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Dynamical large deviations for plasmas below the Debye length and the Landau equation

Ouassim Feliachi · Freddy Bouchet

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Abstract We consider a homogeneous plasma composed of N particles of the same electric charge which interact through a Coulomb potential. In the large plasma parameter limit, classical kinetic theories justify that the empirical density is the solution of the Balescu–Guernsey–Lenard equation, at leading order. This is a law of large numbers. The Balescu–Guernsey–Lenard equation is approximated by the Landau equation for scales much smaller than the Debye length. In order to describe typical and rare fluctuations, we compute for the first time a large deviation principle for dynamical paths of the empirical density, within the Landau approximation. We obtain a large deviation Hamiltonian that describes fluctuations and rare excursions of the empirical density, in the large plasma parameter limit. We obtain this large deviation Hamiltonian either from the Boltzmann large deviation Hamiltonian in the grazing collision limit, or directly from the dynamics, extending the classical kinetic theory for plasmas within the Landau approximation. We also derive the large deviation Hamiltonian for the empirical density of N particles, each of which is governed by a Markov process, and coupled in a mean field way. We explain that the plasma large deviation Hamiltonian is not the one of N particles coupled in a mean-field way.

Keywords Plasma · Landau equation · Balescu–Guernsey–Lenard equation · Large deviation theory · Macroscopic fluctuation theory

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1 Introduction: kinetic theories, dynamical large deviations and equilibrium statistical mechanics

In the field of statistical physics, the literature that describes the static fluctuations of a system around equilibrium and its relaxation to equilibrium is very rich. For instance, working in the appropriate thermodynamic ensemble, we can express the probability of observing a given state of a system as a function of the corresponding thermodynamic potential. Beyond equilibrium, classical kinetic theories describe the relaxation to equilibrium in some asymptotic regimes. For instance the Boltzmann equation describes the relaxation to equilibrium of a dilute gas in the Boltzmann-Grad limit, and the Balescu-Guernsey-Lenard equation in the opposite limit of particles with long range interactions, for instance plasma in the weak coupling limit or self-gravitating systems. The Landau equation is either an approximation of the Balescu-Guernsey-Lenard equation that describes the relaxation of plasma at a scale much smaller than the Debye length, or an approximation of the Boltzmann equation in the weak scattering limit. All those classical kinetic equations describe the relaxation of the empirical distribution $g_N(\mathbf{r}, \mathbf{v}, t) \equiv \frac{1}{N} \sum_{n=1}^N \delta(\mathbf{v} - \mathbf{v}_n(t)) \delta(\mathbf{r} - \mathbf{r}_n(t))$, where δ are Dirac delta functions, t is time, $(\mathbf{r}_n(t), \mathbf{v}_n(t))_{1 \leq n \leq N}$ are the N particle positions and velocities. The six-dimensional space of one-particle position-velocity, with points (\mathbf{r}, \mathbf{v}) , is called the μ -space. g_N is a distribution over the μ -space that evolves with time.

The probability $\mathbf{P}_{eq}(g_N = g^0)$ to observe g_N close to a given distribution g^0 of the μ -space, at some fixed arbitrary time, in the microcanonical ensemble, verifies

$$\mathbf{P}_{eq}(g_N = g^0) \propto e^{N \frac{\mathcal{S}[g^0]}{k_B}}. \quad (1)$$

This is the classical Einstein formula relating the specific entropy $\mathcal{S}[g^0]$ of the macrostate g^0 with its equilibrium probability. k_B is the Boltzmann constant. This can be seen as a definition of the Boltzmann entropy $\mathcal{S}[g^0]$ of the macrostate g^0 . For a dilute gas, because the particles are independent at leading order, or for systems with long range interactions, because the two-body interactions are weak, it is known that \mathcal{S} is the negative of the Boltzmann \mathcal{H} function ($\mathcal{S}[g^0] = -k_B \int d\mathbf{r} d\mathbf{v} g^0 \log g^0$) if the macrostate g^0 satisfies the conservation laws (mass, momentum and energy), and $\mathcal{S}[g^0] = -\infty$ otherwise.

However all those classical works and results in equilibrium statistical mechanics and kinetic theory do not describe the probability of paths that may lead to any macrostate g^0 . More generally, the macroscopic or mesoscopic stochastic process for g_N is not described by classical theories, and dynamical description is restricted to relaxation to equilibrium. In principle, very rarely, the microscopic dynamics can lead the distribution function to follow other paths than the relaxation paths described by the kinetic equation. What is the probability of such rare excursions? How do these probabilities depend on the paths? Those are key questions. Answering them are the starting point for solving many other non-equilibrium problems. Moreover, if the microscopic dynamics is time-reversible (in the sense of dynamical systems), for instance if the microscopic dynamics is Hamiltonian, then we expect the stochastic process for g_N to be also time-reversible (in the sense of stochastic processes). It is a

fundamental question to describe this stochastic process for the empirical distribution g_N .

More precisely we need to estimate the probability $\mathbf{P}(\{g_N(t)\}_{0 \leq t \leq T} = \{g(t)\}_{0 \leq t \leq T})$ to observe the evolution of $\{g_N(t)\}$ close to any prescribed path $\{g(t)\}$, for times $0 \leq t \leq T$, in some asymptotic limit when the kinetic description is valid. The mathematical and theoretical formalism adapted to this problem is large deviation theory. We need to prove the large deviation result

$$\mathbf{P}(\{g_N(t)\}_{0 \leq t \leq T} = \{g(t)\}_{0 \leq t \leq T}) \underset{\varepsilon \rightarrow 0}{\asymp} e^{-\frac{1}{\varepsilon} \int_0^T dt \text{Sup}_p \{ \int \dot{g} p \text{drdv} - H[g, p] \}}, \quad (2)$$

where \dot{g} is the time derivative of g , p is a function over the μ -space and is called the conjugated momentum of \dot{g} , the Hamiltonian H is a functional of g and p that characterizes the dynamical fluctuations, and where the symbol $\underset{\varepsilon \rightarrow 0}{\asymp}$ means a logarithmic equivalence (roughly speaking, the log of the left hand side is equivalent to the log of the right hand side). We note that H is not the Hamiltonian of the microscopic dynamics but H rather defines a statistical field theory that quantifies the probabilities of paths of the empirical distribution. H is associated with a Lagrangian $L[g, \dot{g}] = \text{Sup}_p \{ \int \dot{g} p \text{drdv} - H[g, p] \}$ and an action $\int_0^T dt L(g, \dot{g})$. The large deviation speed ε is a small parameter associated to the kinetic limit. ε could be $1/N$, but more generally it will depend on the physical system under consideration.

In the paper [4], we explained why deriving a dynamical large deviation principle like (2) shed an illuminating perspective on the irreversibility paradox. In a nutshell, if the microscopic dynamics is time-reversible, then H will automatically verify a time-reversal symmetry, relating the microscopic time-reversibility to the time-reversibility of the stochastic process of the empirical distribution. The entropy will be automatically related to the quasipotential, quantifying precisely the relation between the dynamical properties of the field theory determined by H , to the interpretation of the entropy as characterizing the static properties through the Einstein formula (1). The increase of the entropy for relaxation paths will immediately follow as a general property of the quasipotential, as a mere consequence of the convexity of H with respect to the variable p , a property which is always true for a large deviation Hamiltonian. Then (2) characterizes the large deviations of a time-reversible process, and thus does not break the time reversibility. The most probable evolution of this time-reversible process will break time-reversal symmetry because we consider a specific path, and will be the solution of the kinetic equation. This explains why the kinetic equation increases S although the microscopic dynamics is time-reversible. Moreover, (2) characterizes the probability of any paths at the large deviation level, and quantifies very precisely the exponential concentration close to the solution of the kinetic equation.

Several works recently computed the dynamical large deviations for particle systems. One of the firsts was a work by Derrida, Lebowitz and Speer [8] for systems of particles that have a Markovian dynamics, for instance the SEP (Simple Exclusion Process). Following this work, Rome's group derived a consistent general formalism to describe phenomenologically macroscopic fluctuation theories [2] of systems which mesoscopic dynamics is diffusive. Those two complementary approaches nicely describe the dynamical large deviations for a large class of particle systems.

However, it would be interesting to deal with large deviation principles for particle systems with a more physical dynamics than the one considered so far, starting from the Hamiltonian dynamics of atoms or molecules.

This paper is the second of a series of three in which we address the computation of the large deviation Hamiltonian H , and of the large deviation parameter ε , for the three classical kinetic theories associated respectively to the dilute gases (the Boltzmann equation), mean field interactions, plasma and self-gravitating stars (the Balescu–Guernsey–Lenard equation), and plasma at a scale much smaller than the Debye length and in a weak coupling limit (the Landau equation). In our first paper [4], we explained that for dilute gases, ε is the inverse of the number of particles in a volume of the size of the mean free path. In this first paper, we also derived the Boltzmann large deviation Hamiltonian (see formulas (55-58) in section 5 of the present paper) from the natural Boltzmann hypothesis of molecular chaos. Long before our work [4], Rezakhanlou has proven [18] a large deviation result for 1D stochastic dynamics mimicking the hard sphere dynamics. The functional form of the large deviation Hamiltonian we deduced from Boltzmann’s molecular chaos hypothesis is actually the same as Rezakhanlou’s one. Moreover, for the specific case of hard spheres and in the Boltzmann-Grad limit, Bodineau, Gallagher, Saint-Raymond and Simonella [3] have rigorously proven large deviation asymptotics that give an information equivalent to the large deviation formulas (55-58), and which is valid for times of order of one collision time, as Lanford result for the kinetic equation.

The aim of the present paper, is to derive the large deviation Hamiltonian, and the formula for ε , for plasma in the weak coupling limit, and scales much smaller than the Debye length, which kinetic equation is the Landau equation. The aim of our third paper, in preparation, is to derive the large deviation Hamiltonian, and the formula for ε , associated to plasma in the weak coupling limit and systems with long range interactions, independently on the hypothesis that perturbations are at scales much smaller than the Debye length. The kinetic equation for this third case is the Balescu–Guernsey–Lenard [11, 15]. In both the second and third paper, we consider first the case of homogeneous dynamics, for simplicity.

In this paper, we deal with the case of the kinetic theory that leads to the Landau equation [11, 15]. The Landau equation is the law of large numbers for the relaxation to equilibrium of an homogeneous plasma, in the weak coupling limit and for perturbations at scales much smaller than the Debye length. We consider more generally any system with long range interactions at a scale much smaller than the Debye length scale (the scale at which inertia and interaction effects do balance each others). For these systems, we consider the rescaled empirical density $g_\Lambda(\mathbf{r}, \mathbf{v}, t) \equiv \Lambda^{-1} \sum_{n=1}^N \delta(\mathbf{v} - \mathbf{v}_n(t)) \delta(\mathbf{r} - \mathbf{r}_n(t))$, where Λ is plasma parameter, e.g. the number of particles in a box of the size of the Debye length. The main result of this paper is the derivation of the Landau Hamiltonian H_{Landau} that describes the dynamical large deviations for the probability of any homogeneous evolution paths $\{f(t)\}_{0 \leq t \leq T}$ for the empirical density $\{g_\Lambda(t)\}_{0 \leq t \leq T}$. We obtain

$$\mathbf{P}(\{g_\Lambda(t)\}_{0 \leq t \leq T} = \{f(t)\}_{0 \leq t \leq T}) \underset{\Lambda \rightarrow \infty}{\asymp} e^{-\Lambda \int_0^T dt \text{Sup}_p \{ \int d\mathbf{r} d\mathbf{v} \dot{f} p - H_{\text{Landau}}[f, p] \}}, \quad (3)$$

where $p(\mathbf{v}, t)$ is a homogenous function over the μ -space, and where the large deviation Hamiltonian $H_{\text{Landau}}[f, p]$ is

$$H_{\text{Landau}}[f, p] = H_{MF}[f, p] + H_I[f, p], \quad (4)$$

with

$$H_{MF}[f, p] = \int d\mathbf{r} d\mathbf{v} f \left\{ \mathbf{b}[f] \cdot \frac{\partial p}{\partial \mathbf{v}} + \frac{\partial}{\partial \mathbf{v}} \cdot \left(\mathbf{D}[f] \cdot \frac{\partial p}{\partial \mathbf{v}} \right) + \mathbf{D}[f] : \frac{\partial p}{\partial \mathbf{v}} \frac{\partial p}{\partial \mathbf{v}} \right\}, \quad (5)$$

and

$$H_I[f, p] = - \int d\mathbf{r} d\mathbf{v}_1 d\mathbf{v}_2 f(\mathbf{v}_1) f(\mathbf{v}_2) \frac{\partial p}{\partial \mathbf{v}_1} \frac{\partial p}{\partial \mathbf{v}_2} : \mathbf{B}(\mathbf{v}_1, \mathbf{v}_2). \quad (6)$$

The drift \mathbf{b} , diffusion tensor \mathbf{D} , and interaction tensor \mathbf{B} will be defined in the following sections. In particular, in this paper we show that whenever the size of the domain is larger than the Debye length, the relevant large deviation parameter is the plasma parameter Λ , while whenever the size of the domain is smaller than the Debye length, the relevant large deviation parameter is the number of particles.

We give two derivations of this Hamiltonian H_{Landau} . The first derivation starts from the large deviation Hamiltonian H_B [4] of a dilute gas in the Boltzmann–Grad limit (the large deviation for the Boltzmann kinetic theory) and considers the weak scattering limit. Both the Landau equation and the large deviation Hamiltonian H_{Landau} are obtained in the weak scattering limit from the large deviations of the Boltzmann kinetic theory. As a second derivation, we compute the large deviation Hamiltonian H_{Landau} directly from the plasma dynamics.

Independently from these two derivations, we also derive another new and important result: the large deviation Hamiltonian for the empirical density of N particles driven by N independent Markov processes (equation (41)). In the case of N diffusions with mean field interactions we obtain the Hamiltonian (5). One of the conclusions of this paper is that, while the Landau equation can be understood as a diffusion equation for N independent particles (Fokker-Planck interpretation), the large deviation Hamiltonian associated to the Landau equation is not the large deviation Hamiltonian of N independent particles. The weak physical interactions impose a new interaction term (6) which is essential for describing the large deviations. We prove that this interaction term (6) is also crucial for the energy conservation properties of the statistical field theory. Finally, all along the paper we prove the expected properties of the obtained Hamiltonian: conservation law symmetries, time-reversal symmetry, and we prove that the entropy is the negative of the quasipotential up to conservation laws.

We also explain that the path large deviation principle for the empirical distribution implies a gradient structure for the Landau equation. This gradient structure does not involve the Wasserstein distance as in many kinetic theories, but another more intricate distance that takes into account of the effect of weak interaction between particles in the kinetic limit.

The subject of plasma fluctuations is a classical one, see for instance §51 of [11], or chapter 11 of [1], among hundreds of other publications. For instance, the space-time two-point correlations for the fluctuations of the distribution function and potential of a plasma with a non-equilibrium distribution function which is stable for Vlasov dynamics, for times much smaller than the evolution time of the distribution function itself, can be computed either from a Klimontovich approach [11], a truncation of the BBGKY hierarchy [15], or using equipartition of local van Kampen modes [13]. One may wonder how the present work connects to those classical results. First, as will be clear in section 6.2, our derivation starts from the classical formulas for the local in time fluctuations of non-equilibrium stable distributions. Then our approach is fully consistent with the classical results of fluctuations in plasma. However, we address a question of a nature that has never been considered so far: the probability that those local fluctuations lead to a large deviation in the long term evolution of the distribution function. Our main result, the large deviation Hamiltonian that describes the long term path probability for the distribution function, is thus entirely new, as far as we know. It is fully compatible with the classical theories of local fluctuations in plasmas.

In section 2, we present the expected general properties for the dynamical large deviations of a kinetic theory. In section 2, we also present heuristically two important and classical frameworks for dynamical large deviation theory: large deviations due to N independent small increments leading to an effect of order 1, and large deviations for slow-fast systems. In section 3, we present the dynamics of N particles with Coulomb interactions and the related kinetic equations: the Vlasov, the Balescu–Guernsey–Lenard and the Landau equations. Inspired by the structure of the Landau and Balescu–Guernsey–Lenard equation, which can be seen as non-linear Fokker–Planck equations, we compute in section 4 the large deviation Hamiltonian for the empirical density of N particles with diffusions coupled in a mean field way. We show that it cannot be the large deviation Hamiltonian for neither the Balescu–Guernsey–Lenard nor the Landau equation. In section 5, we derive the large deviation Hamiltonian for the kinetic theory associated to the Landau equation, from the one previously obtained for the Boltzmann equation. This Hamiltonian is quadratic in p the conjugated variable to \dot{f} , showing that for the Landau equation Gaussian fluctuations properly describe path large deviations. It is natural to use this Hamiltonian large deviation principle for the Landau equation kinetic theory, to conjecture a Hamiltonian large deviation principle for the kinetic theory leading to the Balescu–Guernsey–Lenard equation, by replacing the Landau collision kernel by the Balescu–Guernsey–Lenard one. We call this Hamiltonian the dressed Landau Hamiltonian. However, we show in section 5 that this dressed Landau Hamiltonian is not the large deviation Hamiltonian associated to the kinetic theory leading to the Balescu–Guernsey–Lenard equation. We argue that the large deviation Hamiltonian for the Balescu–Guernsey–Lenard kinetic theory is not quadratic in the conjugated momentum (the large deviations are driven by non-Gaussian fluctuations). Finally, in section 6, we compute the large deviation Hamiltonian directly from the N particle dynamics. We show that a cumulant expansion coincides with the dressed Landau Hamiltonian, up to a certain truncation in terms of the power of the interaction potential. We explain that this justifies that

the large deviation Hamiltonian for the kinetic theory associated to the Landau equation is quadratic in the conjugated momentum, because of the limit of small scales compared to the Debye length. This result is fully consistent with the one obtained in section 5.

2 Dynamical large deviations and kinetic theories

The aim of many works in statistical mechanics is to describe the evolution of the empirical density of particle dynamics. For instance, in this work, we will consider the rescaled empirical distribution $g_\varepsilon(\mathbf{v}, t) \equiv \varepsilon \sum_{n=1}^N \delta(\mathbf{r} - \mathbf{r}_n(t)) \delta(\mathbf{v} - \mathbf{v}_n(t))$. A large deviation principle for the dynamics of the empirical distribution is a result that reads

$$\mathbf{P}(\{g_\varepsilon(t)\}_{0 \leq t \leq T} = \{g(t)\}_{0 \leq t \leq T}) \underset{\varepsilon \rightarrow 0}{\asymp} e^{-\frac{1}{\varepsilon} \int_0^T dt \text{Sup}_p \{ \int \dot{g} p d\mathbf{r} d\mathbf{v} - H[g, p] \}}, \quad (7)$$

where ε is a small parameter that can be related to N . This section present a set of known results about large deviation theory which are essential for the following discussion. In section 2.1 we describe the expected properties of any such large deviation principle for the kinetic theory of the empirical distribution. A more detailed account of a similar discussion can be found in [4]. In section 2.2, we present two important frameworks that allow to compute dynamical large deviations: on one hand, large deviations due to N independent small increments leading to an effect of order 1, and on the other hand, large deviations for slow-fast systems.

2.1 Large deviation for kinetic theories

2.1.1 General properties of path large deviations and expected properties for large deviations for kinetic theories

Most probable evolution We consider the properties of a stochastic process which rare fluctuations are described, at the level of large deviations, by the action

$$\mathcal{A}[g] = \int_0^T dt L[g, \dot{g}] = \int_0^T dt \text{Sup}_p \left[\int p \dot{g} - H[g, p] \right]. \quad (8)$$

(see equation (7)). The kinetic equation is expected to be the most probable evolution corresponding to the action (8), and with initial condition $g_r(t=0) = g$. It is also called a **relaxation path issued from** g . It solves $\frac{\partial g_r}{\partial t} = R[g_r]$, with initial condition $g_r(t=0) = g$, where $R[g] = \arg \inf_{\dot{g}} L[g, \dot{g}]$. Then one easily proves that

$$\dot{g} = \frac{\delta H}{\delta p} [g, p=0], \quad (9)$$

is the kinetic equation.

Quasipotential and macrostate entropy We assume that the stochastic process g_ε has a stationary distribution P_s which dynamics follows the large deviation principle

$$P_s(g) \equiv \mathbb{E}[\delta(g_\varepsilon - g)] \underset{\varepsilon \downarrow 0}{\asymp} \exp\left(-\frac{U[g]}{\varepsilon}\right), \quad (10)$$

where U is called the *quasipotential*. In order to simplify the following discussion, we also assume that the relaxation equation has a single fixed point g_0 and that any solution to the relaxation equation converges to g_0 . Then the quasipotential verifies

$$U[g] = \inf_{\{\tilde{g}(t)\}_{-\infty \leq t \leq 0} | \tilde{g}(-\infty) = g_0 \text{ and } \tilde{g}(0) = g} \int_{-\infty}^0 dt L[\tilde{g}, \dot{\tilde{g}}].$$

The minimizer of this variational problem, that is the most probable path starting from g_0 and ending at g , is denoted $g_f(t, g)$ and is called the *fluctuation path ending at g* .

For many kinetic theory, we expect from equilibrium statistical mechanics that the quasipotential $U[g]$ is the opposite of the entropy $S[g] = -\int d\mathbf{v} d\mathbf{r} g \log g$ constrained by the conserved quantities

$$U[g] = \begin{cases} -S[g] & \text{if } M[g] = 1, \mathbf{P}[g] = 0, \text{ and } E[g] = E_0 \\ -\infty & \text{otherwise.} \end{cases}$$

We have the following properties which are direct consequences of the definitions of H and L , and whose proofs are classical and given for example in sections 7.2 to 7.4 of [4]:

1. H is a convex function of the variable p and $H[g, p = 0] = 0$, see sec. 7.2.1 of [4].
2. The relaxation paths solve the equation $\frac{\partial g}{\partial t} = R[g]$ with $\inf_{\dot{g}} L[g, \dot{g}] = 0 = L[g, R[g]]$, and $R[g] = \frac{\delta H}{\delta p}[g, 0]$, see sec. 7.2.2 of [4].
3. The quasipotential solves the stationary **Hamilton–Jacobi equation**

$$H\left[g, \frac{\delta U}{\delta g}\right] = 0, \quad (11)$$

see sec. 7.2.3 of [4].

4. **The fluctuation paths** solve

$$\dot{g} = F[g] \equiv \frac{\delta H}{\delta p}\left[g, \frac{\delta U}{\delta g}\right],$$

see sec. 7.2.4 of [4].

5. As H is convex, the quasipotential decreases along the relaxation paths

$$\frac{dU}{dt}[g_r] = H[g_r, 0] - H\left[g_r, \frac{\delta U}{\delta g}[g_r]\right] + \int d\mathbf{r} d\mathbf{v} \frac{\delta H}{\delta p}[g_r, 0] \frac{\delta U}{\delta g}[g_r] \leq 0, \quad (12)$$

see sec. 7.2.5 of [4]. For kinetic theories, because the quasipotential is the entropy whenever the conservation laws are verified, we can immediately conclude that the entropy will increase along the solution of the kinetic equation.

6. As H is convex, the quasipotential increases along the fluctuation paths

$$\frac{dU}{dt}[g_f] = H[g_f, 0] - H\left[g_f, \frac{\delta U}{\delta g}[g_f]\right] + \int \mathbf{dr}d\mathbf{v} \frac{\delta H}{\delta p}\left[g_f, \frac{\delta U}{\delta g}[g_f]\right] \frac{\delta U}{\delta g}[g_f] \geq 0, \quad (13)$$

see sec. 7.2.5 of [4]. For kinetic theories, because the quasipotential is the entropy whenever the conservation laws are verified, we can immediately conclude that the entropy will decrease along the fluctuation paths.

7. **Generalized detailed balance** (see sec. 7.3.2 of [4]). Let I be an involution that characterizes time-reversal symmetry (for instance the map that correspond to velocity or momentum inversion in many systems). We assume that I is self adjoint for the L^2 scalar product, that is $\int \mathbf{dr}d\mathbf{v} I[g]p = \int \mathbf{dr}d\mathbf{v} gI[p]$. The detailed balance conditions for the quasipotential U combined with the involution I are $U[g] = U[I[g]]$ is

$$H[I[g], -I[p]] = H\left[g, p + \frac{\delta U}{\delta g}\right]. \quad (14)$$

For any systems for which the microscopic dynamics is time reversible, we can infer that the stochastic process of the empirical distribution has to be time-reversal symmetric. As a consequence the large deviation principle should verify detailed balance and the symmetry relation has to be verified.

8. As can be easily checked, if either the detailed balance or the generalized detailed balance conditions are verified, then U satisfies the stationary Hamilton-Jacobi equation (11).

9. If the detailed balance condition is verified, and if U is the quasipotential, then for a path $\{g(t)\}_{0 \leq t \leq T}$ and its time reversed one $\{I[g(T-t)]\}_{0 \leq t \leq T}$ we have the symmetry for the path probability

$$P[\{g_\varepsilon(t)\}_{0 \leq t \leq T} = \{g(t)\}_{0 \leq t \leq T}] e^{-\frac{U[g(t=0)]}{\varepsilon}} = P[\{g_\varepsilon(t)\}_{0 \leq t \leq T} = \{I[g(T-t)]\}_{0 \leq t \leq T}] e^{-\frac{U[I[g(t=T)]]}{\varepsilon}},$$

see sec. 7.3.1 of [4].

10. **Conserved quantities** (see sec. 7.2.6 of [4]). At the level of the large deviations, the condition for $C[g]$ to be a conserved quantity is either

$$\text{for any } g \text{ and } p, L[g, \dot{g}] = +\infty \text{ if } \int \mathbf{dr}d\mathbf{v} \frac{\partial g}{\partial t} \frac{\delta C}{\delta g} \neq 0,$$

or

$$\text{for any } g \text{ and } p, \int \mathbf{dr}d\mathbf{v} \frac{\delta H}{\delta p}[g, p] \frac{\delta C}{\delta g} = 0. \quad (15)$$

In general, kinetic theories conserve at least mass, momentum and energy.

11. **A sufficient condition for U to be the quasipotential** (see sec. 7.4 of [4]). If U solves the Hamilton-Jacobi equation, if U has a single minimum g_0 with $U[g_0] = 0$, and if for any g the solution of the reverse fluctuation path dynamics $\frac{\partial \tilde{g}}{\partial t} = -F[\tilde{g}] = -\frac{\delta H}{\delta p}\left[\tilde{g}, \frac{\delta U}{\delta \tilde{g}}\right]$ with $\tilde{g}(0) = g$ converges to g_0 for large times, then U is the quasipotential.

2.2 Dynamical large deviations

When the evolution of a stochastic process is the consequence of the effect of a large number of small amplitude and statistically independent moves, in the limit of a large number of moves, a law of large number naturally follows. It is often very important to understand the large deviations with respect to this law of large number. For continuous time Markov processes, for instance diffusions with small noises, or more generally locally infinitely divisible processes, a general framework can be developed in order to estimate the probability of large deviations. In section 2.2, taken from [4] and initially inspired by [9, 10], we present this framework briefly and the main result: the formula (16) for computing the large deviation Hamiltonian in this case.

Another classical framework for large deviations are large deviations for the effective dynamics of the slow variable in a slow-fast dynamics (time averaging of the fast degrees of freedom). This classical framework is discussed in the case of stochastic processes in [10, 22]. When the slow dynamics is deterministic similar results have been proven for instance by Kifer. A simple heuristic account is given in [5]. [5] discuss also at length the case when the fast variable is an Ornstein-Uhlenbeck and the coupling with the slow variable is through a quadratic form. In this specific case the Hamiltonian can be computed by solving a matrix Riccati equation.

2.2.1 Large deviation rate functions from the infinitesimal generator of a continuous time Markov process

We consider $\{g_\varepsilon(t)\}_{0 \leq t \leq T}$, a family of continuous time Markov processes parametrized by a real number ε . We denote G_ε the infinitesimal generator of the process (the definition of the infinitesimal generator of a continuous time Markov process is given in [4]). We assume that for all $p \in L^2(\mathbb{T}^3 \times \mathbb{R}^3)$ the limit

$$H[g, p] = \lim_{\varepsilon \downarrow 0} \varepsilon G_\varepsilon \left[e^{\frac{1}{\varepsilon} \int \text{drdv } p(\mathbf{r}, \mathbf{v}) g(\mathbf{r}, \mathbf{v})} \right] e^{-\frac{1}{\varepsilon} \int \text{drdv } p(\mathbf{r}, \mathbf{v}) g(\mathbf{r}, \mathbf{v})} \quad (16)$$

exists. Then the family g_ε verifies a large deviation principle with rate ε and rate function

$$L[g, \dot{g}] = \sup_p \{p \dot{g} - H[g, p]\}. \quad (17)$$

This means that the probability that the path $\{g_\varepsilon(t)\}_{0 \leq t < T}$ be in a neighborhood of $\{g(t)\}_{0 \leq t < T}$ verifies

$$P[\{g_\varepsilon(t)\}_{0 \leq t < T} \approx \{g(t)\}_{0 \leq t < T}] \underset{\varepsilon \downarrow 0}{\asymp} \exp\left(-\frac{\int_0^T dt L[g, \dot{g}]}{\varepsilon}\right), \quad (18)$$

where the symbol $\underset{\varepsilon \downarrow 0}{\asymp}$ is a logarithm equivalence ($g_\varepsilon \underset{\varepsilon \downarrow 0}{\asymp} \exp(\varphi/\varepsilon) \iff \lim_{\varepsilon \downarrow 0} \varepsilon \log g_\varepsilon = \varphi$).

