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SEMICLASSICAL LIMIT FOR ALMOST FERMIONIC ANYONS

THÉOTIME GIRARDOT AND NICOLAS ROUGERIE

ABSTRACT. In two-dimensional space there are possibilities for quantum statistics continuously interpolating between the bosonic and the fermionic one. Quasi-particles obeying such statistics can be described as ordinary bosons and fermions with magnetic interactions. We study a limit situation where the statistics/magnetic interaction is seen as a “perturbation from the fermionic end”. We vindicate a mean-field approximation, proving that the ground state of a gas of anyons is described to leading order by a semi-classical, Vlasov-like, energy functional. The ground state of the latter displays anyonic behavior in its momentum distribution. Our proof is based on coherent states, Husimi functions, the Diaconis-Freedman theorem and a quantitative version of a semi-classical Pauli principle.

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1. MODEL AND MAIN RESULTS

1.1. Introduction. In two dimensional systems there are possibilities for quantum statistics other than the bosonic one and the fermionic one, so called intermediate or fractional statistics. Particles following such statistics, termed anyons (as in *anything* in between bosons and fermions), can arise as effective quasi-particles in correlated many-body quantum systems confined to two dimensions. They have been conjectured [2, 43, 55] to be relevant for the fractional quantum Hall effect (see [25, 31, 29] for reviews and [5, 51] for recent experimental developments). They can also be simulated in certain cold atoms systems with synthetic gauge fields [18, 11, 63, 65, 66, 13, 67, 68].

The statistics of anyonic particles is labeled by a single parameter $\alpha \in [0, 2[$ where the cases $\alpha = 0, 1$ correspond to ordinary bosons and fermions, respectively. Since the many-anyon problem is not exactly soluble (even in the absence of interactions other than the statistical ones) except for the latter special values, it makes sense to study limiting parameter regimes where reduced models are obtained as effective descriptions.

The “almost bosonic” limit $\alpha \rightarrow 0$ is studied in [24, 42], in the joint limit of the particle number $N \rightarrow \infty, \alpha \sim N^{-1}$, which turns out to be the relevant one for the non-trivial statistics to have an effect at leading order [14, 13]. Here we study an “almost fermionic limit” $\alpha \rightarrow 1$ in dependence with the particle number. The relevant scaling in dependence of N leads to a problem resembling the usual mean-field/semi-classical limit for interacting fermions [37, 21, 6, 4, 19, 23].

Our starting point is a gas of anyonic particles evolving in \mathbb{R}^2 subjected to a magnetic field

$$\mathbf{B}_e = \text{curl}\mathbf{A}_e$$

and a trapping external potential

$$V(x) \underset{|x| \rightarrow \infty}{\rightarrow} \infty.$$

To model anyonic behavior we introduce the statistical gauge vector potential felt by particle j due to the influence of all the other particles

$$\mathbf{A}(x_j) = \sum_{j \neq k} \frac{(x_j - x_k)^\perp}{|x_j - x_k|^2}. \quad (1.1)$$

The associated magnetic field is

$$\text{curl}\mathbf{A}(x_j) = 2\pi \sum_{k \neq j} \delta(x_j - x_k). \quad (1.2)$$

The statistics parameter is denoted by α , corresponding to a statistical phase $e^{i\pi(1+\alpha)}$ under a continuous simple interchange of two particles. In this so-called “magnetic gauge picture”, 2D anyons are thus described as fermions, each of them carrying an Aharonov-Bohm magnetic flux of strength α . Instead of looking at the ground state of a non-interacting Hamiltonian acting on anyonic wave-functions we thus study the following Hamiltonian

$$H_N = \sum_{j=1}^N \left((-i\hbar\nabla_j + \mathbf{A}_e(x_j) + \alpha\mathbf{A}(x_j))^2 + V(x_j) \right) \text{ acting on } L^2_{\text{asym}}(\mathbb{R}^{2N}). \quad (1.3)$$

Note that in the above we view the anyons statistics parameter α counting from the fermionic representation of anyons. The limit $\alpha \rightarrow 0$ for (1.3) is then formally equivalent to $\alpha \rightarrow 1$ in the bosonic representation alluded to above [24, 42].

We are interested in a joint limit $N \rightarrow \infty$, $\alpha \rightarrow 0$ in which the problem would simplify but still display signatures of anyonic behavior, i.e. a non-trivial dependence on α . Since we perturb around fermions, the Pauli principle implies that the kinetic energy grows faster than N . This behavior is encoded in the Lieb-Thirring inequality [41, 40] which, for any N -particle normalized antisymmetric wave function Ψ_N with support in a bounded domain Ω^N , gives

$$\sum_{j=1}^N \int_{\Omega^N} |\nabla_j \Psi|^2 \geq C |\Omega|^{-\frac{2}{d}} N^{1+\frac{2}{d}} \quad (1.4)$$

where d is the dimension of the physical space. Because of this rapid growth of the kinetic energy, the natural mean-field limit [6, 7, 21] for fermions involves a scaling $\hbar \sim N^{-1/d}$. Here with $d = 2$ we get a kinetic energy proportional to N^2 and thus, in expectation value $-i\nabla_j \sim N^{1/2}$. Investigating the N -dependence of each term of (1.3) we thus expect for any $\Psi_N \in L^2(\mathbb{R}^{2N})$

$$\langle \Psi_N, H_N \Psi_N \rangle \approx C_1 N \left(\hbar N^{1/2} + \alpha N \right)^2 + C_2 N$$

where the terms in parenthesis accounts for kinetic energy, including the influence of (1.1), and the second term is the energy in the external potential V . To obtain a non trivial limit when $\alpha \rightarrow 0$, a natural choice is to take

$$\hbar = \frac{1}{\sqrt{N}} \quad \text{and} \quad \alpha = \frac{\beta}{N} \quad (1.5)$$

$\beta \in \mathbb{R}$ a constant allowing us to keep a track of the anyonic behavior. We shall consider the limit

$$\lim_{N \rightarrow \infty} \frac{E(N)}{N}$$

with

$$E(N) := \inf \left\{ \langle \Psi_N, H_N \Psi_N \rangle, \Psi_N \in L^2_{\text{asym}}(\mathbb{R}^{2N}), \int_{\mathbb{R}^{2N}} |\Psi_N|^2 = 1 \right\}. \quad (1.6)$$

Such an approach combines two limits (see also Remark 1.2 below), the semi-classical $\hbar \rightarrow 0$ and the quasi-fermionic $\alpha \rightarrow 0$.

1.2. Extended anyons: energy convergence. The Hamiltonian (1.3) is actually too singular. Consequently we introduce a length R over which the magnetic flux attached to our particles is smeared. In our approach, $R \rightarrow 0$ will make us recover the point-like anyons point of view. This ‘‘extended anyons’’ model is discussed in [48, 62, 10]. Anyons arising as quasi-particles in condensed-matter systems are not point-like objects. The radius R then corresponds to their size of the quasi-particle, i.e. a length-scale below which the quasi-particle description breaks down.

We introduce the 2D Coulomb potential generated by a unit charge smeared over the disc of radius R

$$w_R(x) = (\log |\cdot| * \chi_R)(x), \quad \text{with the convention } w_0 = \log |\cdot| \quad (1.7)$$

and $\chi_R(x)$ a positive smooth function of unit mass

$$\chi_R = \frac{1}{R^2} \chi\left(\frac{\cdot}{R}\right) \text{ and } \int_{\mathbb{R}^2} \chi = 1, \quad \chi(x) = \begin{cases} 1/\pi^2 & |x| \leq 1 \\ 0 & |x| \geq 2. \end{cases} \quad (1.8)$$

We take χ to be smooth, positive and decreasing between 1 and 2. We will recover the magnetic field (1.2) (in a distributional sense) in the limit $R \rightarrow 0$ by defining

$$\mathbf{A}^R(x_j) = \sum_{k \neq j} \nabla^\perp w_R(x_j - x_k). \quad (1.9)$$

We henceforth work with the regularized Hamiltonian of N anyons of radius R

$$\boxed{H_N^R := \sum_{j=1}^N \left((p_j^{\mathbf{A}} + \alpha \mathbf{A}^R(x_j))^2 + V(x_j) \right)} \quad (1.10)$$

denoting

$$p_j^{\mathbf{A}} = -i\hbar \nabla_j + \mathbf{A}_e(x_j). \quad (1.11)$$

It is essentially self-adjoint on $L_{\text{asym}}^2(\mathbb{R}^{2N})$ (see [52, Theorem X.17] and [3]). The bottom of its spectrum then exists for any fixed $R > 0$, we denote it

$$E^R(N) = \inf \sigma(H_N^R). \quad (1.12)$$

Under our assumptions H_N^R will actually have compact resolvent, so the above is an eigenvalue. However, this operator does not have a unique limit as $R \rightarrow 0$ and the Hamiltonian at $R = 0$ is *not* essentially self-adjoint, see for instance [46, 15, 16, 1, 9]. Nevertheless, in the joint limit $R \rightarrow 0$ and $N \rightarrow \infty$ we recover a unique well-defined (non-linear) model. It is convenient to expand (1.10) and treat summands separately

$$\begin{aligned} H_N^R &= \sum_{j=1}^N \left((p_j^{\mathbf{A}})^2 + V(x_j) \right) \text{ “Kinetic and potential terms”} \\ &+ \alpha \sum_{j \neq k} \left(p_j^{\mathbf{A}} \cdot \nabla^\perp w_R(x_j - x_k) + \nabla^\perp w_R(x_j - x_k) \cdot p_j^{\mathbf{A}} \right) \text{ “Mixed two-body term”} \\ &+ \alpha^2 \sum_{j \neq k \neq l} \nabla^\perp w_R(x_j - x_k) \cdot \nabla^\perp w_R(x_j - x_l) \text{ “Three-body term”} \\ &+ \alpha^2 \sum_{j \neq k} \left| \nabla^\perp w_R(x_j - x_k) \right|^2 \text{ “Singular two-body term”}. \end{aligned} \quad (1.13)$$

The fourth term of the above, being N times smaller than the others in the regime (1.5), will easily be estimated. Further heuristics will only involve the mixed two-body term and the three-body term. These being of the same order, we expect (as in other types of mean-field limits, see e.g. [57] and references therein) that, for large N , particles behave independently, and thus

$$\sum_{j \neq k} \nabla^\perp w_R(x_j - x_k) \approx \int_{\mathbb{R}^2} \nabla^\perp w_R(x_j - y) \rho(y) dy = \nabla^\perp w_R * \rho(x_j)$$

where ρ is the density of particles. Therefore, it is convenient to define, given a one-body density ρ normalized in $L^1(\mathbb{R}^2)$

$$\mathbf{A}[\rho] = \nabla^\perp w_0 * \rho \quad \text{and} \quad \mathbf{A}^R[\rho] = \nabla^\perp w_R * \rho.$$

Natural mean-field functionals associated with (1.13) are then

$$\mathcal{E}_R^{\text{af}}[\gamma] = \frac{1}{N} \text{Tr} \left[(p_1^\mathbf{A} + \alpha \mathbf{A}^R[\rho_\gamma])^2 \gamma \right] + \frac{1}{N} \int_{\mathbb{R}^2} V(x_1) \rho_\gamma(x_1) dx_1 \quad (1.14)$$

$$\mathcal{E}^{\text{af}}[\gamma] = \frac{1}{N} \text{Tr} \left[(p_1^\mathbf{A} + \alpha \mathbf{A}[\rho_\gamma])^2 \gamma \right] + \frac{1}{N} \int_{\mathbb{R}^2} V(x_1) \rho_\gamma(x_1) dx_1 \quad (1.15)$$

where the argument is a trace-class operator

$$\gamma \in \mathfrak{S}^1(L^2(\mathbb{R}^2)), 0 \leq \gamma \leq 1, \text{Tr } \gamma = 1. \quad (1.16)$$

Similar Hartree-like functionals were obtained in the almost-bosonic limit [24, 42]. Here we simplify things one step further by considering a semi-classical analogue

$$\mathcal{E}_{\text{Vla}}[m] = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} |p + \mathbf{A}_e(x) + \beta \mathbf{A}[\rho](x)|^2 m(x, p) dx dp + \int_{\mathbb{R}^2} V(x) \rho_m(x) dx \quad (1.17)$$

with $m(x, p)$ a positive measure on the phase-space \mathbb{R}^4 of positions/momenta and

$$\rho_m(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} m(x, p) dp. \quad (1.18)$$

Our convention is that m satisfies the mass constraint

$$\iint_{\mathbb{R}^2 \times \mathbb{R}^2} m(x, p) dx dp = (2\pi)^2 \quad (1.19)$$

and the semi-classical Pauli principle

$$\boxed{0 \leq m(x, p) \leq 1 \text{ a.e.}} \quad (1.20)$$

The latter is imposed at the classical level by the requirement that a classical state (x, p) cannot be occupied by more than one fermion. By the Bathtub principle [35, Theorem 1.14] we can perform the minimization in p explicitly (this parallels the considerations in [21]). We find minimizers of the form

$$m_\rho(x, p) = \mathbf{1} \left(|p + \mathbf{A}_e(x) + \beta \mathbf{A}[\rho](x)|^2 \leq 4\pi\rho(x) \right)$$

where ρ minimizes the Thomas-Fermi energy

$$\mathcal{E}_{\text{TF}}[\rho] = \mathcal{E}_{\text{Vla}}[m_\rho] = 2\pi \int_{\mathbb{R}^2} \rho^2(x) dx + \int_{\mathbb{R}^2} V(x) \rho(x) dx. \quad (1.21)$$

We define the minimum of the Thomas-Fermi functional

$$e_{\text{TF}} = \inf \left\{ \mathcal{E}_{\text{TF}}[\rho] : 0 \leq \rho \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2), \int_{\mathbb{R}^2} \rho = 1 \right\}. \quad (1.22)$$

Our first purpose is to obtain the above as the limit of the true many-body energy. We shall prove this under the following assumptions

Assumption 1.1 (External Potentials).

