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Kinetics of Contact Processes under Segregation

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Abstract

The kinetics of contact processes are determined by the interplay among local mass transfer mechanisms, spatial heterogeneity, and segregation. Determining the macroscopic behavior of a wide variety of phenomena across the disciplines requires linking reaction times to the statistical properties of spatially fluctuating quantities. We formulate the dynamics of advected agents interacting with segregated immobile components in terms of a chemical continuous time random walk. The inter-reaction times result from the first-passage times of mobile species to and across reactive regions, and available immobile reactants undergo a restart procedure. Segregation leads to memory effects and enhances the role of concentration fluctuations in the large-scale dynamics.

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6 I. INTRODUCTION

7 Mass-action reactions find applicability as population dynamics models of contact pro-
8 cesses between agents, spanning biological processes [1], epidemiology [2], ecology [3], quan-
9 tum molecular dynamics [4], and chemical reactions in geological media [5]. The large-scale
10 dynamics of such processes are determined by the interplay between local mechanisms,
11 spatial heterogeneity, and segregation. Transport limitations under segregation and spa-
12 tial heterogeneity lead to inter-reaction times which are related to the first encounter time
13 between reactants [6–10]. Broad reaction times result in mesoscopic dynamical coupling
14 between transport and reaction due to memory effects [11, 12]. In turn, the large-scale
15 behavior may involve non-classical, fluctuation-driven kinetics [13–15], broad reaction rate
16 distributions [16], time-dependent reaction rates [17], and time-nonlocal kinetics [12]. The
17 concept of residence time and its role in reactive transport have received much attention,
18 in particular in hydrogeological applications [18, 19]. This has led to the formulation of
19 large-scale dynamics in terms of so-called stochastic-convective streamtube models, where
20 reactive transport in the presence of physical and chemical heterogeneity is represented in
21 terms of an ensemble of streamtubes [20–23].

22 The classical picture for stochastic reactions, corresponding to the Gillespie algorithm [24],
23 assumes complete reactant mixing. It leads to exponential inter-reaction times and predicts
24 the classical mass-action rate laws for large reactant numbers [25–27]. The recently proposed
25 chemical continuous time random walk (chCTRW) framework allows for more general inter-
26 reaction times, leading to a broader class of large-scale rate laws including time-nonlocal
27 kinetics [28]. However, linking disorder properties, mass transfer, and inter-reaction times,
28 a fundamental step towards the understanding and quantification of the emergence of large-
29 scale kinetics in the presence of spatial heterogeneity and reactant segregation, remains in
30 general an open problem.

31 This work develops this link for advective transport under spatial chemical disorder. We
32 consider segregated immobile reactants, which react with mobile components. Inter-reaction
33 times relate directly to first-passage times (FPTs) of mobile reactants to and across reactive
34 regions. Upon encountering a reactive region, mobile components come into contact with
35 the locally-available reactants. As we will see, this corresponds to a restart of the immobile
36 reactants according to the original resident copy numbers. Processes under restart have

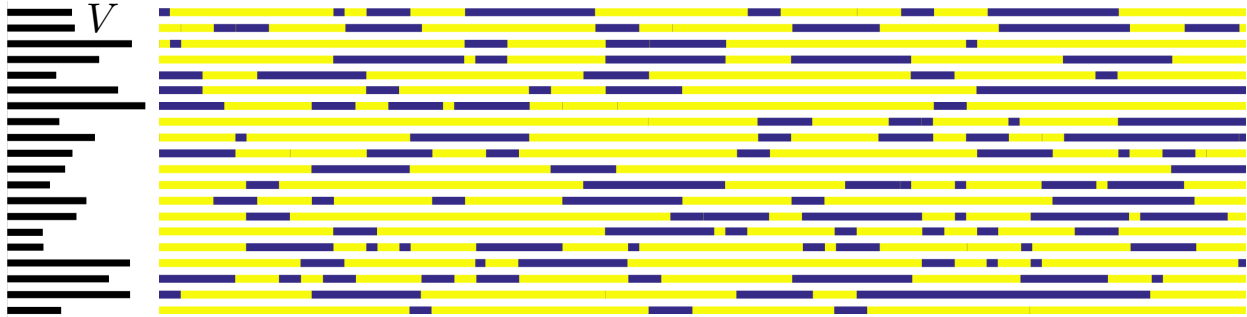


Figure 1. Illustration of the model segregation structure. The velocity V of mobile species (proportional to black bar length) is constant along each separate trajectory and distributed across trajectories. Mass evolves according to reaction in reactive regions (blue), interspersed with non-reactive regions (yellow), both of distributed length.

37 received considerable attention in the context of chemical reactions and in particular as a
 38 framework for optimizing search strategies [29–32].

39 We derive a generalized master equation [33] for the chemical kinetics using the chCTRW
 40 framework, which we generalize to account for restart and parameterize in terms of the het-
 41 erogeneity. The corresponding large-scale kinetics exhibit memory effects, and fluctuations
 42 about the average concentration play a role at large scales.

43 II. MODEL

44 We consider *mobile* reactants advected along separate trajectories. Advective velocity V
 45 is constant in each trajectory, and independent and identically distributed (i.i.d.) accord-
 46 ing to a probability density function (PDF) $\xi(\cdot)$ across trajectories. Additional reactants
 47 are confined to certain regions, see Fig. 1. We term these reactants *immobile* and these
 48 regions *reactive*. The latter are treated as well-mixed batch reactors, such that the chemical
 49 dynamics within proceed according to the classical Gillespie algorithm [24].

50 The length L_r of reactive regions is an i.i.d. random variable distributed according to
 51 the PDF $\rho_r(\cdot)$. The well-mixed assumption is difficult to justify if ρ_r has infinite moments.
 52 Instead, we assume L_r is characterized by a finite mean ℓ_r , and for simplicity we set $\rho_r(\ell) =$
 53 $e^{-\ell/\ell_r}/\ell_r$. Segregation of immobile reactants is characterized by the PDF $\rho_c(\cdot)$ of non-
 54 reactive region lengths L_c between reactive regions, which are also i.i.d. We distinguish the
 55 cases of *mild segregation*, corresponding to finite inter-reactive-region mean distance ℓ_c , and

56 *strong segregation*, corresponding to infinite mean distance, $\rho_c(\ell) \approx (\ell/\ell_c)^{-1-\beta}/[\ell_c|\Gamma(-\beta)|]$
 57 for $\ell \gg \ell_c$, where $\Gamma(\cdot)$ is the gamma function, ℓ_c is a characteristic length, and $0 < \beta < 1$ [34].
 58 We assume in what follows that mobile reactants begin within a reactive region to simplify
 59 the exposition. The general case of start in an arbitrary region, as illustrated in Fig. 1, can
 60 readily be treated in the framework developed below by including an additional waiting time
 61 until the first reaction, corresponding to the time to reach the first reactive region.

62 The idealized model set forth here retains the key features of segregation and will allow us
 63 to develop a quantitative model of its impact on chemical reactions. We consider in general
 64 a set of m_s species \mathcal{S}_j , $j = 1, \dots, m_s$, which are subject to m_r reactions, $\sum_{j=1}^{m_s} r_{ij}\mathcal{S}_j \rightarrow_{\kappa_i}$
 65 $\sum_{j=1}^{m_s} p_{ij}\mathcal{S}_j$, $i = 1, \dots, m_r$. Each reaction is characterized by its stoichiometry, with r_{ij} (p_{ij})
 66 denoting the number of reactants (products) of species j consumed (produced) by reaction
 67 i . Furthermore, each reaction is associated with an intrinsic (microscopic) reaction rate
 68 κ_i which fully determines its behavior under well-mixed conditions. All reactions involve
 69 immobile reactants, so that no reaction occurs outside reactive regions. The chemical state
 70 $\mathbf{N}(t)$ describes the number of mobile and immobile reactants of each species at time t . The
 71 net change in chemical state due to reaction i is described by the stoichiometry vectors
 72 $\mathbf{s}_i = \mathbf{p}_i - \mathbf{r}_i$. Sub- or superscripts M, I mark quantities relating to mobile and immobile
 73 components, respectively. For example, the chemical state is decomposed into mobile and
 74 immobile components as $\mathbf{N} = (\mathbf{N}_M, \mathbf{N}_I)$ and the stoichiometry vectors as $\mathbf{s}_i = (\mathbf{s}_i^M, \mathbf{s}_i^I)$.
 75 Throughout, we denote the Laplace transform by a tilde, the Laplace variable by λ , and
 76 ensemble averages (across trajectories) by angled brackets. A vertical bar is used to denote
 77 conditioning.

78 We introduce also some key quantities governing the dynamics. The mean time spent
 79 in a reactive region at velocity v is $\mu(v) = \ell_r/v$, so that $1/\mu(v)$ is the probability per unit
 80 time for mobile reactants to exit a given reactive region and experience the delay induced
 81 by the subsequent non-reactive region. Reactive patches are associated with a Damköhler
 82 number $\text{Da}(v) = \mu(v)a(\bar{\mathbf{n}})/\bar{n}$, where $\bar{\mathbf{n}}$ are characteristic copy numbers of each species (e.g.,
 83 the initial state) and \bar{n} is an overall characteristic copy number (e.g., the average of $\bar{\mathbf{n}}$ across
 84 components or a component of interest). This dimensionless number is the ratio of the
 85 characteristic transport and reaction times in a single reactive region. Finally, the ratio of
 86 characteristic non-reactive and reactive region lengths is denoted by $\alpha = \ell_c/\ell_r$.

87 III. QUALITATIVE DYNAMICS OF REACTION UNDER SEGREGATION

88 In order to motivate the general theory developed below, we first illustrate the im-
 89 pact of segregation on the large-scale kinetics of the mobile-immobile degradation reaction
 90 $\mathcal{S}_M + \mathcal{S}_I \rightarrow_{\kappa} \emptyset$, where \mathcal{S}_M is mobile and \mathcal{S}_I is immobile. We consider both mild and
 91 strong segregation, specifically with exponential and Lévy-stable [34] inter-reactive region
 92 lengths. For simplicity, we set a fixed velocity $V = v$ and a fixed number of initial immobile
 93 components $\mathbf{n}_{0,I}$ in each reactive region.

