q C_q + 1)C'N^{-k}$, which goes to 0 more quickly than any power of N . Taking ε close to 0, we deduce that for all $\delta < \frac{1}{3}$, for all $p < \infty$, there is some C such that for all $N \in \mathbb{N}^*$,

$$\mathbb{E}\left[\left|ND_N - \frac{1}{2\pi}\right|^p\right] \leq CN^{-2\delta}.$$

This is the announced bound. \square

7. A SIMILAR RESULT FOR YOUNG INTEGRATION

Let $\gamma = (x, y) : [0, 1] \rightarrow \mathbb{R}^2$ be a continuous function. Let us recall that $\theta_\gamma(z)$ denotes the integer winding of γ around z , obtained by closing γ by adding to it a straight segment between the endpoints. When γ is piecewise linear or smooth, the Stokes formula implies

$$\int_0^1 x_t dy_t - \frac{x_1 + x_0}{2}(y_1 - y_0) = \int_{\mathbb{R}^2} \theta_\gamma(z) dz. \quad (40)$$

Our main theorem can be understood as a generalization of this result to the case when γ is a Brownian motion.

We now show Theorem 7.1, which states that (40) can also be generalized (without Cauchy laws involved) when the Young integral $\int_0^1 x_t dy_t$ is well defined. Let us recall that a *dissection* D of $[0, 1]$ is a finite increasing sequence $0 = t_0 < \dots < t_n = 1$ (with n that depends implicitly

on D). Its *mesh* $|D|$ is the positive real $\max\{t_i - t_{i-1} : i \in \{1, \dots, n\}\}$. The p -variation norm $\|x\|_p$ of a continuous function $x : [0, 1] \rightarrow \mathbb{R}$ is the (possibly infinite) quantity

$$\left(\sup_D \sum_{i=1}^n |x_{t_{i-1}} - x_{t_i}|^p \right)^{\frac{1}{p}}$$

where the supremum is over the dissections $D = (t_1 < \dots < t_n)$ of $[0, 1]$. We denote by \mathcal{V}^p the set of continuous functions from $[0, 1]$ to \mathbb{R} with finite p -variation. We also let $\mathcal{V}^{p,q}$ be the set of couples $\gamma = (x, y)$ with $x \in \mathcal{V}^p$, $y \in \mathcal{V}^q$. We identify such a couple γ with the function from $[0, 1]$ to \mathbb{R}^2 that maps t to (x_t, y_t) .

Theorem 7.1. *Let $p, q \geq 1$ be reals such that $\delta = \frac{1}{p} + \frac{1}{q} - 1 > 0$. Let $\gamma \in \mathcal{V}^{p,q}$. Then, the range of γ has zero Lebesgue measure and $\theta_\gamma \in L^r(\mathbb{R}^2, \mathbb{Z})$ for any $r \in [1, 1 + \delta)$. Besides, the equality (40) holds if the left-hand side is interpreted as a Young integral.*

In [2, Theorem 3.2], a similar result is shown in the case where γ is further assumed to be simple.

We recall from the theory of Young integration that a *control* is a function $\omega : \Delta = \{(s, t) : 0 \leq s \leq t \leq 1\} \rightarrow \mathbb{R}$ which is continuous, vanishes on the diagonal, and satisfies $\omega_{s,t} + \omega_{t,u} \leq \omega_{s,u}$ for $s < t < u$. As a preliminary material, we state four previously known results. We give a precise reference for each of them, but we also refer the reader to [5] and [9] as general references. Apart from the first one, all of them are used in the construction of the Young integral. The proof of Theorem 7.1 is partly similar to this construction.