The external potentials entering (1.10) satisfy

- $\mathbf{A}_e \in L^2(\mathbb{R}^2) \cap W^{2,\infty}(\mathbb{R}^2)$

- $|\mathbf{A}_e|^2 \in W^{1,\infty}(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$

and, for some $s > 1$ and $c, C, c', C', c'', C'' > 0$

- $V(x) \geq c|x|^s - C$
- $|\nabla V(x)| \leq c'|x|^{s-1} + C'$
- $|\Delta V(x)| \leq c''|x|^{s-2} + C''$

Theorem 1.1 (Convergence of the ground state energy).

We consider N extended anyons of radius $R = N^{-\eta}$ in an external potential V . Under Assumption (1.1) and setting

$$0 < \eta < \frac{1}{4}$$

we have, in the parameter regime (1.5)

$$\boxed{\lim_{N \rightarrow \infty} \frac{E^R(N)}{N} = e_{\text{TF}}}$$

where the many-anyon ground state energy $E^R(N)$ and the Thomas-Fermi energy are defined in (1.12) and (1.22), respectively.

The study of the regime (1.5) resembles the usual Thomas-Fermi limit for a large fermionic system [37, 21, 61, 34], but with important new aspects:

- The effective interaction comprises a three-body term, and a two-body term (see Equation (1.13)) which mixes position and momentum variables.
- The limit problem (1.17) comprises an effective self-consistent magnetic field $\mathbf{A}[\rho]$.
- One should deal with the limit $R \rightarrow 0$, $\alpha \rightarrow 0$ and $\hbar \rightarrow 0$ at the same time as $N \rightarrow \infty$.

Regarding the last point, we make the

Remark 1.2 (Scaling of parameters).

Extracting a multiplicative factor N from the energy, the parameter regime (1.5) we consider is equivalent to

$$\hbar = 1 \quad \text{and} \quad \alpha = \frac{\beta}{\sqrt{N}} \tag{1.23}$$

provided we replace the external potential $V(x)$ by $NV(x)$ and a similar replacement for the external vector potential \mathbf{A}_e . The scaling (1.23) better highlights the quasi-fermionic character of the limit we take. We however find it preferable technically to work with (1.5), where the semi-classical aspect is more apparent.

Regarding the scaling of R , it would be highly desirable (and challenging) to be able to deal with a dilute regime $\eta > 1/2$ where the radius of the magnetic flux is much smaller than the typical inter-particle distance. We could probably relax our constraint $\eta < 1/4$ to $\eta < 1/2$ if we assume a Lieb-Thirring inequality for extended anyons of the form

$$\left\langle \Psi_N, \sum_{j=1}^N (p_j^{\mathbf{A}} + \alpha \mathbf{A}^R(x_j))^2 \Psi_N \right\rangle \geq C_{\alpha,R} \int_{\mathbb{R}^2} \rho_{\Psi_N}(x)^2 dx$$

for any fermionic wave-function Ψ_N with one-particle density ρ_{Ψ_N} . In the above we would need a $C_{\alpha,R}$ uniformly bounded from below when $N \rightarrow \infty$, $\alpha \rightarrow 0$ and $R \gg N^{-1/2}$. Such

an inequality is made very plausible by the results of [30] on the homogeneous extended anyon gas and the known Lieb-Thirring inequalities for point-like anyons [44, 45, 46, 47]. We plan to return to this matter in the future. \diamond

The limit ground state energy e_{TF} does not reveal any anyonic behavior, i.e. it does not depend on α . This is however deceptive, for the behavior of the system's ground state *does* depend on α and shows the influence of the non-trivial statistics. This will become apparent when we state the second part of our result, the convergence of ground states. It is expressed in terms of Husimi functions, constructed using coherent states. We discuss this next.

1.3. Squeezed coherent states. Let¹

$$f(x) := \frac{1}{\sqrt{\pi}} e^{-\frac{|x|^2}{2}}. \quad (1.24)$$

For every (x, p) in the phase space $\mathbb{R}^2 \times \mathbb{R}^2$ we define the squeezed coherent state

$$F_{x,p}(y) := \frac{1}{\sqrt{\hbar_x}} f\left(\frac{y-x}{\sqrt{\hbar_x}}\right) e^{i\frac{p \cdot y}{\hbar}}. \quad (1.25)$$

It has the property of being localized on a scale $\sqrt{\hbar_x}$ in space and $\sqrt{\hbar_p}$ in momentum, these two scales being related by

$$\hbar = \sqrt{\hbar_x} \sqrt{\hbar_p}. \quad (1.26)$$

The usual coherent state [21, Section 2.1] is the particular case $\sqrt{\hbar_x} = \sqrt{\hbar_p} = \sqrt{\hbar}$. Any such state saturates Heisenberg's uncertainty principle, and is thus as classical as a quantum state can be. Indeed, for the two observables x_1 and $p_1 = -i\partial_{x_1}$ evaluated in this state we get

$$\Delta_{x_1} \Delta_{p_1} = \frac{\hbar}{2} \text{ where } \Delta_a = \langle F_{x,p}, a^2 F_{x,p} \rangle - \langle F_{x,p}, a F_{x,p} \rangle^2.$$

In our proofs we will take $\hbar_x \ll \hbar_p \rightarrow 0$, which is convenient because our Hamiltonian is singular in x but not in p .

We define the \hbar -Fourier transform

$$\mathcal{F}_\hbar[\psi](p) = \frac{1}{2\pi\hbar} \int_{\mathbb{R}^2} \psi(x) e^{-i\frac{p \cdot x}{\hbar}} dx \quad (1.27)$$

and denote

$$F_{\hbar_x}(x) = F_{0,0}(x) = \frac{1}{\sqrt{\hbar_x} \sqrt{\pi}} e^{-\frac{|x|^2}{2\hbar_x}}. \quad (1.28)$$

Its Fourier transform (see the calculation in Appendix C)

$$G_{\hbar_p}(p) = \mathcal{F}_\hbar[F_{\hbar_x}](p) = \frac{1}{\sqrt{\hbar_p} \sqrt{\pi}} e^{-\frac{|p|^2}{2\hbar_p}} \quad (1.29)$$

makes apparent the localization in momentum space we just claimed.

To $F_{x,p}$ we associate the orthogonal projector $P_{x,p}$

$$P_{x,p} := |F_{x,p}\rangle \langle F_{x,p}|. \quad (1.30)$$

Then we have the well-known

¹We make the standard choice that f is a gaussian but any radial L^2 function could be used instead.

Lemma 1.3 (Resolution of the identity on $L^2(\mathbb{R}^2)$).

We have

$$\frac{1}{(2\pi\hbar)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} P_{x,p} \, dx dp = \mathbf{1}_{L^2(\mathbb{R}^2)}. \quad (1.31)$$

The proof is recalled in Appendix C, see also [35, Theorem 12.8].

1.4. Convergence of Husimi functions. The Husimi function $m_{\Psi_N}^{(k)}(x_1, p_1, \dots, x_k, p_k)$ [28, 12, 60] describes how many particles are distributed in the k semi-classical boxes of size $\sqrt{\hbar_x} \times \sqrt{\hbar_p}$ centered at its arguments $(x_1, p_1), \dots, (x_k, p_k)$. A wave function Ψ_N being given we define

$$m_{\Psi_N}^{(k)}(x_1, p_1, \dots, x_k, p_k) = \frac{N!}{(N-k)!} \left\langle \Psi_N, \bigotimes_{j=1}^k P_{x_j, p_j} \otimes \mathbf{1}_{N-k} \Psi_N \right\rangle. \quad (1.32)$$

Alternatively,

$$m_{\Psi_N}^{(k)}(x_1, p_1, \dots, x_k, p_k) = \frac{N!}{(N-k)!} \int_{\mathbb{R}^{2(N-k)}} \left| \left\langle F_{x_1, p_1} \otimes \dots \otimes F_{x_k, p_k}, \Psi_N(\cdot, z) \right\rangle_{L^2(\mathbb{R}^{2k})} \right|^2 dz$$

or, in terms of fermionic annihilation and creation operators (see (2.2) below)

$$m_{\Psi_N}^{(k)}(x_1, p_1, \dots, x_k, p_k) = \langle \Psi_N, a^*(F_{x_1, p_1}) \dots a^*(F_{x_k, p_k}) a(F_{x_k, p_k}) \dots a(F_{x_1, p_1}) \Psi_N \rangle,$$

where a and a^* satisfy the canonical anticommutation relations [59, 26]

$$\begin{aligned} a^*(f)a(g) + a(g)a^*(f) &= \langle g, f \rangle \\ a^*(f)a^*(g) + a^*(g)a^*(f) &= 0. \end{aligned}$$

We will also use the k -particle density matrices of Ψ_N , i.e. the operator $\gamma_{\Psi_N}^{(k)}$ with integral kernel

$$\gamma_{\Psi_N}^{(k)}(x_1, \dots, x_k; x'_1, \dots, x'_k) = \binom{N}{k} \int_{\mathbb{R}^{2(N-k)}} \Psi_N(x_1, \dots, x_N) \overline{\Psi_N(x'_1, \dots, x'_k, x_{k+1}, \dots, x_N)} dx_{k+1} \dots dx_N. \quad (1.33)$$

We can also express the Husimi function in terms of the k -particle density matrices of above

$$m_{\Psi_N}^{(k)}(x_1, p_1, \dots, x_k, p_k) = k! \operatorname{Tr} \left[\bigotimes_{j=1}^k P_{x_j, p_j} \gamma_{\Psi_N}^{(k)} \right]. \quad (1.34)$$

These objects are furthermore related to the density marginals in space and in momentum, respectively defined as

$$\rho_{\Psi_N}^{(k)}(x_1, \dots, x_k) = \binom{N}{k} \int_{\mathbb{R}^{2(N-k)}} |\Psi_N(x_1, \dots, x_N)|^2 dx_{k+1} \dots dx_N \quad (1.35)$$

$$t_{\Psi_N}^{(k)}(p_1, \dots, p_k) = \binom{N}{k} \int_{\mathbb{R}^{2(N-k)}} |\mathcal{F}_\hbar[\Psi_N](p_1, \dots, p_N)|^2 dp_{k+1} \dots dp_N. \quad (1.36)$$

We can now express the convergence of states corresponding to Theorem (1.1) using the above objects. The minimizer of the Vlasov energy (1.17) is

$$m_{\rho^{\text{TF}}}(p, x) = \mathbf{1}(|p + \mathbf{A}_e(x) + \beta \mathbf{A}[\rho^{\text{TF}}](x)|^2 \leq 4\pi \rho^{\text{TF}}(x)). \quad (1.37)$$

Its marginals in space and momentum are respectively given by

$$\rho^{\text{TF}}(x) = \frac{1}{4\pi} (\lambda^{\text{TF}} - V(x))_+ \quad (1.38)$$

where λ^{TF} is a Lagrange multiplier ensuring normalization, and

$$t^{\text{TF}}(p) = \int_{\mathbb{R}^2} \mathbb{1}_{\{|p + \mathbf{A}_e(x) + \beta \mathbf{A}[\rho^{\text{TF}}](x)|^2 \leq 4\pi \rho^{\text{TF}}(x)\}} dx. \quad (1.39)$$

We will prove the

Theorem 1.4 (Convergence of states).