94 We consider an instantaneous, point initial injection of mobile components, and we sim-
 95 ulate their average degradation under segregation for large particle numbers. When char-
 96 acteristic reactant numbers $\bar{n} \rightarrow \infty$, we may define continuous (number) concentrations
 97 $\mathbf{C}(t) = \mathbf{N}(t)/\bar{n}$. We take \bar{n} as the average initial number of reactants. The reaction in reac-
 98 tive patches then proceeds according to the well-mixed rate laws. For $n_{M,I}$ mobile/immobile
 99 particles under well-mixed conditions, there are $n_M n_I$ mobile-immobile particle pairs avail-
 100 able for reaction. The reaction between each pair proceeds at the microscopic reaction rate
 101 κ . The reaction between *some* pair thus proceeds at a rate $a(\mathbf{n}) = \kappa n_M n_I$. In terms of
 102 concentrations, the well-mixed rate laws are thus given by

$$\frac{dc_{M,I}^{wm}(t)}{dt} = -\kappa^C c_M^{wm}(t) c_I^{wm}(t), \quad (1)$$

103 where $\kappa^C = \bar{n}\kappa$ is a macroscopic reaction rate associated with well-mixed concentration de-
 104 cay. The analytical solution of this equation, along with details on the numerical simulations
 105 under segregation, is given in Appendix A.

106 The solid lines in Fig. 2 show the ensemble-averaged mobile concentrations $\bar{c}_M(t)$, nor-
 107 malized by the initial concentration $c_{0,M}$, as a function of time t for different values of
 108 Damköhler number $\text{Da} = \kappa^C \mu c_{0,I}$, where $c_{0,I}$ is the initial immobile concentration in each
 109 reactive region. The evolution of concentration presents qualitative differences in the func-
 110 tional form of the decay with varying Damköhler number. For *mild segregation* (left panel),
 111 the decay at low Da is always exponential, but slower-than-exponential decay is present for
 112 longer times as Da increases. Under *strong segregation* (right panel), we observe power-law
 113 decay for all finite Da , characterized by the exponent β associated with the inter-reactive-
 114 region lengths. The solutions for a single well-mixed patch, corresponding to the $\text{Da} \rightarrow \infty$
 115 limit, are also shown. In this case, the concentration initially decays faster than in the
 116 segregated systems. However, due to depletion of the initial immobile concentration, which

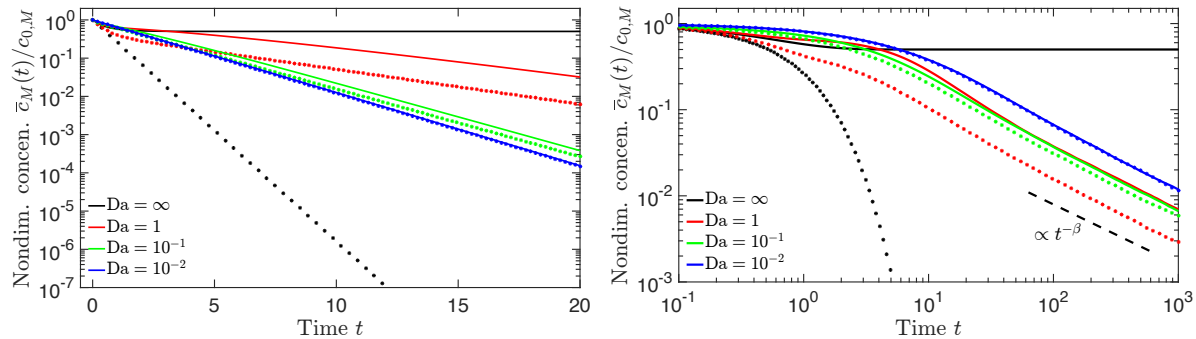


Figure 2. Temporal evolution of concentration under the full degradation reaction $\mathcal{S}_M + \mathcal{S}_I \rightarrow_{\kappa} \emptyset$ (solid lines) and the catalytic degradation reaction $\mathcal{S}_M + \mathcal{S}_I \rightarrow_{\kappa} \mathcal{S}_I$ (markers), for different values of Damköhler number Da . The initial conditions for mobile and immobile concentrations are $c_{0,M} = 4/3$ and $c_{0,I} = 2/3$. Fixed parameters are $\alpha = 2$, $\nu = 4$, and $\kappa^C = 2$, and Da is set by varying ℓ_r . Concentrations are averaged over 10^5 realizations. **Left:** Mild segregation, with exponentially-distributed non-reactive region lengths. **Right:** Strong segregation, with Lévy-stable-distributed non-reactive region lengths with exponent $\beta = 0.7$.

117 in this example is lower than the initial mobile concentration, reaction then slows down and
 118 the mobile concentration approaches equilibrium.

119 The impact of segregation on reaction dynamics, illustrated here for this simple degra-
 120 dation reaction, is in general due to two factors. First, reaction is punctuated by times
 121 spent in non-reactive regions, which leads to an effective reaction slowdown. Second, mo-
 122 bile components react only with the locally-available immobile reactants in a given region.
 123 When a reactive region is left and a subsequent one is entered, there is a restart of available
 124 immobile reactants, according to their initially-available concentration. In order to highlight
 125 the role of immobile reactant depletion and restart, the markers in Fig. 2 show the evolution
 126 of mobile concentrations for the catalytic degradation reaction $\mathcal{S}_M + \mathcal{S}_I \rightarrow_{\kappa} \mathcal{S}_I$, using the
 127 same parameter values as before. In this case, the immobile component acts as a catalyst,
 128 and it is thus neither consumed nor subject to the effect of restart. For a single well-mixed
 129 patch, in the $Da \rightarrow \infty$ limit, the mobile species' decay is then purely exponential and always
 130 faster than under segregation. However, as $Da \rightarrow 0$, corresponding to negligible reaction
 131 within each single region, the decay of mobile concentration becomes identical under the
 132 two reactions.

133 **IV. THEORY**

134 In what follows, we will discuss and quantify the mechanisms discussed in the previous
 135 section. To this end, we develop a general framework for arbitrary reaction dynamics under
 136 spatial segregation.

137 **A. Catalytic reactions**

138 We start by examining the case where the immobile components serve as catalysts, i.e.,
 139 they participate in the reactions but are not consumed or produced ($\mathbf{s}_i^I = 0$). We further
 140 assume for the moment that the immobile reactants are present in identical copy numbers in
 141 each reactive region. In this case, the dynamics proceed identically with the classical well-
 142 mixed scenario, except the periods of availability of reactants are punctuated by waiting
 143 times due to segregation. We proceed to find the chCTRW description of this system in
 144 terms of a generalized master equation, starting from the determination of the inter-reaction
 145 times.

146 *1. Inter-reaction times*

147 The lack of memory of the exponential distribution implies that the distribution of dis-
 148 tances to the end of reactive regions, starting from any point within a reactive region, has the
 149 same length distribution as the full regions [27]. Thus, it is sufficient to study inter-reaction
 150 times starting at the beginning of reactive regions.

151 Consider a given chemical state $\mathbf{N}(t) = \mathbf{n}$ and velocity $V = v$. For advective transport,
 152 the FPT across reactive/conservative regions is given by the crossing time $L_{r,c}/v$. Thus,
 153 the corresponding PDFs are given by $\psi_{r,c}(t|v) = v\rho_{r,c}(vt)$. Following chCTRW theory, we
 154 write the inter-reaction times as $\tau(\mathbf{n}, v) = \tau_r(\mathbf{n}) + \tau_c(\tau_r, v)$. The intrinsic reaction time τ_r
 155 represents reaction in the absence of segregation. It depends on the current chemical state
 156 and corresponds to time spent in reactive regions, where reaction proceeds according to the
 157 classical well-mixed theory. Let $\phi_i^r(t|\mathbf{n}) dt$ represent the joint probability that, in the absence
 158 of segregation, reaction $1 \leq i \leq m_r$ fires next and after a waiting time in $[t, t + dt]$. Then,
 159 $\tau_r(\mathbf{n})$ has PDF $\phi^r(t|\mathbf{n}) = \sum_i \phi_i^r(t|\mathbf{n})$ (see Appendix B for a brief overview of the well-mixed
 160 description). The additional global delay τ_c is caused by segregation. It corresponds to

161 time spent in non-reactive regions, which is fully determined by the current τ_r and does
 162 not depend further on the chemical state. Specifically, a given time $\tau_r = t_r$ corresponds to
 163 fully traversing a certain number $\eta(t_r, v)$ of reactive regions, and $\tau_c(t_r, v) = \sum_{k=1}^{\eta(t_r, v)} L_{c,k}/v$,
 164 where the $L_{c,k}$ are i.i.d. according to ρ_c . For exponential reactive lengths, $\eta(t_r, v)$ is Poisson-
 165 distributed, as shown in Appendix C. Thus, segregation leads to a compound-Poisson inter-
 166 reaction delay. According to [28], the full inter-reaction time density is then related to the
 167 intrinsic inter-reaction time density by

$$\tilde{\phi}_i(\lambda|\mathbf{n}, v) = \tilde{\phi}_i^r[\lambda/\tilde{K}(\lambda|v)|\mathbf{n}, v], \quad (2)$$

168 where

$$\tilde{K}(\lambda|v) = \left[1 + \frac{1 - \tilde{\psi}_c(\lambda|v)}{\mu(v)\lambda} \right]^{-1} \quad (3)$$

169 is the Laplace transform of a memory kernel, as we will see shortly.