Theorem 7.2 (Banchoff-Pohl inequality, [1]). *Let $\gamma : [0, 1] \rightarrow \mathbb{R}^2$ be a continuous function with finite 1-variation. Then, $\theta_\gamma \in L^2$ and*

$$\|\theta_\gamma\|_{L^2}^2 \leq \frac{\|\gamma\|_1^2}{4\pi}.$$

Theorem 7.3 ([6, lemma 6.2]). *Let $\Gamma : \Delta = \{0 \leq s < t \leq 1\} \rightarrow \mathbb{R}$ and assume that*

- *there exists a control $\hat{\omega}$ such that*

$$\lim_{r \rightarrow 0} \sup_{(s,t) \in \Delta: \hat{\omega}(s,t) \leq r} \frac{\Gamma_{s,t}}{r} = 0,$$

- *there exist a control ω and $\theta > 1, \xi > 0$ such that*

$$|\Gamma_{s,u}| \leq |\Gamma_{s,t}| + |\Gamma_{t,u}| + \xi \omega(s, u)^\theta$$

holds for $0 \leq s \leq t \leq u \leq 1$.

Then, for all $0 \leq s < t \leq 1$,

$$|\Gamma_{s,t}| \leq \frac{\xi}{1 - 2^{1-\theta}} \omega(s, t)^\theta.$$

For a function $x : [0, 1] \rightarrow \mathbb{R}$ and a dissection $D = (t_0 < \dots < t_n)$ of $[0, 1]$, let x^D be the piecewise linear function defined by $x_t^D = \frac{t-t_{i-1}}{t_i-t_{i-1}}x_{t_i} + \frac{t_i-t}{t_i-t_{i-1}}x_{t_{i-1}}$, where $i \in \{1, \dots, n\}$ is such that $t_{i-1} \leq t < t_i$.

Theorem 7.4 ([6, theorem 5.25]). *Let $x \in \mathcal{V}^p$. Let (D_n) be a sequence of dissections of $[0, 1]$ with mesh converging to 0. Then, x^{D_n} converges to x in uniform norm and for all n ,*

$$\|x^{D_n}\|_p \leq 3^{1-1/p} \|x\|_p.$$

Theorem 7.5 ([6, theorem 5.33] (Wiener's characterization)). *Let $x \in \mathcal{V}^p$. The following statements are equivalent:*

1. x belongs to the p -variation closure of \mathcal{V}^1 .
2. $\lim_{\varepsilon \rightarrow 0} \sup_{D:|D|<\varepsilon} \sum_{i=1}^n d(x_{t_{i-1}}, x_{t_i})^p = 0$.

To be clear, in the second statement, the supremum is taken over all dissections of $[0, 1]$ with mesh less than ε .

The proof of Theorem 7.1 is organized as follows. First, we show an inequality similar to the Young–Loève estimate: for smooth enough curves, the L^r norm of θ_γ can be controlled by the p (resp. q)-variation of its coordinates. This is Lemma 7.6. We then show that θ_γ is defined almost everywhere (Lemma 7.7), that it lies in L^r for r small enough (Lemma 7.8), and finally that the equality (40) holds.

We fix once and for all p, q and δ as in Theorem 7.1.

Lemma 7.6. *Let $\gamma = (x, y) : [0, 1] \rightarrow \mathbb{R}^2$ be a continuous curve with finite 1-variation. Then, for every $r \in [1, 2]$, $\theta_\gamma \in L^r$. Moreover, for all $\delta \leq 1$ and all $r \in [1, 1 + \delta)$, one has*

$$\|\theta_\gamma\|_{L^r} \leq \frac{\|x\|_p \|y\|_q}{1 - 2^{1-\frac{1+\delta}{r}}}. \quad (41)$$

Proof. For $0 \leq s < t \leq 1$, set $\theta_{s,t} = \theta_{\gamma|_{[s,t]}}$ and $\Gamma_{s,t} = \|\theta_{s,t}\|_{L^r}$. For $0 \leq s < t < u \leq 1$, let $T_{s,t,u}$ be the convex hull of $\{\gamma_s, \gamma_t, \gamma_u\}$.