Let $\{\Psi_N\} \subset L^2_{\text{asym}}(\mathbb{R}^{2N}) = \bigotimes_{\text{asym}}^N L^2(\mathbb{R}^2)$ be any L^2 -normalized sequence such that

$$\langle \Psi_N, H_N \Psi_N \rangle = E^R(N) + o(N)$$

in the limit $\hbar = 1/\sqrt{N}$, $\alpha = \beta/N$, $N \rightarrow \infty$. For any choice of $\hbar_x, \hbar_p \rightarrow 0$, related by (1.26), and under Assumption 1.1, the Husimi functions of Ψ_N converge to those of the Vlasov minimizer

$$m_{\Psi_N}^{(k)}(x_1, p_1, \dots, x_k, p_k) \rightarrow \prod_{j=1}^k m_{\rho^{\text{TF}}}(p_j, x_j)$$

weakly as measure for all $k \geq 1$, i.e.

$$\int_{\mathbb{R}^{4k}} m_{\Psi_N}^{(k)} \phi_k \rightarrow \int_{\mathbb{R}^{4k}} (m_{\rho^{\text{TF}}})^{\otimes k} \phi_k$$

for any bounded continuous function $\phi_k \in C_b(\mathbb{R}^{4k})$.

Consequently we also have the convergence of the k -particles marginals in position and momentum

$$\begin{aligned} \binom{N}{k}^{-1} \rho_{\Psi_N}^{(k)}(x_1, \dots, x_k) &\rightarrow \prod_{j=1}^k \rho^{\text{TF}}(x_j) \\ \binom{N}{k}^{-1} t_{\Psi_N}^{(k)}(p_1, \dots, p_k) &\rightarrow \prod_{j=1}^k t^{\text{TF}}(x_j) \end{aligned}$$

weakly as measures.

The convergence of Wigner functions follows from the Husimi functions convergence, see [21, Theorem 2.1]. The result above does not depend on the choice of function f used to construct the coherent states [21, Lemma 2.8]. Likewise, the result is the same with any choice of scaling of $\hbar_x, \hbar_p \rightarrow 0$ as long as (1.26) is satisfied. In our proof of energy convergence it will however be convenient to make a particular choice, and in particular use squeezed coherent states, $\hbar_x \ll \hbar_p$.

Recall that the semi-classical Thomas-Fermi energy does not depend on α , but the Vlasov minimizer does. Anyonic features are retained e.g. in the limiting object's momentum distribution (1.39).

1.5. Outline and Sketch of the proof. Our general strategy is inspired by works on mean-field limits for interacting fermions [37, 38, 39, 22, 32, 61], in particular by the method of [21]. Several improvements are required to handle the singularity of the anyonic Hamiltonian that emerges in the limit $R \rightarrow 0$. In particular

- we replace the use of the Hewitt-Savage theorem by that of its' quantitative version, the Diaconis-Freedman theorem.
- we implement the Pauli principle quantitatively at the semi-classical level.

To obtain an energy upper bound (Section 2) we use a Slater determinant as trial state, leading by Wick's theorem to a Hartree-Fock-like energy. We then discard exchange terms to get to the simpler Hartree energy (1.14). Finally, a suitable choice of the trial state's one-body density matrix γ allows to take the semi-classical limit and establish an upper bound in terms of the Vlasov energy(1.17).

For the corresponding lower bound we first express (Section 3) the energy semi-classically, in terms of the Husimi functions. The remainder terms are shown to be negligible when $N \rightarrow \infty$. The use of squeezed coherent states with $\hbar_x \ll \hbar_p$ takes into account the singularities of the Hamiltonian, more severe in position than in momentum space.

Section 4 contains the main novelties compared to [21]. There we tackle the mean-field limit of the semi-classical functional expressed in terms of Husimi functions. We use the Diaconis-Freedman theorem [17] to express the latter as statistical superpositions of factorized measures, with a quantitatively controlled error. We then estimate quantitatively the probability for the Diaconis-Freedman measure to violate the Pauli-principle (1.20). This leads to the sought-after energy lower bound and completes the proof of Theorem 1.1. The convergence of states Theorem 1.4 follows as a corollary.

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2. ENERGY UPPER BOUND

In this section we derive an upper bound to the ground state energy (1.12). We first introduce a Hartree-Fock trial state and apply Wick's theorem to calculate its energy. We next discard lower contributions (exchange terms) to obtain a regularized ($R > 0$) Hartree energy in Lemma 2.4 and use Lemma 2.5 to send the regularization to 0. We conclude by using a particular sequence of semi-classical states, which will make the Hartree energy converge to the Thomas-Fermi one. All in all this will prove the

Proposition 2.1 (Energy upper bound).

Under Assumption 1.1, setting $R = N^{-\eta}$ with $\eta < 1/2$, we have

$$\limsup_{N \rightarrow \infty} \frac{E^R(N)}{N} \leq e_{\text{TF}}. \quad (2.1)$$

2.1. Hartree-Fock trial state. We recall the definition of creation and annihilation operators of a one-body state $f \in L^2(\mathbb{R}^2)$ (for more details about second quantization see

[59, 26])

$$\begin{aligned} a(f) \sum_{\sigma \in \Sigma_N} (-1)^{\text{sgn}(\sigma)} f_{\sigma(1)} \otimes \dots \otimes f_{\sigma(N)} &= \sqrt{N} \sum_{\sigma \in \Sigma_N} (-1)^{\text{sgn}(\sigma)} \langle f, f_{\sigma(1)} \rangle f_{\sigma(2)} \otimes \dots \otimes f_{\sigma(N)} \\ a^\dagger(f) \sum_{\sigma \in \Sigma_N} (-1)^{\text{sgn}(\sigma)} f_{\sigma(1)} \otimes \dots \otimes f_{\sigma(N)} &= \frac{1}{\sqrt{N+1}} \sum_{\sigma \in \Sigma_{N+1}} (-1)^{\text{sgn}(\sigma)} f_{\sigma(1)} \otimes \dots \otimes f_{\sigma(N+1)}. \end{aligned} \quad (2.2)$$

Sums are over the permutation group and $\text{sgn}(\sigma)$ is the signature of a permutation. Let now $\{\psi_j\}_{j=1\dots N} \in L^2(\mathbb{R}^2)$ be an orthonormal family. We use it to construct a Slater determinant Ψ_N^{SL} (also called Hartree-Fock state)

$$\Psi_N^{\text{SL}}(x_1, \dots, x_N) = \frac{1}{\sqrt{N!}} \det(\psi_i(x_j)) = \sum_{\sigma \in \Sigma_N} (-1)^{\text{sgn}(\sigma)} \prod_{j=1}^N \psi_{\sigma(j)}(x_j). \quad (2.3)$$

Denoting $a_j^\dagger = a^\dagger(\psi_j)$ and $a_j = a(\psi_j)$ we have:

$$\langle a_j^\dagger a_k \rangle_{\text{SL}} = \langle \Psi_N^{\text{SL}}, a_j^\dagger a_k \Psi_N^{\text{SL}} \rangle = \delta_{jk}.$$

Hence the associated 1-particle reduced density matrix (1.33) and density are

$$\gamma_{\Psi_N^{\text{SL}}}^{(1)} = N \text{Tr}_{2 \rightarrow N} [|\Psi_N^{\text{SL}}\rangle\langle\Psi_N^{\text{SL}}|] = \sum_j |\psi_j\rangle\langle\psi_j|, \quad \rho_\gamma = \sum_j |\psi_j|^2 \quad (2.4)$$

with $\text{Tr}_{2 \rightarrow N}$ the partial trace. Replacing the last formula in (1.34) gives us the 1-particle reduced Husimi measure

$$m_{\Psi_N^{\text{SL}}}^{(1)}(x, p) = \sum_{j=1}^N \left| \langle \psi_j, F_{x,p} \rangle \right|^2.$$

We will use Wick's theorem, see [50, Corollary IV.6] or [59].

Theorem 2.2 (Corollary of Wick's theorem).

Let $a_1^\sharp, \dots, a_{2n}^\sharp$ be creation or annihilation operators. We have that

$$\langle a_1^\sharp \dots a_{2n}^\sharp \rangle_{\text{SL}} = \sum_{\sigma} (-1)^{\text{sgn}(\sigma)} \prod_{j=1}^n \langle a_{\sigma(2j-1)}^\sharp a_{\sigma(2j)}^\sharp \rangle_{\text{SL}}$$

where the sum is over all pairings, i.e. permutations of the $2n$ indices such that $\sigma(2j-1) < \min\{\sigma(2j), \sigma(2j+1)\}$ for all j .

We evaluate

$$\langle \Psi_N^{\text{SL}}, H_N^R \Psi_N^{\text{SL}} \rangle = \langle H_N^R \rangle_{\text{SL}}$$

term by term as in (1.13). Until the end of this section $\langle \cdot \rangle$ means $\langle \cdot \rangle_{\text{SL}}$. We will use the following notation (recall (1.11))

$$W_j = (p_j^{\mathbf{A}})^2 + V(x_j) \quad (2.5)$$

$$W_{jk} = p_j^{\mathbf{A}} \cdot \nabla^\perp w_R(x_j - x_k) + \nabla^\perp w_R(x_j - x_k) \cdot p_j^{\mathbf{A}} \quad (2.6)$$

$$W_{jkl} = \nabla^\perp w_R(x_j - x_k) \cdot \nabla^\perp w_R(x_j - x_\ell). \quad (2.7)$$

Recalling the definition of the Hartree energy functional

$$\mathcal{E}_R^{\text{af}}[\gamma] = \frac{1}{N} \text{Tr} \left[(p_1^{\mathbf{A}} + \alpha \mathbf{A}^R[\rho_\gamma])^2 \gamma \right] + \frac{1}{N} \int_{\mathbb{R}^2} V(x) \rho_\gamma(x) dx$$

we have the following

Lemma 2.3 (Hartree-Fock energy).

With Ψ^{SL} as above and γ the associated one-particle density matrix (2.4) we have

$$\begin{aligned} \langle H_N^R \rangle_{\text{SL}} &= N \mathcal{E}_R^{\text{af}}[\gamma] - \alpha \text{Tr} (W_{12} U_{12} \gamma \otimes \gamma) \\ &+ \alpha^2 \int_{\mathbb{R}^4} \left| \nabla^\perp w_R(x_1 - x_2) \right|^2 \left(\rho_\gamma(x_1) \rho_\gamma(x_2) - |\gamma(x_1, x_2)|^2 \right) dx_1 dx_2 \\ &- \alpha^2 \int_{\mathbb{R}^6} W_{123} \left[\rho_\gamma(x_1) |\gamma(x_2, x_3)|^2 + \rho_\gamma(x_2) |\gamma(x_1, x_3)|^2 + \rho_\gamma(x_3) |\gamma(x_1, x_2)|^2 \right] dx_1 dx_2 dx_3 \\ &+ \alpha^2 \int_{\mathbb{R}^6} W_{123} \left[2\Re \left(\gamma(x_1, x_2) \gamma(x_2, x_3) \gamma(x_3, x_1) \right) \right] dx_1 dx_2 dx_3. \end{aligned} \quad (2.8)$$

In the first line we have denoted by U_{12} the exchange operator

$$(U_{12}\psi)(x_1, x_2) = \psi(x_2, x_1). \quad (2.9)$$

Proof. By definition

$$\left\langle \sum_{j=1}^N (p_j^{\mathbf{A}})^2 + V(x_j) \right\rangle = \text{Tr} \left[((p^{\mathbf{A}})^2 + V) \gamma \right].$$

For the two- and three-body terms we express higher density matrices of Ψ^{SL} using Wick's theorem. To this end, let $(\psi_j)_j \in L^2(\mathbb{R}^2)$ be an orthonormal basis, and a_j, a_j^\dagger the associated annihilation and creation operators.