170 2. Generalized master equation

171 We first define the probabilities $\gamma_i(\mathbf{n}|\mathbf{n}')$ of reaction i yielding the chemical state \mathbf{n} , given
 172 the starting state \mathbf{n}' . The change in state is given by the stoichiometry vector \mathbf{s}_i . Thus,

$$\gamma_i(\mathbf{n}|\mathbf{n}') = \delta_{\mathbf{n}, \mathbf{n}' + \mathbf{s}_i}, \quad (4)$$

173 where $\delta_{\cdot, \cdot}$ is the Kronecker delta. According to the general theory developed in [28], the
 174 generalized master equation for the ensemble-averaged propagator of the chemical state,
 175 given initial state \mathbf{n}_0 and velocity v , is then given by

$$\begin{aligned} \frac{\partial p(\mathbf{n}, t|\mathbf{n}_0, v)}{\partial t} &= \sum_{\mathbf{n}' \geq 0} \sum_{i=1}^{m_r} \int_0^t dt' [\gamma_i(\mathbf{n}|\mathbf{n}') - \delta_{\mathbf{n}, \mathbf{n}'}] \\ &\quad \times M_i(t - t'|\mathbf{n}', v) p(\mathbf{n}', t'|\mathbf{n}_0, v), \end{aligned} \quad (5)$$

176 where the Laplace transforms of the memory functions are given in terms of the inter-reaction
 177 time densities as

$$\tilde{M}_i(\lambda|\mathbf{n}, v) = \frac{\lambda \tilde{\phi}_i(\lambda|\mathbf{n}, v)}{1 - \sum_{i=1}^{m_r} \tilde{\phi}_i(\lambda|\mathbf{n}, v)}. \quad (6)$$

178 Using Eq. (13) together with the well-mixed description, we obtain

$$\tilde{M}_i(\lambda|\mathbf{n}, v) = \tilde{K}(\lambda|v) a_i(\mathbf{n}), \quad (7)$$

179 with $a_i(\mathbf{n}) = \kappa_i h_i(\mathbf{n})$ and $a(\mathbf{n}) = \sum_{i=1}^{m_r} a_i(\mathbf{n})$, where the κ_i are microscopic rate constants
180 and the $h_i(\mathbf{n})$ encode the dependency of the rates on the state. In the absence of segregation,
181 $\psi_c(t) = \delta(t)$. Equation (3) with $\tilde{\psi}_c(\lambda) = 1$ leads to $\tilde{K}(\lambda|v) = 1$, and therefore $K(t|v) = \delta(t)$.
182 This recovers the classical chemical master equation [25]. We thus see that K plays the
183 role of a memory kernel describing reaction slowdown due to segregation. Reactions under
184 compound-Poisson delay, and corresponding simulation techniques to solve the generalized
185 master equation (5), are discussed in general terms in [28]. Note that for catalytic reactions,
186 $\mathbf{s}_i^I = 0$ implies immobile copy numbers do not change, and it is sufficient to consider the
187 evolution of the mobile components.

188 As an example, consider the catalytic degradation reaction introduced in Section III.
189 We have $h_1(\mathbf{n}) = n_M n_I$, the number of pairs of mobile-immobile particles, and $a(\mathbf{n}) =$
190 $a_1(\mathbf{n}) = \kappa n_M n_I$. The intrinsic waiting time density is given by $\phi_1^r(t|\mathbf{n}) = a(\mathbf{n}) \exp[a(\mathbf{n})t]$,
191 and the single memory function is $\tilde{M}_1(\lambda|\mathbf{n}, v) = \tilde{K}(\lambda|v)a(\mathbf{n})$. The stoichiometry vector is
192 $\mathbf{s}_1 = (-1, 0)$. Thus, the master equation for the mobile components becomes

$$\frac{\partial p(n_M, t)}{\partial t} = - \int_0^t dt' K(t-t') \left[a(n_M, n_{0,I}) p(n_M, t') \right. \\ \left. - a(n_M + 1, n_{0,I}) p(n_M + 1, t') \right], \quad (8)$$

193 where we have omitted the dependency on the initial condition and v for notational brevity.

194 The ensemble-averaged probability of a given state at a given time is in general obtained
195 by averaging over velocities and initial conditions. Denote by $\gamma_M(\cdot|v)$ the initial distribution
196 of mobile components at injection and by $\gamma_I(\cdot|v)$ the initial distribution of immobile reactant
197 numbers across reactive regions, given velocity v . The initial copy number distribution at
198 the first reactive region is thus $\gamma(\mathbf{n}|v) = \gamma_M(\mathbf{n}_M|v)\gamma_I(\mathbf{n}_I|v)$. For equal initial immobile
199 component copy numbers $\mathbf{n}_{0,I}$ in each reactive patch, $\gamma_I(\mathbf{n}|v) = \delta_{\mathbf{n}, \mathbf{n}_{0,I}}$. The probability
200 of finding the state $\mathbf{N}(t) = \mathbf{n}$ at time t is $p(\mathbf{n}, t) = \langle p[\mathbf{n}, t | \mathbf{N}_0(V), V] \rangle$, where for each
201 $V = v$ the initial condition $\mathbf{N}_0(v)$ is distributed according to $\gamma(\cdot|v)$. We note also that
202 the propagator contains all necessary information to compute spatial quantities. Spatial
203 distributions may be obtained by multiplying the propagator by the probability that mobile
204 reactants are at position x at time t before averaging, which, assuming mobile species start
205 at $x = 0$ at $t = 0$, is given here by the Dirac delta $\delta(x - Vt)$. Similarly, concentrations fluxes

206 at a control plane at distance x as a function of time t are obtained by multiplying by the
207 FPT to distance x , given here by $\delta(t - x/V)$.

208 **B. General reactions**

209 We now study the general case where reactions may involve net production or consump-
210 tion of immobile components, and where different reactive regions may initially comprise
211 different copy numbers of immobile reactants. In this case, as the mobile reactants reach
212 each reactive region, they encounter the initial resident copy numbers. These are then de-
213 pleted or produced according to reaction. Once the mobile components exit a given reactive
214 region and arrive at the subsequent one, they again encounter resident copy numbers accord-
215 ing to their initial distribution. This leads naturally to the concept of restart of immobile
216 components. Developing a generalized master equation for these dynamics thus requires
217 generalizing the inter-reaction times to account for restart.

218 *1. Inter-reaction times under restart*

219 In order to make use of the techniques developed in [28], we require that the dynamics
220 be a Markov process in reaction step. For exponential reactive region lengths, whenever a
221 reaction fires, the leftover reactive region length is identically distributed with the full region
222 length, as discussed above. However, if the initial number of immobile reactants depends on
223 the reactive region length, the number of immobile reactants then gives information about
224 the region length, and the inter-reaction times are no longer independent of past history.
225 We assume here that this is not the case.

226 Consider a given chemical state $\mathbf{N}(t) = \mathbf{n}$, velocity $V = v$, and lengths of a consecutive
227 reactive/non-reactive region pair $L_{r,c}$. The effect of restart may be treated as a special
228 reaction, which we number $i = 0$. It fires after a time L_r/v and leads to restart of the
229 immobile components after a time $(L_r + L_c)/v$. The next reaction to fire is the one with the
230 minimum inter-reaction time, including restart. The inter-reaction time densities, given the
231 chemical state, velocity, and region lengths, are defined such that $\phi_i(t|\mathbf{n}, v, L_r, L_c) dt$ is the
232 joint probability of reaction i firing and the inter-reaction time being in $[t, t + dt]$. Under

233 restart, we write

$$\phi_i(t|\mathbf{n}, v, L_r, L_c) = b_i(\mathbf{n}, v, L_r)\phi_{|i}(t|\mathbf{n}, v, L_r, L_c), \quad (9)$$

234 where $b_i(\mathbf{n}, v, L_r)$ is the propensity of reaction i , i.e., the probability that it fires next, and
 235 $\phi_{|i}(\cdot|\mathbf{n}, v, L_r, L_c)$ is the inter-reaction time PDF of reaction i given that it will fire next.
 236 Restart occurs if the minimum reaction time is larger than the restart time L_r/v . This
 237 happens with probability $\int_{L_r/v}^{\infty} dt \phi^r(t|\mathbf{n}) = \exp[-a(\mathbf{n})L_r/v]$. Otherwise, with probability
 238 $1 - \exp[-a(\mathbf{n})L_r/v]$, the normal reaction with the minimum inter-reaction time fires. Thus,

$$b_i(\mathbf{n}, v, L_r) = \begin{cases} e^{-a(\mathbf{n})L_r/v} \\ [1 - e^{-a(\mathbf{n})L_r/v}] \frac{a_i(\mathbf{n})}{a(\mathbf{n})} \end{cases}. \quad (10)$$

239 Here and in what follows, the first case refers to $i = 0$ (restart), and the second to $1 \leq i \leq m_r$
 240 (regular reactions). The inter-reaction time PDFs must also be conditioned on $t < L_r/v$ for
 241 the regular reactions, and the waiting time associated with restart is $(L_r + L_c)/v$, so that

$$\phi_{|i}(t|\mathbf{n}, v, L_r, L_c) = \begin{cases} \delta(t - \frac{L_r+L_c}{v}) \\ \frac{a(\mathbf{n}) \exp[-a(\mathbf{n})t]}{1 - \exp[-a(\mathbf{n})L_r/v]} H\left(\frac{L_r}{v} - t\right) \end{cases}, \quad (11)$$

242 where H is the Heaviside step function. Thus, according to Eq. (9),

$$\phi_i(t|\mathbf{n}, v, L_r, L_c) = \begin{cases} e^{-a(\mathbf{n})L_r/v} \delta(t - \frac{L_r+L_c}{v}) \\ a_i(\mathbf{n}) e^{-a(\mathbf{n})t} H\left(\frac{L_r}{v} - t\right) \end{cases}. \quad (12)$$

243 Defining $\phi_i(t|\mathbf{n}, v) = \langle \phi_i(t|\mathbf{n}, v, L_r, L_c) \rangle$, we find the Laplace transforms

$$\tilde{\phi}_i(\lambda|\mathbf{n}, v) = \begin{cases} \tilde{\psi}_r[\lambda + a(\mathbf{n})|v] \tilde{\psi}_c(\lambda|v) \\ a_i(\mathbf{n}) \frac{1 - \tilde{\psi}_r[\lambda + a(\mathbf{n})|v]}{\lambda + a(\mathbf{n})} \end{cases}. \quad (13)$$

244 The inter-reaction times are thus fully determined by first-passage properties together with
 245 the rates $a_i(\mathbf{n})$.