This result is proven for specific cases (diffusions, locally infinitely divisible processes) in the Theorem 2.1, page 127, of the third edition of Freidlin-Wentzell textbook [10]. A general heuristic derivation is given in section 7.1.2 of [4]. Equation

(16) will be the key starting point for several results of this paper. For instance, we apply this framework to the fluctuations of N independent diffusions and N diffusions with mean field coupling in section 4.

In formula (16) the infinitesimal generator is tested through the function $e^{\frac{1}{\varepsilon} \int dr dv pg}$. In the small ε limit, this tests changes of the observable which are of order of ε . The ε prefactor in the right hand side of equation (16) means that the overall effect of these small changes of order ε is expected to be of order $1/\varepsilon$. H in formula (16) thus accounts for the effects of a large number (of order $1/\varepsilon$) of small amplitude statistically independent moves (each one of order ε).

2.2.2 Large deviation for slow-fast systems

We consider the slow-fast dynamics

$$\begin{cases} \frac{dX_\varepsilon}{d\tau} = \alpha(X_\varepsilon, Y_\varepsilon) \\ \frac{dY_\varepsilon}{d\tau} = \frac{1}{\varepsilon} \beta(X_\varepsilon, Y_\varepsilon) + \frac{1}{\sqrt{\varepsilon}} \gamma(X_\varepsilon, Y_\varepsilon) \frac{dW}{d\tau} \end{cases}, \quad (19)$$

where X_ε is the slow variable, Y_ε the fast variable, W a Wiener process, and ε quantifies the time scale separation. We assume that the dynamics for Y_ε is mixing over timescales of order ε . The following discussion would apply for other classes of dynamics for Y_ε , beyond diffusions, with little modifications, for instance for chaotic deterministic systems with mixing hypothesis.

We are interested in the slow dynamics for X_ε . Then for generic hypotheses, we have the large deviation principle

$$\mathbb{P}(X_\varepsilon = x) \underset{\varepsilon \rightarrow 0}{\asymp} e^{-\frac{1}{\varepsilon} \int_0^T \text{Sup}_p \{ \dot{x} \cdot p - H(x, p) \} d\tau} \quad (20)$$

$$\text{with } H(x, p) = \lim_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{E}_x \left\{ \exp \left[p \cdot \int_0^T \alpha(x, Y_x(t)) dt \right] \right\}, \quad (21)$$

where p is conjugated to \dot{x} , the average \mathbb{E}_x is an average over the Y_x process with frozen x (the solution of $\frac{dY_x}{dt} = \beta(x, Y_x) + \gamma(x, Y_x) \frac{dW}{dt}$).

This classical result is proven in the case of stochastic processes in [10, 22]. When the slow dynamics is deterministic, similar results have been proven for instance by Kifer. A simple heuristic account for any Markov dynamics is given in [5]. The result (19-21) is easily heuristically understood as $L(x, \dot{x}) = \text{sup}_p \{ \dot{x} \cdot p - H(x, p) \}$ appears as a large-time large deviations result, of the Freidlin-Wentzell type, for the Newton increment of the slow variable

$$\frac{X_\varepsilon(\tau + \Delta\tau) - x}{\Delta\tau} = \frac{1}{\Delta\tau} \int_0^{\Delta\tau} \alpha(X_\varepsilon(u), Y_\varepsilon(u)) du \simeq \frac{\varepsilon}{\Delta\tau} \int_0^{\frac{\Delta\tau}{\varepsilon}} \alpha(x, Y_x(t)) dt.$$

Then formula (21), with $L(x, \dot{x}) = \text{Sup}_p \{ \dot{x} \cdot p - H(x, p) \}$, appears as a Gärtner–Ellis formula for the large time large deviations

$$\mathbb{E}_x \left[\delta \left(\frac{X_\varepsilon(\tau + \Delta\tau) - x}{\Delta\tau} - \dot{x} \right) \right] \underset{\varepsilon \rightarrow 0}{\asymp} e^{-\frac{L(x, \dot{x}) \Delta\tau}{\varepsilon}}.$$

This last formula is the temporal increment of formula (2.2.2).

We will use formula (19-21) for computing the large deviations of the empirical density, from the microscopic dynamics, in section 6.

3 Dynamics of plasmas

In this section we set up the definitions, and present known results about the kinetic theory of the dynamics of N particles with Coulomb interactions, in limit of a large plasma parameter (or equivalently weak coupling). In section 3.1, we define the Hamiltonian dynamics of N particles coupled by a Coulomb pairwise interaction. In section 3.2, we introduce the Vlasov equation that describes the evolution of the empirical density on timescales of order one. In section 3.3, we introduce the Balescu–Guernsey–Lenard equation that describes the long time relaxation of the empirical density, from Vlasov stationary solutions to the Maxwell-Boltzmann equilibrium distribution, and some of its important physical properties. In section 3.4 we present the Landau equation, which is an approximation of the Balescu–Guernsey–Lenard equation which is valid for scales which are small compared to the Debye length. In section 3.5, we show that these equations can be seen as non-linear Fokker-Planck equations.

3.1 The dynamics of the Coulomb plasma

We consider of a Coulomb plasma of N particles with positions $\{\mathbf{r}_n\}_{1 \leq n \leq N}$ and velocities $\{\mathbf{v}_n\}_{1 \leq n \leq N}$, and with equal charge e and mass m . We consider that \mathbf{r}_n belongs to a 3-dimensional torus \mathbb{T}^3 of size L^3 (doubly periodic boundary conditions), and $\mathbf{v}_n \in \mathbb{R}^3$. However most of our discussion easily generalizables to $\mathbf{r}_n \in \mathbb{R}^3$, with slight modifications. The dynamics is a Hamiltonian one with

$$\begin{cases} \frac{d\mathbf{r}_n}{dt} = \mathbf{v}_n \\ \frac{d\mathbf{v}_n}{dt} = -\frac{e^2}{4\pi\epsilon_0 m} \sum_{m \neq n} \frac{d}{d\mathbf{r}_n} W(\mathbf{r}_n - \mathbf{r}_m) \end{cases} \quad (22)$$

where ϵ_0 is the vacuum permittivity and W is the Coulomb potential. In both a finite box and an infinite space, W can be defined through its Fourier transform

$$\hat{W}(\mathbf{k}) = \int d\mathbf{r} e^{-i\mathbf{k}\cdot\mathbf{r}} W(\mathbf{r}),$$

with

$$\hat{W}(\mathbf{k}) = \frac{1}{k^2},$$

and where $k = |\mathbf{k}|$. We define the Debye length $\lambda_D = \left(\frac{\epsilon_0 k_B T L^3}{e^2 N}\right)^{1/2}$, where k_B is the Boltzmann constant and T the temperature. This length is the typical length beyond which Coulomb interaction are screened [15]. We also define the plasma electron frequency $\omega_{pe} = \left(\frac{e^2 N}{\epsilon_0 m L^3}\right)^{1/2}$, which is the pulsation of the Langmuir waves in a plasma [15], and the thermal velocity $v_T = \lambda_D \omega_{pe} = \sqrt{k_B T / m}$. Then, if we use the dimensionless variables

$$\tilde{\mathbf{r}} = \mathbf{r} / \lambda_D, \quad \tilde{\mathbf{v}} = \mathbf{v} / v_T \quad \text{and} \quad \tilde{t} = \omega_{pe} t,$$

the dimensionless dynamical equations (22) read

$$\begin{cases} \frac{d\tilde{\mathbf{r}}_n}{d\tilde{t}} = \tilde{\mathbf{v}}_n \\ \frac{d\tilde{\mathbf{v}}_n}{d\tilde{t}} = -\frac{1}{\Lambda} \sum_{m \neq n} \frac{d}{d\tilde{\mathbf{r}}_n} \tilde{W}(\tilde{\mathbf{r}}_n - \tilde{\mathbf{r}}_m) \end{cases}$$

where $\Lambda \equiv N(\lambda_D/L)^3$ is the so-called plasma parameter. Λ is the number of particles in a box of size of the Debye length. In this new system of units, called plasma units, $\tilde{\mathbf{r}}_n$ belongs to the 3-dimensional torus $(L/\lambda_D)\mathbb{T}^3$. The dimensionless Coulomb potential \tilde{W} is defined by

$$\hat{W}(\tilde{\mathbf{k}}) = \int d\tilde{\mathbf{r}} e^{-i\tilde{\mathbf{k}} \cdot \tilde{\mathbf{r}}} \tilde{W}(\tilde{\mathbf{r}}),$$

with $\hat{W}(\tilde{\mathbf{k}}) = \frac{1}{k^2}$. For simplicity, in the following we omit the tildes when referring to the dimensionless variables. We will work in dimensionless variables, and give the main results in both dimensionless and physical variables.

We call μ -space the (\mathbf{r}, \mathbf{v}) space. The μ -space is of dimension 6. Let us define g_Λ the μ -space empirical distribution function for the positions and velocities of the N particles rescaled by the plasma parameter

$$g_\Lambda(\mathbf{r}, \mathbf{v}, t) = \frac{1}{\Lambda} \sum_{n=1}^N \delta(\mathbf{r} - \mathbf{r}_n(t)) \delta(\mathbf{v} - \mathbf{v}_n(t)). \quad (23)$$

In the following we will consider the large plasma parameter limit, $\Lambda \rightarrow \infty$. Considering that Λ is the number of particles in a box of size of the Debye length, and that in our non-dimensional units the Debye length is fixed, the scaling $1/\Lambda$ in front of the empirical density (23) is natural.

If the box size L is larger than the Debye length λ_D , the interactions are screened beyond the Debye length and the effective interaction length scale is λ_D . Otherwise, if the size of the box is smaller than the Debye length, then the interactions are not screened in the box and they take place on a length scale L . We call $\ell = \min\{\lambda_D, L\}$ the effective interaction length scale.

In the following, we study the asymptotic dynamics of g_Λ as the number of particles in a box of the size of the effective interaction length scale, e.g. $N\ell^3/L^3$ goes to infinity. If $L > \lambda_D$, this asymptotic regime is the limit of a large plasma parameter Λ ; if $L < \lambda_D$, it is the limit of a large number of particles N . In this paper, we present detailed results for the case $L > \lambda_D$, and we briefly discuss the slight modifications relevant for the case $L < \lambda_D$ at the end of section 6.

3.2 The Vlasov equation

From equation (22), one immediately obtains the Klimontovich equation

$$\frac{\partial g_\Lambda}{\partial t} + \mathbf{v} \cdot \frac{\partial g_\Lambda}{\partial \mathbf{r}} - \frac{\partial V[g_\Lambda]}{\partial \mathbf{r}} \cdot \frac{\partial g_\Lambda}{\partial \mathbf{v}} = 0, \quad (24)$$

where $V[g_\Lambda](\mathbf{r}, t) = \int d\mathbf{v}' d\mathbf{r}' W(\mathbf{r} - \mathbf{r}') g_\Lambda(\mathbf{r}', \mathbf{v}', t)$. This is an exact equation for the evolution of g_Λ , if W is regular enough. For the Coulomb interaction, the formal equation (24) has to be interpreted carefully. In the following, we do not discuss the divergences that might occur related to small scale interactions. At a mathematic level, this would be equivalent to considering a potential which is regularized at small scales, and smooth. The Klimontovich equation (24) contains all the information about the trajectories of the N particles. We would like to build a kinetic theory, that describes the stochastic process for g_Λ at a mesoscopic level.

An important first result is that the sequence $\{g_\Lambda\}$ obeys a law of large numbers when $\Lambda \rightarrow +\infty$. More precisely, if we assume there is a set of initials conditions $\{g_\Lambda^0\}$ such as $\lim_{\Lambda \rightarrow +\infty} \frac{1}{\Lambda} \sum_{n=1}^N g_n^0(\mathbf{r}, \mathbf{v}) = g^0(\mathbf{r}, \mathbf{v})$, then over finite time interval $t \in [0, T]$, the empirical distribution function $g_\Lambda(t)$ converges to $g(t)$ as Λ goes to infinity, where g solves the Vlasov equation

$$\frac{\partial g}{\partial t} + \mathbf{v} \cdot \frac{\partial g}{\partial \mathbf{r}} - \frac{\partial V[g]}{\partial \mathbf{r}} \cdot \frac{\partial g}{\partial \mathbf{v}} = 0 \text{ with } g(\mathbf{r}, \mathbf{v}, t=0) = g^0(\mathbf{r}, \mathbf{v}). \quad (25)$$

As the Klimontovich and the Vlasov equations are formally the same, this is actually a stability result for the Vlasov equation. It has first been proven for smooth interactions by Braun and Hepp [6] for smooth enough potential W . This Vlasov equation has a infinity of Casimir conserved quantities. As a consequence, it has an infinite number of stable stationary states [24]. Any homogeneous distribution $g(\mathbf{r}, \mathbf{v}) = f(\mathbf{v})$ is a stationary solution of the Vlasov equation. In the following, we will consider homogeneous linearly stable stationary solutions of the Vlasov equation $f(\mathbf{v})$. The linear stability of such distributions can be assessed by studying the dielectric susceptibility $\varepsilon[f](\mathbf{k}, \omega)$ [15, 11], defined by

$$\varepsilon[f](\mathbf{k}, \omega) = 1 - \hat{W}(\mathbf{k}) \int d\mathbf{v} \frac{\mathbf{k} \cdot \frac{\partial f}{\partial \mathbf{v}}}{\mathbf{k} \cdot \mathbf{v} - \omega - i\tilde{\varepsilon}}. \quad (26)$$

Equation (26) and every other equations involving $\pm i\tilde{\varepsilon}$ have to be understood as the limit as $\tilde{\varepsilon}$ goes to zero with $\tilde{\varepsilon}$ positive. The dielectric susceptibility function ε plays the role of a dispersion relation in the linearized dynamics, and a solution f is stable if $\varepsilon[f]$ has no poles except for ω on the real line.

From the point of view of dynamical systems, those homogeneous solutions might be attractors of the Vlasov equation, with some sort of asymptotic stability. At a linear level, this convergence for some of the observables, for instance the potential, is called Landau damping [15, 11]. Such a stability might also be true for the full dynamics. Indeed some non-linear Landau damping results have recently been proven [14].

In the following we will study the dynamics of g_Λ , when its initial condition is close to a homogeneous stable state $f(\mathbf{v})$. On time scales of order one, the distribution is stable and remains close to f according to the Vlasov equation. However a slow evolution occurs on a timescale τ of order Λ . For this reason, such f are called quasi-stationary states [24]. In the following section, we explain that this slow evolution is described by the Balescu–Guernsey–Lenard equation for most initial conditions. More precisely, g_Λ converges to the solution of the Balescu–Guernsey–Lenard equation as a law of large numbers.

3.3 The Balescu–Guernsey–Lenard equation

With the rescaling of time $\tau = t/\Lambda$, we expect a law of large numbers in the sense that “for almost all initial conditions” the empirical distribution function g_Λ converges to f , with f that evolves according to the Balescu–Guernsey–Lenard equation

$$\frac{\partial f}{\partial \tau} = \frac{\partial}{\partial \mathbf{v}} \cdot \int d\mathbf{v}_2 \mathbf{B}[f](\mathbf{v}, \mathbf{v}_2) \left(-\frac{\partial f}{\partial \mathbf{v}_2} f(\mathbf{v}) + f(\mathbf{v}_2) \frac{\partial f}{\partial \mathbf{v}} \right), \quad (27)$$

with

$$\mathbf{B}[f](\mathbf{v}_1, \mathbf{v}_2) = \pi \left(\frac{\lambda_D}{L} \right)^3 \int_{-\infty}^{+\infty} d\omega \sum_{\mathbf{k} \in 2\pi(\lambda_D/L)\mathbb{Z}^{*3}} \frac{\hat{W}(\mathbf{k})^2 \mathbf{k}\mathbf{k}}{|\mathcal{E}[f](\omega, \mathbf{k})|^2} \delta(\omega - \mathbf{k} \cdot \mathbf{v}_1) \delta(\omega - \mathbf{k} \cdot \mathbf{v}_2). \quad (28)$$

The tensor \mathbf{B} is called the collision kernel of the Balescu–Guernsey–Lenard equation.

We know no mathematical proof of such a result. In the theoretical physics literature, this equation is derived as an exact consequence of the dynamics once natural hypothesis are made. Two classes of derivations are known, either the BBGKY hierarchy detailed in [15] or the Klimontovich approach presented for instance in [11]. The Klimontovich derivation is the more straightforward from a technical point of view. We now recall the main steps of the Klimontovich derivation, that will be useful later.

In the following we will consider statistical averages over measures of initial conditions for the N particle initial conditions $\{\mathbf{r}_n^0, \mathbf{v}_n^0\}$. We denote \mathbb{E}_S the average with respect to this measure of initial conditions. As an example the measure of initial conditions could be the product measure $\prod_{n=1}^N g^0(\mathbf{r}_n^0, \mathbf{v}_n^0) d\mathbf{r}_n d\mathbf{v}_n$. But we might consider other measures of initial conditions. We recall that Λ is the number of particle in a box of size λ_D . We will consider the limit $\Lambda \rightarrow \infty$, which is a large particle number limit. For this reason the limit $\lim_{\Lambda \rightarrow \infty}$ of the empirical density will be called a law of large numbers¹. We assume that for the statistical ensemble of initial conditions, the law of large numbers $\lim_{\Lambda \rightarrow \infty} \frac{1}{\Lambda} \sum_{n=1}^N g_\Lambda^0(\mathbf{r}, \mathbf{v}) = g^0(\mathbf{r}, \mathbf{v})$ is valid at the initial time. This is true for instance for the product measure. In the following, for simplicity, we restrict the discussions to cases when the initial conditions are statistically homogenous: $g^0(\mathbf{r}, \mathbf{v}) = f^0(\mathbf{v})$. We are then looking for the law of large numbers $\lim_{\Lambda \rightarrow \infty} \frac{1}{\Lambda} \sum_{n=1}^N g_\Lambda^0(\mathbf{r}, \mathbf{v}, t) = f(\mathbf{v}, t)$, valid for any finite time t . Alternatively, we define f as the statistical average of g_Λ over the initial conditions $f(\mathbf{v}, t) = \mathbb{E}_S(g_\Lambda(\mathbf{r}, \mathbf{v}, t))$.

We define the fluctuations δg_Λ by $g_\Lambda(\mathbf{r}, \mathbf{v}, t) = f(\mathbf{v}) + \delta g_\Lambda / \sqrt{\Lambda}$. The scaling $1/\sqrt{\Lambda}$ is natural when we see the Vlasov equation (25) as a law of large numbers for the empirical distribution. For the potential we obtain $V[g_\Lambda] = V[\delta g_\Lambda] / \sqrt{\Lambda}$, as f is homogeneous. If we introduce this decomposition in the Klimontovich equation

¹ In order to have a discussion of the asymptotic behavior that will be independent on the box size L , for instance in order to consider infinite box size, it is more natural to discuss the limit $\Lambda \rightarrow \infty$ than $N \rightarrow \infty$.

(24), we obtain

$$\frac{\partial f}{\partial t} = \frac{1}{\Lambda} \mathbb{E}_S \left(\frac{\partial V[\delta g_\Lambda]}{\partial \mathbf{r}} \cdot \frac{\partial \delta g_\Lambda}{\partial \mathbf{v}} \right) \quad (29)$$

$$\frac{\partial \delta g_\Lambda}{\partial t} + \mathbf{v} \cdot \frac{\partial \delta g_\Lambda}{\partial \mathbf{r}} - \frac{\partial V[\delta g_\Lambda]}{\partial \mathbf{r}} \cdot \frac{\partial f}{\partial \mathbf{v}} = \frac{1}{\sqrt{\Lambda}} \left[\frac{\partial V[\delta g_\Lambda]}{\partial \mathbf{r}} \cdot \frac{\partial \delta g_\Lambda}{\partial \mathbf{v}} - \mathbb{E}_S \left(\frac{\partial V[\delta g_\Lambda]}{\partial \mathbf{r}} \cdot \frac{\partial \delta g_\Lambda}{\partial \mathbf{v}} \right) \right]. \quad (30)$$

In the first equation, the right hand side of the equation $\frac{1}{\Lambda} \mathbb{E}_S \left(\frac{\partial V[\delta g_\Lambda]}{\partial \mathbf{r}} \cdot \frac{\partial \delta g_\Lambda}{\partial \mathbf{v}} \right)$ is called the averaged non linear term and is responsible for the long term evolution of the distribution f . The right hand side of the second equation $\frac{1}{\sqrt{\Lambda}} \left[\frac{\partial V[\delta g_\Lambda]}{\partial \mathbf{r}} \cdot \frac{\partial \delta g_\Lambda}{\partial \mathbf{v}} - \mathbb{E}_S \left(\frac{\partial V[\delta g_\Lambda]}{\partial \mathbf{r}} \cdot \frac{\partial \delta g_\Lambda}{\partial \mathbf{v}} \right) \right]$ describes the fluctuations of the non-linear term. For stable distributions f , and on timescales much smaller than $\sqrt{\Lambda}$, we can neglect this term, following Klimontovich and classical textbooks [11]. This closes the hierarchy of the correlation functions. The Bogoliubov approximation then amounts at using the time scale separation between the evolution of f and δg_Λ . Then for fixed f , the equation for δg_Λ (30) is linear when f is fixed. One computes the correlation function $\mathbb{E}_S \left(\frac{\partial V[\delta g_\Lambda]}{\partial \mathbf{r}} \cdot \frac{\partial \delta g_\Lambda}{\partial \mathbf{v}} \right)$ resulting from (30) with fixed f , and argues that this two point correlation function converges to a stationary quantity on time scales much smaller than $\sqrt{\Lambda}$. Using this quasi-stationary correlation function $\mathbb{E}_S \left(\frac{\partial V[\delta g_\Lambda]}{\partial \mathbf{r}} \cdot \frac{\partial \delta g_\Lambda}{\partial \mathbf{v}} \right)$, one can compute the right hand side of (29) as a function of f . After time rescaling $\tau = t/\Lambda$, the closed equation which is obtained from (29) is the Balescu–Guernsey–Lenard equation (27). We do not reproduce these lengthy and classical computations that can be found in a plasma physics textbook, for instance in the chapter 51 of [11].

Symmetries and conservation properties. The Balescu–Guernsey–Lenard equation (27) has several important physical properties:

1. It conserves the mass $M[f]$, momentum $\mathbf{P}[f]$ and total kinetic energy $E[f]$ defined by

$$M[f] = \int d\mathbf{v} f(\mathbf{v}), \quad \mathbf{P}[f] = \int d\mathbf{v} \mathbf{v} f(\mathbf{v}) \quad \text{and} \quad E[f] = \int d\mathbf{v} \frac{\mathbf{v}^2}{2} f(\mathbf{v}). \quad (31)$$

2. It increases monotonically the entropy $S[f]$ defined by

$$S[f] = - \int d\mathbf{v} f(\mathbf{v}) \log f(\mathbf{v}).$$

3. It converges towards the Boltzmann distribution for the corresponding energy

$$f_B(\mathbf{v}) = \frac{\beta^{3/2}}{(2\pi)^{3/2}} \exp\left(-\beta \frac{\mathbf{v}^2}{2}\right).$$

The Balescu–Guernsey–Lenard is a good approximation to describe the long time evolution of system of particles with mean field interactions but it is quite complicated to handle, especially because the tensor \mathbf{B} depends on the actual distribution f in a non-trivial way. The Balescu–Guernsey–Lenard operator (the right hand side of (27)), is a very complex non-linear functional of f .

3.4 The Landau equation

Neglecting the collective effects in the Balescu–Guernsey–Lenard equation, we obtain the Landau equation

$$\frac{\partial f}{\partial \tau} = \frac{\partial}{\partial \mathbf{v}} \cdot \int d\mathbf{v}_2 \mathbf{B}(\mathbf{v}, \mathbf{v}_2) \left(-\frac{\partial f}{\partial \mathbf{v}_2} f(\mathbf{v}) + f(\mathbf{v}_2) \frac{\partial f}{\partial \mathbf{v}} \right), \quad (32)$$

where \mathbf{B} for the Landau equation is given by the same expression as the one for \mathbf{B} in equation (28), but with $\varepsilon(\mathbf{k}, \omega) = 1$:

$$\mathbf{B}(\mathbf{v}_1, \mathbf{v}_2) = \pi \left(\frac{\lambda_D}{L} \right)^3 \int_{-\infty}^{+\infty} d\omega \sum_{\mathbf{k} \in 2\pi(\lambda_D/L)\mathbb{Z}^{*3}} \hat{W}(\mathbf{k})^2 \mathbf{k} \mathbf{k} \delta(\omega - \mathbf{k} \cdot \mathbf{v}_1) \delta(\omega - \mathbf{k} \cdot \mathbf{v}_2). \quad (33)$$

The Landau approximation of the Balescu–Guernsey–Lenard equation is valid to describe plasma at scales which are much smaller than the Debye length (associated with large wavenumbers compared to $1/\lambda_D$), or globally when the effect of those scales dominate the collision kernel \mathbf{B} . Within this approximation, we can assume that $\varepsilon(\mathbf{k}, \omega) = 1$ which means that the dielectric susceptibility does not depend on the distribution f anymore. This approximation is relevant for many applications in plasma physics.

3.5 The Balescu–Guernsey–Lenard and Landau equations as non-linear Fokker-Planck equations

It is possible to consider the Balescu–Guernsey–Lenard and the Landau equations as non-linear Fokker-Planck equations. Indeed, introducing the drift and the diffusion terms

$$\begin{cases} \mathbf{b}[f](\mathbf{v}) &= \int d\mathbf{v}_2 \mathbf{B}[f](\mathbf{v}, \mathbf{v}_2) \frac{\partial f}{\partial \mathbf{v}_2} \\ \mathbf{D}[f](\mathbf{v}) &= \int d\mathbf{v}_2 \mathbf{B}[f](\mathbf{v}, \mathbf{v}_2) f(\mathbf{v}_2), \end{cases} \quad (34)$$

the Balescu–Guernsey–Lenard and the Landau equations write

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial \mathbf{v}} \cdot \left\{ -f \mathbf{b}[f] + \mathbf{D}[f] \frac{\partial f}{\partial \mathbf{v}} \right\}. \quad (35)$$

This is the functional form of a Fokker-Planck equation, but by contrast with the linear Fokker-Planck equation with constant drift and diffusion coefficient, the drift and diffusion coefficients depend on f .

We remark that this equation could be obtained from the dynamics of N particles governed by the Ito diffusion

$$d\mathbf{v}_n = \mathbf{b}[h_N](\mathbf{v}_n) dt + \frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{D}[h_N](\mathbf{v}_n) dt + \sqrt{2} \boldsymbol{\sigma}[h_N](\mathbf{v}_n) dW_{n,t}, \quad (36)$$

with

$$h_N(\mathbf{v}, t) = \frac{1}{N} \sum_{n=1}^N \delta(\mathbf{v}_n(t) - \mathbf{v}), \quad (37)$$

where σ is such that $\mathbf{D}[h_N](\mathbf{v}_n) = \sigma[h_N](\mathbf{v}_n) \sigma[h_N](\mathbf{v}_n)^\top$, and $W_{n,t}$ are Wiener processes that satisfy $\mathbb{E}(dW_{m,t} dW_{n,t'}) = \delta_{m,n} \delta(t' - t) dt$. In this equation, the drift and diffusion coefficients $\mathbf{b}[h_N]$ and $\mathbf{D}[h_N]$ and the matrix σ depend on a mean field way on the empirical density h_N .

There is a link between h_N and the empirical density g_Λ rescaled by the plasma parameter. We define f_Λ the projection of g_Λ over homogeneous distributions over the μ -space :

$$f_\Lambda(\mathbf{v}, t) = \left(\frac{\lambda_D}{L}\right)^3 \int_{[0, L/\lambda_D]^3} d\mathbf{r} g_\Lambda(\mathbf{r}, \mathbf{v}, t).$$

(both f_Λ and g_Λ are distributions over the μ -space). We note that f_Λ , which is a homogeneous distribution over the μ -space can also be interpreted as a distribution over the velocity space. Then using the relation between N and Λ : $\Lambda L^3 / \lambda_D^3 = N$, one can check that $f_\Lambda = h_N$.

The law of large numbers for the empirical density h_N for these N particles with mean field coupling insures that $\lim_{N \rightarrow \infty} h_N = f$ where f verifies the Balescu–Guernsey–Lenard equation (35). From this remark, a natural question is whether the dynamical large deviations for the empirical distribution h_N in (36-37) are the same as the dynamical large deviations of N particles with Coulomb interactions (the large deviation of the Balescu–Guernsey–Lenard equation). We address this very natural question in the following section.

4 Large deviations for N independent diffusions and N diffusions with mean field coupling

The aim of this section is to address the following question: are the dynamical large deviations (36) for the empirical distribution h_N in (36) the same as the dynamical large deviations of N particles with mean field interactions (the large deviations for the Balescu–Guernsey–Lenard or the Landau equations)? In section 4.1 we derive the large deviation rate function for the empirical density defined as $h_N(\mathbf{v}, t) = \frac{1}{N} \sum_{n=1}^N \delta(\mathbf{v}_n(t) - \mathbf{v})$ of N independent particles, where each $\mathbf{v}_n(t)$ is governed by a Markov dynamics with infinitesimal generator G .

In section 4.2 we apply this to the case when the N independent Markov dynamics are diffusions, and in section 4.3 when the particles are not independent anymore but are coupled in a mean field way, as in (36). For each of these cases we prove that

$$\mathbf{P}(\{h_N(t)\}_{0 \leq t \leq T} = \{h(t)\}_{0 \leq t \leq T}) \underset{N \rightarrow \infty}{\asymp} e^{-N \text{Sup}_p \int_0^T \{ \int d\mathbf{v} \dot{h} p - H[h, p] \}}, \quad (38)$$

where the corresponding H are given by formula (41), (43) and (46), respectively.