Two-body terms. We have

$$\left\langle a_\alpha^\dagger a_\beta^\dagger a_\varepsilon a_\gamma \right\rangle = \delta_{\alpha\gamma} \delta_{\beta\varepsilon} - \delta_{\alpha\varepsilon} \delta_{\beta\gamma}$$

where δ_{ij} is the Kronecker delta. It follows that

$$\gamma_{\Psi^{\text{SL}}}^{(2)} = \gamma \otimes \gamma - U_{12} \gamma \otimes \gamma \quad (2.10)$$

with the exchange operator defined as in (2.9). Using [59, Lemma 7.12] we have

$$\sum_{j \neq k} W_{jk} = \sum_{\alpha, \beta, \gamma, \varepsilon} \langle \psi_\alpha \otimes \psi_\beta, W_{12} \psi_\gamma \otimes \psi_\varepsilon \rangle a_\alpha^\dagger a_\beta^\dagger a_\varepsilon a_\gamma$$

and thus

$$\left\langle \sum_{j \neq k} W_{j,k} \right\rangle = \text{Tr} \left[(p^{\mathbf{A}} \cdot \mathbf{A}^R[\rho_\gamma] + \mathbf{A}^R[\rho_\gamma] \cdot p^{\mathbf{A}}) \gamma \right] - \text{Tr} (W_{12} U_{12} \gamma \otimes \gamma). \quad (2.11)$$

Similarly we obtain

$$\left\langle \sum_{j \neq k} \left| \nabla^\perp w_R(x_j - x_k) \right|^2 \right\rangle = \int_{\mathbb{R}^4} \left| \nabla^\perp w_R(x_1 - x_2) \right|^2 \left(\rho_\gamma(x_1) \rho_\gamma(x_2) - |\gamma(x_1, x_2)|^2 \right) dx_1 dx_2. \quad (2.12)$$

Three-body term. Using [59, Lemma 7.12] again we write

$$\sum_{j \neq k \neq l} W_{jkl} = \sum_{\alpha, \beta, \gamma, \delta, \varepsilon, \zeta} \langle \psi_\alpha \otimes \psi_\beta \otimes \psi_\gamma, W_{123} \psi_\varepsilon \otimes \psi_\zeta \otimes \psi_\eta \rangle a_\alpha^\dagger a_\beta^\dagger a_\gamma^\dagger a_\eta a_\zeta a_\varepsilon.$$

Applying Wick's theorem (2.2) we obtain

$$\langle a_\alpha^\dagger a_\beta^\dagger a_\gamma^\dagger a_\eta a_\zeta a_\varepsilon \rangle = \delta_{\alpha\varepsilon} \delta_{\beta\zeta} \delta_{\gamma\eta} + \delta_{\alpha\zeta} \delta_{\beta\eta} \delta_{\gamma\varepsilon} + \delta_{\alpha\eta} \delta_{\beta\varepsilon} \delta_{\gamma\zeta} - \delta_{\alpha\varepsilon} \delta_{\beta\eta} \delta_{\gamma\zeta} - \delta_{\alpha\zeta} \delta_{\beta\varepsilon} \delta_{\gamma\eta} - \delta_{\alpha\eta} \delta_{\beta\zeta} \delta_{\gamma\varepsilon}$$

and we deduce the expression

$$\gamma_{\Psi^{\text{SL}}}^{(3)} = (\mathbb{1} + U_{12}U_{23} + U_{23}U_{12} - U_{12} - U_{13} - U_{23}) \gamma \otimes \gamma \otimes \gamma \quad (2.13)$$

for the 3-body density matrix of the Slater determinant, where the exchange operators U_{ij} are natural extensions of (2.9) to the three-particles space. Gathering the above expressions yields

$$\begin{aligned} \left\langle \sum_{j \neq k \neq l} W_{jkl} \right\rangle &= \int_{\mathbb{R}^6} W_{123} [\rho_\gamma(x_1) \rho_\gamma(x_2) \rho_\gamma(x_3)] dx_1 dx_2 dx_3 \\ &+ \int_{\mathbb{R}^6} W_{123} [2\Re(\gamma(x_1, x_2) \gamma(x_2, x_3) \gamma(x_3, x_1))] dx_1 dx_2 dx_3 \end{aligned} \quad (2.14)$$

$$- \int_{\mathbb{R}^6} W_{123} [\rho_\gamma(x_1) |\gamma(x_2, x_3)|^2 + \rho_\gamma(x_2) |\gamma(x_1, x_3)|^2 + \rho_\gamma(x_3) |\gamma(x_1, x_2)|^2] dx_1 dx_2 dx_3. \quad (2.15)$$

□

We now want to discard the exchange terms and the singular two-body terms to reduce the above expression to the Hartree functional.

Lemma 2.4 (From Hartree-Fock to Hartree).

Recalling the notation (1.11), assume that

$$\text{Tr}(|p^{\mathbf{A}}|^2 \gamma) \leq CN.$$

Then

$$E^R(N) \leq N \mathcal{E}_R^{\text{af}}[\gamma] + \frac{C}{R^2}. \quad (2.16)$$

Proof. We recall the bound

$$\|\nabla^\perp w_R\|_{L^\infty} \leq \frac{C}{R} \quad (2.17)$$

from [24, Lemma 2.1] and the identities $\gamma = \gamma^2 = \gamma^*$ with

$$\text{Tr} \gamma = \int_{\mathbb{R}^2} \rho_\gamma(x) dx = \int_{\mathbb{R}^2} \gamma(x, x) dx = N$$

for the density matrix (2.4) of a Slater determinant. It follows that

$$\left| \int_{\mathbb{R}^4} |\nabla^\perp w_R(x_1 - x_2)|^2 (\rho_\gamma(x_1) \rho_\gamma(x_2) - |\gamma(x_1, x_2)|^2) dx_1 dx_2 \right| \leq C \frac{N^2}{R^2}.$$

Next we use Cauchy-Schwarz to obtain

$$|W_{12} + W_{21}| \leq \varepsilon (|p_1^{\mathbf{A}}|^2 + |p_2^{\mathbf{A}}|^2) + \frac{2}{\varepsilon} |\nabla^\perp w_R(w_1 - x_2)|^2$$

and hence, since both sides commute with U_{12} ,

$$|\alpha \operatorname{Tr}(W_{12}U_{12}\gamma \otimes \gamma)| \leq \frac{C\varepsilon}{N} \operatorname{Tr}(|p^{\mathbf{A}}|^2\gamma) + \frac{C}{N\varepsilon} \int_{\mathbb{R}^4} \left| \nabla^\perp w_R(x_1 - x_2) \right|^2 |\gamma(x_1, x_2)|^2 dx_1 dx_2$$

where we used that

$$\operatorname{Tr}(|p_1^{\mathbf{A}}|^2 U_{12} \gamma \otimes \gamma) = \operatorname{Tr}(|p^{\mathbf{A}}|^2 \gamma).$$

Optimizing over ε gives

$$|\alpha \operatorname{Tr}(W_{12}U_{12}\gamma \otimes \gamma)| \leq \frac{C}{R}.$$

Next, using (2.17) again,

$$\left| \int_{\mathbb{R}^6} W_{123} \rho_\gamma(x_1) |\gamma(x_2, x_3)|^2 dx_1 dx_2 dx_3 \right| \leq \frac{C}{R^2} \left(\int_{\mathbb{R}^2} \rho_\gamma(x) dx \right) \left(\int_{\mathbb{R}^4} |\gamma(x, y)|^2 dx dy \right) \leq C \frac{N^2}{R^2}$$

and we obtain a similar bound on the last term of (2.8) noticing that

$$\begin{aligned} |\gamma(x_1, x_2)\gamma(x_2, x_3)\gamma(x_3, x_1)| &\leq \sum_j |\gamma(x_1, x_2)\psi_j(x_3)| |\gamma(x_2, x_3)\psi_j(x_1)| \\ &\leq \frac{1}{2} \sum_j |\gamma(x_1, x_2)|^2 |\psi_j(x_3)|^2 + \frac{1}{2} \sum_j |\gamma(x_2, x_3)|^2 |\psi_j(x_1)|^2 \\ &= \frac{1}{2} |\gamma(x_1, x_2)|^2 \rho_\gamma(x_3) + \frac{1}{2} |\gamma(x_2, x_3)|^2 \rho_\gamma(x_1). \end{aligned}$$

□

We now want to take $R \rightarrow 0$ in (2.16) and show that $\mathcal{E}_R^{\text{af}}$ can be replaced by \mathcal{E}^{af} .

Lemma 2.5 (Convergence of the regularized energy).

The functional $\mathcal{E}_R^{\text{af}}$ (1.14) converges pointwise to \mathcal{E}^{af} (1.15) as $R \rightarrow 0$. More precisely, for any fermionic one-particle density matrix γ_N with

$$\operatorname{Tr} \gamma_N = N$$

and associated density ρ_N we have

$$\left| \mathcal{E}_R^{\text{af}}[\gamma_N] - \mathcal{E}^{\text{af}}[\gamma_N] \right| \leq CR \mathcal{E}^{\text{af}}[\gamma_N]^{3/2}. \quad (2.18)$$

Proof. In terms of

$$\tilde{\gamma} = N^{-1} \gamma_N \text{ and } \tilde{\rho} = N^{-1} \rho_{\gamma_N}$$

we have the expression

$$\mathcal{E}_R^{\text{af}}[\gamma_N] = \operatorname{Tr} \left((p^{\mathbf{A}} + \beta \mathbf{A}_R[\tilde{\rho}])^2 \tilde{\gamma} \right) + \int_{\mathbb{R}^2} V \tilde{\rho} \quad (2.19)$$

and similarly for \mathcal{E}^{af} , which is the case $R = 0$. The Lieb-Thirring inequality [36, Theorem 4.3] yields

$$\operatorname{Tr} \left((p^{\mathbf{A}} + \beta \mathbf{A}_R[\tilde{\rho}])^2 \tilde{\gamma} \right) \geq C \int_{\mathbb{R}^2} \tilde{\rho}^2$$

and

$$\operatorname{Tr} \left((p^{\mathbf{A}})^2 \tilde{\gamma} \right) \geq C \int_{\mathbb{R}^2} \tilde{\rho}^2$$

while the weak Young inequality (see e.g. similar estimates in [24, Appendix A], [42, Appendix A] or Appendix A below) gives

$$\mathrm{Tr} \left(|\mathbf{A}_R[\tilde{\rho}]|^2 \tilde{\gamma} \right) = \int_{\mathbb{R}^2} |\mathbf{A}_R[\tilde{\rho}]|^2 \tilde{\rho} \leq C \int_{\mathbb{R}^2} \tilde{\rho}^2.$$

Hence a use of the Cauchy-Schwarz inequality implies

$$\mathcal{E}_R^{\mathrm{af}}[\gamma_N] \geq C \mathrm{Tr} \left((p^{\mathbf{A}})^2 \tilde{\gamma} \right).$$

We now expand the two energies of (2.19) and begin by estimating the squared terms using the Hölder and Young inequalities

$$\begin{aligned} & \left\| \left(|\mathbf{A}^R[\tilde{\rho}]|^2 - |\mathbf{A}[\tilde{\rho}]|^2 \right) \tilde{\rho} \right\|_{L^1} \leq \left\| |\mathbf{A}^R[\tilde{\rho}]|^2 - |\mathbf{A}[\tilde{\rho}]|^2 \right\|_{L^2}^2 \|\tilde{\rho}\|_{L^2} \\ & \leq \|(\nabla w_R - \nabla w_0) * \tilde{\rho}\|_{L^4} \|(\nabla w_R + \nabla w_0) * \tilde{\rho}\|_{L^4} \|\tilde{\rho}\|_{L^2} \\ & \leq \|\nabla w_R - \nabla w_0\|_{L^{4/3}} \|\nabla w_R + \nabla w_0\|_{L^{4/3}} \|\tilde{\rho}\|_{L^2}^3 \leq CR \mathcal{E}^{\mathrm{af}}[\gamma_N]^{3/2} \end{aligned}$$

as per the above estimates, and bounds on ∇w_R following from [24, Lemma 2.1]. As for the mixed term

$$\begin{aligned} \mathrm{Tr} \left[(\mathbf{A}^R[\tilde{\rho}] - \mathbf{A}[\tilde{\rho}]) \cdot p^{\mathbf{A}} \tilde{\gamma} \right] & \leq \left\| p^{\mathbf{A}} \sqrt{\tilde{\gamma}} \right\|_{\mathfrak{S}^2} \left\| (\mathbf{A}^R[\tilde{\rho}] - \mathbf{A}[\tilde{\rho}]) \sqrt{\tilde{\gamma}} \right\|_{\mathfrak{S}^2} \\ & \leq C \mathcal{E}^{\mathrm{af}}[\tilde{\gamma}]^{1/2} \left\| \mathbf{A}^R[\tilde{\rho}] - \mathbf{A}[\tilde{\rho}] \right\|_{L^4} \|\tilde{\rho}\|_{L^2}^{1/2} \\ & \leq C \mathcal{E}^{\mathrm{af}}[\tilde{\gamma}]^{1/2} \|(\nabla w_R - \nabla w_0) * \tilde{\rho}\|_{L^4} \|\tilde{\rho}\|_{L^2}^{1/2} \\ & \leq C \mathcal{E}^{\mathrm{af}}[\tilde{\gamma}]^{1/2} \|\nabla w_R - \nabla w_0\|_{L^{4/3}} \|\tilde{\rho}\|_{L^2}^{3/2} \\ & \leq CR \mathcal{E}^{\mathrm{af}}[\tilde{\gamma}]^{3/2} \end{aligned}$$

where $\|W\|_{\mathfrak{S}^p} = \mathrm{Tr} [|W|^p]^{1/p}$ is the Schatten norm [58, Chapter 2] and where we used Young's inequality. \square

Inserting (2.18) in (2.4) we get

$$\frac{E^R(N)}{N} \leq \mathcal{E}^{\mathrm{af}}[\gamma_N] + CR + \frac{C}{NR^2} \quad (2.20)$$

for a sequence γ_N with uniformly bounded Hartree energy. The last step consists in using constructing such a sequence γ_N , whose energy will behave semi-classically.