246 2. Generalized chemical master equation under restart

247 We turn to the generalized master equation incorporating the effect of restart. The transi-
 248 tion probabilities $\gamma_i(n|n')$ corresponding to the regular reactions remain given by Eq. (4), as

249 the effect of these reactions does not change. The effect of the restart reaction, conditioned
 250 on a given velocity v , is characterized by:

$$\gamma_0(\mathbf{n}|\mathbf{n}', v) = \gamma_I(\mathbf{n}_I|v)\delta_{\mathbf{n}_M, \mathbf{n}'_M}, \quad (14)$$

251 meaning that mobile reactants remain unaffected, and immobile copy numbers are redrawn
 252 from the initial distribution as discussed above. As shown in detail in Appendix D, the
 253 generalized master equation corresponding to the dynamics under restart retains the same
 254 form as Eq. (5), with the memory functions again given in terms of the inter-reaction time
 255 densities according to Eq. (6). However, sums over reactions in both these equations now
 256 extend to $i = 0$, the reaction describing restart, and the inter-reaction time densities are
 257 given by Eq. (13). Direct computation shows that the modified inter-reaction times lead
 258 to the same memory functions for the regular reactions, as given by Eq. (7). Restart is
 259 associated with the memory function

$$M_0(\mathbf{n}|\mathbf{n}', v) = \tilde{K}(\lambda|v)\tilde{\psi}_c(\lambda)/\mu(v). \quad (15)$$

260 These results follow from the fact that normal reactions proceed at rate $a_i(\mathbf{n})$ in reactive
 261 regions, whereas restart occurs at rate $1/\mu(v)$ and is associated with an additional delay
 262 corresponding to traversing a non-reactive region. Realizations of these dynamics may be
 263 simulated with recourse to a generalized Gillespie algorithm under restart, which we outline
 264 in Appendix E.

265 Consider as an example the full degradation of Section III, with equal initial copy numbers
 266 of immobile components $n_{0,I}$ in each reactive region. The stoichiometry vector is $\mathbf{s}_1 =$
 267 $(-1, -1)$, and $\gamma_0 = \delta_{n_I, n_{0,I}}\delta_{n_M, n'_M}$. Similarly to Eq. (8) for the catalytic degradation example,
 268 we find the master equation

$$\begin{aligned} \frac{\partial p(n_M, n_I, t)}{\partial t} = & - \int_0^t dt' K(t-t') \left[a(n_M, n_I) p(n_M, n_I, t') \right. \\ & \left. - a(n_M + 1, n_I + 1) p(n_M + 1, n_I + 1, t') \right] \\ & - \int_0^t dt' K_c(t-t') \left[a(n_M, n_I) p(n_M, n_I, t') \right. \\ & \left. - a(n_M, n_{0,I}) p(n_M, n_{0,I}, t') \right], \end{aligned} \quad (16)$$

269 where the memory kernel associated with restart is given by the convolution $K_c(t) =$
 270 $\int_0^t dt' K(t-t')\rho_c(t')$, and we have again omitted dependencies on the initial condition and
 271 U .

272 C. Restart and catalytic reactions

273 Consider equal initial immobile copy numbers in all reactive regions as in Section IV A.
 274 Since for catalytic reactions $\mathbf{s}_i^I = 0$, immobile copy numbers do not change due to either
 275 restart or regular reactions, the $i = 0$ term in the master equation is null, and we recover
 276 catalytic dynamics, Eq. (5).

277 The catalytic description also plays a role as the limiting behavior for slow reaction. For
 278 small Damköhler number, $Da \ll 1$, the dynamics are transport-dominated at the scale of a
 279 single region, meaning that many reactive regions must be visited before appreciable change
 280 due to reaction can occur. For fixed initial immobile copy numbers, the reset mechanism
 281 ensures there is no appreciable change in immobile copy numbers. The catalytic description
 282 is then valid for arbitrary reactions, and the subordination formulation of [28] holds. Note
 283 also that under these conditions, for arbitrary initial copy numbers, the dynamics are inde-
 284 pendent of the specific reactive region length distribution as long as it has a finite mean, see
 285 Appendix C.

286 D. Large-scale kinetics

287 Next, we obtain the large-scale description corresponding to the mesoscopic master equa-
 288 tions developed in the previous sections. In the large particle number limit, we work in terms
 289 of concentrations $\mathbf{C}(t) = \mathbf{N}(t)/\bar{n}$ as introduced in Section III. Correspondingly, we define
 290 $a_i^C(\mathbf{c}) = \kappa_i^C h_i^C(\mathbf{c})$, $a^C(\mathbf{c}) = \sum_{i=1}^{m_r} a_i^C(\mathbf{c})$, with $\kappa_i^C = \bar{n}^{\sum_{j=1}^{m_s} r_{ij}-1} \kappa_i / \prod_{j=1}^{m_s} r_{ij}!$ the macroscopic
 291 rate constants and $h_i^C(\mathbf{c}) = \prod_{j=1}^{m_s} c_j^{r_{ij}}$. Note that these quantities are fully defined in terms
 292 of their microscopic equivalents. For example, considering again the degradation reaction
 293 of Section III, we have $\kappa^C = \bar{n}\kappa$ and $h_1^C(\mathbf{n}) = c_M c_I$, from which $a^C(\mathbf{c}) = a_1^C(\mathbf{c}) = \kappa^C c_A c_B$,
 294 the usual rate in the well-mixed rate laws for concentration. The average initial numbers of
 295 immobile components for velocity v are denoted by $\bar{\mathbf{c}}_{0,I}(v)$ and the average concentrations
 296 for initial condition \mathbf{c}_0 and velocity v by $\bar{\mathbf{c}}(t|\mathbf{c}_0, v) = \langle \mathbf{C}(t) | \mathbf{c}_0, v \rangle$.

298 In Appendix F, we show that the ensemble-averaged concentrations obey the integro-
299 differential dynamical equations

$$\begin{aligned} \frac{d\bar{\mathbf{c}}_M(t|\mathbf{c}_0, v)}{dt} &= \int_0^t dt' K(t-t'|v) \sum_{i=1}^{m_r} \mathbf{s}_i^M \langle a_i^C[\mathbf{C}(t')] | \mathbf{c}_0, v \rangle, \\ \frac{d\bar{\mathbf{c}}_I(t|\mathbf{c}_0, v)}{dt} &= \sum_{i=1}^{m_r} \mathbf{s}_i^I \langle a_i^C[\mathbf{C}(t)] | \mathbf{c}_0, v \rangle \\ &\quad - \mu(v)^{-1} [\bar{\mathbf{c}}_I(t|\mathbf{c}_0, v) - \bar{\mathbf{c}}_{0,I}(v)]. \end{aligned} \quad (17)$$

300 Segregation induces memory effects in the form of a convolution with a memory kernel for the
301 mobile components. For the immobile components, restart leads to a mean-reverting forcing
302 term. For *mild segregation*, the memory is short-term, and the late-time rate equations are
303 time-local, whereas memory is long-range under *strong segregation* and the late-time rate
304 equations involve non-local fractional derivatives [35].

305 In the classical rate laws, fluctuations vanish in the large-particle-number limit and
306 $\langle a_i^C[\mathbf{C}(t)] | \mathbf{c}_0, v \rangle = a_i^C[\bar{\mathbf{c}}(t|\mathbf{c}_0, v)]$ [27]. Here, this is not the case. Under *strong segrega-*
307 *tion*, realizations of the dynamics break ergodicity weakly due to large fluctuations in the
308 inter-reaction times, leading to persistent fluctuations about the average concentrations [28].
309 Under *mild segregation*, short-term memory effects coupled with the discontinuous changes
310 in immobile concentrations caused by restart also prevent the fluctuations from vanishing.
311 While equations for higher moments of the concentration can be found by appropriate av-
312 eraging of the generalized master equation (5), these depend on still higher moments, and
313 closed rate equations for the average components do not exist in general. In other words, fluc-
314 tuations play an important role in the mean behavior, analogous to Ovchinnikov–Zeldovich
315 segregation in a bimolecular annihilation reaction among diffusing components [13, 36].
316 A common approach is to employ moment closure approximations [37, 38], a technique
317 that must be adapted to specific reactions and which we do not explore here. We employ
318 stochastic algorithms, outlined in Appendix E, to numerically solve for the exact average
319 concentration. We note here that the mean-reverting term in the large-scale description
320 of immobile concentration depends only on average concentrations. Thus, the catalytic
321 description (for catalytic reactions or the low-Damköhler limit of general reactions, see Sec-
322 tions IV A and IV C) holds at the large scale even if initial immobile copy numbers vary

323 across reactive regions.