For $f \in \mathcal{V}^p$, let $\|f\|_{p,[s,t]}$ be the p -variation norm of the restriction of f to $[s, t]$ (linearly reparametrized by $[0, 1]$). We will apply Theorem 7.3 with $\xi = 1$, with the controls $\omega_{s,t} = \|x\|_{p,[s,t]}^{1/(1+\delta)} \|y\|_{q,[s,t]}^{1/(1+\delta)}$ and $\tilde{\omega}_{s,t} = \|x\|_{1,[s,t]} + \|y\|_{1,[s,t]}$. These are the exact same controls that one uses to prove the Young–Loève estimate, and we refer to [6] again for the proof that these are indeed controls (see Proposition 1.15, Exercise 1.10, Proposition 5.8 and page 120).

Since θ takes its values in \mathbb{Z} , one has

$$\Gamma_{s,t} = \left(\int_{\mathbb{R}^2} |\theta_{s,t}(z)|^r dz \right)^{\frac{1}{r}} \leq \left(\int_{\mathbb{R}^2} \theta_{s,t}^2(z) dz \right)^{\frac{1}{r}} \leq \frac{\|\gamma\|_{1,[s,t]}^{\frac{2}{r}}}{4\pi} \leq \frac{\tilde{\omega}_{s,t}^{\frac{2}{r}}}{4\pi}.$$

This allows us to obtain the first assumption of 7.3.

Then, for $s < t < u$, set $\xi_{s,t,u} = \theta_{s,u} - \theta_{s,t} - \theta_{t,u}$, so that

$$|\xi_{s,t,u}| = \mathbb{1}_{T_{s,t,u}}.$$

Thus, $\|\theta_{s,u}\|_{L^r} \leq \|\theta_{s,t}\|_{L^r} + \|\theta_{t,u}\|_{L^r} + \|\mathbb{1}_{T_{s,t,u}}\|_{L^r}$, that is,

$$|\Gamma_{s,u}| \leq |\Gamma_{s,t}| + |\Gamma_{t,u}| + |T_{s,t,u}|^{\frac{1}{r}} \leq |\Gamma_{s,t}| + |\Gamma_{t,u}| + \omega(s, t)^{\frac{\theta}{r}}.$$

This is the second assumption of 7.3, with $\xi = 1$.

We now apply 7.3 to obtain the announced result. \square

Lemma 7.7. *For any $\gamma \in \mathcal{V}^{p,q}$, the range of γ has vanishing Lebesgue measure.*

Proof. Let $\gamma \in \mathcal{V}^{p,q}$. The range of $\gamma|_{[\frac{k}{\varepsilon}, \frac{k+1}{\varepsilon}]}$ is included in a box of length $C\varepsilon^{1/p}$ and width $C\varepsilon^{1/q}$, for some constant C that depends only on γ, p, q . Such a box can be covered by $C'\varepsilon^{1/p+1/q-2}$ balls of diameter ε . Thus, it is possible to cover the whole range of γ with no more than $\varepsilon^{-1}C'\varepsilon^{1/p+1/q-2} = C'\varepsilon^{-(2-\delta)}$ balls of diameter ε . Thus, the range of γ has Hausdorff dimension at most $2 - \delta$, and thus has vanishing Lebesgue measure. \square

Lemma 7.8. *For any $\gamma \in \mathcal{V}^{p,q}$, the function θ_γ lies in $L^r(\mathbb{R}^2, \mathbb{Z})$ for any $r < 1 + \delta$. Besides,*

$$\|\theta_\gamma\|_{L^r} \leq \frac{3^{1-\delta}}{1 - 2^{1-\frac{1+\delta}{r}}} \|x\|_p \|y\|_q. \quad (42)$$

Proof. We set $\gamma^{(\varepsilon)}$ the ε -thickening of the range of γ , that is the set $\{x \in \mathbb{R}^2 : d(x, \text{Range}(\gamma)) < \varepsilon\}$. From the fact that the range of γ has vanishing measure, we deduce that the Lebesgue measure $|\gamma^{(\varepsilon)}|$ of $\gamma^{(\varepsilon)}$ goes to 0 with ε .