2.2. Semi-classical upper bound for the Hartree energy. We use [21, Lemma 3.2], whose statement we reproduce for the convenience of the reader.

Lemma 2.6 (Semi-classical limit of the Hartree energy).

Let $\rho \geq 0$ be a fixed function in $C_c^\infty(\mathbb{R}^2)$ with support in the square $C_r = (-r/2, r/2)^2$ such that $\rho \geq 0$ and $\int_{C_r} \rho = 1$. Let $\mathbf{A} \in L^4(C_r)$ be a magnetic vector potential. If we define

$$\gamma_N = \mathbf{1} \left((-i\hbar\nabla + \mathbf{A})_{C_r}^2 - 4\pi\rho(x) \leq 0 \right) \quad (2.21)$$

where $((-i\hbar\nabla + \mathbf{A})_{C_r}^2)$ is the magnetic Dirichlet Laplacian in the cube and $\hbar = 1/\sqrt{N}$ we have

$$\lim_{N \rightarrow \infty} N^{-1} \mathrm{Tr} \left((-i\hbar\nabla + \mathbf{A})^2 \gamma_N \right) = 2\pi \int_{\mathbb{R}^2} \rho(x)^2 dx \quad (2.22)$$

and

$$\lim_{N \rightarrow \infty} N^{-1} \text{Tr } \gamma_N = \int_{\mathbb{R}^2} \rho(x) dx \quad \text{with } \frac{\rho_{\gamma_N}}{N} \rightarrow \rho \quad (2.23)$$

and weakly-* in $L^\infty(\mathbb{R}^2)$, strongly in $L^1(\mathbb{R}^2)$ and $L^2(\mathbb{R}^2)$. Furthermore, the same properties stay true if we replace γ_N by $\tilde{\gamma}_N$ the projection onto the N lowest eigenvectors of $(-i\hbar\nabla + \mathbf{A})_{C_r}^2 - 4\pi\rho(x)$.

The convergences we claim for $N^{-1}\rho_{\gamma_N}$ are stronger than in the statement of [21, Lemma 3.2]. They easily follow from the proof therein.

We are now able to complete the proof of Proposition 2.1. We take

$$\gamma_N = \mathbb{1} \left((-i\hbar\nabla + \mathbf{A}_e + \beta\mathbf{A}[\rho^{\text{TF}}])_{C_r}^2 - 4\pi\rho^{\text{TF}}(x) \leq 0 \right) \quad (2.24)$$

and $\tilde{\gamma}_N$ the associated rank- N projector, as in the above lemma. Therefore we know by the above theorem that

$$\frac{\rho_{\tilde{\gamma}_N}}{N} \rightarrow \rho^{\text{TF}}$$

weakly-* in $L^\infty(\mathbb{R}^2)$, strongly in $L^1(\mathbb{R}^2)$ and $L^2(\mathbb{R}^2)$. We have that (the definition of the weak L^2 space is recalled in Appendix A below)

$$\nabla^\perp w_0 = \nabla^\perp \log |x| \in L_{\text{loc}}^1(C_r) \cap L^{2,w}(\mathbb{R}^2).$$

We deduce that

$$\alpha\mathbf{A}[\rho_{\tilde{\gamma}_N}] \rightarrow \beta\mathbf{A}[\rho^{\text{TF}}]$$

strongly in $L^\infty(\mathbb{R}^2)$, by the weak Young inequality [35, Chapter 4]. We thus have

$$\mathcal{E}^{\text{af}}[\tilde{\gamma}_N] = \frac{(1 + o_N(1))}{N} \text{Tr} \left[(p^{\mathbf{A}} + \beta\mathbf{A}[\rho^{\text{TF}}])^2 \tilde{\gamma}_N \right] + \frac{1}{N} \int_{\mathbb{R}^2} V(x) \rho_{\tilde{\gamma}_N}(x) dx + o_N(1). \quad (2.25)$$

We now use (2.22) to take the limit of the trace and (2.23) to take the limit of the potential term ($V \in L_{\text{loc}}^1(\mathbb{R}^2)$ by assumption).

Combining with the bounds from Lemmas 2.5 and 2.4 we finally obtain

$$\begin{aligned} \frac{E^R(N)}{N} &\leq \mathcal{E}^{\text{af}}[\tilde{\gamma}_N] + CR + \frac{C}{R^2N} \\ &\leq 2\pi \int_{\mathbb{R}^2} \rho_{\text{TF}}^2(x) dx + \int_{\mathbb{R}^2} V(x) \rho_{\text{TF}}(x) dx + o_N(1) \\ &= e_{\text{TF}} + o_N(1) \end{aligned}$$

provided $R = N^{-\eta}$ with $0 < \eta < \frac{1}{2}$, completing the proof of Proposition 2.1.

3. ENERGY LOWER BOUND: SEMI-CLASSICAL LIMIT

In this section we first derive some useful a priori bounds and properties of the Husimi functions. Next we express our energy in terms of the latter and obtain a classical energy approximation on the phase space, plus some errors terms that we show to be negligible. We will use this expression in Section 4 to construct our lower bound.

3.1. A priori estimates.

Lemma 3.1 (Kinetic energy bound).

Let $\Psi_N \in L_{\text{asym}}(\mathbb{R}^{2N})$ be a fermionic wave-function such that

$$\langle \Psi_N, H_N^R \Psi_N \rangle \leq CN.$$

Then

$$\frac{1}{N^2} \left\langle \Psi_N, \left(\sum_{j=1}^N -\Delta_j \right) \Psi_N \right\rangle \leq \frac{C}{R^2} \quad (3.1)$$

and

$$\int_{\mathbb{R}^2} \left(\frac{\rho_{\Psi_N}^{(1)}}{N} \right)^2 \leq \frac{C}{R^2}. \quad (3.2)$$

Moreover

$$\int_{\mathbb{R}^2} V(x) \rho_{\Psi_N}^{(1)}(x) dx \leq CN. \quad (3.3)$$

Proof. We expand the Hamiltonian and use the Cauchy-Schwarz inequality for operators to get

$$\begin{aligned} H_N^R &= \sum_{j=1}^N \left((p_j^{\mathbf{A}})^2 + \alpha p_j^{\mathbf{A}} \cdot \mathbf{A}_j^R + \alpha \mathbf{A}_j^R \cdot p_j^{\mathbf{A}} + \alpha^2 |\mathbf{A}_j^R|^2 + V(x_j) \right) \\ &\geq \sum_{j=1}^N \left((1 - 2\delta^{-1}) (p_j^{\mathbf{A}})^2 + (1 - 2\delta) \alpha^2 |\mathbf{A}_j^R|^2 + V(x_j) \right) \\ &= \sum_{j=1}^N \left(\frac{1}{2} ((p_j^{\mathbf{A}})^2 + V(x_j)) - 7\alpha^2 |\mathbf{A}_j^R|^2 \right) \end{aligned}$$

choosing $\delta = 4$. Thus we have

$$\text{Tr} \left[\left((p^{\mathbf{A}})^2 + V \right) \gamma_N^{(1)} \right] \leq CN + 7\alpha^2 \left\langle \Psi_N, \sum_{j=1}^N |\mathbf{A}_j^R|^2 \Psi_N \right\rangle$$

where $\gamma_N^{(1)}$ is the 1-particle reduced density matrix

$$N \text{Tr}_{2 \rightarrow N} [|\Psi_N\rangle \langle \Psi_N|].$$

We estimate the second term of the above using [24, Lemma 2.1]:

$$\left\langle \Psi_N, \sum_{j=1}^N |\mathbf{A}_j^R|^2 \Psi_N \right\rangle \leq \frac{CN^3}{R^2}.$$

Recalling that $\alpha = N^{-1}$ we get

$$\text{Tr} \left[\left((p^{\mathbf{A}})^2 + V \right) \gamma_N^{(1)} \right] \leq \frac{CN}{R^2}$$

Now we use that

$$| -i\hbar\nabla + \mathbf{A}_e |^2 \geq -\frac{\hbar^2}{2} \Delta - |\mathbf{A}_e|^2$$

and that $\hbar^2 = N^{-1}$ to obtain

$$\frac{CN}{R^2} \geq \frac{C}{N} \left\langle \Psi_N, \left(\sum_{j=1}^N -\Delta_j \right) \Psi_N \right\rangle - \text{Tr} \left[|\mathbf{A}_e|^2 \gamma_N^{(1)} \right]$$

recalling that $V \geq 0$. But $|\mathbf{A}_e|^2 \in L^\infty(\mathbb{R}^2)$, we thus deduce (3.1). We obtain (3.2) by using the Lieb-Thirring inequality [40, 41]

$$\text{Tr} \left[-\Delta \gamma_N^{(1)} \right] \geq C \int_{\mathbb{R}^2} \left(\rho_{\psi_N}^{(1)}(x) \right)^2 dx$$

combined with (3.1).

Finally (3.3) is a straightforward consequence of the positivity of the full kinetic energy (first term in (1.3)). \square

3.2. Properties of the phase space measures. The following connects the space and momentum densities (1.35) and (1.36) with the Husimi function and is extracted from [21, Lemma 2.4]. The generalization to squeezed states we use does not change the properties of the phase space measures.

Lemma 3.2 (Densities and fermionic semi-classical measures).

Let F_{\hbar_x} , G_{\hbar_p} be defined as in (1.28) and (1.29) (recall that $\hbar = \sqrt{\hbar_x} \sqrt{\hbar_p}$). Let $\Psi_N \in L^2_{\text{asym}}(\mathbb{R}^{2N})$ be any normalized fermionic wave function. We have

$$\frac{1}{(2\pi)^{2k}} \int_{\mathbb{R}^{2k}} m_{\Psi_N}^{(k)}(x_1, p_1, \dots, x_k, p_k) dp_1 \dots dp_k = k! \hbar^{2k} \rho_{\Psi_N}^{(k)} * (|F_{\hbar_x}|^2)^{\otimes k} \quad (3.4)$$

and

$$\frac{1}{(2\pi)^{2k}} \int_{\mathbb{R}^{2k}} m_{\Psi_N}^{(k)}(x_1, p_1, \dots, x_k, p_k) dx_1 \dots dx_k = k! \hbar^{2k} t_{\Psi_N}^{(k)} * (|G_{\hbar_p}|^2)^{\otimes k}. \quad (3.5)$$

The proof is similar to considerations from [21]. We recall it in Appendix C for completeness.

Lemma 3.3 (Properties of the phase space measures).