324 2. Asymptotics

325 We now examine the asymptotic behavior of the large-scale kinetics. For *mild segrega-*
 326 *tion*, under which the inter-reactive-region lengths have a mean, the Laplace transform of
 327 the corresponding FPT is approximated by $\tilde{\psi}_c(\lambda|v) \approx 1 - \alpha\mu(v)\lambda$ for $\lambda \ll 1/[\alpha\mu(v)]$, cor-
 328 responding to large times compared to the mean time to traverse a non-reactive region. To
 329 leading order in $\lambda \ll 1/[(1 + \alpha)\mu(v)]$ (corresponding to large times compared to the mean
 330 time to traverse a reactive and a non-reactive region),

$$\tilde{M}_i(\lambda|\mathbf{n}, v) = \frac{1}{1 + \alpha} \begin{cases} 1/\mu(v) \\ a_i(\mathbf{n}) \end{cases}. \quad (18)$$

331 This leads to time-local equations for the mobile components at late times,

$$\frac{d\bar{\mathbf{c}}_M(t|\mathbf{c}_0, v)}{dt} = (1 + \alpha)\mu(v)^2 \sum_{i=1}^{m_r} \mathbf{s}_i^M \langle a_i^C[\mathbf{C}(t)]|\mathbf{c}_0, v \rangle. \quad (19)$$

332 For *strong segregation*, under which the inter-reactive region lengths do not have a mean,
 333 we have instead the small- λ expansion $\tilde{\psi}_c(\lambda|v) \approx 1 - [\alpha\mu(v)\lambda]^\beta$, $0 < \beta < 1$. To leading order
 334 in $\lambda \ll 1/[(1 + \alpha)\mu(v)]$,

$$\tilde{M}_i(\lambda|\mathbf{n}, v) = \mu(v)\lambda [\alpha\mu(v)\lambda]^{-\beta} \begin{cases} 1/\mu(v) \\ a_i(\mathbf{n}) \end{cases}, \quad (20)$$

335 yielding, at late times, the time-nonlocal equations

$$\frac{d^\beta \bar{\mathbf{c}}_M(t|\mathbf{c}_0, v)}{dt^\beta} = [\alpha\mu(v)]^\beta \mu(v) \sum_{i=1}^{m_r} \mathbf{s}_i^M \langle a_i^C[\mathbf{C}(t)]|\mathbf{c}_0, v \rangle, \quad (21)$$

336 where d^β/dt^β denotes the Riemann–Liouville fractional derivative of order β [35].

337 V. LARGE-SCALE DYNAMICS OF REACTION UNDER SEGREGATION

338 In order to illustrate the main features of the theoretical developments in the context
 339 of a particular reaction, let us return to the large-scale dynamics of mobile concentration
 340 for the degradation reaction $\mathcal{S}_M + \mathcal{S}_I \rightarrow_\kappa \emptyset$ introduced in Section III. As the Damköhler

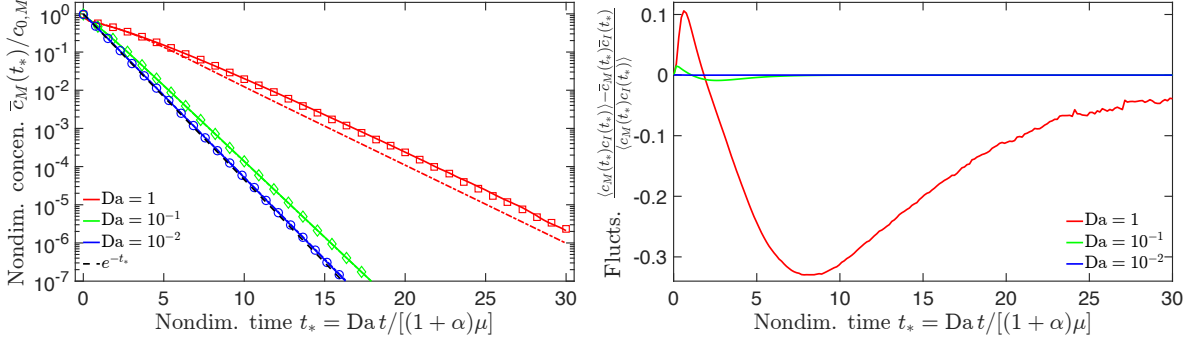


Figure 3. Temporal evolution of concentration for $\mathcal{S}_M + \mathcal{S}_I \rightarrow \emptyset$ under *mild segregation* for different values of Damköhler number Da . Symbols represent stochastic simulations based on the Gillespie algorithm (10^7 initial mobile particles and $5 \cdot 10^6$ immobile particles per reactive region, averaged over 10^5 realizations) and solid lines employ the well-mixed rate equations in reactive regions (10^6 realizations). Non-reactive region lengths are exponentially-distributed. Fixed parameters are $\alpha = 2$, $v = 4$, and $\kappa^C = 2$, and Da is set by varying ℓ_r . Dash-dotted lines (overlapping the solid lines for low Da) are numerical solutions of Eq. (17) using $\langle C_M C_I \rangle = \bar{c}_M \bar{c}_I$. **Left:** Average mobile concentration. The dashed line is the analytical solution in the limit of small Damköhler. **Right:** Concentration fluctuations.

341 number $Da(v) = \kappa^C \mu(v) \bar{c}_{0,I}(v) \rightarrow 0$, the changes in immobile concentration due to reaction
 342 in each region become arbitrarily small, so that $\langle C_I C_M \rangle \approx \bar{c}_{0,I}(v) \langle C_M \rangle$ at all times. Thus,
 343 for small Da , the late-time rate equation for the mobile component under *mild segregation*
 344 is, according to Eq. (19),

$$\frac{d\bar{c}_M(t|\mathbf{c}_0, v)}{dt} = -\frac{Da(v)}{(1 + \alpha)\mu(v)} \bar{c}_M(t|\mathbf{c}_0, v). \quad (22)$$

345 There is no appreciable reaction before the late-time equation is valid, so that the initial
 346 condition $c_{0,M}$ may be employed, and

$$\bar{c}_M(t|\mathbf{c}_0, v) = c_{0,M} \exp \left[-\frac{Da(v)t}{(1 + \alpha)\mu(v)} \right]. \quad (23)$$

347 Under *strong segregation*, the Laplace transform of the late-time equation (21) is, for
 348 $Da(v) \ll 1$,

$$\frac{[\alpha\mu(v)\lambda]^\beta}{Da(v)} [\tilde{c}_M(t|\mathbf{c}_0, v) - \lambda^{-1}c_{0,M}] = -\tilde{c}_M(\lambda|\mathbf{c}_0, v). \quad (24)$$

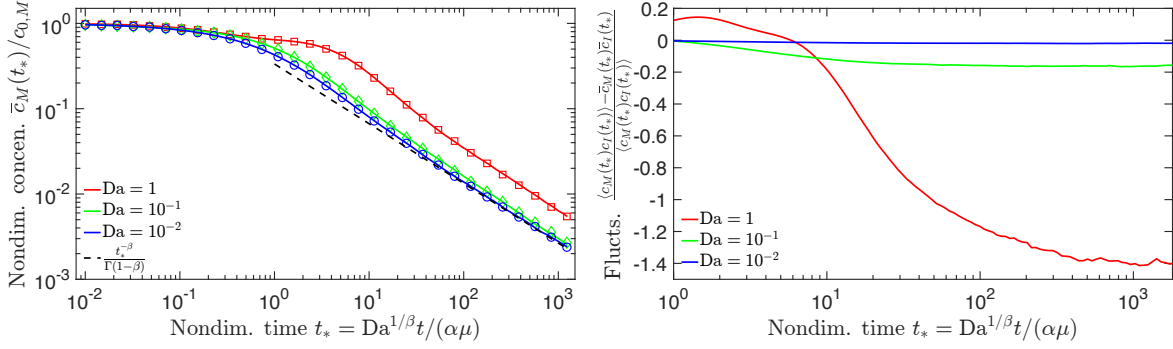


Figure 4. Temporal evolution of concentration for $\mathcal{S}_M + \mathcal{S}_I \rightarrow_{\kappa} \emptyset$ under *strong segregation* for different values of Damköhler number Da . Symbols represent stochastic simulations based on the Gillespie algorithm (10^7 initial mobile particles and $5 \cdot 10^6$ immobile particles per reactive region) and solid lines employ the well-mixed rate equations in reactive regions. All results are averaged over 10^5 realizations. Non-reactive region lengths are Lévy-stable-distributed with exponent $\beta = 0.7$. Fixed parameters are $\alpha = 2$, $v = 4$, and $\kappa^C = 2$, and Da is set by varying ℓ_r . **Left:** Average mobile concentration. The dashed line is the late-time analytical solution in the limit of small Damköhler. **Right:** Concentration fluctuations.

349 Noticing that, for small λ , the initial-condition term on the left-hand side dominates, and
 350 inverting the Laplace transform,

$$\bar{c}_M(t|\mathbf{c}_0, v) \approx \frac{c_{0,M}}{\text{Da}(v)\Gamma(1-\beta)} \left[\frac{t}{\alpha\mu(v)} \right]^{-\beta}. \quad (25)$$

351 Simulation results for mild segregation are shown in Fig. 3 and for strong segregation
 352 in Fig. 4. We consider for concreteness exponential ρ_c for mild and Lévy-stable ρ_c for
 353 strong segregation, as before. Note that, unlike in Fig. 2, time is nondimensionalized so
 354 as to highlight the collapse of the low-Damköhler behavior onto the Da -independent curve
 355 valid for both the full degradation reaction and the catalytic degradation reaction $\mathcal{S}_M +$
 356 $\mathcal{S}_I \rightarrow_{\kappa} \mathcal{S}_I$. Under this nondimensionalization, higher Da leads to slower decay due to the
 357 effect of depletion of the immobile component. We compare a full stochastic algorithm
 358 employing the Gillespie method in reactive regions to a more efficient algorithm, valid for
 359 large particle numbers, which makes use of the well-mixed rate equations as in Section III
 360 (see Appendix E). The results are in very good agreement.

361 For *mild segregation*, simulations suggest that $\langle C_I C_M \rangle \approx \bar{c}_{0,I}(v)\langle C_M \rangle$ holds at late times,
 362 as expected under finite-mean inter-reaction times (see the right panel of Fig. 3). Thus,

363 the late-time concentration for each velocity v decays in general exponentially as argued
 364 above, but the leading coefficient differs because the appropriate initial condition for the
 365 late-time equation depends on the dynamics before it becomes valid. However, for large Da ,
 366 the exponential asymptotic regime is not observed, as the concentration reaches very small
 367 values before it occurs. We show also the solution obtained by numerically integrating the
 368 closed rate equations obtained by setting $\langle C_M C_I \rangle = \bar{c}_M \bar{c}_I$ in Eq. (17) (see Appendix F).
 369 Its breakdown for $Da \gtrsim 1$ is due to the role of concentration fluctuations. Even though the
 370 fluctuations vanish at late times, they have an irreversible impact on the total reaction.