We now fix a sequence $(\delta_n)_{n \geq 0}$ decreasing to 0, and for all n , a dissection D_n with mesh less than δ_n . We set $\gamma_n = (x^{D_n}, y^{D_n})$. We fix $\varepsilon > 0$, and n_0 such that for $n \geq n_0$, the range of γ_n is included in $\gamma^{(\varepsilon)}$. Then, for every $k > 0$ and $r < \delta$,

$$\begin{aligned} \int_{\mathbb{R}^2} \min(|\theta_\gamma(z)|^r, k) \, dz &\leq \int_{\mathbb{R}^2} |\theta_{\gamma_n}(z)|^r \, dz + k|\gamma^\varepsilon| \\ &\leq \left(\frac{\|x_n\|_p \|y_n\|_q}{1 - 2^{1-\frac{\delta}{r}}} \right)^r + k|\gamma^\varepsilon| \quad (\text{using Lemma 7.6}) \\ &\leq \left(\frac{3^{1-\delta} \|x\|_p \|y\|_q}{1 - 2^{1-\frac{1+\delta}{r}}} \right)^r + k|\gamma^\varepsilon| \quad (\text{using Lemma 7.4}). \end{aligned}$$

We let ε go to zero and then k go to infinity to conclude. \square

Finally we are ready to show the theorem.

Proof of Theorem 7.1. Let $\gamma = (x, y) \in \mathcal{V}^{p,q}$. From the previous lemma, we know that both sides of (40) are well defined.

In the case when γ is piecewise-linear, an easy recursion on the number of vertices shows the equality stated by Theorem 7.1.

According to [6, Corollary 5.35], since x has finite p -variation, it belongs to the closure of \mathcal{V}^1 in p' -variation norm for all $p' > p$. Together with the Wiener's characterization, this implies

$$\lim_{\varepsilon \rightarrow 0} \sup_{D: |D| < \varepsilon} \sum_{i=1}^n |x_{t_i} - x_{t_{i-1}}|^{p'} = 0.$$

Let $D = (t_0 < \dots < t_n)$ be a dissection of $[0, 1]$. A consequence of Wiener's characterization is that when the mesh of D is small enough, the maximum over $i \in \{1, \dots, n\}$ of $\|x\|_{p', t_{i-1}, t_i}$ (resp. $\|y\|_{q', t_{i-1}, t_i}$) is less than 1. We assume this condition to be satisfied.

We have the equality almost everywhere:

$$\theta_\gamma = \theta_{\gamma^D} + \sum_{i=1}^n \theta_{\gamma|_{[t_{i-1}, t_i]}}.$$

With $\delta' = \frac{1}{p'} + \frac{1}{q'} - 1 > 0$, and $p'' > p' > p$, $q'' > q' > q$ such that $\frac{1}{p''} + \frac{1}{q''} = 1$, we have:

$$\begin{aligned} \int_{\mathbb{R}^2} |\theta_\gamma(z) - \theta_{\gamma^D}(z)| \, dz &\leq \sum_{i=1}^n \int_{\mathbb{R}^2} |\theta_{\gamma|_{[t_{i-1}, t_i]}}(z)| \, dz \\ &\leq \frac{3^{1-\delta'}}{1-2^{\delta'}} \sum_{i=1}^n \|x\|_{p', t_{i-1}, t_i} \|y\|_{q', t_{i-1}, t_i} \quad (\text{using (42) with } r=1) \\ &\leq \frac{3^{1-\delta'}}{1-2^{\delta'}} \left(\sum_{i=1}^n \|x\|_{p'', t_{i-1}, t_i} \right)^{\frac{1}{p''}} \left(\sum_{i=1}^n \|y\|_{q'', t_{i-1}, t_i} \right)^{\frac{1}{q''}} \\ &\leq \frac{3^{1-\delta'}}{1-2^{\delta'}} \left(\sum_{i=1}^n \|x\|_{p', t_{i-1}, t_i} \right)^{\frac{1}{p''}} \left(\sum_{i=1}^n \|y\|_{q', t_{i-1}, t_i} \right)^{\frac{1}{q''}} \\ &\xrightarrow{|D| \rightarrow 0} 0. \end{aligned}$$