Let $\Psi_N \in L^2_{\text{asym}}(\mathbb{R}^{2N})$ be a normalized fermionic wave function. For every $1 \leq k \leq N$ the function $m_{\Psi_N}^{(k)}$ defined in (1.32) is symmetric and satisfies

$$0 \leq m_{\Psi_N}^{(k)} \leq 1 \quad \text{a.e. on } \mathbb{R}^{4k}. \quad (3.6)$$

In addition

$$\frac{1}{(2\pi)^{2k}} \int_{\mathbb{R}^{4k}} m_{\Psi_N}^{(k)}(x_1, p_1, \dots, x_k, p_k) dx_1 \dots dp_k = N(N-1) \dots (N-k+1) \hbar^{2k} \rightarrow 1 \quad (3.7)$$

when N tends to infinity and

$$\frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} m_{\Psi_N}^{(k)}(x_1, p_1, \dots, x_k, p_k) dx_k dp_k = \hbar^2 (N-k+1) m_{\Psi_N}^{(k-1)}(x_1, p_1, \dots, x_{k-1}, p_{k-1}).$$

We refer to [21, Lemma 2.2] for the proof.

3.3. Semi-classical energy. We now define a semi-classical analogue of the original energy functional, given in terms of Husimi functions. We denote $dx_{ij} = dx_i dx_j$ and $x_{ij} = (x_i, x_j)$:

$$\begin{aligned} \mathcal{E}_C^R [m_{\Psi_N}^{(3)}] &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} (|p + \mathbf{A}_e(x)|^2 + V(x_1)) m_{\Psi_N}^{(1)}(x, p) dx dp \\ &+ \frac{2\beta}{(2\pi)^4} \int_{\mathbb{R}^8} (p_1 + \mathbf{A}_e(x_1)) \cdot \nabla^\perp w_R(x_1 - x_2) m_{\Psi_N}^{(2)}(x_{12}, p_{12}) dx_{12} dp_{12} \\ &+ \frac{\beta^2}{(2\pi)^6} \int_{\mathbb{R}^{12}} \nabla^\perp w_R(x_1 - x_2) \cdot \nabla^\perp w_R(x_1 - x_3) m_{\Psi_N}^{(3)}(x_{123}, p_{123}) dx_{123} dp_{123}. \end{aligned} \quad (3.8)$$

Our aim is to show that it correctly captures the leading order of the full quantum energy.

Proposition 3.4 (Semi-classical energy functional).

Pick some $\varepsilon > 0$ and set

$$\sqrt{\hbar_x} = N^{-1/2+\varepsilon}, \quad \sqrt{\hbar_p} = N^{-\varepsilon}. \quad (3.9)$$

Pick a sequence Ψ_N of fermionic wave-functions such that

$$\langle \Psi_N, H_N^R \Psi_N \rangle \leq CN. \quad (3.10)$$

Under Assumption 1.1 we have

$$\frac{\langle \Psi_N, H_N^R \Psi_N \rangle}{N} \geq (1 - o_N(1)) \mathcal{E}_C^R [m_{\Psi_N}^{(3)}] - CN^{4\eta-1+2\varepsilon} - CN^{2\eta-1} - CN^{-2\varepsilon}. \quad (3.11)$$

We first derive an exact expression, whose extra unwanted terms will be estimated below.

Lemma 3.5 (Semi-classical energy with errors).

Let Ψ_N be a fermionic wave-function satisfying the bound (3.10), with $\langle \cdot \rangle$ the corresponding expectation values:

$$\begin{aligned} &\left\langle \sum_{j=1}^N (p_j^A + \alpha \mathbf{A}^R(x_j))^2 + V(x_j) \right\rangle \geq N \mathcal{E}_C^R [m_{\Psi_N}^{(3)}] \\ &+ 2\alpha \Re \left\langle \sum_{j=1}^N \sum_{k \neq j} \left(\nabla^\perp w_R - \nabla^\perp w_R * |F_{\hbar_x}|^2 * |F_{\hbar_x}|^2 \right) (x_j - x_k) \cdot (-i\hbar \nabla_j) \right\rangle \end{aligned} \quad (3.12)$$

$$\begin{aligned} &+ 2\alpha \left\langle \sum_{j=1}^N \sum_{k \neq j} \mathbf{A}_e(x_j) \cdot \nabla^\perp w_R(x_j - x_k) \right. \\ &\quad \left. - \int_{\mathbb{R}^2} \mathbf{A}_e(x_j - u) \cdot (\nabla^\perp w_R * |F_{\hbar_x}|^2)(x_j - x_k - u) |F(u)|^2 du \right\rangle \end{aligned} \quad (3.13)$$

$$\begin{aligned} &+ \alpha^2 \sum_{j=1 \neq k \neq l}^N \left\langle \nabla^\perp w_R(x_j - x_k) \cdot \nabla^\perp w_R(x_j - x_l) - \int_{\mathbb{R}^2} (\nabla^\perp w_R * |F_{\hbar_x}|^2)(x_j - x_k - u) \right. \\ &\quad \left. \cdot (\nabla^\perp w_R * |F_{\hbar_x}|^2)(x_j - x_l - u) |F_{\hbar_x}(u)|^2 du \right\rangle \end{aligned} \quad (3.14)$$

$$+ \text{Err}_1 + \text{Err}_2 \quad (3.15)$$

with

$$\text{Err}_1 = 2\Re \left\langle \sum_{j=1}^N (\mathbf{A}_e - \mathbf{A}_e * |F_{\hbar_x}|^2)(x_j) \cdot (-i\hbar \nabla_j) \right\rangle \quad (3.16)$$

$$+ \left\langle \sum_{j=1}^N (|\mathbf{A}_e|^2 - |\mathbf{A}_e|^2 * |F_{\hbar_x}|^2)(x_j) + V - V * |F_{\hbar_x}|^2(x_j) \right\rangle \quad (3.17)$$

and

$$\text{Err}_2 = \alpha^2 \left\langle \sum_{j=1}^N \sum_{k \neq j} |\nabla^\perp w_R(x_j - x_k)|^2 \right\rangle - N\hbar_p \int_{\mathbb{R}^2} |\nabla f(y)|^2 dy. \quad (3.18)$$

Proof. We expand the whole energy as in (1.10) and express each term using Husimi's function (1.32). We begin with calculations involving the p variable. Using (1.36) and (3.5) we can write, for the purely kinetic term,

$$\begin{aligned} & \left\langle \Psi_N, \left(\sum_{j=1}^N -\hbar^2 \Delta_j \right) \Psi_N \right\rangle = \int_{\mathbb{R}^2} |p|^2 t_{\Psi_N}^{(1)}(p) dp \\ &= \frac{1}{(2\pi\hbar)^2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} m_{\Psi_N}^{(1)}(x, p) |p|^2 dx dp + \int_{\mathbb{R}^2 \times \mathbb{R}^2} t_{\Psi_N}^{(1)}(p) |G_{\hbar_p}(q-p)|^2 (|p|^2 - |q|^2) dp dq \\ &= \frac{1}{(2\pi\hbar)^2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} m_{f, \Psi_N}^{(1)}(x, p) |p|^2 dx dp - \int_{\mathbb{R}^2 \times \mathbb{R}^2} t_{\Psi_N}^{(1)}(p) |G_{\hbar_p}(q-p)|^2 (|q-p|^2) dp dq \\ & \quad - 2 \left(\int_{\mathbb{R}^2} p t_{\Psi_N}^{(1)}(p) dp \right) \cdot \left(\int_{\mathbb{R}^2} p |G_{\hbar_p}(p)|^2 dp \right). \end{aligned}$$

We now use that $G_{\hbar_p}(-p) = \overline{G_{\hbar_p}(p)}$ which makes $|G_{\hbar_p}(p)|^2$ an even function (recall that G_{\hbar_p} is the Fourier transform of F_{\hbar_x} (1.28)) to discard the last term of the above. On the other hand, using (1.36) we find

$$\begin{aligned} & \int_{\mathbb{R}^2 \times \mathbb{R}^2} t_{\Psi_N}^{(1)}(p) |G_{\hbar_p}(q-p)|^2 (|q-p|^2) dp dq = N \int_{\mathbb{R}^2} |G_{\hbar_p}(p)|^2 |p|^2 dp \\ &= \frac{N}{(2\pi\hbar)^2} \int_{\mathbb{R}^6} \frac{|p|^2}{\hbar_x} f\left(\frac{x}{\sqrt{\hbar_x}}\right) f\left(\frac{x'}{\sqrt{\hbar_x}}\right) e^{\frac{ip \cdot x}{\hbar}} e^{-\frac{ip \cdot x'}{\hbar}} dp dx dx' \\ &= \frac{N\hbar^2}{(2\pi\hbar)^2} \int_{\mathbb{R}^6} \nabla f(u) \nabla f(v) e^{\frac{ip \cdot (u-v)}{\sqrt{\hbar_p}}} dp du dv = -N\hbar_p \int_{\mathbb{R}^2} |\nabla f(y)|^2 dy, \end{aligned}$$

thus concluding that

$$\left\langle \Psi_N, \left(\sum_{j=1}^N -\hbar^2 \Delta_j \right) \Psi_N \right\rangle = \frac{1}{(2\pi\hbar)^2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} m_{f, \Psi_N}^{(1)}(x, p) |p|^2 dx dp - N\hbar_p \int_{\mathbb{R}^2} |\nabla f(y)|^2 dy.$$

For the magnetic cross term (3.16) we use (C.2) to obtain, for any $u \in L^2(\mathbb{R}^2)$,

$$\begin{aligned} & \frac{1}{(2\pi\hbar)^2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} p \cdot \mathbf{A}_e(x) |\langle u, F_{x,p} \rangle|^2 dx dp = \int_{\mathbb{R}^2 \times \mathbb{R}^2} p \cdot \mathbf{A}_e(x) |\mathcal{F}_\hbar[F_{x,0}u](p)|^2 dp dx \\ &= \int_{\mathbb{R}^2} \mathbf{A}_e(x) \cdot \int_{\mathbb{R}^2} p |\mathcal{F}_\hbar[F_{x,0}u](p)|^2 dp dx = \hbar \int_{\mathbb{R}^2} \mathbf{A}_e(x) \cdot \Im \int_{\mathbb{R}^2} F_{x,0}(y) \bar{u}(y) \nabla[F_{x,0}u](y) dy dx. \end{aligned}$$

Then since $F_{x,0}$ and $|u|^2$ are real we obtain

$$\begin{aligned} \frac{1}{(2\pi\hbar)^2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} p \cdot \mathbf{A}_e(x) |\langle u, F_{x,p} \rangle|^2 dx dp &= \hbar \int_{\mathbb{R}^2} \mathbf{A}_e(x) \cdot \int_{\mathbb{R}^2} |F_{\hbar x}(y-x)|^2 \Im [\bar{u}(y) \nabla u(y)] dy dx \\ &= \hbar \int_{\mathbb{R}^2} (\mathbf{A}_e(x) * |F_{\hbar x}|^2)(y) \cdot \Im [\bar{u}(y) \nabla u(y)] dy. \end{aligned}$$

Combining the spectral decomposition of the one-particle reduced density matrix with the above we get

$$\hbar \sum_{j=1}^N \langle p_j \cdot \mathbf{A}_e(x_j) \rangle = \frac{1}{(2\pi\hbar)^2} \int_{\mathbb{R}^4} p \cdot \mathbf{A}_e dm_{\Psi_N}^{(1)} + \Re \left\langle \sum_{j=1}^N (\mathbf{A}_e - \mathbf{A}_e * |F_{\hbar x}|^2)(x_j) \cdot (-i\hbar \nabla_j) \right\rangle.$$