371 Under *strong segregation*, the leading coefficient varies with Da for two reasons. First,
 372 the initial condition to be used with the asymptotic equation differs as above. Second,
 373 the weak ergodicity breaking displayed by the inter-reaction times impedes $\langle C_I(t) C_M(t) \rangle \rightarrow$
 374 $\bar{c}_{0,I} \langle C_M(t) \rangle$ for late times, because there is a sufficiently high probability that $C_M(t)$ remains
 375 large due to long non-reactive regions. Nonetheless, simulations suggest that $\langle C_I(t) C_M(t) \rangle \propto$
 376 $\bar{c}_I(t) \bar{c}_M(t) \approx c_{0,I} \bar{c}_M(t)$ at late times for all values of Da , see the right panel of Fig. 4, so that
 377 the power-law behavior $\propto t^{-\beta}$ remains unaffected.

378 VI. CONCLUSIONS

379 The link between first-passage and inter-reaction times connects the kinetics of contact
 380 processes to spatial heterogeneity and segregation. We have quantified this link for advective
 381 transport under spatial segregation of immobile components and formulated the reaction
 382 dynamics in terms of a generalized master equation. The evolution of total mass may be
 383 obtained from this description, and mass fluxes and spatial reactant distributions may also
 384 be easily computed.

385 In contrast to the classical picture for well-mixed reactions, the resulting large-scale ki-
 386 netics cannot be fully quantified in terms of the dynamical equations obtained by averaging
 387 over the chemical master equation. This is due to the presence of concentration fluctuations
 388 on the order of the average values, which result from a combination of the restart mech-
 389 anism and memory effects, both caused by segregation. For this reason, closed-form rate
 390 laws valid for all times do not exist in general. In the case of strong segregation, character-
 391 ized by infinite-mean inter-reactive-region distances, memory is long-range and induces weak
 392 ergodicity breaking across trajectories, a typical feature of anomalous transport [39–44].

393 Future work will focus on extending this approach to more complex transport processes,
 394 including effects such as variable velocity within each trajectory [45, 46] and local mixing
 395 (e.g., diffusion). Moment closure approximations for the rate equations will also be the
 396 subject of further study.

397 ACKNOWLEDGMENTS

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 399 the project MHetScale (617511).

400 Appendix A: Well-mixed degradation kinetics

401 Consider the degradation reaction $\mathcal{S}_M + \mathcal{S}_I \rightarrow_{\kappa} \emptyset$ introduced in the main text. In a
 402 well-mixed reactor, the kinetic rate laws are given by Eq. (1). For equal initial conditions,
 403 $\mathbf{c}_0 = (c_{0,M}, c_{0,I})$ with $c_{0,M} = c_{0,I} = c_0$, the solution is

$$c_{M,I}^{wm}(t|\mathbf{c}_0) = \frac{c_0}{1 + \kappa^C c_0 t}. \quad (\text{A1})$$

404 For $c_{0,M} \neq c_{0,I}$, setting $c_{\max} = \max\{c_M^{wm}, c_I^{wm}\}$ and $c_{\min} = \min\{c_M^{wm}, c_I^{wm}\}$,

$$\begin{aligned} c_{\min}(t|\mathbf{c}_0) &= c_{0,\min} g(t, \mathbf{c}_0) e^{-\kappa^C t (c_{0,\max} - c_{0,\min})}, \\ c_{\max}(t|\mathbf{c}_0) &= c_{0,\max} g(t, \mathbf{c}_0), \end{aligned} \quad (\text{A2})$$

405 where

$$g(t, \mathbf{c}_0) = \frac{c_{0,\max} - c_{0,\min}}{c_{0,\max} - c_{0,\min} e^{-\kappa^C t (c_{0,\max} - c_{0,\min})}}. \quad (\text{A3})$$

406 The simulations of the evolution of concentration due to this reaction in the presence
 407 of segregation proceed as follows. A reactive region length ℓ is generated according to the
 408 PDF ρ_r . The solution of the well-mixed rate laws is then applied to obtain the evolution
 409 of concentrations for a time interval ℓ/v . Then, a non-reactive region length ℓ' is generated
 410 according to ρ_c , and no reaction occurs for a time interval ℓ'/v . This procedure is iterated up
 411 to a desired time, with the initial condition for the well-mixed rate laws in each reactive patch
 412 being set according to the current mobile concentration and the initial resident immobile
 413 concentration.

414 If one considers instead the catalytic degradation reaction $\mathcal{S}_M + \mathcal{S}_I \rightarrow_{\kappa} \mathcal{S}_I$ in a well-mixed
 415 reactor, the immobile species concentration does not change and remains equal to $c_{0,I}$. The
 416 rate law for the mobile component is the same as before, and we obtain

$$c_M^{wm}(t|\mathbf{c}_0) = e^{-\kappa^C c_{0,I} t}. \quad (\text{A4})$$

417 The simulations under segregation proceed in the same manner as above, using this solution
 418 for the mobile concentration.

419 Appendix B: Intrinsic inter-reaction times

420 This Appendix provides a brief review of the intrinsic inter-reaction times, which char-
 421 acterize reactions in the absence of segregation. These correspond to the classical stochas-
 422 tic theory of well-mixed reaction [24]. Consider a given chemical state \mathbf{n} . Each reac-
 423 tion $i = 1, \dots, m_r$ considered in isolation has an exponential inter-reaction time with rate
 424 $a_i(\mathbf{n}) = \kappa_i h_i(\mathbf{n})$, where κ_i is a (microscopic) rate constant and $h_i(\mathbf{n})$ encodes the dependency
 425 on the chemical state. For mass-action reactions,

$$h_i(\mathbf{n}) = \prod_{j=1}^{m_s} \frac{n_j!}{r_{ij}!(n_j - r_{ij})!}. \quad (\text{B1})$$

426 It follows from the exponential character of reaction times, and the fact that the next reaction
 427 to fire is the one with the minimum waiting time, that the inter-reaction time density is

$$\phi_i^r(t|\mathbf{n}) = \frac{a_i(\mathbf{n})}{a(\mathbf{n})} \phi^r(t|\mathbf{n}), \quad (\text{B2})$$

428 where $a(\mathbf{n}) = \sum_{i=1}^{m_r} a_i(\mathbf{n})$, and

$$\phi^r(t|\mathbf{n}) = a(\mathbf{n}) e^{-a(\mathbf{n})t}, \quad (\text{B3})$$

429 with $\phi^r(t|\mathbf{n}) dt$ the probability that the inter-reaction time is in $[t, t + dt]$. The Laplace
 430 transform of the inter-reaction time density is thus

$$\tilde{\phi}_i^r(\lambda|\mathbf{n}) = \frac{a_i(\mathbf{n})}{\lambda + a(\mathbf{n})}. \quad (\text{B4})$$

431 Appendix C: Number of traversed reactive regions

432 Here, we determine the distribution $\nu_r(\cdot|t_r, v)$ of the number $\eta_r(t_r, v)$ of fully traversed
 433 reactive regions between reactions, given time t_r spent in reactive regions and velocity v .

434 The length traversed in time t_r is vt_r . The number $\eta_r(t_r, v)$ of reactive regions traversed in
 435 this time is such that their total length is smaller than vt_r , but the total length of $\eta_r(t_r, v) + 1$
 436 regions is larger than vt_r . Thus,

$$\nu_r(k|t_r, v) = \left\langle H \left(vt_r - \sum_{k'=1}^k L_{r,k'} \right) \times H \left(\sum_{k'=1}^{k+1} L_{r,k'} - vt_r \right) \right\rangle, \quad (\text{C1})$$

437 where H is the Heaviside step function and the $L_{r,k'}$ are i.i.d. according to ρ_r . Conditioning
 438 on the total length of the first k regions, we obtain

$$\nu_r(k|t_r, v) = \int_0^{vt_r} d\ell \left\langle \delta \left(\ell - \sum_{k'=1}^k L_{r,k} \right) \right\rangle \times \int_{vt_r-\ell}^{\infty} d\ell' \rho_r(\ell'). \quad (\text{C2})$$

439 Taking the Laplace transform with respect to t_r ,

$$\tilde{\nu}_r(k|\lambda, v) = \frac{1 - \tilde{\psi}_r(\lambda|v)}{\lambda} \tilde{\psi}_r(\lambda|v)^k. \quad (\text{C3})$$

440 Thus, using $\rho_r(\ell) = e^{-\ell/\ell_r}/\ell_r$ and $\psi_r(t|v) = v\rho_r(vt)$, we have $\tilde{\nu}_r(k|\lambda, v) = \mu(v)[1 + \mu(v)\lambda]^{-k}$,
 441 with $\mu(v) = \ell_r/v$. Inverting the Laplace transform,

$$\nu_r(k|t_r, v) = \frac{[t_r/\mu(v)]^k}{k!} e^{-t_r/\mu(v)}, \quad (\text{C4})$$

442 so that $\eta_r(t_r, v)$ is Poisson-distributed with mean $t_r/\mu(v)$.

443 Note that, as long as reactive region lengths have a finite mean ℓ_r , we have, for small
 444 $\lambda \ll 1/\mu(v)$ (corresponding to the large $t_r \gg \mu(v)$ limit), $\tilde{\psi}_r(\lambda|v) \approx 1 - \mu(v)\lambda \approx 1/[1 + \mu(v)\lambda]$,
 445 the Laplace transform of the exponential density. Thus, the distribution of the number of
 446 traversed reactive patches is always approximately Poisson for large t_r . Since typical reaction
 447 times are large when the Damköhler number is low, as explained in the main text, the specific
 448 distribution of reactive lengths does not play a role in that case.

449 **Appendix D: Generalized master equation under restart**

450 Consider the process $\mathcal{K}(t)$, which describes the number of reactions as a function of
 451 time, and write $\mathbf{N}_{\mathcal{K}(t)} = \mathbf{N}(t)$. We have $\mathcal{K}(t) = \sup\{k \mid T_k < t\}$, where T_k is the time

452 of the k th reaction. The propagator can then be written as $p(\mathbf{n}, t|\mathbf{n}_0, v) = \langle \delta_{\mathbf{n}, \mathbf{N}_{\mathcal{K}(t)}} | \mathbf{n}_0, v \rangle$.