In section 4.3, we prove that the large deviations of the Balescu–Guernsey–Lenard or the Landau equations are **not** the large deviations of N diffusing particles with mean field coupling (36), as might have been naturally hypothesized.

4.1 Large deviations for the empirical density of N independent Markov processes

We consider N continuous time independent Markov processes $\{\mathbf{v}_n(t)\}_{t \in [0, T], 1 \leq n \leq N}$, where each $\mathbf{v}_n(t)$ is governed by a Markov dynamics with infinitesimal generator G . G acts on functions $\phi : \mathbb{R}^3 \rightarrow \mathbb{R}$ and is defined by

$$G[\phi](\mathbf{v}_0) = \lim_{\Delta T \rightarrow 0} \frac{\mathbb{E}_{\mathbf{v}_0}[\phi(\mathbf{v}(\Delta T))] - \phi(\mathbf{v}_0)}{\Delta T}. \quad (39)$$

Then, the empirical density h_N verifies a large deviation principle

$$\mathbf{P}(h_N = h) \underset{N \rightarrow \infty}{\asymp} e^{-N \text{Sup}_p \int_0^T \int d\mathbf{v} \dot{h} p - H[h, p]} \quad (40)$$

where

$$H[h, p] = \int d\mathbf{v} h(\mathbf{v}) G \left[e^{p(\cdot)} \right] (\mathbf{v}) e^{-p(\mathbf{v})}, \quad (41)$$

in this expression, the variable p is the conjugate momentum to h , and it is a scalar function of the velocity \mathbf{v} .

Formal proof The empirical density h_N is also itself a continuous time Markov process. We denote G_h its infinitesimal generator, defined by

$$G_h[\psi](h_0) = \lim_{\Delta T \rightarrow 0} \frac{\mathbb{E}_{h_0}[\psi(h(\Delta T))] - \psi(h_0)}{\Delta T},$$

where ψ is a functional. Then, from the result (16) in section 2.2.1, we know that if the limit

$$H[h, p] = \lim_{N \rightarrow \infty} \frac{1}{N} e^{-N \int d\mathbf{v} p h} G_h \left[e^{N \int d\mathbf{v} p h} \right],$$

exists, then we have the large deviation principle (40). Using the definition of the empirical density, we find

$$\begin{aligned} G_h \left[e^{N \int d\mathbf{v} p h_N} \right] &= G_h \left[e^{\sum_{n=1}^N p(\mathbf{v}_n)} \right] \\ &= \lim_{\Delta T \rightarrow 0} \frac{1}{\Delta T} \left(\mathbb{E} \left(e^{\sum_{n=1}^N p(\mathbf{v}_n(\Delta T))} \right) - e^{\sum_{n=1}^N p(\mathbf{v}_n(0))} \right). \end{aligned}$$

Then, using that the particles are independent

$$H[h, p] = \lim_{N \rightarrow \infty} \lim_{\Delta T \rightarrow 0} \frac{1}{N \Delta T} \left(\prod_{n=1}^N \mathbb{E} \left(e^{\Delta p(\mathbf{v}_n)} \right) - 1 \right),$$

where $\mathbb{E} \left(e^{\Delta p(\mathbf{v}_n)} \right) = \mathbb{E} \left(e^{p(\mathbf{v}_n(\Delta T))} \right) e^{-p(\mathbf{v}_n(0))}$. Furthermore, using the definition of the infinitesimal generator for the diffusion process (39), we have

$$\mathbb{E} \left(e^{\Delta p(\mathbf{v}_n)} \right) = 1 + \Delta T G \left[e^{p(\mathbf{v}_n(0))} \right] e^{-p(\mathbf{v}_n(0))} + o(\Delta T) \quad (\Delta T \rightarrow 0).$$

To the same precision we can compute the product for $1 \leq n \leq N$

$$\prod_{n=1}^N \mathbb{E} \left(e^{\Delta p(\mathbf{v}_n)} \right) - 1 = \Delta T \sum_{n=1}^N G \left[e^{p(\mathbf{v}_n(0))} \right] e^{-p(\mathbf{v}_n(0))} + o(\Delta T) \quad (\Delta T \rightarrow 0).$$

From this expansion, it is possible to compute the limit as ΔT goes to 0

$$\lim_{\Delta T \rightarrow 0} \frac{1}{N\Delta T} \left(\prod_{n=1}^N \mathbb{E} \left(e^{\Delta p(\mathbf{v}_n)} \right) - 1 \right) = \sum_{n=1}^N G \left[e^{p(\mathbf{v}_n(0))} \right] e^{-p(\mathbf{v}_n(0))}.$$

It is important to note that the order of the limits $N \rightarrow \infty$ and $\Delta T \rightarrow 0$ is crucial. From there, it comes easily that

$$H[h, p] = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N G \left[e^{p(\mathbf{v}_n(0))} \right] e^{-p(\mathbf{v}_n(0))} = \int d\mathbf{v} h(\mathbf{v}) G \left[e^{p(\cdot)} \right] (\mathbf{v}) e^{-p(\mathbf{v})}.$$

We remark that the Hamiltonian (41) is in general not quadratic in p , reflecting the fact that the large deviations are not Gaussian, although they arise from the sum of N independent contributions.

4.2 Large deviations for the empirical density of N independent diffusions

From equation (41), it is straightforward to compute the Hamiltonian that describes the large deviations for the empirical density of N particles with independent diffusions.

Let us consider N particles with velocities $\{\mathbf{v}_n\}_{1 \leq n \leq N}$ with the following Ito diffusion dynamics

$$d\mathbf{v}_n = \left[\mathbf{b}(\mathbf{v}_n) + \frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{D}(\mathbf{v}_n) \right] dt + \sqrt{2} \boldsymbol{\sigma}(\mathbf{v}_n) dW_{n,t}. \quad (42)$$

We define \mathbf{D} the diffusion tensor as $\mathbf{D} = \boldsymbol{\sigma} \boldsymbol{\sigma}^\top$. We call h the probability density function of \mathbf{v}_n for some n . It does not depend on n as we consider N non-interacting particles, we can write the Fokker-Planck equation associated with the diffusion of a particle

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial \mathbf{v}} \cdot \left\{ -h\mathbf{b} + \mathbf{D} \frac{\partial h}{\partial \mathbf{v}} \right\}.$$

Now, we define h_N the empirical density of the velocity distribution

$$h_N(\mathbf{v}, t) = \frac{1}{N} \sum_{n=1}^N \delta(\mathbf{v}_n(t) - \mathbf{v}).$$

We want to compute $H[h, p]$ the Hamiltonian associated with the large deviation principle for the empirical density

$$\mathbf{P}(h_N = h) \underset{N \rightarrow \infty}{\asymp} e^{-N \text{Sup}_p \int_0^T \{ \int d\mathbf{v} \dot{h} p - H[h, p] \}}.$$

We showed in section 4.1 that $H[h, p]$ is given by

$$H[h, p] = \int d\mathbf{v} h(\mathbf{v}) G \left[e^{p(\cdot)} \right] (\mathbf{v}) e^{-p(\mathbf{v})}.$$

It is a classical result in stochastic analysis that the infinitesimal generator G of the diffusion stochastic process is

$$G = \mathbf{b} \cdot \frac{\partial}{\partial \mathbf{v}} + \frac{\partial}{\partial \mathbf{v}} \cdot \left(\mathbf{D} \frac{\partial}{\partial \mathbf{v}} \right),$$

the adjoint of the Fokker-Planck operator. This leads to the Hamiltonian associated with the empirical density of N particles diffusing independently

$$H[h, p] = \int d\mathbf{v} h \left\{ \mathbf{b} \cdot \frac{\partial p}{\partial \mathbf{v}} + \frac{\partial}{\partial \mathbf{v}} \left(\mathbf{D} \frac{\partial p}{\partial \mathbf{v}} \right) + \mathbf{D} : \frac{\partial p}{\partial \mathbf{v}} \frac{\partial p}{\partial \mathbf{v}} \right\}, \quad (43)$$

where the symbol “:” means the contraction of two second order symmetric tensors : $\mathbf{M} : \mathbf{N} = \text{Tr}(\mathbf{MN}) = \sum_{ij} M_{ij} N_{ij}$.

We remark that the Hamiltonian (41) is quadratic in p . This means that the large deviations are Gaussian. This reflects the fact that the large deviations arise from the sum of N independent Gaussian increments. Because of this property, we can also recover from the Hamiltonian an equivalent stochastic differential equation for the empirical h_N that involves a Gaussian noise. More precisely, a quadratic Hamiltonian

$$H[h, p] = \int d\mathbf{v} A[h](\mathbf{v}) p(\mathbf{v}) + \iint d\mathbf{v} d\mathbf{v}' p(\mathbf{v}) C[h](\mathbf{v}, \mathbf{v}') p(\mathbf{v}')$$

is the Hamiltonian that describes the dynamical large deviations of the stochastic differential equation

$$\frac{\partial h_N}{\partial t} = A[h_N](\mathbf{v}) + \sqrt{\frac{2}{N}} \eta(\mathbf{v}, t)$$

with

$$\mathbb{E}(\eta(\mathbf{v}, t) \eta(\mathbf{v}', t')) = C[h_N](\mathbf{v}, \mathbf{v}').$$

Using partial integration, we can identify $A[h]$ and $C[h](\mathbf{v}, \mathbf{v}')$ for the Hamiltonian (43). The associated stochastic differential equation for the empirical density is

$$\frac{\partial h_N}{\partial t} = \frac{\partial}{\partial \mathbf{v}} \cdot \left\{ -h_N \mathbf{b} + \mathbf{D} \frac{\partial h_N}{\partial \mathbf{v}} \right\} + \sqrt{\frac{2}{N}} \eta(\mathbf{v}, t) \quad (44)$$

with,

$$\mathbb{E}(\eta(\mathbf{v}, t) \eta(\mathbf{v}', t')) = \frac{\partial^2}{\partial \mathbf{v} \partial \mathbf{v}'} : (h_N(\mathbf{v}) \delta(\mathbf{v} - \mathbf{v}') \mathbf{D}) \delta(t - t').$$

Recalling that $\mathbf{D} = \boldsymbol{\sigma}\boldsymbol{\sigma}^\top$, we can rewrite equation (44) as a conservative equation

$$\frac{\partial h_N}{\partial t} = \frac{\partial}{\partial \mathbf{v}} \cdot \left\{ -h_N \mathbf{b} + \mathbf{D} \frac{\partial h_N}{\partial \mathbf{v}} + \sqrt{\frac{2}{N}} h_N \boldsymbol{\sigma} \xi(\mathbf{v}, t) \right\},$$

with ξ a tridimensional Gaussian noise that satisfies

$$\mathbb{E}(\xi^i(\mathbf{v}, t) \xi^j(\mathbf{v}', t')) = \delta^{ij} \delta(\mathbf{v} - \mathbf{v}') \delta(t - t').$$

4.3 Large deviations for N diffusions with mean field coupling

In the previous section, we have derived the large deviation Hamiltonian for the empirical density of N independent particles driven by the diffusion (42). We now consider the case when the drift and diffusion coefficients depend on the empirical density itself:

$$d\mathbf{v}_n = \mathbf{b}[h_N](\mathbf{v}_n) dt + \sqrt{2} \boldsymbol{\sigma}[h_N](\mathbf{v}_n) dW_{n,t}, \quad (45)$$

with $h_N(\mathbf{v}, t) = \frac{1}{N} \sum_{n=1}^N \delta(\mathbf{v}_n(t) - \mathbf{v})$. We denote $\mathbf{D}[h] = \boldsymbol{\sigma}[h] \boldsymbol{\sigma}[h]^\top$. For this case, the particles are no more statistically independent. However, for such a mean field coupling, it is an easy exercise to adapt the derivation that leads to the Hamiltonian (41) in section 4.1 to this specific case. We find that the Hamiltonian that describes the large deviation of the empirical density is

$$H_{MF,h}[h, p] = \int d\mathbf{v} h \left\{ \mathbf{b}[h] \cdot \frac{\partial p}{\partial \mathbf{v}} + \frac{\partial}{\partial \mathbf{v}} \left(\mathbf{D}[h] \frac{\partial p}{\partial \mathbf{v}} \right) + \mathbf{D}[h] : \frac{\partial p}{\partial \mathbf{v}} \frac{\partial p}{\partial \mathbf{v}} \right\}. \quad (46)$$

The subscript MF, h denotes that this is the Hamiltonian for a mean field dynamics without spatial structure. We note that this Hamiltonian is the same as (43), but with drift and diffusion constant that depend of h . The corresponding stochastic dynamics is

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial \mathbf{v}} \cdot \left\{ -h \mathbf{b}[h] + \mathbf{D}[h] \frac{\partial h}{\partial \mathbf{v}} + \sqrt{\frac{2}{N}} h \boldsymbol{\sigma}[h] \xi(\mathbf{v}, t) \right\}, \quad (47)$$

with ξ a tridimensional Gaussian noise that satisfies

$$\mathbb{E}(\xi^i(\mathbf{v}, t) \xi^j(\mathbf{v}', t')) = \delta^{ij} \delta(\mathbf{v} - \mathbf{v}') \delta(t - t').$$

Now let us get back to the remark of section 3.5. In section 3.5, we have noticed that one could see the Balescu–Guernsey–Lenard and Landau equations as non-linear Fokker–Planck equation for N diffusions with mean field coupling defined by (36). As we already stated, the law of large numbers for the empirical density indicates $\lim_{N \rightarrow \infty} h_N = h$, where h solves the Balescu–Guernsey–Lenard equation. For this dynamics and for the empirical measure h_N , we can derive a large deviation principle with computations that are analogous to the one we did to obtain the large deviation principle (38)-(46). The result reads

$$\mathbf{P}(\{h_N(t)\}_{0 \leq t \leq T} \approx \{h(t)\}_{0 \leq t \leq T}) \underset{N \rightarrow \infty}{\asymp} e^{-N \int_0^T dt \text{Sup}_p \{ \int d\mathbf{v} \dot{h} p - H_{MF,h}[h, p] \}}. \quad (48)$$

In the following, we examine the properties of this large deviation Hamiltonian and we conclude that it cannot describe the large deviations associated to the Balescu–Guernsey–Lenard kinetic theory.

Relaxation paths and most probable evolutions For the dynamics (45), we also noted at the end of section 3.5 that the evolution of the average of the empirical density is given asymptotically by the Balescu–Guernsey–Lenard. As a consequence we expect the most probable evolution for the Hamiltonian (46), also called a relaxation path to be the Balescu–Guernsey–Lenard equation (27). Using the equation for relaxation paths (equation (9) in section 2.1) we check that indeed

$$\frac{\partial h}{\partial t} = \frac{\delta H_{MF,h}}{\delta p}[h, p = 0] = \frac{\partial}{\partial \mathbf{v}} \left\{ -h \mathbf{b}[h] + \mathbf{D}[h] \frac{\partial h}{\partial \mathbf{v}} \right\}. \quad (49)$$

Relative entropy and quasipotential In section 2.2.1 we define the quasipotential. For the empirical distribution h_N it is defined as $\mathbb{P}(h_N = h) \underset{N \rightarrow \infty}{\asymp} e^{-NU[h]}$. As the N particles are coupled only in a mean field way, in view of Sanov’s theorem adapted for this case, it is natural to conjecture that the quasipotential for the dynamics of the empirical density is $U[h] = -S[h]$ where S is the relative entropy

$$S_{\text{rel}}[h] = - \int d\mathbf{v} h \log(h/h_{\text{eq}}),$$

where h_{eq} is the stationary solution of the Balescu–Guernsey–Lenard equation. A necessary condition for the $-S$ to be the quasipotential is the stationary Hamilton–Jacobi equation $H_{MF,h}[h, -\delta S/\delta h] = 0$. We check in the appendix A that this stationary Hamilton–Jacobi equation is indeed verified when $\mathbf{b}[h] = \mathbf{b}$ and $\mathbf{D}[h] = \mathbf{D}$ do not depend on h , i.e. when the N diffusions are independent from each other. However, we also check that this is no more the case in general if $\mathbf{b}[h]$ and $\mathbf{D}[h]$ actually depend on h . This remark is enough to conclude that the Hamiltonian (46) cannot be the correct Hamiltonian system for empirical measure of N interacting particles with mean field interactions, or for particles with Coulomb interactions.

Moreover an easy direct computation shows that

$$\int d\mathbf{v} \frac{\delta H_{MF,h}}{\delta p}[f, p] \frac{\delta E}{\delta f} \neq 0,$$

where E is the kinetic energy (31). As explained in section 2.2.1, $\int d\mathbf{v} \frac{\delta H_{MF,h}}{\delta p}[f, p] \frac{\delta E}{\delta f} = 0$ is the energy conservation formula. Equivalently the noise in equation (47) is not an energy conserving noise, and thus cannot describe the empirical density of particles with Coulomb interactions (22).

We thus conclude the Hamiltonian (46) is not the Hamiltonian for the large deviations of systems of particles for N interacting particles with mean field interactions, or for particles with Coulomb interactions. In the next sections we derive in two different ways the large deviation Hamiltonian for the Landau equation.

5 Large deviations associated with the Landau kinetic theory from the Boltzmann kinetic theory

The Landau equation has been presented in section 3.4 as an approximation of the Balescu–Guernsey–Lenard equation. However it also has a strong link with the Boltzmann equation that describes a dilute gas of particles in the Boltzmann–Grad limit. One can look for instance in [11] for this connection. Moreover, the large deviation Hamiltonian for the Boltzmann equation has already been obtained, for toy models which are analogue to the dilute gas dynamics [18] or for the dilute gas dynamics [3,4]. The aim of this section is to derive the large deviation Hamiltonian associated with the Landau equation from the large deviation Hamiltonian associated with the Boltzmann equation.

In section 5.1, we introduce the notations for the Boltzmann equation and the large deviation Hamiltonian for a dilute gas in the Boltzmann–Grad limit. In section 5.2, following [11], we derive the Landau equation from the Boltzmann equation using the grazing collision limit. Using the same limit but for the large deviation Hamiltonian, rather than for the kinetic equation, we derive the large deviation Hamiltonian for the Landau equation (64) in section 5.3. In section 5.4, we show that this Hamiltonian satisfies all the expected symmetries and conservation properties. In section 5.5, we derive the gradient flow structure of the Landau equation associated with this Hamiltonian. In section 5.6, we conjecture the Hamiltonian associated with Balescu–Lenard–Guernsey equation from the Landau equation Hamiltonian.

5.1 The Boltzmann equation for a dilute gas

We consider the dynamics of a dilute gas composed of atoms or molecules. We neglect any internal degrees of freedom. We assume that the N particles evolve through a Hamiltonian dynamics with short range two body interactions, for instance hard sphere collisions.

Let us first define the collision kernel and the collision cross-section. We consider a thread of particles with velocities \mathbf{v}_1 that meets a thread of particles with velocities \mathbf{v}_2 . We assume that particles of each velocity type are distributed according to a homogeneous point Poisson process with densities $\rho(\mathbf{v}_1)d\mathbf{v}_1$ and $\rho(\mathbf{v}_2)d\mathbf{v}_2$, respectively. These particle distributions will give rise to collisions where $(\mathbf{v}_1, \mathbf{v}_2)$ particle pairs undergo a random change towards pairs of the type $(\mathbf{v}'_1, \mathbf{v}'_2)$, up to $(d\mathbf{v}'_1, d\mathbf{v}'_2)$. This occurs at a rate per unit of time and unit of volume which is proportional to the \mathbf{v}_1 incident particle number $\rho(\mathbf{v}_1)d\mathbf{v}_1$, the \mathbf{v}_2 incident particle number $\rho(\mathbf{v}_2)d\mathbf{v}_2$, $d\mathbf{v}'_1$, and $d\mathbf{v}'_2$. The proportionality coefficient is called the collision kernel and is denoted

$$w_0(\mathbf{v}'_1, \mathbf{v}'_2; \mathbf{v}_1, \mathbf{v}_2) / 2. \quad (50)$$

The local conservation of momentum and energy implies that

$$w_0(\mathbf{v}'_1, \mathbf{v}'_2; \mathbf{v}_1, \mathbf{v}_2) = \sigma_0(\mathbf{v}'_1, \mathbf{v}'_2; \mathbf{v}_1, \mathbf{v}_2) \delta(\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}'_1 - \mathbf{v}'_2) \delta(\mathbf{v}_1^2 + \mathbf{v}_2^2 - \mathbf{v}'_1^2 - \mathbf{v}'_2^2), \quad (51)$$

where σ_0 is the diffusion cross-section. σ_0 is of the order of a^2 where a is a typical atom size. We detail the different symmetry properties of the collision kernel in annex C.

Several length scales are important to describe a dilute gas: a typical atom size a , that we will define more precisely below in relation with the diffusion cross-section, a typical interparticle distance $1/\rho^{1/3}$ where ρ is the averaged gas density, the mean free path which is the averaged length a particle travels between two collisions, and a typical box size L . The mean free path is given by $l = c/a^2\rho$, where c is a non-dimensional number that depends on the collision kernel. The gas is said dilute if we have the following relation between those scales

$$a \ll \frac{1}{\rho^{1/3}} \ll l.$$

A limit in which those inequalities are satisfied is called a Boltzmann–Grad limit. We consider the 4 physically independent parameters a, L, N and the inverse temperature β ($\rho = N/L^3$). From those four, we can choose two independent non-dimensional parameters. In the following we choose N and the Knudsen number $\alpha = l/L$ as those two independent parameters. The inverse of the number of particles in a volume of the size l is then $\varepsilon = 1/l^3\rho = a^2/l^2 = a^6\rho^2$ and is another non-dimensional parameter.

We will use the large deviation result in the limit $N \rightarrow \infty$ with fixed Knudsen number α . In this limit, from $l = c/a^2\rho$ we see that $a^2 = c/\alpha N$. As the diffusion cross-section σ_0 is of the order of a^2 , in the limit $N \rightarrow \infty$, it is thus natural to consider the rescaled cross-section $\sigma = N\sigma_0$. Moreover, in the following it will be convenient to consider momentum exchange. We thus use the following definition of w

$$w\left(\mathbf{v}_1 + \frac{1}{2}\mathbf{q}, \mathbf{v}_2 - \frac{1}{2}\mathbf{q}; \mathbf{q}\right) = \gamma N w_0(\mathbf{v}_1 + \mathbf{q}, \mathbf{v}_2 - \mathbf{q}; \mathbf{v}_1, \mathbf{v}_2), \quad (52)$$

where \mathbf{q} is the momentum transfer between the incident particles with momenta $(\mathbf{v}_1, \mathbf{v}_2)$ and the scattered particles with momenta $(\mathbf{v}_1 + \mathbf{q}, \mathbf{v}_2 - \mathbf{q})$. Writing the collision kernel this way automatically takes into account momentum conservation during the collision process. In this reasoning, the coefficient γ is any non-dimensional coefficient which is held fixed in the limit $N \rightarrow \infty$. In the following sections, for the specific case of the Coulomb interaction, we will consider

$$\gamma = \left(\frac{\lambda_D}{L}\right)^3,$$

where λ_D is the Debye length and L the size of the box.

We define a rescaled empirical density

$$g_\gamma(\mathbf{r}, \mathbf{v}, t) = (\gamma N)^{-1} \sum_{n=1}^N \delta(\mathbf{v} - \mathbf{v}_n(t)) \delta(\mathbf{r} - \mathbf{r}_n(t)). \quad (53)$$

We note that with $\gamma = (\lambda_D/L)^3$, g_γ coincides with $g_\Lambda(\mathbf{r}, \mathbf{v}, t) = \Lambda^{-1} \sum_{n=1}^N \delta(\mathbf{v} - \mathbf{v}_n(t)) \delta(\mathbf{r} - \mathbf{r}_n(t))$ (see (23), page 13). When these N particles undergo a dilute gas dynamics, the empirical density g_γ has a law of a large numbers. More precisely,

if we assume that for a set of initial conditions, an initial law of large numbers holds: $\lim_{N \rightarrow \infty} (\gamma N)^{-1} \sum_{n=1}^N g_\gamma^0(\mathbf{r}, \mathbf{v}) = g^0(\mathbf{r}, \mathbf{v})$, then we have at a time t the law of large numbers $\lim_{N \rightarrow \infty} (\gamma N)^{-1} \sum_{n=1}^N g_\gamma(\mathbf{r}, \mathbf{v}, t) = g(\mathbf{r}, \mathbf{v}, t)$, where g is a solution of the Boltzmann equation

$$\frac{\partial g}{\partial t} + \mathbf{v} \cdot \frac{\partial g}{\partial \mathbf{r}} = \int d\mathbf{v}_2 d\mathbf{q} w \left(\mathbf{v} + \frac{1}{2} \mathbf{q}, \mathbf{v}_2 - \frac{1}{2} \mathbf{q}; \mathbf{q} \right) [g(\mathbf{v} + \mathbf{q}, \mathbf{r}) g(\mathbf{v}_2 - \mathbf{q}, \mathbf{r}) - g(\mathbf{v}, \mathbf{r}) g(\mathbf{v}_2, \mathbf{r})], \quad (54)$$

with initial condition $g(\mathbf{r}, \mathbf{v}, 0) = g^0(\mathbf{r}, \mathbf{v})$. We refer to classical textbooks in kinetic theories, for instance [11], or [4] for a detailed presentation of an heuristic derivation of the Boltzmann equation.

In [4], a large deviation principle for the empirical density is derived (equations (1) to (3) in [4]). This large deviation is derived in the limit $\varepsilon = 1/N\alpha^3 \rightarrow 0$. In this paper, we will consider the limit $\gamma N \rightarrow \infty$, with fixed Knudsen number and fixed γ . In this limit, we have $\varepsilon = 1/N\alpha^3 \rightarrow 0$. Then the large deviation result justified in [4] can be directly used in this paper. After adapting equations (1) to (3) in [4] to the notation (52) and (53), we have

$$\mathbf{P} \left(\{g_\gamma(\mathbf{r}, \mathbf{v}, t)\}_{0 \leq t \leq T} = \{g(\mathbf{r}, \mathbf{v}, t)\}_{0 \leq t \leq T} \right) \underset{N \rightarrow \infty}{\asymp} e^{-\gamma N \int_0^T \text{Sup}_p \{ \int d\mathbf{r} d\mathbf{v} \dot{g} p - H_B[g, p] \}}, \quad (55)$$

where

$$H_B[g, p] = H_C[g, p] + H_T[g, p], \quad (56)$$

and with the collision Hamiltonian

$$H_C[g, p] = \frac{1}{2} \int d\mathbf{v}_1 d\mathbf{v}_2 d\mathbf{q} d\mathbf{r} w \left(\mathbf{v}_1 + \frac{1}{2} \mathbf{q}, \mathbf{v}_2 - \frac{1}{2} \mathbf{q}; \mathbf{q} \right) \times g(\mathbf{r}, \mathbf{v}_1) g(\mathbf{r}, \mathbf{v}_2) \left\{ e^{[-p(\mathbf{r}, \mathbf{v}_1) - p(\mathbf{r}, \mathbf{v}_2) + p(\mathbf{r}, \mathbf{v}_1 + \mathbf{q}) + p(\mathbf{r}, \mathbf{v}_2 - \mathbf{q})]} - 1 \right\}, \quad (57)$$

and the free transport Hamiltonian

$$H_T[g, p] = - \int d\mathbf{r} d\mathbf{v} p(\mathbf{r}, \mathbf{v}) \mathbf{v} \cdot \frac{\partial g}{\partial \mathbf{r}}(\mathbf{r}, \mathbf{v}). \quad (58)$$

5.2 From the Boltzmann to the Landau equations

In the case of long-range interactions between particles, e.g. Coulomb type interactions, the two-particle collisions are dominated by small-angle scattering events. This allows some simplification. The related limit is called the **grazing collision limit**. In this section we justify that in the grazing collision limit and for a homogeneous gas, from the Boltzmann equation one obtains the Landau equation

$$\frac{\partial f}{\partial t} = \frac{1}{\Lambda} \frac{\partial}{\partial \mathbf{v}} \int d\mathbf{v}_2 \mathbf{B}(\mathbf{v}, \mathbf{v}_2) \left(- \frac{\partial f}{\partial \mathbf{v}_2} f(\mathbf{v}) + \frac{\partial f}{\partial \mathbf{v}} f(\mathbf{v}_2) \right), \quad (59)$$

where the tensor \mathbf{B} is defined by (33), page 17. In equation (32) of section (3.4), we expressed this equation with the time variable $\tau = t/\Lambda$ rescaled by the plasma parameter. This is why there is no factor Λ^{-1} in the right hand side of equation (32).