Now for the mixed two-body term (3.12):

$$\begin{aligned} M &:= \frac{1}{N(N-1)} \frac{1}{(2\pi\hbar)^2} \int_{\mathbb{R}^4 \times \mathbb{R}^4} m_{\Psi_N}^{(2)}(x_1, x_2, p_1, p_2) \left(p_1 \cdot \nabla^\perp w_R(x_1 - x_2) \right) dx_1 dx_2 dp_1 dp_2 \\ &= - \int_{\mathbb{R}^{2(N-2)}} \int_{\mathbb{R}^4 \times \mathbb{R}^4} p_1 \cdot \nabla^\perp w_R(x_1 - x_2) |\langle F_{x_1, p_1} \otimes F_{x_2, p_2}(\cdot) \Psi_N(\cdot, y) \rangle|_{L^2(\mathbb{R}^4)}^2 dx_{12} dp_{12} dy \\ &= - \int_{\mathbb{R}^{2(N-2)}} \frac{i\hbar}{2} \int_{\mathbb{R}^4} \nabla^\perp w_R(x_1 - x_2) \cdot \int_{\mathbb{R}^8} [\nabla_{y_1} - \nabla_{z_1}] \left(F_{x_1,0}(y_1) F_{x_1,0}(z_1) \bar{\Psi}(y_1, y_2, y) \right. \\ &\quad \left. \Psi_N(z_1, z_2, y) F_{x_2,0}(y_2) F_{x_2,0}(z_2) \right) \delta(y_2 - z_2) \delta(y_1 - z_1) dx_{12} dy_{12} dz_{12} dy \end{aligned}$$

and since $-\frac{i}{2}(u - \bar{u}) = \Im[u]$, we get

$$\begin{aligned} M &= -\hbar \int_{\mathbb{R}^{2(N-2)}} \int_{\mathbb{R}^4 \times \mathbb{R}^4} \nabla^\perp w_R(x_1 - x_2) |F_{\hbar x}(y_1 - x_1)|^2 |F_{\hbar x}(y_2 - x_2)|^2 \cdot \Im \left[\bar{\Psi}_N \nabla_{y_1} \Psi_N \right] dx_{12} dy_{12} dy \\ &= \int_{\mathbb{R}^{2(N-2)}} \int_{\mathbb{R}^4} \left((\nabla^\perp w_R * |F_{\hbar x}|^2) * |F_{\hbar x}|^2 \right) (x_1 - x_2) \Re \left[-i\hbar \nabla_{x_1} \Psi_N \bar{\Psi}_N \right] dx_1 dx_2 dy \end{aligned}$$

and conclude

$$\begin{aligned} \hbar \sum_{j=1}^N \sum_{k \neq j} \langle p_j \cdot \nabla^\perp w_R(x_j - x_k) \rangle &= \frac{1}{(2\pi\hbar)^4} \int_{\mathbb{R}^4 \times \mathbb{R}^4} \left(p_1 \cdot \nabla^\perp w_R(x_1 - x_2) \right) dm_{\Psi_N}^{(2)} \\ &+ \Re \left\langle \sum_{j=1}^N \sum_{k \neq j} \left(\nabla^\perp w_R - \nabla^\perp w_R * |F_{\hbar x}|^2 * |F_{\hbar x}|^2 \right) (x_j - x_k) \cdot (-i\hbar \nabla_j) \right\rangle. \end{aligned}$$

For the terms which do not involve the variable p we only discuss the three-body term (3.14), the others being treated in the same way. We apply Lemma (3.2)

$$\begin{aligned}
& \frac{1}{(2\pi\hbar)^3} \int_{\mathbb{R}^6 \times \mathbb{R}^6} W_{123}(x_1, x_2, x_3) m_{\Psi_N}^{(3)}(x_1, p_1, \dots, x_3, p_3) dx_{123} dp_{123} \\
&= \int_{\mathbb{R}^6 \times \mathbb{R}^6} W_{123}(x_1, x_2, x_3) \rho_{\Psi_N}^{(3)} * (|F_{\hbar_x}|^2)^{\otimes 3} dx_{123} \\
&= \int_{\mathbb{R}^{12}} \nabla^\perp w_R(u - v + y_1 - y_2) \cdot \nabla^\perp w_R(u - w + y_1 - y_3) \\
&\quad \rho_{\Psi_N}^{(3)}(u, v, w) |F_{\hbar_x}|^2(y_1) |F_{\hbar_x}|^2(y_2) |F_{\hbar_x}|^2(y_3) dy_{123} dudvdw \\
&= \int_{\mathbb{R}^6} \left(\nabla^\perp w_R * |F_{\hbar_x}|^2 \right)(v - u - y_1) \cdot \left(\nabla^\perp w_R * |F_{\hbar_x}|^2 \right)(w - u - y_1) |F_{\hbar_x}(y_1)|^2 \rho_{\Psi_N}^{(3)} dudvdwdy_1
\end{aligned}$$

and then

$$\begin{aligned}
& \sum_{j=1}^N \sum_{k \neq j} \sum_{l \neq k} \left\langle \nabla^\perp w_R(x_j - x_k) \nabla^\perp w_R(x_j - x_l) \right\rangle = \frac{1}{(2\pi\hbar)^6} \int_{\mathbb{R}^{12}} \left(\nabla^\perp w_R \cdot \nabla^\perp w_R \right) dm_{\Psi_N}^{(3)} \\
&+ \sum_{j=1}^N \sum_{k \neq l} \left\langle \nabla^\perp w_R(x_j - x_k) \cdot \nabla^\perp w_R(x_j - x_l) - \int_{\mathbb{R}^2} \left(\nabla^\perp w_R * |F_{\hbar_x}|^2 \right)(x_j - x_k - u) \right. \\
&\quad \left. \cdot \left(\nabla^\perp w_R * |F_{\hbar_x}|^2 \right)(x_j - x_l - u) |F_{\hbar_x}(u)|^2 du \right\rangle.
\end{aligned}$$

□

We now show that all errors in (3.15) are smaller than the classical energy \mathcal{E}^C . This will conclude the proof of Proposition 3.4. The following lemma will deal with all convolutions involved in the estimates.

Lemma 3.6 (Convolution terms).

For any function W on $L^\infty(\mathbb{R}^2)$, consider the n -fold convoluted product of W with $|F_{\hbar_x}|^2$.

$$W *_{*n} |F_{\hbar_x}|^2 = W * |F_{\hbar_x}|^2 * |F_{\hbar_x}|^2 * \dots * |F_{\hbar_x}|^2.$$

We have the estimate

$$\|W *_{*n} |F_{\hbar_x}|^2 - W\|_{L^\infty} \leq C_n \hbar_x \|\Delta W\|_{L^\infty}. \quad (3.19)$$

Proof. For any $x, y \in \mathbb{R}^2$ there exists some $z \in \mathbb{R}^2$ such that

$$W\left(x - \sqrt{\hbar_x} y\right) = W(x) - \sqrt{\hbar_x} \nabla W(x) \cdot y + \frac{\hbar_x^2}{2} \langle y, \text{Hess} W(z) y \rangle$$

and hence

$$\begin{aligned}
\|W - W * |F_{\hbar_x}|^2(x)\|_{L^\infty} &= \left| W(x) - \frac{1}{\pi \hbar_x} \int_{\mathbb{R}^2} W(x - y) e^{-\frac{|y|^2}{\hbar_x}} dy \right| \\
&= \left| W(x) - \frac{1}{\pi} \int_{\mathbb{R}^2} W\left(x - \sqrt{\hbar_x} y\right) e^{-|y|^2} dy \right| \\
&\leq C \hbar_x \|\Delta W\|_{L^\infty}
\end{aligned}$$

using the radial symmetry of $y \mapsto e^{-|y|^2}$ to discard the term of order $\sqrt{\hbar_x}$. It follows that for any $n \geq 2$

$$\|W *_{n-1} |F_{\hbar_x}|^2 - W\|_{L^\infty} \leq C_n \hbar_x \|\Delta W\|_{L^\infty}$$

because

$$\|\Delta W *_{n-1} |F_{\hbar_x}|^2\|_{L^\infty} = \|(\Delta W) *_{n-1} |F_{\hbar_x}|^2\|_{L^\infty} \leq C \|\Delta W\|_{L^\infty}.$$

□

Proof of Proposition (3.4). Let Ψ_N be a sequence of fermionic wave functions such that

$$\langle \Psi_N, H_N^R \Psi_N \rangle = O(N).$$

We will systematically use the bounds (3.2) and (3.1)

$$\|\rho_{\gamma_N}\|_{L^2(\mathbb{R}^2)}^2 \leq \frac{CN^2}{R^2} \quad \text{and} \quad \left\langle \Psi_N, \left(\sum_{j=1}^N -\Delta_j \right) \Psi_N \right\rangle \leq \frac{CN^2}{R^2}$$

to deal with our error terms. Recall that the main term in the energy (3.15) is of order N .

Estimate of (3.16). We have

$$\begin{aligned} |\mathrm{Tr}((\mathbf{A}_e - \mathbf{A}_e * |F_{\hbar_x}|^2) \cdot (-i\hbar\nabla) \gamma_N)| &\leq \|(-i\hbar\nabla)\sqrt{\gamma_N}\|_{\mathfrak{S}^2} \|(\mathbf{A}_e - \mathbf{A}_e * |F_{\hbar_x}|^2)\sqrt{\gamma_N}\|_{\mathfrak{S}^2} \\ &\leq \hbar\sqrt{\mathrm{Tr}(-\Delta)\gamma_N} \|(\mathbf{A}_e - \mathbf{A}_e * |F_{\hbar_x}|^2)\sqrt{\rho_{\gamma_N}}\|_{L^2(\mathbb{R}^2)} \\ &\leq C\hbar\frac{N}{R} \|\mathbf{A}_e - \mathbf{A}_e * |F_{\hbar_x}|^2\|_{L^\infty(\mathbb{R}^2)} \|\rho_{\gamma_N}\|_{L^1(\mathbb{R}^2)}^{1/2} \\ &\leq \frac{CN}{R} \|\mathbf{A}_e - \mathbf{A}_e * |F_{\hbar_x}|^2\|_{L^\infty(\mathbb{R}^2)} \end{aligned}$$

where $\|W\|_{\mathfrak{S}^p} = \mathrm{Tr}[|W|^p]^{1/p}$ is the Schatten p -norm. We used (3.1) to bound the kinetic term and we now use Lemma 3.6 to deduce

$$|\mathrm{Tr}((\mathbf{A}_e - \mathbf{A}_e * |F_{\hbar_x}|^2) \cdot (-i\hbar\nabla) \gamma_N)| \leq \frac{CN\hbar_x}{R} \|\Delta\mathbf{A}_e\|_{L^\infty} \leq CN^{\eta+2\varepsilon}. \quad (3.20)$$

Estimate of (3.17). We have

$$\begin{aligned} \|(|\mathbf{A}_e|^2 - |\mathbf{A}_e|^2 * |F_{\hbar_x}|^2)\rho_{\gamma_N}\|_{L^1(\mathbb{R}^2)} &\leq \|(|\mathbf{A}_e|^2 - |\mathbf{A}_e|^2 * |F_{\hbar_x}|^2)\|_{L^\infty(\mathbb{R}^2)} \|\rho_{\gamma_N}\|_{L^1(\mathbb{R}^2)} \\ &\leq CN \|(|\mathbf{A}_e|^2 - |\mathbf{A}_e|^2 * |F_{\hbar_x}|^2)\|_{L^\infty(\mathbb{R}^2)} \\ &\leq CN\hbar_x \|\Delta|\mathbf{A}_e|^2\|_{L^\infty(\mathbb{R}^2)} \leq CN^{2\varepsilon} \end{aligned} \quad (3.21)$$

where we used Cauchy-Schwarz's inequality and the estimate (3.19). For the second term of (3.17) we use Assumption 1.1 and Lemma 3.6 to write

$$\begin{aligned} \|(V - V * |F_{\hbar_x}|^2)\rho(x)\|_{L^1} &\leq C\hbar_x \int_{\mathbb{R}^2} |\Delta V(x)| \rho(x) dx \\ &\leq C\hbar_x \int_{\mathbb{R}^2} (|x|^{s-2} + 1) \rho(x) dx \\ &\leq C\hbar_x \int_{\mathbb{R}^2} (|x|^s + 1) \rho(x) \leq CN^{2\varepsilon} \end{aligned} \quad (3.22)$$

by using Young's inequality

$$r^{s-2} \leq \frac{s-2}{s} r^s + \frac{2}{s}$$

the choice (3.9) and the a priori bound (3.3).

Estimate of (3.18). We have

$$\alpha^2 \left\langle \sum_{j=1}^N \sum_{k \neq j} |\nabla^\perp w_R(x_j - x_k)|^2 \right\rangle \leq \frac{C}{R^2} \leq CN^{2\eta}$$

and

$$\hbar_p \int_{\mathbb{R}^2} |\nabla f(y)|^2 dy \leq CN^{-2\varepsilon}$$

recalling that f is a fixed function and the choice (3.9).