453 Conditioning on $K(t) = k$ and $T_k = t'$,

$$p(\mathbf{n}, t|\mathbf{n}_0, v) = \int_0^{\infty} dt' \sum_{k \geq 0} \langle \delta_{\mathbf{n}, \mathbf{N}_k} \delta(T_k - t') \times H(t - t') H[\tau_k - (t - t')] | \mathbf{n}_0, v \rangle, \quad (\text{D1})$$

454 with the inter-reaction times $\tau_k = T_{k+1} - T_k$ independent of T_k . Thus,

$$p(\mathbf{n}, t|\mathbf{n}_0, v) = \int_0^t dt' \sum_{k \geq 0} R_k(\mathbf{n}, t'|\mathbf{n}_0, v) \times \sum_{i=0}^{m_r} \int_{t-t'}^{\infty} dt'' \phi_i(t''|\mathbf{n}, v), \quad (\text{D2})$$

455 where $R_k(\mathbf{n}, t|\mathbf{n}_0, v) = \langle \delta_{\mathbf{n}, \mathbf{N}_k} \delta(T_k - t) | \mathbf{n}_0, v \rangle$ is the probability per time of arriving at state
456 \mathbf{n} at time t given k reaction steps. The dynamics are Markov in reaction step number k ,

457 and $R_k(\mathbf{n}, t|\mathbf{n}_0, v)$ solves the Chapman-Kolmogorov equation

$$R_{k+1}(\mathbf{n}, t|\mathbf{n}_0, v) = \int_0^t dt' \sum_{\mathbf{n}' \geq 0} \sum_{i=0}^{m_r} \gamma_i(\mathbf{n}|\mathbf{n}', v) \times \phi_i(t - t'|\mathbf{n}', v) R_k(\mathbf{n}', t'), \quad (\text{D3})$$

458 with $R_0(\mathbf{n}, t|\mathbf{n}_0, v) = \delta_{\mathbf{n}, \mathbf{n}_0} \delta(t)$, $\gamma_i(\mathbf{n}|\mathbf{n}', v) = \gamma_i(\mathbf{n}|\mathbf{n}')$ given by Eq. (4), and $\gamma_0(\mathbf{n}|\mathbf{n}', v)$ given

459 by Eq. (14). Laplace transforming Eqs. (D2) and (D3) summed over k ,

$$\begin{aligned} \tilde{R}(\mathbf{n}, \lambda|\mathbf{n}_0, v) &= \delta_{\mathbf{n}, \mathbf{n}_0} \\ &+ \sum_{\mathbf{n}' \geq 0} \sum_{i=0}^{m_r} \gamma_i(\mathbf{n}|\mathbf{n}', v) \tilde{\phi}_i(\lambda|\mathbf{n}', v) \tilde{R}(\mathbf{n}', \lambda|\mathbf{n}_0, v), \\ \tilde{p}(\mathbf{n}, \lambda|\mathbf{n}_0, v) &= \tilde{R}(\mathbf{n}, \lambda|\mathbf{n}_0, v) \frac{1 - \sum_{i=0}^{m_r} \tilde{\phi}_i(\lambda|\mathbf{n}, v)}{\lambda}, \end{aligned} \quad (\text{D4})$$

460 where $R(\mathbf{n}, t|\mathbf{n}_0, v) = \sum_{k \geq 0} R_k(\mathbf{n}, t|\mathbf{n}_0, v)$. Eliminating \tilde{R} , we find

$$\begin{aligned} \lambda \tilde{p}(\mathbf{n}, \lambda|\mathbf{n}_0, v) - \delta_{\mathbf{n}, \mathbf{n}_0} &= \sum_{\mathbf{n}' \geq 0} \sum_{i=0}^{m_r} [\gamma_i(\mathbf{n}|\mathbf{n}', v) - \delta_{\mathbf{n}, \mathbf{n}'}] \\ &\times \tilde{M}_i(\lambda|\mathbf{n}', v) \tilde{p}(\mathbf{n}', \lambda|\mathbf{n}_0, v), \end{aligned} \quad (\text{D5})$$

461 where $\tilde{M}_i(\lambda|\mathbf{n}, v)$ is given according to Eq. (6), for all $0 \leq i \leq m_r$ and with the sum extending

462 to $i = 0$. Laplace inversion leads to the same form as the generalized master equation (5),

463 with the sum extending to $i = 0$.

464 **Appendix E: Stochastic simulation algorithms**

465 We describe a generalized Gillespie algorithm that takes into account restart as described
 466 in the main text. Velocity v is to be sampled from the PDF $\xi(\cdot)$. In order to simulate
 467 dynamics up to time t_m (or distance ℓ_m), the following algorithm should be repeated for a
 468 prescribed number of realizations:

- 469 1. Set time $t = 0$ (or distance $\ell = 0$). Generate \mathbf{n} according to the initial state distribu-
 470 tion.
- 471 2. Generate ℓ_1 according to ρ_r and Δt_r according to $\phi^r(\cdot|\mathbf{n})$, see Eq. (B3).
- 472 3. Find the next reaction i : If $\Delta t_r > \ell_1/v$ set $i = 0$. Else generate $0 < r \leq a(\mathbf{n})$ from
 473 the uniform distribution and set i such that $a_i(\mathbf{n}) < r \leq a_{i+1}(\mathbf{n})$, see Eq. (B3).
- 474 4. If $i = 0$ generate ℓ_2 according to ρ_c and set $\Delta t = (\ell_1 + \ell_2)/v$. Else set $\Delta t = \Delta t_r$.
 475 Increment t by Δt (ℓ by $v\Delta t$).
- 476 5. If $t < t_m$ ($\ell < \ell_m$) update \mathbf{n} according to $\gamma_i(\cdot|\mathbf{n}, v)$, see Eq. (4), and go to 2. Else set
 477 $t = t_m$ ($\ell = \ell_m$) and end.

478 For large particle numbers, this procedure may be replaced by a more efficient method to
 479 find the concentrations:

- 480 1. Set time $t = 0$ (or distance $\ell = 0$). Generate \mathbf{c} according to the initial concentration
 481 distribution.
- 482 2. Generate ℓ_1 according to ρ_r and Δt_r according to $\phi^r(\cdot|\mathbf{n})$, see Eq. (B3). Set $\Delta t =$
 483 $\min\{\Delta t_r, t_m - t\}$.
- 484 3. Update \mathbf{c} according to the well-mixed rate equations over the interval $[t, t + \Delta t]$.
 485 Increment t by Δt (ℓ by $v\Delta t$). If $t = t_m$ ($\ell = \ell_m$) end.
- 486 4. Generate ℓ_2 according to ρ_c . Increment t by $\Delta t = \min\{(\ell_1 + \ell_2)/v, t - t_m\}$ (ℓ by $v\Delta t$).
- 487 5. If $t < t_m$ ($\ell < \ell_m$) generate \mathbf{c}_I according to the initial concentration distribution and
 488 go to 2. Else end.

489 Note that this algorithm reduces to the one outlined in Appendix A for the reactions con-
 490 sidered therein.

491 **Appendix F: Large-scale kinetics**

492 Following [28], we define $p^C(\mathbf{c}, t|\mathbf{c}_0, v) = \bar{n}p(\bar{n}\mathbf{c}, t|\bar{n}\mathbf{c}_0, v)$, $M_i^C(t|\mathbf{c}, v) = M_i(t|\bar{n}\mathbf{c}, v)/\bar{n}$,
 493 $i = 1, \dots, m_r$, and $M_0^C(t|\mathbf{c}, v) = M_0(t|\bar{n}\mathbf{c}, v)$. We multiply Eq. (D5) by, and sum over, \mathbf{n} ,
 494 and write $\sum_{\mathbf{n}} = \bar{n} \int d\mathbf{c}$ in the limit $\bar{n} \rightarrow \infty$. Integrating by parts, we obtain

$$\begin{aligned} \lambda \tilde{\mathbf{c}}(\lambda|\mathbf{c}_0, v) - \mathbf{c}_0 &= \sum_{i=1}^{m_r} \mathbf{s}_i \mathcal{L}_{t \rightarrow \lambda} \langle \tilde{M}_i^C[\lambda|\mathbf{C}(t)]|\mathbf{c}_0, v \rangle \\ &- \mathcal{L}_{t \rightarrow \lambda} \langle [\mathbf{C}(t) - (\mathbf{C}_M(t), \bar{\mathbf{c}}_{0,I}(v))] \tilde{M}_0^C[\lambda|\mathbf{C}(t)]|\mathbf{c}_0, v \rangle, \end{aligned} \quad (\text{F1})$$

495 where $\mathcal{L}_{t \rightarrow \lambda} f(t) = \tilde{f}(\lambda)$. Substituting Eq. (6) for the memory functions, we have,

$$\begin{aligned} \lambda \tilde{\mathbf{c}}_M(\lambda|\mathbf{c}_0, v) - \mathbf{c}_{0,M} &= \\ &\sum_{i=1}^{m_r} \mathbf{s}_i^M \tilde{K}(\lambda|v) \mathcal{L}_{t \rightarrow \lambda} \langle a_i^C[\mathbf{C}(t)]|\mathbf{c}_0, v \rangle, \\ [\lambda \tilde{\mathbf{c}}_I(\lambda|\mathbf{c}_0, v) - \bar{\mathbf{c}}_{0,I}(v)] &\left[1 + \frac{\tilde{K}(\lambda|v) \tilde{\psi}_c(\lambda|v)}{\mu(v)\lambda} \right] = \\ &+ \sum_{i=1}^{m_r} \mathbf{s}_i^I \tilde{K}(\lambda|v) \mathcal{L}_{t \rightarrow \lambda} \langle a_i^C[\mathbf{C}(t)]|\mathbf{c}_0, v \rangle. \end{aligned} \quad (\text{F2})$$