The following derivation of the Landau equation from the Boltzmann equation is strongly inspired by the paragraph §42 of [11]. However, here we present a slightly different derivation. First, we consider homogenous solutions of the Boltzmann equation $g(\mathbf{r}, \mathbf{v}, t) = f(\mathbf{v}, t)$ that do not depend on the position variable. The homogeneous Boltzmann equation reads

$$\frac{\partial f}{\partial t} = \underbrace{\int d\mathbf{v}_2 d\mathbf{q} w\left(\mathbf{v} + \frac{1}{2}\mathbf{q}, \mathbf{v}_2 - \frac{1}{2}\mathbf{q}; \mathbf{q}\right)}_{I(\mathbf{v})} [f(\mathbf{v} + \mathbf{q})f(\mathbf{v}_2 - \mathbf{q}) - f(\mathbf{v})f(\mathbf{v}_2)]. \quad (60)$$

From there, we will work in the grazing collision limit, meaning that we will only take into account collisions that imply small transfer of momentum. More precisely, we consider only collisions with $|\mathbf{q}| \ll |\mathbf{v}|, |\mathbf{v}_2|$. This approximation is relevant and often used in plasma physics, where Coulomb interactions tend to make collisions with small scattering angles more numerous and more influential than the other ones, see the first chapter of [15] for quantitative arguments. In order to understand at which precision we shall use this approximation, let us first give the relation between \mathbf{B} and the collision kernel:

$$\mathbf{B}(\mathbf{v}_1, \mathbf{v}_2) = \frac{1}{2}\Lambda \int d\mathbf{q} w(\mathbf{v}_1, \mathbf{v}_2; \mathbf{q}) \mathbf{q} \otimes \mathbf{q}, \quad (61)$$

where $\mathbf{q}_1 \otimes \mathbf{q}_2$ is the tensor product of the two vectors \mathbf{q}_1 and \mathbf{q}_2 (a tensor of rank 2). In appendix B, we prove that for Coulomb interaction the two expressions for \mathbf{B} , (61) and (33) are equal. In the following, we will omit the tensor product symbol, and a product of vector without a dot should be understood as a tensor product: $\mathbf{q}_1 \mathbf{q}_2 \equiv \mathbf{q}_1 \otimes \mathbf{q}_2$. In the case of the Landau equation, the tensor \mathbf{B} is well known and has a list of properties related to the geometry and the physics of the collisions (conservation laws and symmetry properties). For our study, we will retain that \mathbf{B} is a symmetric tensor, that \mathbf{B} is symmetric with respect to the exchange of its two arguments: $\mathbf{B}(\mathbf{v}_1, \mathbf{v}_2) = \mathbf{B}(\mathbf{v}_2, \mathbf{v}_1)$, and that $\mathbf{B}(\mathbf{v}_1, \mathbf{v}_2) \cdot (\mathbf{v}_1 - \mathbf{v}_2) = \vec{0}$, we prove these properties in appendix C.2. We will make a link between those properties and the symmetries of the Landau equation (59) in section 5.4.2.

In appendix D.1, we develop I in the Boltzmann equation (60) at order 2 in \mathbf{q} and we obtain the Landau equation (59). We have thus justified the Landau equation as an approximation of the Boltzmann equation in the grazing collision limit.

5.3 Deriving Landau's large deviation principle from Boltzmann's large deviation principle

In this section we derive the Hamiltonian for the path large deviations of the Landau equation from the Hamiltonian for the path large deviations of the Boltzmann equation, using the grazing collision limit.

We start from the large deviation principle discussed in section (5.1). Adapting the discussion of section (5.1), with

$$g_\Lambda(\mathbf{r}, \mathbf{v}, t) = \Lambda^{-1} \sum_{n=1}^N \delta(\mathbf{v} - \mathbf{v}_n(t)) \delta(\mathbf{r} - \mathbf{r}_n(t)),$$

and with $\gamma = (\lambda_D/L)^3$, we have

$$\mathbf{P}(\{g_\Lambda(\mathbf{r}, \mathbf{v}, \tau)\}_{0 \leq \tau \leq T} = \{g(\mathbf{r}, \mathbf{v}, \tau)\}_{0 \leq \tau \leq T}) \underset{\Lambda \rightarrow \infty}{\asymp} e^{-\Lambda \int_0^T \text{Sup}_p \{ \int \text{drd}\mathbf{v} \dot{g} p - \Lambda H_B[g, p] \} d\tau},$$

where H_B is given by (56) and where we used the rescaled time variable $\tau = t/\Lambda$ by the plasma parameter Λ in the large deviation action.

In the following we will be interested in the case of homogeneous distributions, i.e. distributions that only depend on the velocity variable, denoted by the letter f : $g(\mathbf{r}, \mathbf{v}, t) = f(\mathbf{v}, t)$. Then the large deviation principle reads

$$\mathbf{P}(g_\Lambda = f) \underset{N \rightarrow \infty}{\asymp} e^{-\Lambda \int_0^T \text{Sup}_p \{ \int \text{drd}\mathbf{v} \dot{f} p - H[f, p] \} d\tau}, \quad (62)$$

with

$$H[f, p] = \frac{\Lambda}{2} \int \text{drd}\mathbf{v}_1 \text{d}\mathbf{v}_2 \text{d}\mathbf{q} w \left(\mathbf{v}_1 + \frac{1}{2} \mathbf{q}, \mathbf{v}_2 - \frac{1}{2} \mathbf{q}; \mathbf{q} \right) \times f(\mathbf{v}_1) f(\mathbf{v}_2) \left\{ e^{[-p(\mathbf{v}_1) - p(\mathbf{v}_2) + p(\mathbf{v}_1 + \mathbf{q}) + p(\mathbf{v}_2 - \mathbf{q})]} - 1 \right\}. \quad (63)$$

The idea to obtain the large deviation Hamiltonian for the Landau equation, is to use the same hypothesis of grazing collisions used in section (5.2). As in section (5.2), we will make a Taylor expansion in \mathbf{q} at order 2. Rather than doing this expansion for the Boltzmann equation, we do it in the large deviation Hamiltonian (63). The full computation is detailed in appendix D.2, and we find that the large deviation Hamiltonian $H_{\text{Landau}}[f, p]$ for the Landau equation is

$$H_{\text{Landau}}[f, p] = H_{MF}[f, p] + H_I[f, p], \quad (64)$$

with

$$H_{MF}[f, p] = \int \text{drd}\mathbf{v}_1 f \left\{ \mathbf{b}[f] \cdot \frac{\partial p}{\partial \mathbf{v}_1} + \frac{\partial}{\partial \mathbf{v}_1} \left(\mathbf{D}[f] \frac{\partial p}{\partial \mathbf{v}_1} \right) + \mathbf{D}[f] : \frac{\partial p}{\partial \mathbf{v}_1} \frac{\partial p}{\partial \mathbf{v}_1} \right\},$$

and

$$H_I[f, p] = - \int \text{drd}\mathbf{v}_1 \text{d}\mathbf{v}_2 f(\mathbf{v}_1) f(\mathbf{v}_2) \frac{\partial p}{\partial \mathbf{v}_1} \frac{\partial p}{\partial \mathbf{v}_2} : \mathbf{B}(\mathbf{v}_1, \mathbf{v}_2),$$

where $\mathbf{b}[f]$ and $\mathbf{D}[f]$ are defined in equation (34), and in which we recognize $H_{MF} = \int \text{dr} H_{MF,h}$ where $H_{MF,h}$ is the mean field Hamiltonian (46) and a new additional term H_I .

We have thus justified a large deviation principle for the rescaled empirical density g_Λ in the limit of a large plasma parameter Λ . It reads

$$\mathbf{P}(\{g_\Lambda(\mathbf{r}, \mathbf{v}, \tau)\}_{0 \leq \tau \leq T} = \{f(\mathbf{v}, \tau)\}_{0 \leq \tau \leq T}) \underset{\Lambda \rightarrow \infty}{\asymp} e^{-\Lambda \text{Sup}_p \int_0^T d\tau \{ \int d\mathbf{r} d\mathbf{v} \dot{f} p - H_{\text{Landau}}[f, p] \}}, \quad (65)$$

where H_{Landau} is defined in (64).

We note that this Hamiltonian is quadratic in its conjugate momentum p . Then, in the grazing collision limit, the large deviations are Gaussian. This is a consequence of neglecting the collisions that involve large changes of velocity for the particles. This constrains the fluctuations of the empirical density g_Λ in a reduced range where they can be considered as Gaussian fluctuations. As mentioned in section 4.2, a quadratic Hamiltonian can be associated with a stochastic differential equation involving a Gaussian noise. In this case,

$$\frac{\partial g_\Lambda}{\partial \tau} = \frac{\partial}{\partial \mathbf{v}} \cdot \left\{ -g_\Lambda \mathbf{b} + \mathbf{D} \frac{\partial g_\Lambda}{\partial \mathbf{v}} \right\} + \sqrt{\frac{2}{\Lambda}} \eta(\mathbf{v}, \tau), \quad (66)$$

with

$$\mathbb{E}(\eta(\mathbf{r}, \mathbf{v}, \tau) \eta(\mathbf{r}', \mathbf{v}', \tau')) = \frac{\partial^2}{\partial \mathbf{v} \partial \mathbf{v}'} : (g_\Lambda(\mathbf{v}) \delta(\mathbf{v} - \mathbf{v}') \mathbf{D} - g_\Lambda(\mathbf{v}) g_\Lambda(\mathbf{v}') \mathbf{B}(\mathbf{v}, \mathbf{v}')) \delta(\mathbf{r} - \mathbf{r}') \delta(\tau - \tau').$$

The Gaussian fluctuations have a non-trivial correlation structure.

5.4 Verifications of the properties of the Hamiltonian

Let us check all the expected properties for the Hamiltonian (64).

5.4.1 Most probable evolution

First, we should verify that the most probable evolution associated with this Hamiltonian is the Landau equation, i.e. that

$$\frac{\partial f}{\partial \tau} = \frac{\delta H_{\text{Landau}}}{\delta p}[f, p = 0] = \frac{\partial}{\partial \mathbf{v}_1} \left\{ -f \mathbf{b}[f] + \mathbf{D}[f] \frac{\partial f}{\partial \mathbf{v}_1} \right\}. \quad (67)$$

We already know from equation (49)

$$\frac{\delta H_{MF}}{\delta p}[f, p = 0] = \frac{\partial}{\partial \mathbf{v}_1} \left\{ -f \mathbf{b}[f] + \mathbf{D}[f] \frac{\partial f}{\partial \mathbf{v}_1} \right\}.$$

In addition to this,

$$\frac{\delta H_I}{\delta p}[f, p] = -2 \frac{\partial}{\partial \mathbf{v}_1} \left\{ \int d\mathbf{v}_2 f(\mathbf{v}_1) f(\mathbf{v}_2) \mathbf{B}(\mathbf{v}_1, \mathbf{v}_2) \frac{\partial p}{\partial \mathbf{v}_2} \right\},$$

in particular, $\frac{\delta H_I}{\delta p}[f, p = 0] = 0$. Thus, property (67) is verified. It is important to notice that, since we rescaled the time variable $\tau = t/\Lambda$ by the plasma parameter, there is no factor Λ^{-1} in the right hand side of (67).

5.4.2 Conservation laws

From the result (15) of section 2, we know that a functional $C[f]$ is a conserved quantity if and only if $\int \mathbf{dr} \mathbf{dv} \frac{\delta H_{\text{Landau}}}{\delta p} \frac{\delta C}{\delta f} = 0$ or equivalently, if for any f, p and α : $H_{\text{Landau}}[f, p] = H_{\text{Landau}}[f, p + \alpha \frac{\delta C}{\delta f}]$.

Mass conservation It is easily checked that the mass $M[f]$ defined as $M[f] = \int \mathbf{dv} f$ is conserved. Indeed, $\frac{\delta M}{\delta f} = 1$ and $H_{\text{Landau}}[f, p + \alpha] = H_{\text{Landau}}[f, p]$ as H does not depend explicitly on p but only on its derivatives.

Momentum conservation Let us check the conservation of \mathbf{P} the momentum defined as $\mathbf{P}[f] = \int \mathbf{dv} \mathbf{v} f$. First, we notice that $\frac{\delta \mathbf{P}}{\delta f} = \mathbf{v}$. The functional derivative of H is

$$\frac{\delta H_{\text{Landau}}}{\delta p} = \int \mathbf{dv}_2 \frac{\partial}{\partial \mathbf{v}} \left\{ -\mathbf{B}(\mathbf{v}, \mathbf{v}_2) \left[\frac{\partial f}{\partial \mathbf{v}_2} f(\mathbf{v}) - \frac{\partial f}{\partial \mathbf{v}} f(\mathbf{v}_2) + 2f(\mathbf{v})f(\mathbf{v}_2) \left(\frac{\partial p}{\partial \mathbf{v}} - \frac{\partial p}{\partial \mathbf{v}_2} \right) \right] \right\}.$$

Hence, integrating by parts we have

$$\int \mathbf{dr} \mathbf{dv} \frac{\delta H_{\text{Landau}}}{\delta p} \frac{\delta \mathbf{P}}{\delta f} = \int \mathbf{dr} \mathbf{dv} \mathbf{dv}_2 \mathbf{B}(\mathbf{v}, \mathbf{v}_2) \left[\frac{\partial f}{\partial \mathbf{v}_2} f(\mathbf{v}) - \frac{\partial f}{\partial \mathbf{v}} f(\mathbf{v}_2) + 2f(\mathbf{v})f(\mathbf{v}_2) \left(\frac{\partial p}{\partial \mathbf{v}} - \frac{\partial p}{\partial \mathbf{v}_2} \right) \right].$$

Then, using the fact that $\mathbf{B}(\mathbf{v}, \mathbf{v}_2) = \mathbf{B}(\mathbf{v}_2, \mathbf{v})$, we find

$$\int \mathbf{dr} \mathbf{dv} \frac{\delta H_{\text{Landau}}}{\delta p} \frac{\delta \mathbf{P}}{\delta f} = 0.$$

This means that the total momentum \mathbf{P} is conserved by the dynamics. During this calculation, it is interesting to notice that both the first two terms and the last two terms of H preserve the momentum independently. This means that both the deterministic part of H and the noise part of H preserve the momentum independently. More precisely, the last term that came up with our approach, which did not appear in the naive mean field approach, compensates the contribution of the last term of H_{MF} . Another interesting property, is that a necessary condition for the deterministic part of the Hamiltonian to conserve the momentum is the following relation between the deterministic drift \mathbf{b} and the deterministic diffusion coefficient \mathbf{D} : $\int \mathbf{dv} f(\mathbf{v}) \left\{ \mathbf{b}[f] + \frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{D}[f] \right\} = 0$.

Energy conservation Now we should check that the total kinetic energy E is conserved, with $E[f] = \frac{1}{2} \int \mathbf{dv} \mathbf{v}^2 f$. Here, $\frac{\delta E}{\delta f} = \frac{1}{2} \mathbf{v}^2$. Using an integration by part we can write

$$\int \mathbf{dv} \frac{\delta H_{\text{Landau}}}{\delta p} \frac{\delta E}{\delta f} = \int \mathbf{dv} \mathbf{dv}_2 \mathbf{B}(\mathbf{v}, \mathbf{v}_2) \left\{ \left(\frac{\partial f}{\partial \mathbf{v}_2} f(\mathbf{v}) - \frac{\partial f}{\partial \mathbf{v}} f(\mathbf{v}_2) \right) \cdot \mathbf{v} + 2 \left(f(\mathbf{v})f(\mathbf{v}_2) \left(\frac{\partial p}{\partial \mathbf{v}} - \frac{\partial p}{\partial \mathbf{v}_2} \right) \right) \cdot \mathbf{v} \right\},$$

and because $\mathbf{B}(\mathbf{v}, \mathbf{v}_2) = \mathbf{B}(\mathbf{v}_2, \mathbf{v})$, we have

$$\int d\mathbf{v} \frac{\delta H_{\text{Landau}}}{\delta p} \frac{\delta E}{\delta f} = \int d\mathbf{v} d\mathbf{v}_2 \left\{ \frac{\partial f}{\partial \mathbf{v}_2} f(\mathbf{v}) + 2f(\mathbf{v})f(\mathbf{v}_2) \frac{\partial p}{\partial \mathbf{v}} \right\} \mathbf{B}(\mathbf{v}, \mathbf{v}_2) \cdot (\mathbf{v} - \mathbf{v}_2).$$

We have seen in appendix (C.2), that $\mathbf{B}(\mathbf{v}, \mathbf{v}_2) \cdot (\mathbf{v} - \mathbf{v}_2) = \vec{0}$, as a consequence of energy conservation in each collision. Then the integrand of the last formula is zero and we find that the total kinetic energy is conserved. Here too, both the deterministic part and the noise part of H preserve energy independently.

5.4.3 Entropy, quasipotential and time reversal symmetry

Entropy and quasipotential We define $S[f]$ the entropy functional:

$$S[f] = - \int d\mathbf{v} f \log f \quad (68)$$

Using results from section 2, we are going to check that $-S$ is a quasipotential as long as the conservation laws of mass, momentum and energy hold. Here, we only check the necessary condition which is that $-S$ satisfies the Hamilton-Jacobi equation, more precisely that: $H_{\text{Landau}} \left[f, -\frac{\delta S}{\delta f} \right] = 0$.

Given the definition of S , $\frac{\delta S}{\delta f} = -\log f + c$ where c is a constant which, because of the mass conservation, has no effect and we have

$$H_{\text{Landau}} \left[f, -\frac{\delta S}{\delta f} \right] = \int d\mathbf{r} d\mathbf{v} d\mathbf{v}_2 \left(f(\mathbf{v})f(\mathbf{v}_2) \frac{\partial^2 \mathbf{B}}{\partial \mathbf{v} \partial \mathbf{v}_2} - \frac{\partial f}{\partial \mathbf{v}} \frac{\partial f}{\partial \mathbf{v}_2} \mathbf{B} \right).$$

Integrating by parts twice the second term, we find out that the integrand is zero and that $-S$ satisfies the Hamilton-Jacobi equation: $H \left[f, -\frac{\delta S}{\delta f} \right] = 0$.

Time reversal symmetry We define the time reversal operator I by $I[f](\mathbf{v}) = f(-\mathbf{v})$. One can easily check that $H_{\text{Landau}} [I[f], -I[p]] = H_{\text{Landau}} \left[f, p - \frac{\delta S}{\delta f} \right]$. The computation is very close to the one above, that was performed to prove that the entropy is the negative of the quasipotential up to conservation laws.

We stated in section 2 that $H_{\text{Landau}} [I[f], -I[p]] = H_{\text{Landau}} \left[f, p - \frac{\delta S}{\delta f} \right]$ implies a time reversal symmetry of the path $\{f(t)\}_{0 \leq t \leq T}$ at the level of large deviations. The fluctuation paths are thus the time reversed of the relaxation paths.

Moreover, from results (13) and (12) of section 2, we deduce that entropy increases along the relaxation paths. Thanks to the time reversal symmetry of the large deviation structure, we can also conclude that the entropy decreases along the fluctuation paths.

As a conclusion, we have derived the Hamiltonian for the Landau equation and we have checked all its expected properties.

5.5 The gradient flow structure of the Landau equation derived from the large deviation Hamiltonian

It is customary and classical to observe that many dynamical models related to kinetic theories and mesoscopic systems in interaction with thermal baths have a gradient-transverse structure

$$\frac{\partial f}{\partial t} = -\text{Grad}_f \mathcal{U}[f] + \mathcal{G}[f], \quad (69)$$

where \mathcal{U} might be the free energy or minus the entropy, where for any f ($\text{Grad}_f \mathcal{U}, \mathcal{G}$) = 0. Grad_f is the gradient with respect to a f -dependent norm $(p, C[f]p)$, where C is a quadratic form: $\text{Grad}_f \mathcal{U}[f] = C[f] \frac{\delta \mathcal{U}}{\delta f}$. \mathcal{G} is often associated to the microscopic reversible dynamics or the free transport.

For example, for the Fourier law $\frac{\partial \rho}{\partial t} = D \frac{\partial^2 \rho}{\partial \mathbf{r}^2}$, has this structure [16,23], where $\mathcal{U} = \int d\mathbf{r} \rho \log \rho$ is the negative of the relative entropy, the metric used to compute the gradient is the Wasserstein distance with $C[f](\mathbf{r}, \mathbf{r}') = D \frac{\partial^2}{\partial \mathbf{r}^2} (\rho(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}'))$, and $\mathcal{G} = 0$. Another classical example is the McKean-Vlasov equation [16,23].

For the Landau equation, such a gradient structure has recently been described by [7]. In this section, we explain the connection of this structure with the large deviation formalism.

Even if this gradient-transverse structure is customarily observed, it is not always easy to determine the quadratic form C . Moreover a general explanation of the source of this structure is of interest. In section 5 of [4], we explain simply, following [12], that there is a close relation between the large deviations of the empirical density of particle system with detailed balance, and the gradient-transverse flow structure of the partial differential equations that describe kinetic theories. Whenever the detailed balance condition (14) is satisfied at the large deviations level, and whenever the large deviation Hamiltonian is quadratic in p , \mathcal{U} is the quasipotential, and the metric used to compute the gradient in (69) is given by the quadratic part of the large deviation Hamiltonian.

If we apply this general result to the Landau equation, using the large deviation principle that we just derived (equations (64)-(65)), we can conclude that the Landau equation has a gradient flow structure $\frac{\partial f}{\partial t} = -\text{Grad}_f \mathcal{U}[f]$ (in this case $\mathcal{G} = 0$ for homogeneous distribution). It reads

$$\frac{\partial f}{\partial t} = \int d\mathbf{v}' C[f](\mathbf{v}, \mathbf{v}') \frac{\delta S}{\delta f}(\mathbf{v}') \quad (70)$$

where $S[f] = -\int d\mathbf{v} f \log f$ is the Boltzmann entropy functional (the negative of the quasipotential), and $C[f]$ is the quadratic part of the Hamiltonian (64) and reads

$$C[f](\mathbf{v}, \mathbf{v}') = \frac{\partial^2}{\partial \mathbf{v} \partial \mathbf{v}'} : (f(\mathbf{v}) \delta(\mathbf{v} - \mathbf{v}') \mathbf{D}[f](\mathbf{v}) - f(\mathbf{v}) f(\mathbf{v}') \mathbf{B}[f](\mathbf{v}, \mathbf{v}')). \quad (71)$$

As discussed before, for independent particles, for instance independent Brownian motion leading to the Fourier law, the gradient is computed with respect to the Wasserstein distance. For particles with mean field interactions, for instance leading to the

McKean–Vlasov equation, the relevant metric is still the Wasserstein one. More generally for particles with mean field interaction with a diffusion coefficient that might be non-uniform and f dependent, as described in section 4.3, from the quadratic part of the Hamiltonian one finds $C[f](\mathbf{v}, \mathbf{v}') = \frac{\partial^2}{\partial \mathbf{v} \partial \mathbf{v}'} : (f(\mathbf{v}) \delta(\mathbf{v} - \mathbf{v}') \mathbf{D}[f](\mathbf{v}))$. This metric is still a kind of deformed Wasserstein one, that involves a f dependent diffusion coefficient. However for plasma in the weak coupling limit, and the Landau equation, one can see from equation (71) that the metric is no more simply related to the Wasserstein distance. One see in equation (71), that to the Wasserstein like term linear in f associated to independent motion of particles, one has to add a quadratic term in f related to the weak two-body interactions. This is an interesting remark.

5.6 A possible candidate for the large deviations Hamiltonian for the Balescu–Guernsey–Lenard equation

The Landau equation is also an approximation of the Balescu–Guernsey–Lenard equation

$$\frac{\partial f}{\partial \tau} = \frac{\partial}{\partial \mathbf{v}} \int d\mathbf{v}_2 \mathbf{B}[f](\mathbf{v}, \mathbf{v}_2) \left(-\frac{\partial f}{\partial \mathbf{v}_2} f(\mathbf{v}) + \frac{\partial f}{\partial \mathbf{v}} f(\mathbf{v}_2) \right),$$

which only differs from the Landau equation by the definition of the tensor \mathbf{B} :

$$\mathbf{B}(\mathbf{v}_1, \mathbf{v}_2) = \pi \left(\frac{\lambda_D}{L} \right)^3 \int_{-\infty}^{+\infty} d\omega \sum_{\mathbf{k}} \frac{\mathbf{k} \mathbf{k} \hat{W}(\mathbf{k})^2}{|\varepsilon[f](\omega, \mathbf{k})|^2} \delta(\omega - \mathbf{k} \cdot \mathbf{v}_1) \delta(\omega - \mathbf{k} \cdot \mathbf{v}_2). \quad (72)$$

where $\varepsilon[f](\omega, \mathbf{k})$ is the dielectric susceptibility defined by

$$\varepsilon[f](\mathbf{k}, \omega) = 1 - \hat{W}(\mathbf{k}) \int d\mathbf{v} \frac{\mathbf{k} \cdot \frac{\partial f}{\partial \mathbf{v}}}{\mathbf{k} \cdot \mathbf{v} - \omega - i\tilde{\varepsilon}},$$

which depends on the actual distribution of f . We recover the Landau equation by setting $\varepsilon(\omega, \mathbf{k})$ to 1. It is easy to check that the tensor \mathbf{B} for the Balescu–Guernsey–Lenard equation satisfies the same property as the tensor \mathbf{B} for the Landau equation:

1. $\mathbf{B}(\mathbf{v}_1, \mathbf{v}_2)$ is a symmetric tensor for all pair of momenta $(\mathbf{v}_1, \mathbf{v}_2)$,
2. $\mathbf{B}(\mathbf{v}_1, \mathbf{v}_2) = \mathbf{B}(\mathbf{v}_2, \mathbf{v}_1)$,
3. $\mathbf{B}(\mathbf{v}_1, \mathbf{v}_2) \cdot (\mathbf{v}_1 - \mathbf{v}_2) = \vec{0}$.

Furthermore, in our derivation of the Hamiltonian for the Landau equation, we never used the explicit expression of \mathbf{B} or the fact it did not depend on f .

All these remarks lead to the natural conjecture that the Hamiltonian (64) could describe the large deviations for the Balescu–Guernsey–Lenard equation as long as we replace the Landau expression of tensor \mathbf{B} (61) by the Lenard-Balescu expression of tensor \mathbf{B} (72). In other words, we might conjecture that the large deviations of

the Balescu–Guernsey–Lenard equation are described by the Hamiltonian $H_{BGL}^{(\text{conjecture})}$ that reads

$$H_{BGL}^{(\text{conjecture})}[f, p] = \int \mathrm{d}\mathbf{r} \mathrm{d}\mathbf{v}_1 f \left\{ \mathbf{b}[f] \cdot \frac{\partial p}{\partial \mathbf{v}_1} + \frac{\partial}{\partial \mathbf{v}_1} \left(\mathbf{D}[f] \frac{\partial p}{\partial \mathbf{v}_1} \right) + \mathbf{D}[f] : \frac{\partial p}{\partial \mathbf{v}_1} \frac{\partial p}{\partial \mathbf{v}_1} \right\} \quad (73)$$

$$- \int \mathrm{d}\mathbf{r} \mathrm{d}\mathbf{v}_1 \mathrm{d}\mathbf{v}_2 f(\mathbf{v}_1) f(\mathbf{v}_2) \frac{\partial p}{\partial \mathbf{v}_1} \frac{\partial p}{\partial \mathbf{v}_2} : \mathbf{B}[f](\mathbf{v}_1, \mathbf{v}_2).$$

One can check that this large deviation Hamiltonian has all the expected properties: it has the conservation law symmetries and the negative of the entropy $-S$ (see (68)) solves the stationary Hamilton–Jacobi equation. However, we will prove in section 6 that the correct Hamiltonian for the Balescu–Guernsey–Lenard equation is not quadratic in p , and that a quadratic Hamiltonian in p is obtained only in the Landau limit $k\lambda_D \gg 1$ (or $k \gg 1$ in our set of non-dimensional variables).

We thus conclude that although very natural, $H_{BGL}^{(\text{conjecture})}$ is not the Hamiltonian for the large deviations associated with the Balescu–Guernsey–Lenard equation.

6 Large deviations associated to the Landau kinetic theory from the microscopic dynamics

In this section, we compute the Hamiltonian for the large deviations of the empirical density for plasma directly from the dynamics (22). We use the formalism of large deviations for slow–fast system presented in section 2.2.2. Our result is a series representation of the large deviation Hamiltonian for the empirical density of N particles coupled through Coulomb interactions. We compute explicitly the terms of this series only to order four. This expansion can be truncated at order two, and then fluctuations are Gaussian, either the limit of a large plasma parameter ($\Lambda \rightarrow \infty$) when $L > \lambda_D$ or in the limit of large N , when $L < \lambda_D$.

We then discuss the Landau approximation. The Landau approximation is valid to describe the contributions to the kinetic equation or to the large deviation Hamiltonian of very large wavevectors compared to the inverse of the Debye length ($k \gg 1$ in non-dimensional plasma variables or equivalently $k\lambda_D \gg 1$ for physical variables). We show that within the Landau equation, this series expansion of the Hamiltonian can be truncated at order two, with larger order terms being negligible. Through this truncation and noting that some terms of order two in p are also negligible, we obtain the large deviation Hamiltonian for the Landau equation. The Landau large deviation Hamiltonian indeed coincide with the one computed in section 5 from the Boltzmann equation, as expected. In this Landau limit, the large deviation Hamiltonian describes locally Gaussian fluctuations. We note however, that beyond the Landau limit, when one cannot assume that $k \gg 1$ in non-dimensional plasma variables (or equivalently $k\lambda_D \gg 1$ for physical variables), the full expression for the Gaussian fluctuations does not coincide with the Landau Gaussian fluctuations.

In section 6.1, we introduce the quasi-linear dynamics of the empirical density of N particles coupled with a Coulomb interaction, for which the law of large numbers is the Balescu–Guernsey–Lenard kinetic theory. We also explain that this quasi-linear

dynamics of the empirical density can be seen as a slow-fast system. We can use the slow-fast large deviation formalism, as presented in section 2.2.2. In section 6.2, we characterize the stationary process of the fast variables, which is the fluctuating part of the empirical density. We also perform the computation of the two first terms of its cumulant series expansion. In section 6.5 we show that the terms of this cumulant series expansion are naturally ordered as powers of the wavevectors. As a consequence, from the two first cumulants, we can deduce the expression of the large deviation Hamiltonian for the Landau equation. In section 6.6, we show that the large deviation Hamiltonian for the Landau equation, obtained either from the microscopic dynamics or from the Boltzmann equation, are the same. In section 6.7, we discuss the large deviation result for the case where the size of the domain is smaller than the Debye length: $L < \lambda_D$. In section 6.8, we switch back to dimensional variables and we express the large deviation principle associated with the Landau equation in physical units.