Thus, θ_{γ^D} converges in L^1 to θ_γ . Since the Young integral is also continuous and since (40) holds for the piecewise linear curve γ^D , the integral $\int_{\mathbb{R}^2} \theta_{\gamma^D}(z) \, dz$ converges to both the left-hand side and the right-hand sides of (40). This concludes the proof. \square

8. FURTHER DISCUSSION

For a real number x and a positive number k , recall that we denote by $(x)_k$ the quantity $\max(\min(x, k), -k)$. The main theorem of this paper implies that we can use, as a definition for the integral $\int_0^1 X \, dY$, the almost surely defined quantity

$$\lim_{k \rightarrow +\infty} \int_{\mathbb{R}^2} (\theta_B(z))_k \, dx \wedge dy. \quad (43)$$

This definition would have, from our point of view, three important advantages that we briefly discuss now.

The first one is that the expression (43) does not really require to know the structure of vector space on \mathbb{R}^2 , nor actually its structure of Riemannian manifold. Given the differential structure and orientation of \mathbb{R}^2 , the data of a closed curve γ with vanishing measure (a property that does not depend on a specific choice of volume form), and of a smooth differential 1-form α , we can as well form the quantity

$$\int_{\mathbb{R}^2} (\theta_\gamma(z))_k \, d\alpha$$

and hope that it will have a limit as $k \rightarrow +\infty$. The invariance of the integral under diffeomorphisms is then granted by definition. A similar property seems rather difficult to obtain for other integration theories, and even *false* for the simple case of the Young integral: even rotations of the plane can mix up the regularities, and turn a well-posed integral into an ill-posed one. Any

integration theory that would rely on approximations by piecewise geodesic curves is likely to suffer from this metric dependency that we avoid here.

The second advantage is also linked with the approximations. The usual definition of the stochastic integral imposes to choose at some point a sequence of dissections. One example for which this dependence appears is the following. Let $\Delta = (\Delta_{t,n})_{t \in [0,1], n \in \mathbb{N}}$, with each $\Delta_{t,n}$ a dissections of $[0, t]$. Then, the set $\Gamma(\Delta)$ of couples (ω, t) for which the Riemann sums associated with $\Delta_{t,n}$ converge (as $n \rightarrow +\infty$) depends on Δ . For two such families Δ, Δ' , even if we assume that Δ_t and Δ'_t are increasing sequences (for all t), there is no clear relation between $\Gamma(\Delta)$ and $\Gamma(\Delta')$. This forbids to define a universal ‘good’ set Γ of couples (ω, t) for which the integral $\int_0^t X dY$ is well-defined. When we replace the limit of the Riemann sums with a continuous modification of it, we simply eliminate this set and we cannot study it anymore. As opposed to that, our approach allows to define such a ‘good’ set Γ without any additional data involved.

The last advantage is the generality to which this definition extends. We will discuss this precise point further in a forthcoming paper. We intend to show that one can recover the Young integration and a large part of stochastic calculus (not restricted to semi-martingales). In particular, some ‘weak’ diffeomorphism invariance of those integral theories can then be proved. We also intend to show that one can define new integrals, for which both the 1-form α and the path γ are random and very irregular.

There are nonetheless three important drawbacks. The first one is technical: as we have seen, it seems rather difficult to show the existence of the limit (43), even for the most simple cases, for which the stochastic integral is almost trivially shown to be well defined.