496 Using Eq. (3) for the memory kernel, we find that $1 - \tilde{K}(\lambda|v)[1 - \tilde{\psi}_c(\lambda|v)]/[\mu(v)\lambda] = \tilde{K}(\lambda|v)$,
 497 so that the second equation may be rearranged and divided by $\tilde{K}(\lambda|v)$ to give

$$\begin{aligned} \lambda \tilde{\mathbf{c}}_I(\lambda|\mathbf{c}_0, v) - \bar{\mathbf{c}}_{0,I}(v) &= -\frac{\lambda \tilde{\mathbf{c}}_I(\lambda|\mathbf{c}_0, v) - \bar{\mathbf{c}}_{0,I}(v)}{\mu(v)\lambda} \\ &+ \sum_{i=1}^{m_r} \mathbf{s}_i^I \mathcal{L}_{t \rightarrow \lambda} \langle a_i^C[\mathbf{C}(t)]|\mathbf{c}_0, v \rangle. \end{aligned} \quad (\text{F3})$$

498 Inverse-Laplace-transforming yields Eq. (17).

499 In order to integrate Eqs. (17) for the degradation reaction $\mathcal{S}_M + \mathcal{S}_I \rightarrow \emptyset$, under the
 500 assumption $\langle C_I C_M \rangle = \bar{c}_I \bar{c}_M$ and mild segregation, note that the memory kernel (3) reads
 501 $\tilde{K}(\lambda|v) = 1 - \alpha/[1 + \alpha + \alpha\mu(v)\lambda]$, so that, inverting the Laplace transform, $K(t|v) =$

502 $\delta(t) = \exp\{-(1 + \alpha)t/[\alpha\mu(v)]\}/\mu(v)$. Equations (17) then read

$$\begin{aligned}
 \frac{d\bar{c}_M(t|\mathbf{c}_0, v)}{dt} &= -\kappa^C \bar{c}_M(t|\mathbf{c}_0, v) \bar{c}_I(t|\mathbf{c}_0, v) \\
 &+ \frac{\kappa^C}{\mu(v)} \int_0^t dt' e^{-\frac{1+\alpha}{\alpha\mu(v)}(t-t')} \bar{c}_M(t'|\mathbf{c}_0, v) \bar{c}_I(t'|\mathbf{c}_0, v), \\
 \frac{d\bar{c}_I(t|\mathbf{c}_0, v)}{dt} &= -\kappa^C \bar{c}_M(t|\mathbf{c}_0, v) \bar{c}_I(t|\mathbf{c}_0, v) \\
 &- \frac{\bar{c}_I(t|\mathbf{c}_0, v) - c_{0,I}}{\mu(v)}.
 \end{aligned}
 \tag{F4}$$

503 We solve these equations numerically using a combination of the forward Euler method for
 504 the derivative with the trapezoidal rule for the convolution integral. We use a discretization
 505 time step $\Delta t = 10^{-2} \min\{\mu, 1/(\kappa^C c_{0,M}), 1/(\kappa^C c_{0,I})\}$.

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- 506 [1] N. Barkai and S. Leibler, *Nature* **403**, 267 (2000).
 507 [2] D. Mollison, *J. R. Stat. Soc. B* **39**, 283 (1977).
 508 [3] R. J. Williams and N. D. Martinez, *Nature* **404**, 180 (2000).
 509 [4] M. P. Nightingale and C. J. Umrigar, *Quantum Monte Carlo Methods in Physics and Chem-*
 510 *istry* (Springer Science & Business Media, New York, 1998).
 511 [5] C. I. Steefel, D. J. DePaolo, and P. C. Lichtner, *Earth Planet. Sci. Lett.* **240**, 539 (2005).
 512 [6] D. ben Avraham and S. Havlin, *Diffusion and Reactions in Fractals and Disordered Systems*
 513 (Cambridge University Press, Cambridge, 2005).
 514 [7] Y. Meroz, I. M. Sokolov, and J. Klafter, *Phys. Rev. E* **83**, 020104(R) (2011).
 515 [8] A. M. Tartakovsky, D. M. Tartakovsky, and P. Meakin, *Phys. Rev. Lett.* **101**, 044502 (2008).
 516 [9] O. Bénichou, C. Chevalier, J. Klafter, B. Meyer, and R. Voituriez, *Nat. Chem.* **2**, 472 (2010).
 517 [10] A. Godec and R. Metzler, *Phys. Rev. X* **6**, 041037 (2016).
 518 [11] M. Schmidt, F. Sagués, and I. Sokolov, *J. Phys.-Condens. Mat.* **19**, 065118 (2007).
 519 [12] G. J. Lapeyre and M. Dentz, *Phys. Chem. Chem. Phys.* **19**, 18863 (2017).
 520 [13] A. Ovchinnikov and Y. B. Zeldovich, *Chem. Phys.* **28**, 215 (1978).
 521 [14] J. M. Sancho, A. Romero, K. Lindenberg, F. Sagués, R. Reigada, and A. Lacasta, *J. Phys.*
 522 *Chem.* **100**, 19066 (1996).
 523 [15] R. Reigada, F. Sagués, I. M. Sokolov, J. M. Sancho, and A. Blumen, *Phys. Rev. Lett.* **78**,

- 524 741 (1997).
- 525 [16] G. Srinivasan, D. M. Tartakovsky, B. A. Robinson, and A. Aceves, *Water Resour. Res.* **43**,
526 [W12415](#) (2007).
- 527 [17] P. C. Lichtner and D. Tartakovsky, *Stoch. Env. Res. Risk. A.* **17**, 419 (2003).
- 528 [18] T. R. Ginn, *Water Resour. Res.* **35**, 1395 (1999).
- 529 [19] P. Grathwohl, H. Rügner, T. Wöhling, K. Osenbrück, M. Schwientek, S. Gayler,
530 U. Wollschläger, B. Selle, M. Pause, J.-O. Delfs, *et al.*, *Environ. Earth Sci.* **69**, 317 (2013).
- 531 [20] C. Simmons, T. Ginn, and B. Wood, *Water Resour. Res.* **31**, 2675 (1995).
- 532 [21] T. Ginn, C. Simmons, and B. Wood, *Water Resour. Res.* **31**, 2689 (1995).
- 533 [22] U. Seeboonruang and T. R. Ginn, *J. Contam. Hydrol.* **84**, 127 (2006).
- 534 [23] A. Sanz-Prat, C. Lu, R. T. Amos, M. Finkel, D. W. Blowes, and O. A. Cirpka, *J. Contam.*
535 *Hydrol.* **192**, 35 (2016).
- 536 [24] D. T. Gillespie, *J. Phys. Chem.* **81**, 2340 (1977).
- 537 [25] D. T. Gillespie, *Physica A* **188**, 404 (1992).
- 538 [26] J. L. Doob, *Trans. Am. Math. Soc.* **52**, 37 (1942).
- 539 [27] N. G. van Kampen, *Stochastic Processes in Physics and Chemistry* (Elsevier, Amsterdam,
540 1992).
- 541 [28] T. Aquino and M. Dentz, *Phys. Rev. Lett.* **119**, 230601 (2017).
- 542 [29] M. J. D. Powell, *Math. Program.* **12**, 241 (1977).
- 543 [30] H. Kautz, E. Horvitz, Y. Ruan, C. Gomes, and B. Selman, *Proc. 18th Nat. Conf. on Artificial*
544 *Intelligence* **97**, 674 (2002).
- 545 [31] S. Reuveni, M. Urbakh, and J. Klafter, *Proc. Natl. Acad. Sci. U.S.A.* **111**, 4391 (2014).
- 546 [32] A. Pal, I. Eliazar, and S. Reuveni, *Phys. Rev. Lett.* **122**, 020602 (2019).
- 547 [33] V. Kenkre, E. Montroll, and M. Shlesinger, *J. Stat. Phys.* **9**, 45 (1973).
- 548 [34] W. Feller, *An Introduction to Probability Theory and Its Applications*, Vol. 2 (John Wiley &
549 Sons, New York, 2008).
- 550 [35] M. M. Meerschaert and A. Sikorskii, *Stochastic Models for Fractional Calculus*, Vol. 43 (Walter
551 de Gruyter, Amsterdam, 2012).
- 552 [36] P. De Anna, T. Le Borgne, M. Dentz, D. Bolster, and P. Davy, *The Journal of chemical*
553 *physics* **135**, 174104 (2011).
- 554 [37] R. Grima, *J. Chem. Phys.* **136**, 04B616 (2012).

- 555 [38] A. Paster, D. Bolster, and D. A. Benson, *J. Comput. Phys.* **263**, 91 (2014).
- 556 [39] R. Klages, G. Radons, and I. M. Sokolov, *Anomalous Transport: Foundations and Applica-*
557 *tions* (John Wiley & Sons, New York, 2008).
- 558 [40] G. Bel and E. Barkai, *Phys. Rev. Lett.* **94**, 240602 (2005).
- 559 [41] S. Burov, J.-H. Jeon, R. Metzler, and E. Barkai, *Phys. Chem. Chem. Phys.* **13**, 1800 (2011).
- 560 [42] M. A. Lomholt, I. M. Zaid, and R. Metzler, *Phys. Rev. Lett.* **98**, 200603 (2007).
- 561 [43] A. V. Weigel, B. Simon, M. M. Tamkun, and D. Krapf, *Proc. Natl. Acad. Sci. U.S.A.* **108**,
562 6438 (2011).
- 563 [44] J.-H. Jeon, V. Tejedor, S. Burov, E. Barkai, C. Selhuber-Unkel, K. Berg-Sørensen, L. Odder-
564 shede, and R. Metzler, *Phys. Rev. Lett.* **106**, 048103 (2011).
- 565 [45] M. Dentz, P. K. Kang, A. Comolli, T. Le Borgne, and D. R. Lester, *Phys. Rev. Fluids* **1**,
566 074004 (2016).
- 567 [46] A. Puyguiraud, P. Gouze, and M. Dentz, *Water Resour. Res.* **55**, 1196 (2019).