6.1 The Klimontovich approach, quasilinear and slow-fast dynamics

We consider the empirical density

$$g_\Lambda(\mathbf{r}, \mathbf{v}, t) = \frac{1}{\Lambda} \sum_{n=1}^N \delta(\mathbf{v} - \mathbf{v}_n(t)) \delta(\mathbf{r} - \mathbf{r}_n(t)),$$

rescaled by the plasma parameter Λ , of N particles interacting via a Coulomb potential according to the dynamics (22). From these equations of motion, we can deduce the Klimontovich equation

$$\frac{\partial g_\Lambda}{\partial t} + \mathbf{v} \cdot \frac{\partial g_\Lambda}{\partial \mathbf{r}} - \frac{\partial V[g_\Lambda]}{\partial \mathbf{r}} \cdot \frac{\partial g_\Lambda}{\partial \mathbf{v}} = 0. \quad (74)$$

We consider the decomposition

$$g_\Lambda(\mathbf{r}, \mathbf{v}, t) = f_\Lambda(\mathbf{v}) + \frac{1}{\sqrt{\Lambda}} \delta g_\Lambda(\mathbf{r}, \mathbf{v}, t),$$

where $f_\Lambda(\mathbf{v}, t) = \left(\frac{\lambda_D}{L}\right)^3 \int_{[0, L/\lambda_D]^3} d\mathbf{r} g_\Lambda(\mathbf{r}, \mathbf{v}, t)$ is the projection of g_Λ on homogeneous distributions (distributions that only depend on the velocity). From the Klimontovich equation (74), we straightforwardly write

$$\frac{\partial f_\Lambda}{\partial t} = \frac{1}{\Lambda} \left(\frac{\lambda_D}{L}\right)^3 \int d\mathbf{r} \left(\frac{\partial V[\delta g_\Lambda]}{\partial \mathbf{r}} \cdot \frac{\partial \delta g_\Lambda}{\partial \mathbf{v}} \right), \quad (75)$$

$$\begin{aligned} \frac{\partial \delta g_\Lambda}{\partial t} = & -\mathbf{v} \cdot \frac{\partial \delta g_\Lambda}{\partial \mathbf{r}} + \frac{\partial V[\delta g_\Lambda]}{\partial \mathbf{r}} \cdot \frac{\partial f_\Lambda}{\partial \mathbf{v}} \\ & + \frac{1}{\sqrt{\Lambda}} \left[\frac{\partial V[\delta g_\Lambda]}{\partial \mathbf{r}} \cdot \frac{\partial \delta g_\Lambda}{\partial \mathbf{v}} - \frac{\lambda_D^3}{\Lambda L^3} \int d\mathbf{r} \left(\frac{\partial V[\delta g_\Lambda]}{\partial \mathbf{r}} \cdot \frac{\partial \delta g_\Lambda}{\partial \mathbf{v}} \right) \right]. \end{aligned} \quad (76)$$

Those equations are similar to (29-30), but we do not take statistical averages. We will study in this section the complete statistics of the right hand side of (75) and not just its average as in (29-30).

We now assume the validity of the quasi-linear approximation, which amount at neglecting the terms of order $\Lambda^{-1/2}$ in the evolution equation for δg_Λ . We also change the timescale $\tau = t/\Lambda$ and obtain the quasilinear dynamics

$$\frac{\partial f_\Lambda}{\partial \tau} = \left(\frac{\lambda_D}{L}\right)^3 \int d\mathbf{r} \left(\frac{\partial V[\delta g_\Lambda]}{\partial \mathbf{r}} \cdot \frac{\partial \delta g_\Lambda}{\partial \mathbf{v}} \right), \quad (77)$$

$$\frac{\partial \delta g_\Lambda}{\partial \tau} = \Lambda \left\{ -\mathbf{v} \cdot \frac{\partial \delta g_\Lambda}{\partial \mathbf{r}} + \frac{\partial V[\delta g_\Lambda]}{\partial \mathbf{r}} \cdot \frac{\partial f_\Lambda}{\partial \mathbf{v}} \right\}. \quad (78)$$

When Λ goes to infinity, we observe that the equation for δg_Λ is a fast process, with timescales for τ of order $1/\Lambda$, while the equation for f_Λ is a slow one with timescales for τ of order 1. For such slow-fast dynamics, it is natural to consider f_Λ fixed (frozen) in equation (78) on time scales for τ of order $\tau = 1/\Lambda$. For fixed f_Λ , the dynamics for δg_Λ is linear and can be solved. Computing the average of the term $\int d\mathbf{r} \frac{\partial V[\delta g_\Lambda]}{\partial \mathbf{r}} \cdot \frac{\partial \delta g_\Lambda}{\partial \mathbf{v}}$, for the asymptotic process for δg_Λ for fixed f_Λ leads to the Balescu–Guernsey–Lenard equation, as explained in section 3, or its Landau approximation whenever small length scales dominate the collision kernel of the Balescu–Guernsey–Lenard equation. Those computations can be found in classical textbooks [15, 11, 19].

In the following we want to go beyond these classical computations, by estimating not just the average of the right hand side in (77) $\int d\mathbf{r} \frac{\partial V[\delta g_\Lambda]}{\partial \mathbf{r}} \cdot \frac{\partial \delta g_\Lambda}{\partial \mathbf{v}}$, but all the cumulants of the time averages $\int_0^T \int d\mathbf{r} \frac{\partial V[\delta g_\Lambda]}{\partial \mathbf{r}} \cdot \frac{\partial \delta g_\Lambda}{\partial \mathbf{v}}$ in order to describe the large deviations for the process f_Λ . Using the classical result of large deviations for slow-fast dynamics, as explained in section 2.2.2 (see equations (20-21)), we conclude that

$$\mathbf{P}(f_\Lambda = f) \underset{\Lambda \rightarrow \infty}{\asymp} e^{-\Lambda \text{Sup}_p \int_0^T \{ \int d\mathbf{r} d\mathbf{v} j p - H[f, p] \}}. \quad (79)$$

where

$$H[f, p] = \lim_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{E}_f \left[\exp \left(\int_0^T d\tau \int d\mathbf{v} p(\mathbf{v}) \int d\mathbf{r}' \frac{\partial V[\delta g_\Lambda]}{\partial \mathbf{r}'} \cdot \frac{\partial \delta g_\Lambda}{\partial \mathbf{v}} \right) \right] \quad (80)$$

and where \mathbb{E}_f denotes the expectation on the process for δg_Λ , where δg_Λ evolves according to

$$\frac{\partial \delta g_\Lambda}{\partial t} = \left\{ -\mathbf{v} \cdot \frac{\partial \delta g_\Lambda}{\partial \mathbf{r}} + \frac{\partial V[\delta g_\Lambda]}{\partial \mathbf{r}} \cdot \frac{\partial f}{\partial \mathbf{v}} \right\}. \quad (81)$$

In this equation $f_\Lambda = f$ is frozen and time independent.

We note that to obtain equation (80) from equation (21), we have considered f_Λ as a function of the μ -space. Then the conjugated momentum $p(\mathbf{r}, \mathbf{v})$ should also be a function of the μ -space and the scalar product be the one of the μ -space. However, recognizing that for homogeneous f , p should also be homogeneous ($p(\mathbf{r}, \mathbf{v}) = p(\mathbf{v})$), and performing trivial integration over \mathbf{r} leads to (80).

The goal of the following subsections is to compute (80).

6.2 The quasi-stationary Gaussian process for δg_A

In order to compute (80), we first note that for frozen f , equation (81) is linear. It thus describe a Gaussian process, for instance when the initial conditions are distributed according to a Gaussian. Moreover, as explained in §51 of [11] such a process is expected to converge to a stationary Gaussian process irrespective of the initial condition. The properties of this process are determined by the fact that we are dealing with a dynamics with discrete particles. In the following we will thus consider averages in equation (80) as averages over this stationary Gaussian process. Such stationary averages are denoted \mathbb{E}_S .

We do not reproduce the classical computations of the correlation functions of this stationary process, but just report the formulas which can be found for instance in §51 of [11]. The potential autocorrelation function is homogeneous because of the space translation symmetry. Then

$$\mathbb{E}_S (V [\delta g_A] (\mathbf{r}_1, t_1) V [\delta g_A] (\mathbf{r}_2, t_2)) = \mathcal{C}_{VV} (\mathbf{r}_1 - \mathbf{r}_2, t_1 - t_2),$$

We define $\tilde{\varphi}$ the Fourier-Laplace transform of a function φ as

$$\tilde{\varphi} (\mathbf{k}, \omega) = \int_{[0, L/\lambda_D]^3} d\mathbf{r} \int_0^\infty dt e^{-i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \varphi (\mathbf{r}, t), \quad (82)$$

following the same convention as in [11]. The autocorrelation function of the Fourier-Laplace transform of the potential then reads

$$\mathbb{E}_S \left(V \left[\widetilde{\delta g_A} \right] (\mathbf{k}_1, \omega_1) V \left[\widetilde{\delta g_A} \right] (\mathbf{k}_2, \omega_2) \right) = 2\pi \left(\frac{L}{\lambda_D} \right)^3 \delta_{\mathbf{k}_1, -\mathbf{k}_2} \delta (\omega_1 + \omega_2) \widetilde{\mathcal{C}_{VV}} (\mathbf{k}_1, \omega_1), \quad (83)$$

where $\widetilde{\mathcal{C}_{VV}}$ is the space-time Fourier transform of \mathcal{C}_{VV} . Equation (51.20), §51 of [11], with the identifications $V = \varphi$, $\hat{W} (\mathbf{k}) = 1/k^2$, and with the dimensionless variables defined in section 3.1, gives

$$\widetilde{\mathcal{C}_{VV}} (\mathbf{k}, \omega) = 2\pi \left[\int d\mathbf{v}' f (\mathbf{v}') \delta (\omega - \mathbf{k}\cdot\mathbf{v}') \right] \frac{\hat{W} (\mathbf{k})^2}{|\mathcal{E} [f] (\mathbf{k}, \omega)|^2}, \quad (84)$$

Similarly the time stationary correlation functions between the potential and the distribution fluctuations is space-time homogeneous $\mathbb{E}_S (V [\delta g_N] (\mathbf{r}_1, t_1) \delta g_N (\mathbf{r}_2, \mathbf{v}, t_2)) = \mathcal{C}_{VG} (\mathbf{r}_1 - \mathbf{r}_2, t_1 - t_2, \mathbf{v})$, with space-time Fourier transform

$$\mathbb{E}_S \left(V \left[\widetilde{\delta g_A} \right] (\mathbf{k}_1, \omega_1) \widetilde{\delta g_A} (\mathbf{k}_2, \omega_2) \right) = 2\pi \left(\frac{L}{\lambda_D} \right)^3 \delta_{\mathbf{k}_1, -\mathbf{k}_2} \delta (\omega_1 + \omega_2) \widetilde{\mathcal{C}_{VG}} (\mathbf{k}_1, \omega_1, \mathbf{v}). \quad (85)$$

Similarly $\mathbb{E}_S (\delta g_A (\mathbf{r}_1, \mathbf{v}_1, t_1) \delta g_A (\mathbf{r}_2, \mathbf{v}_2, t_2)) = \mathcal{C}_{GG} (\mathbf{r}_1 - \mathbf{r}_2, t_1 - t_2, \mathbf{v}_1, \mathbf{v}_2)$, with

$$\mathbb{E}_S \left(\widetilde{\delta g_A} (\mathbf{k}_1, \omega_1) \widetilde{\delta g_A} (\mathbf{k}_2, \omega_2) \right) = 2\pi \left(\frac{L}{\lambda_D} \right)^3 \delta_{\mathbf{k}_1, -\mathbf{k}_2} \delta (\omega_1 + \omega_2) \widetilde{\mathcal{C}_{GG}} (\mathbf{k}_1, \omega_1, \mathbf{v}_1, \mathbf{v}_2). \quad (86)$$

The formulas for $\widetilde{\mathcal{C}_{VG}}$ are given by equation (51.21) and (51.23) respectively, in [11], with the identifications $V = \varphi$, $\hat{W} (\mathbf{k}) = 1/k^2$. They are

$$\widetilde{\mathcal{C}}_{VG}(\mathbf{k}, \omega, \mathbf{v}) = -\frac{\mathbf{k}}{\omega - \mathbf{k} \cdot \mathbf{v} - i\tilde{\varepsilon}} \cdot \frac{\partial f}{\partial \mathbf{v}}(\mathbf{v}) \widetilde{\mathcal{C}}_{VV}(\mathbf{k}, \omega) + \frac{2\pi\hat{W}(\mathbf{k})}{\varepsilon[f](\mathbf{k}, \omega)} f(\mathbf{v}) \delta(\omega - \mathbf{k} \cdot \mathbf{v}), \quad (87)$$

and

$$\begin{aligned} \widetilde{\mathcal{C}}_{GG}(\mathbf{k}, \omega, \mathbf{v}_1, \mathbf{v}_2) &= 2\pi\delta(\mathbf{v}_1 - \mathbf{v}_2) f(\mathbf{v}_1) \delta(\omega - \mathbf{k} \cdot \mathbf{v}_1) \\ &+ \frac{\widetilde{\mathcal{C}}_{VV}(\mathbf{k}, \omega)}{(\omega - \mathbf{k} \cdot \mathbf{v}_1 + i\tilde{\varepsilon})(\omega - \mathbf{k} \cdot \mathbf{v}_2 - i\tilde{\varepsilon})} \mathbf{k} \cdot \frac{\partial f}{\partial \mathbf{v}}(\mathbf{v}_1) \mathbf{k} \cdot \frac{\partial f}{\partial \mathbf{v}}(\mathbf{v}_2) \\ &- 2\pi\hat{W}(\mathbf{k}) \mathbf{k} \cdot \frac{\partial f}{\partial \mathbf{v}}(\mathbf{v}_1) \frac{f(\mathbf{v}_2) \delta(\omega - \mathbf{k} \cdot \mathbf{v}_2)}{\varepsilon(\mathbf{k}, \omega)(\omega - \mathbf{k} \cdot \mathbf{v}_1 + i\tilde{\varepsilon})} \\ &- 2\pi\hat{W}(\mathbf{k}) \mathbf{k} \cdot \frac{\partial f}{\partial \mathbf{v}}(\mathbf{v}_2) \frac{f(\mathbf{v}_1) \delta(\omega - \mathbf{k} \cdot \mathbf{v}_1)}{\varepsilon^*(\mathbf{k}, \omega)(\omega - \mathbf{k} \cdot \mathbf{v}_2 - i\tilde{\varepsilon})}. \end{aligned} \quad (88)$$

We note that the order in the correlation functions for V and δg_Λ matters. We have $\mathbb{E}_S(\delta g_\Lambda(\mathbf{r}_1, \mathbf{v}, t_1) V[\delta g_\Lambda](\mathbf{r}_2, t_2)) = \mathcal{C}_{GV}(\mathbf{r}_1 - \mathbf{r}_2, t_1 - t_2, \mathbf{v})$. Then $\widetilde{\mathcal{C}}_{VG}(\mathbf{k}, \omega, \mathbf{v}) = \widetilde{\mathcal{C}}_{GV}(-\mathbf{k}, -\omega, \mathbf{v}) = \widetilde{\mathcal{C}}_{GV}^*(\mathbf{k}, \omega, \mathbf{v})$. We also note the symmetry property of \mathcal{C}_{GG} : $\mathcal{C}_{GG}(\mathbf{k}, \omega, \mathbf{v}_1, \mathbf{v}_2) = \mathcal{C}_{GG}(-\mathbf{k}, -\omega, \mathbf{v}_2, \mathbf{v}_1)$, which is a consequence of the symmetry $\mathcal{C}_{GG}(\mathbf{r}, t, \mathbf{v}_1, \mathbf{v}_2) = \mathcal{C}_{GG}(-\mathbf{r}, -t, \mathbf{v}_2, \mathbf{v}_1)$.

From this stationary Gaussian process, we are now ready to compute the large deviation Hamiltonian through a cumulant expansion in the two following sections.

6.3 Computation of a series expansion of large deviation Hamiltonian

In order to have explicit formulas for (80), in this section we first compute the two first cumulants for

$$X[f] = -\int_0^T dt \int d\mathbf{v} \frac{\partial p}{\partial \mathbf{v}} \int d\mathbf{r} \frac{\partial V[\delta g_\Lambda]}{\partial \mathbf{r}} \cdot \delta g_\Lambda. \quad (89)$$

If we expand a cumulant generating function for a random variable X , we obtain $\log \mathbb{E} \exp(X) = \mathbb{E}(X) + \mathbb{E}\mathbb{E}(X^2)/2 + H^{>2}$, where for the second order cumulant we use the notation $\mathbb{E}\mathbb{E}(X^2) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2$, and where $H^{>2}$ is the contribution of all cumulants of order larger than 2. We thus have

$$H = H^{(1)} + H^{(2)} + H^{>2}. \quad (90)$$

If X is given by (89), we have

$$H^{(1)} = \int d\mathbf{r} \int d\mathbf{v} \frac{\partial p}{\partial \mathbf{v}}(\mathbf{v}) \cdot \mathbf{C}^{(1)}(\mathbf{v}), \quad \text{where } \mathbf{C}^{(1)}(\mathbf{v}) = -\mathbb{E}_S \left(\frac{\partial V[\delta g_\Lambda]}{\partial \mathbf{r}} \delta g_\Lambda \right). \quad (91)$$

and

$$H^{(2)} = \int d\mathbf{r} d\mathbf{v}_1 d\mathbf{v}_2 \frac{\partial p}{\partial \mathbf{v}}(\mathbf{v}_1) \frac{\partial p}{\partial \mathbf{v}}(\mathbf{v}_2) : \mathbf{C}(\mathbf{v}_1, \mathbf{v}_2), \quad (92)$$

where

$$\mathbf{C}(\mathbf{v}_1, \mathbf{v}_2) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_0^T dt_1 \int_0^T dt_2 \int d\mathbf{r}_1 \int d\mathbf{r}_2 \mathbb{E} \mathbb{E} \left[\frac{\partial V[\delta g_\Lambda]^{(1)}}{\partial \mathbf{r}} \delta g_\Lambda^{(1)} \frac{\partial V[\delta g_\Lambda]^{(2)}}{\partial \mathbf{r}} \delta g_\Lambda^{(2)} \right]. \quad (93)$$

We note that \mathbf{C} is a second order tensor and that in the formula for $H^{(2)}$, the symbol “:” means the contraction of two second order tensors. In the formula for \mathbf{C} , the superscripts (1) or (2) mean that the quantities are evaluated at either (\mathbf{r}_1, t_1) and (\mathbf{r}_2, t_2) , respectively, or $(\mathbf{r}_1, \mathbf{v}_1, t_1)$ and $(\mathbf{r}_2, \mathbf{v}_2, t_2)$, respectively.

We note that a truncation at second order of the cumulant expansion gives a Hamiltonian which is quadratic in p .

Computation of the first cumulant Using (89) and (87) one can compute $\mathbf{C}^{(1)}$. The computations can be found in appendix F.1. The computations are not exactly the same, but really similar to the one in §51 of [11]. One obtains

$$\begin{aligned} \mathbf{C}^{(1)}(\mathbf{v}) &= \int d\mathbf{v}_2 \mathbf{B}[f](\mathbf{v}, \mathbf{v}_2) \left(-\frac{\partial f}{\partial \mathbf{v}_2} f(\mathbf{v}) + f(\mathbf{v}_2) \frac{\partial f}{\partial \mathbf{v}} \right) \\ &= \mathbf{b}[f](\mathbf{v}) f(\mathbf{v}) - \mathbf{D}[f](\mathbf{v}) \cdot \frac{\partial f}{\partial \mathbf{v}}(\mathbf{v}), \end{aligned}$$

where \mathbf{B} is the tensor defined in equation (28). Integrating over \mathbf{r} in equation (91), we find that $H^{(1)}$ is then given by

$$H^{(1)} = \int d\mathbf{r} d\mathbf{v} f(\mathbf{v}) \left\{ \mathbf{b}[f] \cdot \frac{\partial p}{\partial \mathbf{v}} + \frac{\partial}{\partial \mathbf{v}} \left(\mathbf{D}[f] \frac{\partial p}{\partial \mathbf{v}} \right) \right\}, \quad (94)$$

where $\mathbf{b}[f]$ and $\mathbf{D}[f]$ are defined in equation (34). $H^{(1)}$, which is the linear part with respect to p , gives the formula that corresponds to the Balescu–Guernsey–Lenard operator, as expected.

Computation of the second cumulant Now, the more challenging and new part is to compute $H^{(2)}$ the second cumulant. In order to compute (93) using (89), we see that we will have to evaluate four-point correlation functions. As the fluctuations are locally Gaussian, we can use Wick’s theorem in order to express the four-points correlation functions $\mathbb{E}_S \left(\frac{\partial V[\delta g_N]^{(1)}}{\partial \mathbf{r}} \frac{\partial V[\delta g_N]^{(2)}}{\partial \mathbf{r}} \delta g_N^{(1)} \delta g_N^{(2)} \right)$ as a sum of products of two-points correlation functions, and use the formulas for the two point correlation functions. After some lengthy computations reported in appendix F.2, we obtain the result

$$\begin{aligned} H^{(2)} &= \int d\mathbf{r} d\mathbf{v}_1 \frac{\partial p}{\partial \mathbf{v}}(\mathbf{v}_1) \frac{\partial p}{\partial \mathbf{v}}(\mathbf{v}_1) \mathbf{D}(\mathbf{v}_1) f(\mathbf{v}_1) \\ &\quad - \int d\mathbf{r} d\mathbf{v}_1 d\mathbf{v}_2 \frac{\partial p}{\partial \mathbf{v}}(\mathbf{v}_1) \frac{\partial p}{\partial \mathbf{v}}(\mathbf{v}_2) : \mathbf{B}[f](\mathbf{v}_1, \mathbf{v}_2) f(\mathbf{v}_1) f(\mathbf{v}_2) \\ &\quad + \int d\mathbf{r} d\mathbf{v}_1 d\mathbf{v}_2 d\mathbf{v}_3 d\mathbf{v}_4 \frac{\partial p}{\partial \mathbf{v}_1} \frac{\partial p}{\partial \mathbf{v}_2} \mathbf{B}^{(2)}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) \left\{ f(\mathbf{v}_1) f(\mathbf{v}_2) \frac{\partial f}{\partial \mathbf{v}_3} \frac{\partial f}{\partial \mathbf{v}_4} \right. \\ &\quad \left. - 2f(\mathbf{v}_1) \frac{\partial f}{\partial \mathbf{v}_2} f(\mathbf{v}_3) \frac{\partial f}{\partial \mathbf{v}_4} + \frac{\partial f}{\partial \mathbf{v}_1} \frac{\partial f}{\partial \mathbf{v}_2} f(\mathbf{v}_3) f(\mathbf{v}_4) \right\}, \quad (95) \end{aligned}$$

with

$$\mathbf{B}^{(2)}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) = 2\pi^3 \left(\frac{\lambda_D}{L}\right)^3 \sum_{\mathbf{k}} \int d\omega \mathbf{k}\mathbf{k}\mathbf{k}\mathbf{k} \frac{\hat{W}(\mathbf{k})^4}{|\mathcal{E}(\mathbf{k}, \omega)|^4} \prod_{i=1}^4 \delta(\omega - \mathbf{k} \cdot \mathbf{v}_i), \quad (96)$$

being a fully symmetric order-4 tensor.

6.4 Computation of higher order cumulants

Using this same method, it is possible to compute, by induction, every terms of the cumulant expansion. However, there is a infinity of them, and a priori they are non-zero and, in general, they are of the same order of magnitude as the second order one. Nevertheless, we can recognize a pattern in the cumulant expansion. To understand it better, let us compute the following term in the cumulant expansion of the large deviation Hamiltonian .

In this subsection, we compute the third cumulant, but the procedure would be exactly the same if we were to compute the n -th cumulant. We already computed the first two cumulants $H^{(1)}$ and $H^{(2)}$ associated to this cumulant generating function. Now let us compute the third cumulant $H^{(3)}$ which can be expressed as a combination of moments of the random variable X (89):

$$H^{(3)} = \lim_{T \rightarrow \infty} \frac{1}{T} \left(\mathbb{E}(X^3) - 3\mathbb{E}(X^2)\mathbb{E}(X) + 2\mathbb{E}(X)^3 \right).$$

Similarly, we denote $H^{(n)}$ the term of the large deviation Hamiltonian (80) accounting for the contribution of the n -th cumulant of the random variable X (89).

Because the process for δg_Λ is Gaussian, we can compute all the moments of X from the two-points correlation functions (84, 87, 88) and the Wick theorem. To express the result, let us introduce the fully symmetric order- $2n$ tensor $\mathbf{B}^{(n)}$ defined as

$$\mathbf{B}^{(n)}(\mathbf{v}_1, \dots, \mathbf{v}_{2n}) = \frac{(2\pi)^{2n}}{4\pi n} \left(\frac{\lambda_D}{L}\right)^3 \sum_{\mathbf{k}} \int_{\Gamma} d\omega \frac{\mathbf{k}^{\otimes 2n} \hat{W}(\mathbf{k})^{2n}}{|\mathcal{E}(\mathbf{k}, \omega)|^{2n}} \prod_{i=1}^{2n} \delta(\omega - \mathbf{k} \cdot \mathbf{v}_i), \quad (97)$$

where $\mathbf{k}^{\otimes 2n}$ is the tensor $\mathbf{k} \otimes \dots \otimes \mathbf{k}$, such as $\mathbf{B}^{(1)} = \mathbf{B}$, and it is consistent with the definition of $\mathbf{B}^{(2)}$ (132). Then, the third cumulant reads

$$\begin{aligned} H^{(3)} = & \int d\mathbf{r} d\mathbf{v}_1 d\mathbf{v}_2 d\mathbf{v}_3 d\mathbf{v}_4 f(\mathbf{v}_2) f(\mathbf{v}_3) \left(f(\mathbf{v}_1) \frac{\partial f}{\partial \mathbf{v}_4} - \frac{\partial f}{\partial \mathbf{v}_1} f(\mathbf{v}_4) \right) \mathbf{B}^{(2)} \frac{\partial p}{\partial \mathbf{v}_1} \frac{\partial p}{\partial \mathbf{v}_2} \left\{ \frac{\partial p}{\partial \mathbf{v}_2} - \frac{\partial p}{\partial \mathbf{v}_3} \right\} \\ & + \int d\mathbf{r} d\mathbf{v}_1 \dots d\mathbf{v}_6 \frac{\partial p}{\partial \mathbf{v}_1} \frac{\partial p}{\partial \mathbf{v}_2} \frac{\partial p}{\partial \mathbf{v}_3} \mathbf{B}^{(3)} \left\{ f(\mathbf{v}_1) f(\mathbf{v}_2) f(\mathbf{v}_3) \frac{\partial f}{\partial \mathbf{v}_4} \frac{\partial f}{\partial \mathbf{v}_5} \frac{\partial f}{\partial \mathbf{v}_6} \right. \\ & \quad \left. - 3 \frac{\partial f}{\partial \mathbf{v}_1} f(\mathbf{v}_2) f(\mathbf{v}_3) f(\mathbf{v}_4) \frac{\partial f}{\partial \mathbf{v}_5} \frac{\partial f}{\partial \mathbf{v}_6} \right. \\ & \quad \left. + 3 f(\mathbf{v}_1) \frac{\partial f}{\partial \mathbf{v}_2} \frac{\partial f}{\partial \mathbf{v}_3} f(\mathbf{v}_4) \frac{\partial f}{\partial \mathbf{v}_5} \frac{\partial f}{\partial \mathbf{v}_6} - \frac{\partial f}{\partial \mathbf{v}_1} \frac{\partial f}{\partial \mathbf{v}_2} \frac{\partial f}{\partial \mathbf{v}_3} f(\mathbf{v}_4) f(\mathbf{v}_5) f(\mathbf{v}_6) \right\}. \quad (98) \end{aligned}$$

We note that $H^{(3)}$ involves a term which is proportional to $\mathbf{B}^{(3)}$, but also a term which is proportional to $\mathbf{B}^{(2)}$.

We do not report the detailed computation, but we have explicitly computed $H^{(4)}$ (see appendix G). We observe that it involves terms which are proportional to $\mathbf{B}^{(4)}$ and $\mathbf{B}^{(3)}$, but no terms which are proportional to either $\mathbf{B}^{(2)}$ or $\mathbf{B}^{(1)}$. Based on this remark, we conjecture that $H^{(n)}$ contains only terms which are proportional to $\mathbf{B}^{(n)}$ and $\mathbf{B}^{(n-1)}$, and that the tensor $\mathbf{B}^{(k)}$ for $k < n - 1$ are not involved in the expression for $H^{(n)}$.

As we will explain in the next subsection, in the context of the Landau approximation, there is a natural hierarchy between the tensors $\mathbf{B}^{(n)}$ and the cumulant expansion can be simply truncated.

6.5 Hierarchy of the series expansion within the Landau approximation

Let us first recall that we can obtain the Landau equation from the Balescu–Guernsey–Lenard equation. The collision kernel for the Balescu–Guernsey–Lenard equation converges to the Landau collision kernel in the limit where all the wavevectors in (28) satisfy $k\lambda_D \gg 1$. In our system of plasma unit, where the length unit is renormalized by the Debye length, this means that the Balescu–Guernsey–Lenard collision kernel converges toward the Landau collision kernel in the limit of infinitely large wavevectors. In a similar way, we obtain the large deviation Hamiltonian for the Landau equation H_{Landau} from the large deviation Hamiltonian H (90) of the empirical density of N Coulomb interacting particles using the same limit.