The second downside is that Chasles’ relation is not satisfied in full generality. We had a glimpse of that problem when we computed the position parameter. Though, we still managed to circumvent the difficulty, which gives hope for this drawback not to be critical. In practice, for a given set of curves, we should most often be able to show that Chasles’ relation holds *for these curves*. The defect on Chasles’ relation is similar to the fact that the sum of two Cauchy laws might be a Cauchy law with a position parameter different from the sum of the two previous position parameters.

The third drawback is that since the definition does not depend on the linear structure of \mathbb{R}^2 , the map ‘ $\gamma \mapsto \int_\gamma \alpha$ ’ does not seem to be linear in general. This is nonetheless not very surprising if we consider it as an integral map from curves on a manifold.

In a different direction, it is possible to extend the main result of this paper to study asymptotically the monodromy of the Brownian motion when we consider a flat G -bundle over $\mathbb{R}^2 \setminus \mathcal{P}$ and the random points on \mathcal{P} carry curvature of the order of the inverse of the intensity of \mathcal{P} . More details should be given in another forthcoming paper.

ACKNOWLEDGEMENTS

The author would like to thank his PhD advisor Thierry Lévy for his continual help, and Pierre Perruchaud for the fruitful discussions concerning this work.

REFERENCES

- [1] T.F. Banchoff and W.F. Pohl. A generalization of the isoperimetric inequality. *J. Differential Geom.*, 6(2):175–192, 12 1971.
- [2] H. Boedihardjo and X. Geng. Simple piecewise geodesic interpolation of simple and Jordan curves with applications. *Constr. Approx.*, 42(1):161–180, 2015.
- [3] R. Chen and L.A. Shepp. On the sum of symmetric random variables. *Amer. Statist.*, 37(3):237, 1983.
- [4] W. Feller. *An introduction to probability theory and its applications. Vol II. 2nd ed.* John Wiley & Sons, Hoboken, NJ, 1971.
- [5] P. K. Friz and M. Hairer. *A course on rough paths. With an introduction to regularity structures. 2nd updated edition.* Cham: Springer, 2nd updated edition edition, 2020.
- [6] P. K. Friz and N. B. Victoir. *Multidimensional stochastic processes as rough paths. Theory and applications.*, volume 120. Cambridge: Cambridge University Press, 2010.
- [7] C. Garban and J. Ferreras. The expected area of the filled planar brownian loop is $\pi/5$. *Communications in mathematical physics*, 264(3):797–810, 2006.
- [8] O. Johnson and R. Samworth. Central limit theorem and convergence to stable laws in mallows distance. *Bernoulli*, 11(5):829–845, 2005.
- [9] Terry J. Lyons, Michael Caruana, and Thierry Lévy. *Differential equations driven by rough paths*, volume 1908 of *Lecture Notes in Mathematics*. Springer, Berlin, 2007. Lectures from the 34th Summer School on Probability Theory held in Saint-Flour, July 6–24, 2004, With an introduction concerning the Summer School by Jean Picard.
- [10] Roger Mansuy and Marc Yor. *Aspects of Brownian motion.* Universitext. Springer-Verlag, Berlin, 2008.
- [11] Z. Shi. Windings of brownian motion and random walks in the plane. *The Annals of Probability*, 26(1):112–131, 1998.
- [12] F. Spitzer. Some theorems concerning 2-dimensional Brownian motion. *Trans. Am. Math. Soc.*, 87:187–197, 1958.
- [13] W. Wendelin. Sur les points autour desquels le mouvement brownien plan tourne beaucoup. *Probability Theory and Related Fields*, 99(1):111–144, 1994.
- [14] W. Werner. Formule de Green, lacet brownien plan et aire de Lévy. *Stochastic Processes Appl.*, 57(2):225–245, 1995.
- [15] M. Yor. Loi de l’indice du lacet Brownien, et distribution de Hartman-Watson. *Z. Wahrscheinlichkeitstheor. Verw. Geb.*, 53:71–85, 1980.

ISAO SAUZEDDE – LPSM, SORBONNE UNIVERSITÉ, PARIS
 Email address: `isao.sauzedde@lpsm.paris`