In the expression of the tensor $\mathbf{B}^{(1)} = \mathbf{B}$

$$\mathbf{B}^{(1)} = \mathbf{B}(\mathbf{v}_1, \mathbf{v}_2) = \pi \left(\frac{\lambda_D}{L} \right)^3 \int_{-\infty}^{+\infty} d\omega \sum_{\mathbf{k}} \frac{\mathbf{k}\mathbf{k}}{k^{4n} |\varepsilon[f](\omega, \mathbf{k})|^2} \delta(\omega - \mathbf{k} \cdot \mathbf{v}_1) \delta(\omega - \mathbf{k} \cdot \mathbf{v}_2),$$

the Landau approximation implies that $k \gg 1$. In this context, we can consider that the dielectric function ε is equal to one. From there, a clear hierarchy appears in the cumulant series expansion (90). For $n \geq 2$, the terms involving

$$\mathbf{B}^{(n)}(\mathbf{v}_1, \dots, \mathbf{v}_{2n}) = \frac{(2\pi)^{2n}}{4\pi n} \left(\frac{\lambda_D}{L} \right)^3 \sum_{\mathbf{k}} \int_{\Gamma} d\omega \frac{\mathbf{k}^{\otimes 2n}}{k^{4n} |\varepsilon(\mathbf{k}, \omega)|^{2n}} \prod_{i=1}^{2n} \delta(\omega - \mathbf{k} \cdot \mathbf{v}_i)$$

will be negligible with respect to the terms involving $\mathbf{B}^{(1)} = \mathbf{B}$.

Let us define

$$\mathbf{B}_{\mathbf{k}}^{(n)}(\mathbf{v}_1, \dots, \mathbf{v}_{2n}) = \int_{\Gamma} d\omega \frac{\mathbf{k}^{\otimes 2n}}{k^{4n} |\varepsilon(\mathbf{k}, \omega)|^{2n}} \prod_{i=1}^{2n} \delta(\omega - \mathbf{k} \cdot \mathbf{v}_i),$$

such that

$$\mathbf{B}^{(n)}(\mathbf{v}_1, \dots, \mathbf{v}_{2n}) = \frac{(2\pi)^{2n}}{4\pi n} \sum_{\mathbf{k}} \left(\frac{\lambda_D}{L} \right)^3 \mathbf{B}_{\mathbf{k}}^{(n)}(\mathbf{v}_1, \dots, \mathbf{v}_{2n}).$$

Let us evaluate the size of $\mathbf{B}_{\mathbf{k}}^{(n)}$ in terms of the wavevectors \mathbf{k} . We have,

$$\mathbf{B}_{\mathbf{k}}^{(n)}(\mathbf{v}_1, \dots, \mathbf{v}_{2n}) = k^{1-4n} \frac{\mathbf{m}^{\otimes 2n}}{|\varepsilon(\mathbf{k}, \omega)|^{2n}} \prod_{i=2}^{2n} \delta(\mathbf{m} \cdot (\mathbf{v}_1 - \mathbf{v}_i)),$$

where $\mathbf{m} = \mathbf{k}/k$. Then,

$$\left(\frac{\lambda_D}{L}\right)^3 \mathbf{B}_{\mathbf{k}}^{(n)} \underset{k \gg 1}{=} \mathcal{O}\left(\left(\frac{\lambda_D}{Lk}\right)^3 \left(\frac{1}{k}\right)^{4n-4}\right), \quad (99)$$

where $\mathcal{O}(k^m)$ means that the term is of order k^m .

Furthermore, we note the wavevectors \mathbf{k} are of the form $2\pi(\lambda_D/L)\mathbf{l}$ with $\mathbf{l} \in \mathbb{Z}^3$. Then $\left(\frac{\lambda_D}{Lk}\right)^3$ is of order one at most. Thus, we can conclude that within the Landau approximation ($k \gg 1$ in our non-dimensional plasma variables) all the tensors $\mathbf{B}^{(n)}$ are negligible except for $\mathbf{B}^{(1)} = \mathbf{B}$. We have presented all the computation and this estimation in a finite box of length L . However similar reasoning generalize easily to an infinite box.

As a conclusion, at leading order, we can just keep the terms involving $\mathbf{B}^{(1)}$ in the cumulant series expansion, and the large deviations Hamiltonian for the Landau equation reads

$$\begin{aligned} H_{\text{Landau}}[f, p] = & \int d\mathbf{r} d\mathbf{v}_1 f \left\{ \mathbf{b}[f] \cdot \frac{\partial p}{\partial \mathbf{v}_1} + \frac{\partial}{\partial \mathbf{v}_1} \left(\mathbf{D}[f] \frac{\partial p}{\partial \mathbf{v}_1} \right) + \mathbf{D}[f] : \frac{\partial p}{\partial \mathbf{v}_1} \frac{\partial p}{\partial \mathbf{v}_1} \right\} \\ & - \int d\mathbf{r} d\mathbf{v}_1 d\mathbf{v}_2 f(\mathbf{v}_1) f(\mathbf{v}_2) \frac{\partial p}{\partial \mathbf{v}_1} \frac{\partial p}{\partial \mathbf{v}_2} : \mathbf{B}(\mathbf{v}_1, \mathbf{v}_2). \end{aligned} \quad (100)$$

This is exactly the Hamiltonian we derived from the Boltzmann equation large deviation Hamiltonian in section 5.3.

6.6 Large deviations for the Landau equation

In the previous section 6.5, we have established a large deviation principle for the homogeneous projection of the empirical density of N particles submitted to pairwise Coulomb interactions in the Landau approximation. It describes dynamical fluctuations beyond the Landau equation. More precisely, if we consider N particles evolving according to the dynamics (22), in a 3-dimensional torus of size $(L/\lambda_D)^3$ where λ_D is the Debye length, f_Λ the homogeneous projection of the empirical density

$$f_\Lambda(\mathbf{v}, t) = \frac{1}{\Lambda} \left(\frac{\lambda_D}{L}\right)^3 \sum_{n=1}^N \delta(\mathbf{v} - \mathbf{v}_n(t)),$$

follows the large deviation principle

$$\mathbf{P}(f_\Lambda = f) \underset{\Lambda \rightarrow \infty}{\asymp} e^{-\Lambda \text{Sup}_p \int_0^T \{ \int d\mathbf{r} d\mathbf{v} \dot{f} p - H_{\text{Landau}}[f, p] \}}, \quad (101)$$

where the large deviation Hamiltonian H_{Landau} is given by (100).

Although this Hamiltonian is exactly the one we derived in section 5 from the large deviation Hamiltonian associated to the Boltzmann equation, the large deviation principle (101) is slightly different. Indeed, the large deviation principle (65) describes large deviations of the empirical density g_Λ , whereas the large deviation

principle (101) only describes the large deviations for f_Λ which is the projection of g_Λ over homogeneous distributions. However, it is possible to obtain (101) from (65) through the use of the contraction principle. In large deviation theory, the contraction principle states that if we know a large deviation principle for a random variable X with a large deviation function $I(x)$ it is possible to obtain a large deviation principle for any function $\varphi(X)$ of this random variable and the associated large deviation function is $I_\varphi(y) = \inf_{\varphi(x)=y} I(x)$. The two results are thus fully consistent.

Based on the discussion of section 5.4, the large deviation Hamiltonian H_{Landau} satisfies all the expected properties of the large deviation Hamiltonian for the Landau equation: mass, momentum and energy conservation, as well as entropy as the opposite of the quasipotential and time-reversal symmetry.

6.7 Large deviations for the Landau equation when $L < \lambda_D$

Whenever the size of the domain is smaller than the Debye length, the relevant large deviation parameter is the number of particles in a box of the size of the effective interaction length scale $\ell = L$; i.e. the relevant large deviation parameter is N . We can then study the asymptotics of the empirical density g_Λ and its homogeneous projection as N goes to infinity. Because $\Lambda = (\lambda_D/L)^3 N$, when $L < \lambda_D$ the large N limit implies the large Λ limit, which is responsible for the kinetic behavior of the empirical density. In order to make explicit that N is the natural large deviation rate, we perform the trivial integral on the positions in the large deviation principle (101). It is then possible to rephrase the large deviation principle (101) as following

$$\mathbf{P}(f_\Lambda = f) \underset{N \rightarrow \infty}{\asymp} e^{-N \text{Sup}_p \int_0^T \{ \int d\mathbf{v} \dot{f} p - H_{\text{Landau},h}[f,p] \}}, \quad (102)$$

by defining $H_{\text{Landau},h}$ as the large deviation Hamiltonian divided by the volume of the domain, such that

$$H_{\text{Landau}} = \int d\mathbf{r} H_{\text{Landau},h} = \left(\frac{L}{\lambda_D} \right)^3 H_{\text{Landau},h},$$

and

$$\begin{aligned} H_{\text{Landau},h}[f,p] = & \int d\mathbf{v}_1 f \left\{ \mathbf{b}[f] \cdot \frac{\partial p}{\partial \mathbf{v}_1} + \frac{\partial}{\partial \mathbf{v}_1} \cdot \left(\mathbf{D}[f] \frac{\partial p}{\partial \mathbf{v}_1} \right) + \mathbf{D}[f] : \frac{\partial p}{\partial \mathbf{v}_1} \frac{\partial p}{\partial \mathbf{v}_1} \right\} \\ & - \int d\mathbf{v}_1 d\mathbf{v}_2 f(\mathbf{v}_1) f(\mathbf{v}_2) \frac{\partial p}{\partial \mathbf{v}_1} \frac{\partial p}{\partial \mathbf{v}_2} : \mathbf{B}(\mathbf{v}_1, \mathbf{v}_2). \end{aligned}$$

Using this same relation between N and Λ , we already have remarked that

$$f_\Lambda(\mathbf{v}, t) = \frac{1}{\Lambda} \left(\frac{\lambda_D}{L} \right)^3 \sum_{n=1}^N \delta(\mathbf{v} - \mathbf{v}_n(t)) = \frac{1}{N} \sum_{n=1}^N \delta(\mathbf{v} - \mathbf{v}_n(t)) = h_N(\mathbf{v}, t),$$

where h_N is the velocity empirical density rescaled by the number of particles defined in section 3.5. Then, we have the following large deviation principle for h_N

$$\mathbf{P}(h_N = f) \underset{N \rightarrow \infty}{\asymp} e^{-N \text{Sup}_p \int_0^T \{ \int d\mathbf{v} \dot{f} p - H_{\text{Landau},h}[f,p] \}},$$

which is very similar to the large deviation principle (48) we established for the velocities empirical distribution of N diffusing particles coupled in a mean field way, except that the large deviation Hamiltonian $H_{\text{Landau},h}$ contains an additional term in addition to $H_{MF,h}$ (46), accounting for the weak interactions between the particles.

If in addition to $L < \lambda_D$ we have $L \ll \lambda_D$, then, because the wavevectors \mathbf{k} are in $2\pi(\lambda_D/L)\mathbb{Z}^3$ we have for all scales $k \gg 1$. This amounts at saying that the Landau approximation holds at all scales and that the large deviations described by (102) are Gaussian regardless of the scale of the fluctuations.

6.8 Large deviations for the Landau equation expressed in physical variables

In section 5.3, we established a large deviation principle (equations (64)-(65)) that describes the large deviations of the probability of homogeneous evolution paths for the empirical density $g_\Lambda(\mathbf{r}, \mathbf{v}, t) = \Lambda^{-1} \sum_{n=1}^N \delta(\mathbf{v} - \mathbf{v}_n(t)) \delta(\mathbf{r} - \mathbf{r}_n(t))$. As discussed in section 6.6, this result is consistent with the large deviation principle for the projection of the empirical density on homogeneous paths $f_\Lambda(\mathbf{v}, t) = \Lambda^{-1} (\lambda_D/L)^3 \sum_{n=1}^N \delta(\mathbf{v} - \mathbf{v}_n(t))$. So far, we expressed those results in a set of non-dimensional variables adapted to Coulomb plasmas.

We can express this large deviation result in physical variables, with the change of variables

$$\mathbf{v}_\varphi = v_T \mathbf{v}, \mathbf{k}_\varphi = \mathbf{k}/\lambda_D, t_\varphi = \Lambda \tau / \omega_{pe},$$

where v_T the thermal velocity, λ_D the Debye length, and ω_{pe} the plasma electron frequency are defined in section 3.1., and we denoted dimensional variables expressed in physical units with a subscript φ .

In the following we omit the subscript φ . The result is a large deviation principle for the empirical density in physical units

$$g_\Lambda(\mathbf{r}, \mathbf{v}, t) = \frac{1}{\Lambda} \sum_{n=1}^N \delta(\mathbf{v} - \mathbf{v}_n(t)) \delta(\mathbf{r} - \mathbf{r}_n(t))$$

which reads

$$\mathbf{P}(\{g_\Lambda\}_{0 \leq t \leq T} = \{f\}_{0 \leq t \leq T}) \underset{\Lambda \rightarrow \infty}{\asymp} e^{-\Lambda \text{Sup}_p \int_0^T dt \{ \int d\mathbf{r} d\mathbf{v} \dot{f} p - H_{\text{Landau}}[f, p] \}},$$

where

$$H_{\text{Landau}}[f, p] = \int d\mathbf{r} d\mathbf{v}_1 f \left\{ \mathbf{b}[f] \cdot \frac{\partial p}{\partial \mathbf{v}_1} + \frac{\partial}{\partial \mathbf{v}_1} \cdot \left(\mathbf{D}[f] \frac{\partial p}{\partial \mathbf{v}_1} \right) + \mathbf{D}[f] : \frac{\partial p}{\partial \mathbf{v}_1} \frac{\partial p}{\partial \mathbf{v}_1} \right\} \\ - \int d\mathbf{r} d\mathbf{v}_1 d\mathbf{v}_2 f(\mathbf{v}_1) f(\mathbf{v}_2) \frac{\partial p}{\partial \mathbf{v}_1} \frac{\partial p}{\partial \mathbf{v}_2} : \mathbf{B}(\mathbf{v}_1, \mathbf{v}_2).$$

with

$$\begin{cases} \mathbf{b}[f](\mathbf{v}) &= \int d\mathbf{v}_2 \mathbf{B}(\mathbf{v}, \mathbf{v}_2) \frac{\partial f}{\partial \mathbf{v}_2} \\ \mathbf{D}[f](\mathbf{v}) &= \int d\mathbf{v}_2 \mathbf{B}(\mathbf{v}, \mathbf{v}_2) f(\mathbf{v}_2), \end{cases} \quad (103)$$

and

$$\mathbf{B}(\mathbf{v}_1, \mathbf{v}_2) = \frac{\Lambda q^4 \pi}{m^2 \epsilon_0^2 L^3} \sum_{\mathbf{k} \in (2\pi/L)\mathbb{Z}^{*3}} (\hat{W}(\mathbf{k}))^2 \mathbf{k} \mathbf{k} \delta(\mathbf{k} \cdot \mathbf{v}_2 - \mathbf{k} \cdot \mathbf{v}_1). \quad (104)$$

And the associated Landau equation reads

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial \mathbf{v}} \cdot \int d\mathbf{v}_2 \mathbf{B}(\mathbf{v}_1, \mathbf{v}_2) \left(-\frac{\partial f}{\partial \mathbf{v}_2} f(\mathbf{v}) + f(\mathbf{v}_2) \frac{\partial f}{\partial \mathbf{v}} \right). \quad (105)$$

This differs slightly with the Landau equation one can find in the plasma literature [19, 11, 15] by a factor Λ in the tensor \mathbf{B} (104). Typically, in those references, the Landau equation is an evolution equation for the average of the non-rescaled empirical density. Here, we rescaled the empirical density by the plasma parameter Λ . In order to recover the Landau equation of [19, 11, 15], one should replace f in equation (105) by f_0/Λ . The resulting evolution equation for f_0 would be the usual Landau equation, where $f_0 = \mathbb{E}(\Lambda g_\Lambda)$ is the distribution function typically used in plasma textbooks.

Conclusions

The main result of this paper is the large deviation principle for the dynamics of the empirical density of a homogeneous Coulomb plasma of N equal charges particles. More precisely, we have shown that the probability of a homogeneous evolution path $\{f(t)\}_{0 \leq t \leq T}$ for the empirical density $g_\Lambda(\mathbf{r}, \mathbf{v}, t) = \Lambda^{-1} \sum_{n=1}^N \delta(\mathbf{v} - \mathbf{v}_n(t)) \delta(\mathbf{r} - \mathbf{r}_n(t))$ follows a large deviation principle

$$\mathbf{P}(\{g_\Lambda(t)\}_{0 \leq t \leq T} \approx \{f(t)\}_{0 \leq t \leq T}) \underset{\Lambda \rightarrow \infty}{\asymp} e^{-\Lambda \int_0^T dt \text{Sup}_p \{ \int d\mathbf{r} d\mathbf{v} \dot{f} p - H_{\text{Landau}}[f, p] \}},$$

where the large deviation Hamiltonian $H_{\text{Landau}}[f, p]$ is

$$H_{\text{Landau}}[f, p] = H_{MF}[f, p] + H_I[f, p],$$

with

$$H_{MF}[f, p] = \int d\mathbf{r} d\mathbf{v} f \left\{ \mathbf{b}[f] \cdot \frac{\partial p}{\partial \mathbf{v}} + \frac{\partial}{\partial \mathbf{v}} \cdot \left(\mathbf{D}[f] \cdot \frac{\partial p}{\partial \mathbf{v}} \right) + \mathbf{D}[f] : \frac{\partial p}{\partial \mathbf{v}} \frac{\partial p}{\partial \mathbf{v}} \right\},$$

and

$$H_I[f, p] = - \int d\mathbf{r} d\mathbf{v}_1 d\mathbf{v}_2 f(\mathbf{v}_1) f(\mathbf{v}_2) \frac{\partial p}{\partial \mathbf{v}_1} \frac{\partial p}{\partial \mathbf{v}_2} : \mathbf{B}(\mathbf{v}_1, \mathbf{v}_2).$$

where $\mathbf{D}[f]$, $\mathbf{b}[f]$ are defined in equation (103), and \mathbf{B} is defined in equation (104). This result has been obtained both from the large deviation Hamiltonian associated with the Boltzmann equation, and directly from the dynamics. This result is expressed in physical variables, but throughout the paper we worked with a non-dimensional set of variables adapted to plasmas. The connection is made between these two sets of variable in section 6.8.

This result is valid only for fluctuations at wavenumbers k such that $k\lambda_D \gg 1$ in physical units. This large deviation Hamiltonian is quadratic in its conjugate momentum meaning that large deviations are Gaussian. It also satisfies all the expected properties: conservation laws, time-reversal symmetry and consistency with equilibrium thermodynamics.

This paper also contains a set of complementary results. It contains the expression for the Hamiltonian for the path large deviations of the empirical density of N independent Markov processes (41), of N independent diffusions (43), and of N diffusions coupled in a mean field way (46). It also contains an explicit gradient flow structure for the Landau equation (70), deduced from the large deviation Hamiltonian. We also obtained results for the empirical density of N particles with long-range interactions without the Landau approximation. In this general case, we established a cumulant generating function representation of the large deviation Hamiltonian for the empirical density (80). We computed a cumulant expansion of this cumulant generating function up to order four.

Our results are exact computations, once natural hypothesis are made. The first main hypothesis is the validity of the quasilinear approximation. The second one is convergence of the Gaussian process of fluctuation to a stationary process. The third one is the validity of the classical expression for the large deviation Hamiltonian, in this context. The quasilinear approximation is very natural and is obtained naturally as the leading order contribution in a series expansion. The second hypothesis is partly justified in classical textbook, although a rigorous proof is missing. Actually, from a mathematical point of view, the type of convergence to consider is not clear. About the third one, we note that classical theorem for large deviations for slow-fast systems use sufficient ergodicity hypothesis which are probably wrong for this problem. A mathematical proof would thus require interesting mathematical developments. Actually the second and third hypothesis are strongly connected. In order to obtain a theorem, these three hypothesis should be proven. As far as we understand such a task seems out of reach of the best mathematicians, currently. However it might be achievable in the future, which would be a fascinating perspective.

A natural extension of this work would be to compute the large deviation Hamiltonian associated with the Balescu–Guernsey–Lenard equation. This would be a large deviation principle for the empirical density of N particles which interact through long-range interactions, for instance through Coulomb interactions, but without the Landau approximation. This will be the subject of an upcoming paper.

In this paper, we obtained results quantifying the dynamical fluctuations of the empirical density of a Coulomb plasma in the large plasma parameter limit. A series of mathematical papers [20,21,17] focus on fluctuations of stationary observables for Coulomb gases without using the large plasma parameter limit. This raises the question of whether it would be possible to obtain results about the dynamical fluctuations of the empirical density without the hypothesis of a large plasma parameter limit. This is an interesting perspective that would extend our present work, and at the same time would extend the static picture discussed in [20,21,17].

Another perspective and extension would be to obtain a large deviation principle and Hamiltonian for the evolution of inhomogeneous distribution functions. This would be particularly relevant for systems that are naturally inhomogeneous, for instance self-gravitating systems with application in galactic and globular cluster dynamics.

Finally, a large part of the computations and reasonings of this paper can be formulated beyond the framework of Coulomb plasmas. One of the first generalization we think about is to investigate the large deviations for the empirical density of particles which interact through long range potential and are stochastically forced out-of-equilibrium. Beyond interacting particles system, it will be interesting to use this tool to investigate two-dimensional and geostrophic turbulence. The dynamics of those hydrodynamical systems has deep analogies with systems with long-range interactions.

A The relative entropy for N independent diffusions solves the stationary Hamilton–Jacobi equation

We consider the relative entropy

$$S_{\text{rel}}[h] = - \int d\mathbf{v} h \log \left(\frac{h}{h_{\text{eq}}} \right),$$

where h_{eq} is the equilibrium distribution. In this appendix, we shows that $-S_{\text{rel}}$ solves the stationary Hamilton-Jacobi equation ($H_{MF,h} \left[h, -\frac{\delta S_{\text{rel}}}{\delta h} \right] = 0$), for the case of N independent diffusions (42). We recall that $H_{MF,h} \left[h, -\frac{\delta S_{\text{rel}}}{\delta h} \right] = 0$ is a necessary condition for $-S_{\text{rel}}$ to be the quasipotential. By contrast, when those N diffusions are coupled in a mean field way (in (45)) and the drift and diffusion coefficients depend actually on h , we are no more able to conclude that $H_{MF,h} \left[h, -\frac{\delta S_{\text{rel}}}{\delta h} \right] = 0$ and we believe this is actually wrong in general.

In both cases, the large deviation Hamiltonian for the empirical density h_N reads

$$H_{MF,h}[h, p] = \int d\mathbf{v} h \left\{ \mathbf{b}[h] \cdot \frac{\partial p}{\partial \mathbf{v}} + \frac{\partial}{\partial \mathbf{v}} \left(\mathbf{D}[h] \frac{\partial p}{\partial \mathbf{v}} \right) + \mathbf{D}[h] : \frac{\partial p}{\partial \mathbf{v}} \frac{\partial p}{\partial \mathbf{v}} \right\}. \quad (106)$$

In the simple case where the N diffusions are independent, the drift and the diffusion coefficients do not depend on the actual distribution h : $\mathbf{b}[h] = \mathbf{b}$ and $\mathbf{D}[h] = \mathbf{D}$. In order to check that the relative entropy S_{rel} is the opposite of the quasipotential, according to property 11 from section 2.1, we shall check that it solves the stationary Hamilton–Jacobi equation

$$H_{MF,h} \left[h, -\frac{\delta S_{\text{rel}}}{\delta h} \right] = 0. \quad (107)$$

We have

$$-\frac{\partial}{\partial \mathbf{v}} \left(\frac{\delta S_{\text{rel}}}{\delta h} \right) = \frac{1}{h} \frac{\partial h}{\partial \mathbf{v}} - \frac{1}{h_{\text{eq}}} \frac{\partial h_{\text{eq}}}{\partial \mathbf{v}}, \quad (108)$$

and h_{eq} solves the stationary Fokker–Planck equation

$$\frac{\partial}{\partial \mathbf{v}} \left(\mathbf{D}[h_{\text{eq}}] \frac{\partial h_{\text{eq}}}{\partial \mathbf{v}} - \mathbf{b}[h_{\text{eq}}] h_{\text{eq}} \right) = 0. \quad (109)$$

Using (108) we have

$$H_{MF,h} \left[h, -\frac{\delta S_{\text{rel}}}{\delta h} \right] = \int d\mathbf{v} \left\{ \mathbf{b}[h] \frac{\partial h}{\partial \mathbf{v}} - \frac{h}{h_{\text{eq}}} \frac{\partial h_{\text{eq}}}{\partial \mathbf{v}} \mathbf{b}[h] + \mathbf{D}[h] \frac{\partial h_{\text{eq}}}{\partial \mathbf{v}} \frac{1}{h_{\text{eq}}^2} \left(\frac{\partial h}{\partial \mathbf{v}} h_{\text{eq}} - \frac{\partial h_{\text{eq}}}{\partial \mathbf{v}} h \right) \right\}.$$

Now, we integrate by parts the first and the last term of the expression above, noting that

$$\frac{1}{h_{\text{eq}}^2} \left(\frac{\partial h}{\partial \mathbf{v}} h_{\text{eq}} - \frac{\partial h_{\text{eq}}}{\partial \mathbf{v}} h \right) = \frac{\partial}{\partial \mathbf{v}} \left(\frac{h}{h_{\text{eq}}} \right),$$

and

$$-h \frac{\partial \mathbf{b}[h]}{\partial \mathbf{v}} - \frac{h}{h_{\text{eq}}} \frac{\partial h_{\text{eq}}}{\partial \mathbf{v}} \mathbf{b}[h] = -\frac{h}{h_{\text{eq}}} \frac{\partial}{\partial \mathbf{v}} (\mathbf{b}[h] h_{\text{eq}}).$$

We obtain

$$H_{MF,h} \left[h, -\frac{\delta S_{\text{rel}}}{\delta h} \right] = \int d\mathbf{v} \frac{h}{h_{\text{eq}}} \frac{\partial}{\partial \mathbf{v}} \left(\mathbf{D}[h] \frac{\partial h_{\text{eq}}}{\partial \mathbf{v}} - \mathbf{b}[h] h_{\text{eq}} \right).$$

We see that if for any h

$$\frac{\partial}{\partial \mathbf{v}} \left(\mathbf{D}[h] \frac{\partial h_{\text{eq}}}{\partial \mathbf{v}} - \mathbf{b}[h] h_{\text{eq}} \right) = 0, \quad (110)$$

then $H_{MF,h} \left[h, -\frac{\delta S_{\text{rel}}}{\delta h} \right] = 0$ for any h . When $\mathbf{b}[h] = \mathbf{b}$ and $\mathbf{D}[h] = \mathbf{D}$ do not depend of f , i.e. when the N diffusions are independent, this identity is equivalent to the stationary Fokker–Planck equation (109). It thus holds. It follows that the Hamilton–Jacobi equation (107) is verified and that the negative of the relative entropy solves the stationary Hamilton–Jacobi equation, for the case of N independent diffusions.

However, when the drift and the diffusion coefficient do depend on the distribution, (110) is no more true for any h . Then, we cannot conclude anymore that the relative entropy solves the stationary Hamilton–Jacobi equation.

B Consistence of the two definitions of the tensor \mathbf{B}

We prove that for Coulomb interaction the two expressions for \mathbf{B} , (61) and (33) are equal.

The first expression for \mathbf{B} , (61), is

$$\mathbf{B}(\mathbf{v}_1, \mathbf{v}_2) = \frac{1}{2} \Lambda \int d\mathbf{q} w(\mathbf{v}_1, \mathbf{v}_2; \mathbf{q}) \mathbf{q} \otimes \mathbf{q},$$

Expressing w in terms of the cross-section σ_0 through (52) with $\gamma = (\lambda_D/L)^3$, using (51), and choosing for σ_0 the Rutherford diffusion cross-section

$$\sigma_0(\mathbf{v}_1 + \mathbf{q}, \mathbf{v}_2 - \mathbf{q}; \mathbf{v}_1, \mathbf{v}_2) = \frac{1}{4\pi^2 \Lambda^2 q^4},$$

for two-body collisions of particles with electrostatic interactions [19], we obtain

$$\mathbf{B}(\mathbf{v}_1, \mathbf{v}_2) = \int d\mathbf{q} \frac{\mathbf{q} \otimes \mathbf{q}}{8\pi^2 q^4} \delta(2\mathbf{q} \cdot (\mathbf{v}_2 - \mathbf{v}_1)). \quad (111)$$

We perform the integration over \mathbf{q} angle in (111) to get

$$\mathbf{B}(\mathbf{v}_1, \mathbf{v}_2) = C \frac{g^2 \text{Id} - \mathbf{g}\mathbf{g}}{g^3},$$

with $C = (8\pi)^{-1} \int_0^\infty q^{-1} dq$, $\mathbf{g} = \mathbf{v}_2 - \mathbf{v}_1$, and where Id is the identity matrix in three-dimension. We note that $\mathbf{B}(\mathbf{v}_1, \mathbf{v}_2)$ is proportional to $g^2 \text{Id} - \mathbf{g} \otimes \mathbf{g}$, which is the projector on the plane orthogonal to $\mathbf{v}_2 - \mathbf{v}_1$. This should have been expected as a consequence of symmetries.

In order to obtain the proportionality coefficient C we follow equations (6.3.15-6.3.21) in chapter 6.3 of Schram's textbook [19]. This chapter explains how one can deal with the logarithmic divergence arising in the computation of C . Briefly, one has to regularize the Coulomb interaction at large and small scales by introducing cut-offs, justified by the geometry of grazing collision at small scales, and by the Debye shielding at large scales. The final result reads

$$\mathbf{B}(\mathbf{v}_1, \mathbf{v}_2) = \frac{1}{8\pi} \ln \Lambda \frac{g^2 \text{Id} - \mathbf{g}\mathbf{g}}{g^3}. \quad (112)$$

Following the computations in chapter 8.4 of Schram's textbook [19], we can show in a similar way that the definition of \mathbf{B} given by (33) is also equal to (112). We have thus conclude that the two expression for \mathbf{B} , (61) and (33) are equal.

C Symmetries and conservation laws associated with the collision kernels

C.1 The Boltzmann collision kernel

The time reversal symmetry of the microscopic Hamiltonian dynamics imposes that

$$w_0(\mathbf{v}'_1, \mathbf{v}'_2; \mathbf{v}_1, \mathbf{v}_2) = w_0(-\mathbf{v}_1, -\mathbf{v}_2; -\mathbf{v}'_1, -\mathbf{v}'_2). \quad (113)$$

The space rotation symmetry imposes that for any rotation \mathbf{R} that belongs to the orthogonal group $SO(3)$

$$w_0(\mathbf{v}'_1, \mathbf{v}'_2; \mathbf{v}_1, \mathbf{v}_2) = w_0(\mathbf{R}\mathbf{v}_1, \mathbf{R}\mathbf{v}_2; \mathbf{R}\mathbf{v}'_1, \mathbf{R}\mathbf{v}'_2).$$

The combination of the time reversal symmetry and of the space rotation symmetry for $\mathbf{R} = -\mathbf{I}$, where \mathbf{I} is the identity operator, implies the inversion symmetry

$$w_0(\mathbf{v}'_1, \mathbf{v}'_2; \mathbf{v}_1, \mathbf{v}_2) = w_0(\mathbf{v}_1, \mathbf{v}_2; \mathbf{v}'_1, \mathbf{v}'_2). \quad (114)$$

The local conservation of momentum and energy implies that

$$w_0(\mathbf{v}'_1, \mathbf{v}'_2; \mathbf{v}_1, \mathbf{v}_2) = \sigma(\mathbf{v}'_1, \mathbf{v}'_2; \mathbf{v}_1, \mathbf{v}_2) \delta(\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}'_1 - \mathbf{v}'_2) \delta(v_1^2 + v_2^2 - v'^2_1 - v'^2_2), \quad (115)$$

where σ is the diffusion cross-section. σ is of the order of a^2 where a is a typical atom size.

C.2 The Landau collision kernel

The tensor \mathbf{B} defined by

$$\mathbf{B}(\mathbf{v}_1, \mathbf{v}_2) = \frac{\Lambda}{2} \int d\mathbf{q} w(\mathbf{v}_1, \mathbf{v}_2; \mathbf{q}) \mathbf{q} \otimes \mathbf{q}, \quad (116)$$

involved in the Landau equation (59) has properties related to the symmetry and conservation properties of the collision process. In equation (116), $w(\mathbf{v}_1, \mathbf{v}_2; \mathbf{q})$ is an approximation at order zero of the collision kernel $w(\mathbf{v}_1 + \mathbf{q}/2, \mathbf{v}_2 - \mathbf{q}/2; \mathbf{q})$ associated with the collision of two particles with momenta $(\mathbf{v}_1, \mathbf{v}_2)$ that exchange a momentum \mathbf{q} . We have:

1. $w(\mathbf{v}_1, \mathbf{v}_2; \mathbf{q}) = w(\mathbf{v}_2, \mathbf{v}_1; \mathbf{q})$ because the incident particles are indiscernible,
2. $\mathbf{q} \cdot (\mathbf{v}_1 - \mathbf{v}_2) = 0$ at leading order in \mathbf{q} because of the energy conservation condition $v_1^2 + v_2^2 = (\mathbf{v}_1 + \mathbf{q})^2 + (\mathbf{v}_2 - \mathbf{q})^2$,
3. $w(\mathbf{v}_1, \mathbf{v}_2; \mathbf{q}) = w(\mathbf{v}_1, \mathbf{v}_2; -\mathbf{q})$, which is a direct consequence of (114) and the definition of w (52).

We notice that the momentum conservation is already built-in in the definition of w . The first property implies $\mathbf{B}(\mathbf{v}_1, \mathbf{v}_2) = \mathbf{B}(\mathbf{v}_2, \mathbf{v}_1)$. The second property implies $\mathbf{B}(\mathbf{v}_1, \mathbf{v}_2) \cdot (\mathbf{v}_1 - \mathbf{v}_2) = 0$. In addition to that, $\mathbf{B}(\mathbf{v}_1, \mathbf{v}_2)$ is by construction a symmetric tensor for every pair $(\mathbf{v}_1, \mathbf{v}_2)$.

D Asymptotic expansions leading to the Landau equation and its large deviation Hamiltonian

D.1 Asymptotic expansions leading to the Landau equation

In this appendix, we start from the collision operator of the Boltzmann equation (the right hand side of equation (60)), we develop it at order 2 in \mathbf{q} , and we prove that we recover the collision term of the Landau equation (59).

We start from the expression of I in equation (60). Noting that $[f(\mathbf{v} + \mathbf{q})f(\mathbf{v}_2 - \mathbf{q}) - f(\mathbf{v})f(\mathbf{v}_2)]$ has no term of order zero, in order to compute an expansion at order 2 in $q = |\mathbf{q}|$, it will be sufficient to work with the expansions:

$$\begin{cases} w(\mathbf{v} + \frac{1}{2}\mathbf{q}, \mathbf{v}_2 - \frac{1}{2}\mathbf{q}; \mathbf{q}) & = w(\mathbf{v}, \mathbf{v}_2; \mathbf{q}) + \frac{1}{2} \left(\frac{\partial w}{\partial \mathbf{v}} - \frac{\partial w}{\partial \mathbf{v}_2} \right) \cdot \mathbf{q} + \mathcal{O}(q^2), \text{ and} \\ f(\mathbf{v} + \mathbf{q})f(\mathbf{v}_2 - \mathbf{q}) - f(\mathbf{v})f(\mathbf{v}_2) & = \left(\frac{\partial f}{\partial \mathbf{v}} f(\mathbf{v}_2) - \frac{\partial f}{\partial \mathbf{v}_2} f(\mathbf{v}) \right) \cdot \mathbf{q} + \\ & + \left(\frac{\partial^2 f}{\partial \mathbf{v} \partial \mathbf{v}} f(\mathbf{v}_2) + \frac{\partial^2 f}{\partial \mathbf{v}_2 \partial \mathbf{v}_2} f(\mathbf{v}) - 2 \frac{\partial f}{\partial \mathbf{v}} \frac{\partial f}{\partial \mathbf{v}_2} \right) : \mathbf{q}\mathbf{q} + \mathcal{O}(q^3). \end{cases}$$

Let us now compute the collision integral $I(\mathbf{v})$ order by order. We directly notice that there is no term of order zero in \mathbf{q} . Let us compute $I^{(1)}(\mathbf{v})$ the term of order 1 of the collision integral

$$I^{(1)}(\mathbf{v}) = \Lambda \int d\mathbf{v}_2 d\mathbf{q} w(\mathbf{v}, \mathbf{v}_2; \mathbf{q}) \left(\frac{\partial f}{\partial \mathbf{v}} f(\mathbf{v}_2) - \frac{\partial f}{\partial \mathbf{v}_2} f(\mathbf{v}) \right) \cdot \mathbf{q}.$$

We use that $w(\mathbf{v}, \mathbf{v}_2; \mathbf{q})$ is an even function of \mathbf{q} (point 3 of appendix (C.2)). This makes the integrand an odd function of \mathbf{q} , and implies that $I^{(1)}(\mathbf{v}) = 0$.

At order 2 in \mathbf{q} we have

$$I(\mathbf{v}) = \frac{\Lambda}{2} \int d\mathbf{v}_2 d\mathbf{q} \left\{ \left(\frac{\partial w}{\partial \mathbf{v}} - \frac{\partial w}{\partial \mathbf{v}_2} \right) \left(\frac{\partial f}{\partial \mathbf{v}} f(\mathbf{v}_2) - \frac{\partial f}{\partial \mathbf{v}_2} f(\mathbf{v}) \right) + w \left(\frac{\partial^2 f}{\partial \mathbf{v} \partial \mathbf{v}} f(\mathbf{v}_2) + \frac{\partial^2 f}{\partial \mathbf{v}_2 \partial \mathbf{v}_2} f(\mathbf{v}) - 2 \frac{\partial f}{\partial \mathbf{v}} \frac{\partial f}{\partial \mathbf{v}_2} \right) \right\} : \mathbf{q}\mathbf{q}.$$

To obtain the Landau equation, we have to write $I(\mathbf{v})$ as a divergence involving the tensor \mathbf{B} . In order to do so, we integrate by parts the term involving $\frac{\partial w}{\partial \mathbf{v}_2}$ while keeping the terms involving $\frac{\partial w}{\partial \mathbf{v}}$. This gives

$$I(\mathbf{v}) = \frac{\Lambda}{2} \int d\mathbf{v}_2 d\mathbf{q} \left\{ \frac{\partial w}{\partial \mathbf{v}} \left(\frac{\partial f}{\partial \mathbf{v}} f(\mathbf{v}_2) - \frac{\partial f}{\partial \mathbf{v}_2} f(\mathbf{v}) \right) + w \frac{\partial}{\partial \mathbf{v}} \left(\frac{\partial f}{\partial \mathbf{v}} f(\mathbf{v}_2) - \frac{\partial f}{\partial \mathbf{v}_2} f(\mathbf{v}) \right) \right\} : \mathbf{q}\mathbf{q}.$$

Now, by noting that $I(\mathbf{v})$ can be written as a total divergence with respect to \mathbf{v} and using equation 61 we obtain

$$I(\mathbf{v}) = \frac{\partial}{\partial \mathbf{v}} \int d\mathbf{v}_2 \mathbf{B}(\mathbf{v}, \mathbf{v}_2) \left(-\frac{\partial f}{\partial \mathbf{v}_2} f(\mathbf{v}) + \frac{\partial f}{\partial \mathbf{v}} f(\mathbf{v}_2) \right) + o(q^2), \quad (117)$$

with $\mathbf{B}(\mathbf{v}, \mathbf{v}_2) = \Lambda \int d\mathbf{q} w(\mathbf{v}, \mathbf{v}_2; \mathbf{q}) \otimes \mathbf{q}/2$ (see equation (61)), and $o(q^2)$ means that we omitted terms of order larger than 2. The term of order 2 is the collision operator of the Landau equation (59).

D.2 Asymptotic expansions leading to the large deviation Hamiltonian associated to the Landau equation

In this section, we detail the computation of the large deviation Hamiltonian for the Landau equation starting from the Hamiltonian (63) for the Boltzmann equation and using the grazing collision limit.

First, let us rewrite this Hamiltonian

$$H[f, p] = \frac{\Lambda}{2} \int d\mathbf{r} d\mathbf{v}_1 d\mathbf{v}_2 d\mathbf{q} w \left(\mathbf{v}_1 + \frac{1}{2}\mathbf{q}, \mathbf{v}_2 - \frac{1}{2}\mathbf{q}; \mathbf{q} \right) f(\mathbf{v}_1) f(\mathbf{v}_2) \left\{ e^{[-p(\mathbf{v}_1) - p(\mathbf{v}_2) + p(\mathbf{v}_1 + \mathbf{q}) + p(\mathbf{v}_2 - \mathbf{q})]} - 1 \right\}.$$

In order to obtain a Hamiltonian associated with the Landau equation, we will use the same hypothesis of grazing collisions and a Taylor expansion in \mathbf{q} to the same order

$$\begin{cases} w(\mathbf{v}_1 + \frac{1}{2}\mathbf{q}, \mathbf{v}_2 - \frac{1}{2}\mathbf{q}; \mathbf{q}) & = w(\mathbf{v}_1, \mathbf{v}_2; \mathbf{q}) + \frac{1}{2} \left(\frac{\partial w}{\partial \mathbf{v}_1} - \frac{\partial w}{\partial \mathbf{v}_2} \right) \cdot \mathbf{q} + \mathcal{O}(q^2) \\ e^{[-p(\mathbf{v}_1) - p(\mathbf{v}_2) + p(\mathbf{v}_1 + \mathbf{q}) + p(\mathbf{v}_2 - \mathbf{q})]} - 1 & = \left(\frac{\partial p}{\partial \mathbf{v}_1} - \frac{\partial p}{\partial \mathbf{v}_2} \right) \cdot \mathbf{q} + \\ & + \frac{1}{2} \left\{ \frac{\partial^2 p}{\partial \mathbf{v}_1 \partial \mathbf{v}_1} + \frac{\partial^2 p}{\partial \mathbf{v}_2 \partial \mathbf{v}_2} + \left(\frac{\partial p}{\partial \mathbf{v}_1} - \frac{\partial p}{\partial \mathbf{v}_2} \right) \left(\frac{\partial p}{\partial \mathbf{v}_1} - \frac{\partial p}{\partial \mathbf{v}_2} \right) \right\} : \mathbf{q}\mathbf{q} + \mathcal{O}(q^3). \end{cases}$$

We evaluate the terms of H order by order. There is no term of order zero. The term of order one in \mathbf{q} is

$$\frac{\Lambda}{2} \int \mathrm{d}\mathbf{r} \mathrm{d}\mathbf{v}_1 \mathrm{d}\mathbf{v}_2 \mathrm{d}\mathbf{q} w(\mathbf{v}_1, \mathbf{v}_2; \mathbf{q}) f(\mathbf{v}_1) f(\mathbf{v}_2) \left(\frac{\partial p}{\partial \mathbf{v}_1} - \frac{\partial p}{\partial \mathbf{v}_2} \right) \cdot \mathbf{q},$$

which is zero because $w(\mathbf{v}_1, \mathbf{v}_2; \mathbf{q})$ is an even function of \mathbf{q} (see point 3 of appendix (C.2)). At second order in \mathbf{q} the Hamiltonian reads

$$H_{\text{Landau}}[f, p] = \frac{\Lambda}{4} \int \mathrm{d}\mathbf{r} \mathrm{d}\mathbf{v}_1 \mathrm{d}\mathbf{v}_2 \mathrm{d}\mathbf{q} f(\mathbf{v}_1) f(\mathbf{v}_2) \left\{ w \left[\frac{\partial^2 p}{\partial \mathbf{v}_1 \partial \mathbf{v}_1} + \frac{\partial^2 p}{\partial \mathbf{v}_2 \partial \mathbf{v}_2} + \left(\frac{\partial p}{\partial \mathbf{v}_1} - \frac{\partial p}{\partial \mathbf{v}_2} \right) \left(\frac{\partial p}{\partial \mathbf{v}_1} - \frac{\partial p}{\partial \mathbf{v}_2} \right) \right] + \left(\frac{\partial p}{\partial \mathbf{v}_1} - \frac{\partial p}{\partial \mathbf{v}_2} \right) \left(\frac{\partial w}{\partial \mathbf{v}_1} - \frac{\partial w}{\partial \mathbf{v}_2} \right) \right\} : \mathbf{q} \mathbf{q}.$$

In this expression, in order to make appear the tensor $\mathbf{B}(\mathbf{v}, \mathbf{v}_2) = \Lambda \int \mathrm{d}\mathbf{q} w(\mathbf{v}, \mathbf{v}_2; \mathbf{q}) \mathbf{q} \mathbf{q} / 2$ (see equation (61)), we integrate by parts the terms involving $\frac{\partial w}{\partial \mathbf{v}_1}$ and $\frac{\partial w}{\partial \mathbf{v}_2}$, we develop the derivatives of products generated by partial integration, we use equation (61) and we obtain

$$H_{\text{Landau}}[f, p] = \frac{1}{2} \int \mathrm{d}\mathbf{r} \mathrm{d}\mathbf{v}_1 \mathrm{d}\mathbf{v}_2 \mathbf{B}(\mathbf{v}_1, \mathbf{v}_2) \left\{ f(\mathbf{v}_1) f(\mathbf{v}_2) \left(\frac{\partial p}{\partial \mathbf{v}_1} - \frac{\partial p}{\partial \mathbf{v}_2} \right) \left(\frac{\partial p}{\partial \mathbf{v}_1} - \frac{\partial p}{\partial \mathbf{v}_2} \right) + \left(\frac{\partial p}{\partial \mathbf{v}_1} - \frac{\partial p}{\partial \mathbf{v}_2} \right) \left(\frac{\partial f}{\partial \mathbf{v}_2} f(\mathbf{v}_1) - \frac{\partial f}{\partial \mathbf{v}_1} f(\mathbf{v}_2) \right) \right\}.$$

Using the property that $\mathbf{B}(\mathbf{v}_1, \mathbf{v}_2) = \mathbf{B}(\mathbf{v}_2, \mathbf{v}_1)$ (see appendix C.2), we have for every function g of $(\mathbf{v}_1, \mathbf{v}_2)$: $\int \mathrm{d}\mathbf{v}_1 \mathrm{d}\mathbf{v}_2 \mathbf{B}(\mathbf{v}_1, \mathbf{v}_2) g(\mathbf{v}_1, \mathbf{v}_2) = \int \mathrm{d}\mathbf{v}_1 \mathrm{d}\mathbf{v}_2 \mathbf{B}(\mathbf{v}_1, \mathbf{v}_2) g(\mathbf{v}_2, \mathbf{v}_1)$. Using this property we have

$$H_{\text{Landau}}[f, p] = \int \mathrm{d}\mathbf{r} \mathrm{d}\mathbf{v}_1 \mathrm{d}\mathbf{v}_2 \mathbf{B}(\mathbf{v}_1, \mathbf{v}_2) \left\{ f(\mathbf{v}_1) f(\mathbf{v}_2) \left(\frac{\partial p}{\partial \mathbf{v}_1} \frac{\partial p}{\partial \mathbf{v}_1} - \frac{\partial p}{\partial \mathbf{v}_1} \frac{\partial p}{\partial \mathbf{v}_2} \right) + \frac{\partial p}{\partial \mathbf{v}_1} \frac{\partial f}{\partial \mathbf{v}_2} f(\mathbf{v}_1) - \frac{\partial p}{\partial \mathbf{v}_1} \frac{\partial f}{\partial \mathbf{v}_1} f(\mathbf{v}_2) \right\}.$$

We integrate by parts the last term with respect to \mathbf{v}_1 to obtain

$$H_{\text{Landau}}[f, p] = \int \mathrm{d}\mathbf{r} \mathrm{d}\mathbf{v}_1 \mathrm{d}\mathbf{v}_2 f(\mathbf{v}_1) \left\{ \frac{\partial p}{\partial \mathbf{v}_1} \mathbf{B}(\mathbf{v}_1, \mathbf{v}_2) \frac{\partial f}{\partial \mathbf{v}_2} + \frac{\partial p}{\partial \mathbf{v}_1} \frac{\partial p}{\partial \mathbf{v}_1} \mathbf{B}(\mathbf{v}_1, \mathbf{v}_2) f(\mathbf{v}_2) + \frac{\partial}{\partial \mathbf{v}_1} \left(\mathbf{B}(\mathbf{v}_1, \mathbf{v}_2) f(\mathbf{v}_2) \frac{\partial p}{\partial \mathbf{v}_1} \right) \right\} - \int \mathrm{d}\mathbf{r} \mathrm{d}\mathbf{v}_1 \mathrm{d}\mathbf{v}_2 f(\mathbf{v}_1) f(\mathbf{v}_2) \frac{\partial p}{\partial \mathbf{v}_1} \frac{\partial p}{\partial \mathbf{v}_2} \mathbf{B}(\mathbf{v}_1, \mathbf{v}_2).$$

From here, using equation (34) we obtain

$$H_{\text{Landau}}[f, p] = H_{MF}[f, p] + H_I[f, p], \quad (118)$$

with

$$H_{MF}[f, p] = \int \mathrm{d}\mathbf{r} \mathrm{d}\mathbf{v}_1 f \left\{ \mathbf{b}[f] \cdot \frac{\partial p}{\partial \mathbf{v}_1} + \frac{\partial}{\partial \mathbf{v}_1} \left(\mathbf{D}[f] \frac{\partial p}{\partial \mathbf{v}_1} \right) + \mathbf{D}[f] : \frac{\partial p}{\partial \mathbf{v}_1} \frac{\partial p}{\partial \mathbf{v}_1} \right\},$$

and

$$H_I[f, p] = - \int \mathrm{d}\mathbf{r} \mathrm{d}\mathbf{v}_1 \mathrm{d}\mathbf{v}_2 f(\mathbf{v}_1) f(\mathbf{v}_2) \frac{\partial p}{\partial \mathbf{v}_1} \frac{\partial p}{\partial \mathbf{v}_2} : \mathbf{B}(\mathbf{v}_1, \mathbf{v}_2).$$

E Useful formulas

In this appendix, we list and prove some formulas used in section 6.

E.1 Sokhotski–Plemelj formula

We have

$$\lim_{\varepsilon \rightarrow 0^+} \frac{1}{x - i\varepsilon} = P\left(\frac{1}{x}\right) + i\pi\delta(x), \quad (119)$$

from which we also have

$$\Im\left(\frac{1}{x - i\varepsilon}\right) = \pi\delta(x), \quad (120)$$

where $\Im(z)$ is the imaginary part of the complex number z .

E.2 Some properties of the dielectric function

We discuss a few useful properties of the dielectric function. By definition the dielectric function (26) is

$$\varepsilon[f](\mathbf{k}, \omega) = 1 - \hat{W}(\mathbf{k}) \int d\mathbf{v} \frac{\mathbf{k} \cdot \frac{\partial f}{\partial \mathbf{v}}}{\mathbf{k} \cdot \mathbf{v} - \omega - i\varepsilon}, \quad (121)$$

Using the definition of the dielectric function (121) and (120), we have

$$\Im[\varepsilon[f](\mathbf{k}, \omega)] = -\pi \hat{W}(\mathbf{k}) \int d\mathbf{v} \mathbf{k} \cdot \frac{\partial f}{\partial \mathbf{v}} \delta(\omega - \mathbf{k} \cdot \mathbf{v}). \quad (122)$$

From (121), we readily see that

$$\varepsilon^*[f](\mathbf{k}, \omega) = \varepsilon[f](-\mathbf{k}, -\omega) \quad (123)$$

where \bar{z} is the imaginary part of the complex number z .

E.3 Double integral of a homogeneous kernel

E.3.1 Symmetric kernel

Let f be a function for which $|\int_0^\infty f(t) dt| < \infty$. Then,

$$\frac{1}{2T} \int_0^T \int_0^T dt_1 dt_2 f(|t_1 - t_2|) \xrightarrow{T \rightarrow \infty} \int_0^\infty d\tau f(\tau) = \frac{1}{2} \int_{-\infty}^\infty d\tau f(|\tau|).$$

Proof Using the parity of $f(|\cdot|)$, we get

$$\frac{1}{2T} \int_0^T \int_0^T dt_1 dt_2 f(|t_1 - t_2|) = \frac{1}{T} \int_0^T dt_1 \int_0^{t_1} dt_2 f(|t_1 - t_2|).$$

Rewriting the integrals with the change of variable $(t_1, t_2) \rightarrow (\tau, \tau') = (t_1, t_1 - t_2)$ leads to

$$\frac{1}{2T} \iint_0^T dt_1 dt_2 f(|t_1 - t_2|) = \frac{1}{T} \int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2 f(|\tau_2|).$$

Defining the function g by $g(\tau) = \int_0^\tau d\tau_2 f(|\tau_2|)$, noting that g has a finite limit for large τ and that the integral on $[0, \infty[$ of g does diverge, we obtain asymptotically,

$$\int_0^T g(\tau) d\tau \underset{T \rightarrow \infty}{\sim} T \int_0^\infty f(\tau) d\tau.$$

Combining the last two equations gives the result we wanted to prove

$$\frac{1}{2T} \iint_0^T dt_1 dt_2 f(|t_1 - t_2|) \xrightarrow{T \rightarrow \infty} \frac{1}{2} \int_{-\infty}^\infty d\tau f(|\tau|).$$

E.3.2 General kernel

Let f be a function for which $|\int_0^\infty f(t)dt| < \infty$ and $|\int_{-\infty}^0 f(t)dt| < \infty$. Then,

$$\frac{1}{T} \iint_0^T dt_1 dt_2 f(t_1 - t_2) \xrightarrow{T \rightarrow \infty} \frac{1}{T} \int_{-\infty}^\infty d\tau f(\tau). \quad (124)$$

Proof First, let us rewrite the integral on t_2 using the additivity of integration on intervals

$$\frac{1}{T} \iint_0^T dt_1 dt_2 f(t_1 - t_2) = \frac{1}{T} \int_0^T dt_1 \int_0^{t_1} dt_2 f(t_1 - t_2) + \frac{1}{T} \int_0^T dt_1 \int_{t_1}^T dt_2 f(t_1 - t_2). \quad (125)$$

Rewriting the first term of (125) with the change of variable $(t_1, t_2) \rightarrow (\tau, \tau') = (t_1, t_1 - t_2)$ leads to

$$\frac{1}{T} \int_0^T dt_1 \int_0^{t_1} dt_2 f(t_1 - t_2) = \frac{1}{T} \int_0^T d\tau \int_0^\tau d\tau' f(\tau'),$$

and using the Fubini theorem and the change of variable $(t_1, t_2) \rightarrow (\tau', \tau) = (t_2 - t_1, t_2)$ leads to

$$\frac{1}{T} \int_0^T dt_1 \int_{t_1}^T dt_2 f(t_1 - t_2) = \frac{1}{T} \int_0^T d\tau \int_0^\tau d\tau' f(-\tau').$$

We noticed during the previous proof that

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T d\tau \int_0^\tau d\tau' f(\tau') = \int_0^\infty f(\tau) d\tau.$$

With a similar computation, we can show that

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T d\tau \int_0^\tau d\tau' f(-\tau') = \int_0^\infty f(-\tau) d\tau = - \int_{-\infty}^0 f(\tau) d\tau.$$

Then, gathering the two terms of (125) and taking the limit as T goes to infinity, we find that

$$\frac{1}{T} \iint_0^T dt_1 dt_2 f(t_1 - t_2) \xrightarrow{T \rightarrow \infty} \frac{1}{T} \int_{-\infty}^\infty d\tau f(\tau),$$

which is what we wanted to prove.

E.4 Fourier–Laplace representation of a product

Let ϕ and ψ and two functions that admit Fourier–Laplace transforms $\tilde{\phi}$ and $\tilde{\psi}$ as defined in equation (82). Then,

$$\int d\mathbf{r} \int_{-\infty}^\infty dt \phi(\mathbf{r}, t) \psi(\mathbf{r}, t) = \frac{1}{2\pi} \left(\frac{\lambda_D}{L} \right)^3 \sum_{\mathbf{k} \in (2\pi\lambda_D/L)\mathbb{Z}^3} \int_\Gamma d\omega \tilde{\phi}(\mathbf{k}, \omega) \tilde{\psi}(-\mathbf{k}, -\omega). \quad (126)$$

Proof

Given the definition (82) of the Fourier–Laplace transform, the inversion formula is

$$\phi(\mathbf{r}, t) = \frac{1}{2\pi} \left(\frac{\lambda_D}{L} \right)^3 \sum_{\mathbf{k} \in (2\pi\lambda_D/L)\mathbb{Z}^3} \int_\Gamma d\omega e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \tilde{\phi}(\mathbf{k}, \omega),$$

where Γ is a contour to be chosen to insure the convergence. Using the inversion formula, the proof of the result is straightforward.

F Computation of the linear part and the quadratic part of the large deviation Hamiltonian

F.1 Computation of the first cumulant (linear part)

In this appendix, we explicit the computations of $\mathbf{C}^{(1)}$, using (91) and (87). This computation is different, but analogous to the one in §51 of [11]. We start from (91) which leads to

$$\mathbf{C}^{(1)}(\mathbf{v}) = -\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T dt \mathbb{E} \left(\frac{\partial V[\delta g_\Lambda]}{\partial \mathbf{r}} \delta g_\Lambda \right).$$

We notice that over time t long enough to forget the information about the initial condition, but short enough such that the velocity distribution has not changed much, $\mathbb{E} \left(\frac{\partial V[\delta g_\Lambda]}{\partial \mathbf{r}} \delta g_\Lambda \right)$ reaches a finite limit. In this limit, we simply obtain

$$\mathbf{C}^{(1)}(\mathbf{v}) = -\mathbb{E}_S \left(\frac{\partial V[\delta g_\Lambda]}{\partial \mathbf{r}} \delta g_\Lambda \right).$$

We express each of the two terms $\frac{\partial V[\delta g_\Lambda]}{\partial \mathbf{r}}(\mathbf{r}, t)$ and $\delta g_\Lambda(\mathbf{r}, t)$ through their Fourier–Laplace transforms, and we apply \mathbb{E}_S using (85) to get

$$\mathbf{C}^{(1)}(\mathbf{v}) = \frac{-i}{2\pi} \left(\frac{\lambda_D}{L} \right)^3 \sum_{\mathbf{k}} \int_{\Gamma} d\omega \mathbf{k} \widetilde{\mathcal{E}}_{VG}(\mathbf{k}, \omega, \mathbf{v}).$$

Using (87), we obtain

$$\mathbf{C}^{(1)}(\mathbf{v}) = \left(\mathbf{b}'[f](\mathbf{v}) f(\mathbf{v}) - \mathbf{D}'[f](\mathbf{v}) \cdot \frac{\partial f}{\partial \mathbf{v}}(\mathbf{v}) \right)$$

with

$$\mathbf{D}'[f](\mathbf{v}) = i \left(\frac{\lambda_D}{L} \right)^3 \sum_{\mathbf{k}} \int_{\Gamma} d\omega \int d\mathbf{v}_2 \frac{\mathbf{k}\mathbf{k}}{\mathbf{k}\cdot\mathbf{v} - \omega + i\tilde{\varepsilon}} f(\mathbf{v}_2) \delta(\omega - \mathbf{k}\cdot\mathbf{v}_2) \frac{\hat{W}(\mathbf{k})^2}{|\varepsilon[f](\omega, \mathbf{k})|^2},$$

and

$$\mathbf{b}'[f](\mathbf{v}) = -i \left(\frac{\lambda_D}{L} \right)^3 \sum_{\mathbf{k}} \int_{\Gamma} d\omega \frac{\mathbf{k}\hat{W}(\mathbf{k})}{\varepsilon(\mathbf{k}, \omega)} \delta(\omega - \mathbf{k}\cdot\mathbf{v}).$$

Using (120), we compute the real part of $\overleftarrow{\mathbf{D}}'$ and get

$$\Re(\mathbf{D}'[f](\mathbf{v})) = \mathbf{D}[f](\mathbf{v}) = \int d\mathbf{v}_2 \mathbf{B}[f](\mathbf{v}, \mathbf{v}_2) f(\mathbf{v}_2),$$

where

$$\mathbf{B}(\mathbf{v}, \mathbf{v}_2) = \pi \left(\frac{\lambda_D}{L} \right)^3 \int_{-\infty}^{+\infty} d\omega \sum_{\mathbf{k}} \delta(\omega - \mathbf{k}\cdot\mathbf{v}) \delta(\omega - \mathbf{k}\cdot\mathbf{v}_2) \frac{\mathbf{k}\hat{W}(\mathbf{k})^2}{|\varepsilon[f](\omega, \mathbf{k})|^2}$$

is the tensor defined in equation (28).

Using (122), we compute the real part of \mathbf{b}' , and get

$$\Re(\mathbf{b}'[f](\mathbf{v})) = \mathbf{b}[f](\mathbf{v}) = \int d\mathbf{v}_2 \mathbf{B}[f](\mathbf{v}, \mathbf{v}_2) \frac{\partial f}{\partial \mathbf{v}_2}.$$

It is also easily checked that $\Im[\mathbf{C}^{(1)}(\mathbf{v})] = 0$. We have thus justified that

$$\mathbf{C}^{(1)}(\mathbf{v}) = \int d\mathbf{v}_2 \mathbf{B}[f](\mathbf{v}, \mathbf{v}_2) \left(\frac{\partial f}{\partial \mathbf{v}_2} f(\mathbf{v}) - f(\mathbf{v}_2) \frac{\partial f}{\partial \mathbf{v}} \right) = \left(\mathbf{b}[f](\mathbf{v}) f(\mathbf{v}) - \mathbf{D}[f](\mathbf{v}) \cdot \frac{\partial f}{\partial \mathbf{v}}(\mathbf{v}) \right),$$

where \mathbf{B} is the tensor defined in equation (28).

F.2 Computation of the second cumulant (quadratic part)

In this appendix, in order to compute the second order cumulant of the large deviation Hamiltonian we use Wick's theorem to express the four-points correlation functions $\mathbb{E}_S \left(\frac{\partial V[\delta g_\Lambda]^{(1)}}{\partial \mathbf{r}} \frac{\partial V[\delta g_\Lambda]^{(2)}}{\partial \mathbf{r}} \delta g_\Lambda^{(1)} \delta g_\Lambda^{(2)} \right)$ as a sum of products of two-point correlation functions. In such formulas, the superscripts (1) or (2) mean that the quantities are evaluated at either (\mathbf{r}_1, t_1) and (\mathbf{r}_2, t_2) , respectively, or $(\mathbf{r}_1, \mathbf{v}_1, t_1)$ and $(\mathbf{r}_2, \mathbf{v}_2, t_2)$, respectively. We obtain

$$\begin{aligned} & \mathbb{E}_S \left(\frac{\partial V[\delta g_\Lambda]^{(1)}}{\partial \mathbf{r}} \frac{\partial V[\delta g_\Lambda]^{(2)}}{\partial \mathbf{r}} \delta g_\Lambda^{(1)} \delta g_\Lambda^{(2)} \right) - \mathbb{E}_S \left(\frac{\partial V[\delta g_\Lambda]^{(1)}}{\partial \mathbf{r}} \delta g_\Lambda^{(1)} \right) \mathbb{E}_S \left(\frac{\partial V[\delta g_\Lambda]^{(2)}}{\partial \mathbf{r}} \delta g_\Lambda^{(2)} \right) = \\ & \mathbb{E}_S \left(\frac{\partial V[\delta g_\Lambda]^{(1)}}{\partial \mathbf{r}} \frac{\partial V[\delta g_\Lambda]^{(2)}}{\partial \mathbf{r}} \right) \mathbb{E}_S \left(\delta g_\Lambda^{(1)} \delta g_\Lambda^{(2)} \right) + \mathbb{E}_S \left(\frac{\partial V[\delta g_\Lambda]^{(1)}}{\partial \mathbf{r}} \delta g_\Lambda^{(2)} \right) \mathbb{E}_S \left(\frac{\partial V[\delta g_\Lambda]^{(2)}}{\partial \mathbf{r}} \delta g_\Lambda^{(1)} \right) \end{aligned}$$

Using (93), we thus obtain

$$\mathbf{C} = \mathbf{C}_\alpha + \mathbf{C}_\beta$$

with

$$\mathbf{C}_\alpha = \lim_{N \rightarrow \infty} \frac{1}{2T} \int d\mathbf{r}_1 d\mathbf{r}_2 \iint_0^T dt_1 dt_2 \mathbb{E}_S \left(\frac{\partial V[\delta g_\Lambda]^{(1)}}{\partial \mathbf{r}} \frac{\partial V[\delta g_\Lambda]^{(2)}}{\partial \mathbf{r}} \right) \mathbb{E}_S \left(\delta g_\Lambda^{(1)} \delta g_\Lambda^{(2)} \right),$$

and

$$\mathbf{C}_\beta = \lim_{N \rightarrow \infty} \frac{1}{2T} \int d\mathbf{r}_1 d\mathbf{r}_2 \iint_0^T dt_1 dt_2 \mathbb{E}_S \left(\frac{\partial V[\delta g_\Lambda]^{(1)}}{\partial \mathbf{r}} \delta g_\Lambda^{(2)} \right) \mathbb{E}_S \left(\frac{\partial V[\delta g_\Lambda]^{(2)}}{\partial \mathbf{r}} \delta g_\Lambda^{(1)} \right).$$

Due to spatial and temporal homogeneity, the correlations functions only depend on the difference of the positions and times on which they are computed: $\mathbb{E}_S (V[\delta g_\Lambda](\mathbf{r}_1, t_1) V[\delta g_\Lambda](\mathbf{r}_2, t_2)) = \mathcal{C}_{VV}(\mathbf{r}_1 - \mathbf{r}_2, t_1 - t_2)$, and $\mathbb{E}_S (\delta g_\Lambda(\mathbf{r}_1, \mathbf{v}_1, t_1) \delta g_\Lambda(\mathbf{r}_2, \mathbf{v}_2, t_2)) = \mathcal{C}_{GG}(\mathbf{r}_1 - \mathbf{r}_2, t_1 - t_2, \mathbf{v}_1, \mathbf{v}_2)$. We use

$$\mathbb{E}_S \left(\frac{\partial V[\delta g_\Lambda]^{(1)}}{\partial \mathbf{r}} \frac{\partial V[\delta g_\Lambda]^{(2)}}{\partial \mathbf{r}} \right) = - \frac{\partial}{\partial \mathbf{r}} \frac{\partial}{\partial \mathbf{r}} [\mathcal{C}_{VV}](\mathbf{r}_1 - \mathbf{r}_2, t_1 - t_2),$$

and apply the result (124) from annex E.3 to find

$$\mathbf{C}_\alpha(\mathbf{v}_1, \mathbf{v}_2) = - \frac{1}{2} \int d\mathbf{r}_1 d\mathbf{r}_2 \int_{-\infty}^{\infty} dt \frac{\partial}{\partial \mathbf{r}} \frac{\partial}{\partial \mathbf{r}} [\mathcal{C}_{VV}](\mathbf{r}_1 - \mathbf{r}_2, t) \mathcal{C}_{GG}(\mathbf{r}_1 - \mathbf{r}_2, t, \mathbf{v}_1, \mathbf{v}_2).$$

Then, we apply the change of variables $(\mathbf{r}_1, \mathbf{r}_2) \rightarrow (\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2, \mathbf{r}' = \mathbf{r}_2)$, integrate over \mathbf{r}' , apply the result (126) from annex E.3, to obtain

$$\mathbf{C}_\alpha(\mathbf{v}_1, \mathbf{v}_2) = \frac{1}{2(2\pi)} \sum_{\mathbf{k}} \int_{\Gamma} d\omega \mathbf{k} \mathbf{k} \widetilde{\mathcal{C}}_{VV}(\mathbf{k}, \omega) \widetilde{\mathcal{C}}_{GG}(-\mathbf{k}, -\omega, \mathbf{v}_1, \mathbf{v}_2),$$

Similarly for \mathbf{C}_β , one obtains

$$\mathbf{C}_\beta(\mathbf{v}_1, \mathbf{v}_2) = \frac{-1}{2(2\pi)} \sum_{\mathbf{k}} \int_{\Gamma} d\omega \mathbf{k} \mathbf{k} \widetilde{\mathcal{C}}_{VG}(\mathbf{k}, \omega, \mathbf{v}_1) \widetilde{\mathcal{C}}_{VG}(\mathbf{k}, \omega, \mathbf{v}_2).$$

Summing these two terms we obtain

$$\mathbf{C} = \frac{1}{2(2\pi)} \sum_{\mathbf{k}} \int_{\Gamma} d\omega \mathbf{k} \mathbf{k} \left\{ \widetilde{\mathcal{C}}_{VV}(\mathbf{k}, \omega) \widetilde{\mathcal{C}}_{GG}(\mathbf{k}, \omega, \mathbf{v}_1, \mathbf{v}_2) - \widetilde{\mathcal{C}}_{VG}(\mathbf{k}, \omega, \mathbf{v}_1) \widetilde{\mathcal{C}}_{VG}(\mathbf{k}, \omega, \mathbf{v}_2) \right\}. \quad (127)$$

Let us define \mathcal{A} and \mathcal{B} as

$$\mathcal{A} \equiv \widetilde{\mathcal{C}}_{VV}(\mathbf{k}, \omega) \widetilde{\mathcal{C}}_{GG}(\mathbf{k}, \omega, \mathbf{v}_1, \mathbf{v}_2),$$

and

$$\mathcal{B} \equiv \widetilde{\mathcal{C}}_{VG}(\mathbf{k}, \omega, \mathbf{v}_1) \widetilde{\mathcal{C}}_{VG}(\mathbf{k}, \omega, \mathbf{v}_2).$$

From (84) and (88), \mathcal{A} reads

$$\begin{aligned} \mathcal{A} = & 2\pi\delta(\mathbf{v}_1 - \mathbf{v}_2) f(\mathbf{v}_1) \delta(\omega - \mathbf{k} \cdot \mathbf{v}_1) \widetilde{\mathcal{E}}_{VV}(\mathbf{k}, \omega) \\ & + \frac{\left(\widetilde{\mathcal{E}}_{VV}(\mathbf{k}, \omega)\right)^2}{(\omega - \mathbf{k} \cdot \mathbf{v}_1 + i\bar{\varepsilon})(\omega - \mathbf{k} \cdot \mathbf{v}_2 - i\bar{\varepsilon})} \mathbf{k} \cdot \frac{\partial f}{\partial \mathbf{v}_1} \mathbf{k} \cdot \frac{\partial f}{\partial \mathbf{v}_2} \\ & - 2\pi \widetilde{\mathcal{E}}_{VV}(\mathbf{k}, \omega) \hat{W}(\mathbf{k}) \mathbf{k} \cdot \left\{ \frac{\partial f}{\partial \mathbf{v}_1} \frac{f(\mathbf{v}_2) \delta(\omega - \mathbf{k} \cdot \mathbf{v}_2)}{\varepsilon(\mathbf{k}, \omega)(\omega - \mathbf{k} \cdot \mathbf{v}_1 + i\bar{\varepsilon})} + \frac{\partial f}{\partial \mathbf{v}_2} \frac{f(\mathbf{v}_1) \delta(\omega - \mathbf{k} \cdot \mathbf{v}_1)}{\bar{\varepsilon}(\mathbf{k}, \omega)(\omega - \mathbf{k} \cdot \mathbf{v}_2 - i\bar{\varepsilon})} \right\}. \end{aligned}$$

Similarly, from (87), we can deduce an expression for \mathcal{B}

$$\begin{aligned} \mathcal{B} = & \left(\frac{2\pi \hat{W}(\mathbf{k})}{\varepsilon(\mathbf{k}, \omega)} \right)^2 f(\mathbf{v}_1) f(\mathbf{v}_2) \delta(\omega - \mathbf{k} \cdot \mathbf{v}_1) \delta(\omega - \mathbf{k} \cdot \mathbf{v}_2) \\ & + \frac{\left(\widetilde{\mathcal{E}}_{VV}(\mathbf{k}, \omega)\right)^2}{(\omega - \mathbf{k} \cdot \mathbf{v}_1 - i\bar{\varepsilon})(\omega - \mathbf{k} \cdot \mathbf{v}_2 - i\bar{\varepsilon})} \mathbf{k} \cdot \frac{\partial f}{\partial \mathbf{v}_1} \mathbf{k} \cdot \frac{\partial f}{\partial \mathbf{v}_2} \\ & - 2\pi \widetilde{\mathcal{E}}_{VV}(\mathbf{k}, \omega) \hat{W}(\mathbf{k}) \mathbf{k} \cdot \left\{ \frac{\partial f}{\partial \mathbf{v}_1} \frac{f(\mathbf{v}_2) \delta(\omega - \mathbf{k} \cdot \mathbf{v}_2)}{\varepsilon(\mathbf{k}, \omega)(\omega - \mathbf{k} \cdot \mathbf{v}_1 - i\bar{\varepsilon})} + \frac{\partial f}{\partial \mathbf{v}_2} \frac{f(\mathbf{v}_1) \delta(\omega - \mathbf{k} \cdot \mathbf{v}_1)}{\varepsilon(\mathbf{k}, \omega)(\omega - \mathbf{k} \cdot \mathbf{v}_2 - i\bar{\varepsilon})} \right\}. \end{aligned}$$

To compute the second cumulant from (127), we are specifically interested in the difference $\mathcal{A} - \mathcal{B}$. Let us split this difference into five terms, labelled $\Delta_1, \dots, \Delta_5$ with

$$\begin{aligned} \Delta_1 = & 2\pi\delta(\mathbf{v}_1 - \mathbf{v}_2) f(\mathbf{v}_1) \delta(\omega - \mathbf{k} \cdot \mathbf{v}_1) \widetilde{\mathcal{E}}_{VV}(\mathbf{k}, \omega), \\ \Delta_2 = & \frac{\left(\widetilde{\mathcal{E}}_{VV}(\mathbf{k}, \omega)\right)^2}{\omega - \mathbf{k} \cdot \mathbf{v}_2 - i\bar{\varepsilon}} \mathbf{k} \cdot \frac{\partial f}{\partial \mathbf{v}_1} \mathbf{k} \cdot \frac{\partial f}{\partial \mathbf{v}_2} \left\{ \frac{1}{\omega - \mathbf{k} \cdot \mathbf{v}_1 + i\bar{\varepsilon}} - \frac{1}{\omega - \mathbf{k} \cdot \mathbf{v}_1 - i\bar{\varepsilon}} \right\}, \\ \Delta_3 = & 2\pi \widetilde{\mathcal{E}}_{VV}(\mathbf{k}, \omega) \hat{W}(\mathbf{k}) \mathbf{k} \cdot \frac{\partial f}{\partial \mathbf{v}_1} \frac{f(\mathbf{v}_2) \delta(\omega - \mathbf{k} \cdot \mathbf{v}_2)}{\varepsilon(\mathbf{k}, \omega)} \left\{ \frac{1}{\omega - \mathbf{k} \cdot \mathbf{v}_1 - i\bar{\varepsilon}} - \frac{1}{\omega - \mathbf{k} \cdot \mathbf{v}_1 + i\bar{\varepsilon}} \right\}, \\ \Delta_4 = & 2\pi \widetilde{\mathcal{E}}_{VV}(\mathbf{k}, \omega) \hat{W}(\mathbf{k}) \mathbf{k} \cdot \frac{\partial f}{\partial \mathbf{v}_2} \frac{f(\mathbf{v}_1) \delta(\omega - \mathbf{k} \cdot \mathbf{v}_1)}{\omega - \mathbf{k} \cdot \mathbf{v}_2 - i\bar{\varepsilon}} \left\{ \frac{1}{\varepsilon(\mathbf{k}, \omega)} - \frac{1}{\bar{\varepsilon}(\mathbf{k}, \omega)} \right\}, \end{aligned}$$

and

$$\Delta_5 = - \left(\frac{2\pi \hat{W}(\mathbf{k})}{\varepsilon(\mathbf{k}, \omega)} \right)^2 f(\mathbf{v}_1) f(\mathbf{v}_2) \delta(\omega - \mathbf{k} \cdot \mathbf{v}_1) \delta(\omega - \mathbf{k} \cdot \mathbf{v}_2),$$

such that

$$\mathbf{C} = \frac{1}{2(2\pi)} \sum_{\mathbf{k}} \int_{\Gamma} d\omega \mathbf{k} \mathbf{k} \{ \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 + \Delta_5 \}. \quad (128)$$

The first term Δ_1 is already explicit, there is nothing more to do. For the other terms, we will use the fact that for any complex number z , we have $z - \bar{z} = 2i\Im(z)$. For Δ_2 , using the Sokhotski-Plemelj formula (119), we have

$$\frac{1}{\omega - \mathbf{k} \cdot \mathbf{v}_1 + i\bar{\varepsilon}} - \frac{1}{\omega - \mathbf{k} \cdot \mathbf{v}_1 - i\bar{\varepsilon}} = -2i\pi\delta(\omega - \mathbf{k} \cdot \mathbf{v}_1),$$

and then,

$$\Delta_2 = -2i\pi\delta(\omega - \mathbf{k} \cdot \mathbf{v}_1) \frac{\left(\widetilde{\mathcal{E}}_{VV}(\mathbf{k}, \omega)\right)^2}{(\omega - \mathbf{k} \cdot \mathbf{v}_2 - i\bar{\varepsilon})} \mathbf{k} \cdot \frac{\partial f}{\partial \mathbf{v}_1} \mathbf{k} \cdot \frac{\partial f}{\partial \mathbf{v}_2}.$$

We can notice that the imaginary part of Δ_2 is odd in (\mathbf{k}, ω) , whereas its real part is even. Given how Δ_2 comes into play in the expression of the second cumulant (128), only the real (and even) part of Δ_2 will contribute to the second cumulant, and

$$\Re(\Delta_2) = 2\pi^2 \left(\widetilde{\mathcal{E}}_{VV}(\mathbf{k}, \omega)\right)^2 \mathbf{k} \cdot \frac{\partial f}{\partial \mathbf{v}_1} \mathbf{k} \cdot \frac{\partial f}{\partial \mathbf{v}_2} \delta(\omega - \mathbf{k} \cdot \mathbf{v}_1) \delta(\omega - \mathbf{k} \cdot \mathbf{v}_2).$$

In a similar way, we can prove that only the real parts of Δ_3 and Δ_4 contribute to (128) and that

$$\Re(\Delta_3) = -4\pi^3 \widetilde{\mathcal{E}}_{VV}(\mathbf{k}, \omega) \frac{\hat{W}(\mathbf{k})^2}{|\varepsilon(\mathbf{k}, \omega)|^2} \mathbf{k} \cdot \frac{\partial f}{\partial \mathbf{v}_1} f(\mathbf{v}_2) \delta(\omega - \mathbf{k} \cdot \mathbf{v}_1) \delta(\omega - \mathbf{k} \cdot \mathbf{v}_2) \int d\mathbf{v}' \mathbf{k} \cdot \frac{\partial f}{\partial \mathbf{v}'} \delta(\omega - \mathbf{k} \cdot \mathbf{v}'),$$

and

$$\Re(\Delta_4) = -4\pi^3 \widetilde{\mathcal{E}}_{VV}(\mathbf{k}, \omega) \frac{\hat{W}(\mathbf{k})^2}{|\varepsilon(\mathbf{k}, \omega)|^2} \mathbf{k} \cdot \frac{\partial f}{\partial \mathbf{v}_2} f(\mathbf{v}_1) \delta(\omega - \mathbf{k} \cdot \mathbf{v}_1) \delta(\omega - \mathbf{k} \cdot \mathbf{v}_2) \int d\mathbf{v}' \mathbf{k} \cdot \frac{\partial f}{\partial \mathbf{v}'} \delta(\omega - \mathbf{k} \cdot \mathbf{v}').$$

To compute the contribution of Δ_5 , let us use this simple identity

$$\frac{1}{\varepsilon^2} = \frac{1}{\varepsilon^2} - \frac{1}{|\varepsilon|^2} + \frac{1}{|\varepsilon|^2}.$$

And in a similar way that we did for the other terms, we notice that

$$\frac{1}{\varepsilon^2} - \frac{1}{|\varepsilon|^2} = \frac{1}{\varepsilon |\varepsilon|^2} \{\bar{\varepsilon} - \varepsilon\} = \frac{-2i\Im(\varepsilon)}{\varepsilon |\varepsilon|^2}.$$

Then, the fifth term Δ_5 reads

$$\begin{aligned} \Delta_5 &= \frac{2i\Im(\varepsilon)}{\varepsilon |\varepsilon|^2} (2\pi \hat{W}(\mathbf{k}))^2 f(\mathbf{v}_1) f(\mathbf{v}_2) \delta(\omega - \mathbf{k} \cdot \mathbf{v}_1) \delta(\omega - \mathbf{k} \cdot \mathbf{v}_2) \\ &\quad - \left(\frac{2\pi \hat{W}(\mathbf{k})}{|\varepsilon(\mathbf{k}, \omega)|^2} \right)^2 f(\mathbf{v}_1) f(\mathbf{v}_2) \delta(\omega - \mathbf{k} \cdot \mathbf{v}_1) \delta(\omega - \mathbf{k} \cdot \mathbf{v}_2). \end{aligned}$$

Once again, with parity arguments we can show that only the real part of Δ_5 contributes to (128), that is to say

$$\begin{aligned} \Re(\Delta_5) &= \frac{2\Im(\varepsilon)^2}{|\varepsilon|^4} (2\pi \hat{W}(\mathbf{k}))^2 f(\mathbf{v}_1) f(\mathbf{v}_2) \delta(\omega - \mathbf{k} \cdot \mathbf{v}_1) \delta(\omega - \mathbf{k} \cdot \mathbf{v}_2) \\ &\quad - \left(\frac{2\pi \hat{W}(\mathbf{k})}{|\varepsilon(\mathbf{k}, \omega)|^2} \right)^2 f(\mathbf{v}_1) f(\mathbf{v}_2) \delta(\omega - \mathbf{k} \cdot \mathbf{v}_1) \delta(\omega - \mathbf{k} \cdot \mathbf{v}_2). \end{aligned}$$

Thanks to this analysis, we can compute \mathbf{C} as following

$$\mathbf{C} = \frac{1}{2(2\pi)} \sum_{\mathbf{k}} \int_{\Gamma} d\omega \mathbf{k} \mathbf{k} \Re \{ \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 + \Delta_5 \}. \quad (129)$$

Furthermore, the quadratic term of the Hamiltonian $H^{(2)}$ is linked to this cumulant via the following formula

$$H^{(2)} = \int d\mathbf{r} d\mathbf{v}_1 d\mathbf{v}_2 \frac{\partial p}{\partial \mathbf{v}}(\mathbf{v}_1) \frac{\partial p}{\partial \mathbf{v}}(\mathbf{v}_2) : \mathbf{C}(\mathbf{v}_1, \mathbf{v}_2). \quad (130)$$

Using equations (122, 84, 129, 130), we can show that

$$\begin{aligned} H^{(2)} &= \int d\mathbf{r} d\mathbf{v}_1 \frac{\partial p}{\partial \mathbf{v}}(\mathbf{v}_1) \frac{\partial p}{\partial \mathbf{v}}(\mathbf{v}_1) \mathbf{D}(\mathbf{v}_1) f(\mathbf{v}_1) \\ &\quad - \int d\mathbf{r} d\mathbf{v}_1 d\mathbf{v}_2 \frac{\partial p}{\partial \mathbf{v}}(\mathbf{v}_1) \frac{\partial p}{\partial \mathbf{v}}(\mathbf{v}_2) : \mathbf{B}[f](\mathbf{v}_1, \mathbf{v}_2) f(\mathbf{v}_1) f(\mathbf{v}_2) \\ &\quad + \int d\mathbf{r} d\mathbf{v}_1 d\mathbf{v}_2 d\mathbf{v}_3 d\mathbf{v}_4 \frac{\partial p}{\partial \mathbf{v}_1} \frac{\partial p}{\partial \mathbf{v}_2} \mathbf{B}^{(2)}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) \left\{ f(\mathbf{v}_1) f(\mathbf{v}_2) \frac{\partial f}{\partial \mathbf{v}_3} \frac{\partial f}{\partial \mathbf{v}_4} \right. \\ &\quad \left. - 2f(\mathbf{v}_1) \frac{\partial f}{\partial \mathbf{v}_2} f(\mathbf{v}_3) \frac{\partial f}{\partial \mathbf{v}_4} + \frac{\partial f}{\partial \mathbf{v}_1} \frac{\partial f}{\partial \mathbf{v}_2} f(\mathbf{v}_3) f(\mathbf{v}_4) \right\}, \quad (131) \end{aligned}$$

with

$$\mathbf{B}^{(2)}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) = 2\pi^3 \left(\frac{\lambda_D}{L} \right)^3 \sum_{\mathbf{k}} \int d\omega \mathbf{k} \mathbf{k} \mathbf{k} \mathbf{k} \frac{\hat{W}(\mathbf{k})^4}{|\varepsilon(\mathbf{k}, \omega)|^4} \prod_{i=1}^4 \delta(\omega - \mathbf{k} \cdot \mathbf{v}_i), \quad (132)$$

being a fully symmetric order-4 tensor.

G Expression of the fourth cumulant

In this appendix, we report the result of the computation of $H^{(4)}$ the contribution of the fourth-cumulant to the large deviation Hamiltonian :

$$\begin{aligned}
H^{(4)} = & \int \text{drd}\mathbf{v}_1 \dots \text{d}\mathbf{v}_6 f(\mathbf{v}_2) f(\mathbf{v}_3) \mathbf{B}^{(3)} \frac{\partial p}{\partial \mathbf{v}_1} \frac{\partial p}{\partial \mathbf{v}_2} \frac{\partial p}{\partial \mathbf{v}_3} \left(4 \left\{ \frac{\partial p}{\partial \mathbf{v}_4} - \frac{\partial p}{\partial \mathbf{v}_3} \right\} \frac{\partial f}{\partial \mathbf{v}_1} f(\mathbf{v}_2) f(\mathbf{v}_3) f(\mathbf{v}_4) f(\mathbf{v}_5) \frac{\partial f}{\partial \mathbf{v}_6} \right. \\
& + \left. \left\{ \frac{\partial p}{\partial \mathbf{v}_4} + \frac{\partial p}{\partial \mathbf{v}_3} \right\} \left(\frac{\partial f}{\partial \mathbf{v}_1} \frac{\partial f}{\partial \mathbf{v}_2} f(\mathbf{v}_3) f(\mathbf{v}_4) f(\mathbf{v}_5) f(\mathbf{v}_6) - f(\mathbf{v}_1) f(\mathbf{v}_2) f(\mathbf{v}_3) f(\mathbf{v}_4) \frac{\partial f}{\partial \mathbf{v}_5} \frac{\partial f}{\partial \mathbf{v}_6} \right) \right) \\
& + \int \text{drd}\mathbf{v}_1 \dots \text{d}\mathbf{v}_8 \frac{\partial p}{\partial \mathbf{v}_1} \frac{\partial p}{\partial \mathbf{v}_2} \frac{\partial p}{\partial \mathbf{v}_3} \frac{\partial p}{\partial \mathbf{v}_4} \mathbf{B}^{(4)} \left\{ f(\mathbf{v}_1) f(\mathbf{v}_2) f(\mathbf{v}_3) f(\mathbf{v}_4) \frac{\partial f}{\partial \mathbf{v}_5} \frac{\partial f}{\partial \mathbf{v}_6} \frac{\partial f}{\partial \mathbf{v}_7} \frac{\partial f}{\partial \mathbf{v}_8} \right. \\
& \quad - 4 \frac{\partial f}{\partial \mathbf{v}_1} f(\mathbf{v}_2) f(\mathbf{v}_3) f(\mathbf{v}_4) f(\mathbf{v}_5) \frac{\partial f}{\partial \mathbf{v}_6} \frac{\partial f}{\partial \mathbf{v}_7} \frac{\partial f}{\partial \mathbf{v}_8} \\
& \quad + 6 \frac{\partial f}{\partial \mathbf{v}_1} \frac{\partial f}{\partial \mathbf{v}_2} f(\mathbf{v}_3) f(\mathbf{v}_4) f(\mathbf{v}_5) f(\mathbf{v}_6) \frac{\partial f}{\partial \mathbf{v}_7} \frac{\partial f}{\partial \mathbf{v}_8} \\
& \quad \left. - 4 \frac{\partial f}{\partial \mathbf{v}_1} \frac{\partial f}{\partial \mathbf{v}_2} \frac{\partial f}{\partial \mathbf{v}_3} f(\mathbf{v}_4) f(\mathbf{v}_5) f(\mathbf{v}_6) f(\mathbf{v}_7) \frac{\partial f}{\partial \mathbf{v}_8} \right. \\
& \quad \left. + \frac{\partial f}{\partial \mathbf{v}_1} \frac{\partial f}{\partial \mathbf{v}_2} \frac{\partial f}{\partial \mathbf{v}_3} \frac{\partial f}{\partial \mathbf{v}_4} f(\mathbf{v}_5) f(\mathbf{v}_6) f(\mathbf{v}_7) f(\mathbf{v}_8) \right\}. \quad (133)
\end{aligned}$$

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