Estimate of (3.12):

$$\begin{aligned} & 4\alpha \left| \text{Tr} \left[\left(\nabla^\perp w_R - \nabla^\perp w_R * |F_{\hbar_x}|^2 * |F_{\hbar_x}|^2 \right) \cdot (-i\hbar \nabla_1) \gamma_N^{(2)} \right] \right| \\ & \leq C\alpha \left\| (-i\hbar \nabla) \sqrt{\gamma_N} \right\|_{\mathfrak{S}^2} \left\| \left(\nabla^\perp w_R - \nabla^\perp w_R * |F_{\hbar_x}|^2 * |F_{\hbar_x}|^2 \right) \sqrt{\gamma_N^{(2)}} \right\|_{\mathfrak{S}^2} \\ & \leq C\alpha \hbar \frac{N}{R} \left\| \rho_{\gamma_N}^{(2)} \right\|_{L^1}^{1/2} \left\| \left(\nabla^\perp w_R - \nabla^\perp w_R * |F_{\hbar_x}|^2 * |F_{\hbar_x}|^2 \right) \right\|_{L^\infty} \\ & \leq C\alpha \hbar \frac{N^2}{R} \left\| \left(\nabla^\perp w_R - \nabla^\perp w_R * |F_{\hbar_x}|^2 * |F_{\hbar_x}|^2 \right) \right\|_{L^\infty} \end{aligned} \quad (3.23)$$

where we used (3.2). We then use Lemma 3.6 and the estimate (B.2) to get

$$4\alpha \left| \text{Tr} \left[\left(\nabla^\perp w_R - \nabla^\perp w_R * |F_{\hbar_x}|^2 * |F_{\hbar_x}|^2 \right) \cdot (-i\hbar \nabla_1) \gamma_N^{(2)} \right] \right| \leq CN \frac{\hbar \hbar_x}{R^4} \leq CN^{4\eta+2\varepsilon-1/2}.$$

Estimate of (3.13). We denote

$$I = \mathbf{A}_e(x_j) \cdot \nabla^\perp w_R(x_j - x_k) - \int_{\mathbb{R}^2} \mathbf{A}_e(x_j - u) \cdot \left(\nabla^\perp w_R * |F_{\hbar_x}|^2 \right)(x_j - x_k - u) |F_{\hbar_x}(u)|^2 du. \quad (3.24)$$

Then

$$\begin{aligned} 4\alpha \text{Tr} \left[\mathbf{K} \gamma_N^{(2)} \right] & \leq C\alpha \left\| \rho_{\gamma_N}^{(2)} \right\|_{L^1} \left\| \mathbf{A}_e \right\|_{L^\infty} \left\| \nabla^\perp w_R - \nabla^\perp w_R * |F_{\hbar_x}|^2 * |F_{\hbar_x}|^2 \right\|_{L^\infty} \\ & \leq C\alpha N^2 \hbar_x \left\| \Delta \nabla^\perp w_R \right\|_{L^\infty} \\ & \leq \frac{CN \hbar_x}{R^3} \leq CN^{3\eta+2\varepsilon} \end{aligned} \quad (3.25)$$

where we used (3.2) and (3.6) combined with the estimate (B.2).

Estimate of (3.14). We denote

$$\begin{aligned} \text{II} & = W_{123}(x_1, x_2, x_3) \\ & - \int_{\mathbb{R}^2} \left(\nabla^\perp w_R * |F_{\hbar_x}|^2 \right)(x_1 - x_2 - u) \cdot \left(\nabla^\perp w_R * |F_{\hbar_x}|^2 \right)(x_1 - x_3 - u) |F_{\hbar_x}(u)|^2 du. \end{aligned} \quad (3.26)$$

Then

$$\begin{aligned}
6\alpha^2 \operatorname{Tr} \left[L\gamma_N^{(3)} \right] &\leq \frac{CN\alpha^2}{R} \left\| \left(\nabla^\perp w_R - \nabla^\perp w_R * |F_{\hbar_x}|^2 * |F_{\hbar_x}|^2 \right) \rho_{\gamma_N}^{(2)} \right\|_{L^1} \\
&\leq \frac{CN\alpha^2}{R} \left\| \rho_{\gamma_N}^{(2)} \right\|_{L^1} \left\| \nabla^\perp w_R - \nabla^\perp w_R * |F_{\hbar_x}|^2 * |F_{\hbar_x}|^2 \right\|_{L^\infty} \\
&\leq C\alpha^2 \frac{N^3 \hbar_x}{R} \left\| \Delta \nabla^\perp w_R \right\|_{L^\infty} \\
&\leq \frac{CN\hbar_x}{R^4} \leq CN^{4\eta+2\varepsilon}
\end{aligned} \tag{3.27}$$

where we used [24, Lemma 2.1] and (3.19) combined with (B.2) to bound the last norm. Collecting all the previous estimates leads to the result. \square

4. ENERGY LOWER BOUND: MEAN-FIELD LIMIT

This section is dedicated to a lower bound on (1.6). We have reduced the problem to a classical one in the previous section. We now perform a mean-field approximation of the classical energy (3.8), showing that its infimum is (asymptotically) attained by a factorized Husimi function. The crucial point is to keep track of the semi-classical Pauli principle (1.20).

We first recall the Diaconis-Freedman theorem. We use it to rewrite many-particle probability measures as statistical superpositions of factorized ones. We then estimate quantitatively the probability for the measures in the superposition to violate the Pauli principle. This is the main novelty compared to the approach of [21]. We finally prove the convergence of the energy and states, Theorems 1.1 and 1.4.

4.1. The Diaconis-Freedman measure. For $Z_N \in \mathbb{R}^{4N}$ a set of points in the phase space, we define the associated empirical measure

$$\operatorname{Emp}_{Z_N} = \frac{1}{N} \sum_{i=1}^N \delta_{z_i}. \tag{4.1}$$

The following originates in [17].

Theorem 4.1 (Diaconis-Freedman).

Let $\mu_N \in \mathcal{P}_{\operatorname{sym}}(\mathbb{R}^{4N})$ the set of symmetric probability measures over \mathbb{R}^{4N} . Define the probability measure P_{μ_N} over $\mathcal{P}(\mathbb{R}^4)$

$$P_{\mu_N}(\sigma) := \int_{\mathbb{R}^{4N}} \delta_{\sigma = \operatorname{Emp}_{Z_N}} d\mu_N(z_1, \dots, z_N) \tag{4.2}$$

and set

$$\tilde{\mu}_N := \int_{\mathcal{P}(\mathbb{R}^4)} \rho^{\otimes N} dP_{\mu_N}(\rho)$$

with associated marginals

$$\tilde{\mu}_N^{(n)} = \int_{\rho \in \mathcal{P}(\mathbb{R}^4)} \rho^{\otimes n} dP_{\mu_N}^{\operatorname{DF}}(\rho). \tag{4.3}$$

Then

$$\left\| \mu_N^{(n)} - \tilde{\mu}_N^{(n)} \right\|_{\operatorname{TV}} \leq \frac{2n(n-1)}{N} \tag{4.4}$$

in total variation norm

$$\|\mu\|_{\text{TV}} = \sup_{\phi \in C_b(\mathbb{R}^4), \|\phi\|_{L^\infty} \leq 1} \left| \int_{\mathbb{R}^4} \phi d\mu \right|$$

with $C_b(\Omega)$ the set of continuous bounded functions on Ω .

Moreover, if the sequence $(\mu^{(1)})_N$ is tight, there exists a subsequence and a unique probability measure $P \in \mathcal{P}(\mathcal{P}(\mathbb{R}^4))$ such that (up to extraction)

$$P_{\mu_N} \rightharpoonup P$$

and, for any fixed $n \in \mathbb{N}^*$

$$\tilde{\mu}_N^{(n)} \rightharpoonup \int_{\rho \in \mathcal{P}(\Omega)} \rho^{\otimes n} dP(\rho). \quad (4.5)$$

The measure P obtained in the limit is that appearing in the Hewitt-Savage theorem [27]. See [56, 54, Theorems 2.1 and 2.2] for proofs and references. We will use the following consequence of the Diaconis-Freedman construction.

Lemma 4.2 (First marginals of the Diaconis-Freedman measure).

Let $\mu_N \in \mathcal{P}_{\text{sym}}(\mathbb{R}^{4N})$ and the associated Diaconis-Freedman approximation $\tilde{\mu}_N$ be given by the previous theorem. Using the notation $X_i = (x_i, p_i)$ we have

$$\tilde{\mu}_N^{(1)}(x_1, p_1) = \mu_N^{(1)}(x_1, p_1) \quad (4.6)$$

$$\tilde{\mu}_N^{(2)}(x_1, p_1, x_2, p_2) = \frac{N-1}{N} \mu_N^{(2)}(X_1, X_2) + \frac{1}{N} \mu_N^{(1)}(X_1) \delta_{(x_1, p_1) = (x_2, p_2)} \quad (4.7)$$

$$\begin{aligned} \tilde{\mu}_N^{(3)}(X_1, X_2, X_3) &= \frac{N(N-1)(N-2)}{N^3} \mu_N^{(3)}(X_1, X_2, X_3) \\ &+ \frac{(N-1)}{N^2} \left[\mu_N^{(2)}(X_1, X_3) \delta_{X_1=X_2} + \mu_N^{(2)}(X_2, X_1) \delta_{X_2=X_3} + \mu_N^{(2)}(X_3, X_2) \delta_{X_3=X_1} \right] \\ &+ \frac{1}{N^2} \mu_N^{(1)}(X_1) \delta_{X_1=X_2} \delta_{X_2=X_3} \end{aligned} \quad (4.8)$$

Proof. One can find (4.6) and (4.7) in [54, Remark 2.3]. It follows from [56, 54, Equation 2.16] that

$$\tilde{\mu}_N(X_N) = \int_{\mathbb{R}^{4N}} \mu_N(Z) \sum_{\gamma \in \Gamma_N} N^{-N} \delta_{X=Z_\gamma} dZ = \int_{\mathbb{R}^{4N}} \mu_N(Z) \left(N^{-1} \sum_{j=1}^N \delta_{z_j=(x_j, p_j)} \right)^{\otimes N} dZ \quad (4.9)$$

where Z_σ is the $4N$ -uple $(x_{\sigma(1)}, p_{\sigma(1)}, \dots, x_{\sigma(N)}, p_{\sigma(N)}) = (z_{\sigma(1)}, \dots, z_{\sigma(N)})$, $(z_i = (x_i, p_i))$ and Γ_N the set of maps from $\{1, \dots, N\}$ to itself (which allows repeated indices). We look at the n -th marginal

$$\tilde{\mu}_N^{(n)}(X_N) = \int_{\mathbb{R}^{4N}} \mu_N(Z) \left(N^{-1} \sum_{j=1}^N \delta_{z_j=X_j} \right)^{\otimes n} dZ. \quad (4.10)$$

In particular

$$\begin{aligned}\tilde{\mu}_N^{(1)}(x_1, p_1) &= N^{-1} \int_{\mathbb{R}^{4N}} \mu_N(Z) \left(\sum_{j=1}^N \delta_{z_j=X_1} \right) dZ \\ \tilde{\mu}_N^{(2)}(x_1, p_1, x_2, p_2) &= N^{-2} \int_{\mathbb{R}^{4N}} \mu_N(Z) \left(\sum_{j=1}^N \delta_{z_j=X_1} \right) \left(\sum_{j=1}^N \delta_{z_j=X_2} \right) dZ \\ \tilde{\mu}_N^{(3)}(X_1, X_2, X_3) &= N^{-3} \int_{\mathbb{R}^{4N}} \mu_N(Z) \left(\sum_{j=1}^N \delta_{z_j=X_1} \right) \left(\sum_{j=1}^N \delta_{z_j=X_2} \right) \left(\sum_{j=1}^N \delta_{z_j=X_3} \right) dZ.\end{aligned}$$

Expanding the n -th tensor power yields $\tilde{\mu}_N^{(n)}$ in terms of the marginals of μ_N by an explicit calculation. \square

We now apply the above constructions to the probability measure

$$m_N := \frac{m_{\Psi_N}^{(N)}(x_1, p_1, \dots, x_N, p_N)}{N!(2\pi\hbar)^{2N}} \quad (4.11)$$

to obtain an approximation of the marginals (cf Lemma 3.3)

$$m_N^{(k)} = \frac{m_{\Psi_N}^{(k)}(x_1, p_1, \dots, x_k, p_k)}{N(N-1)\dots(N-k+1)(2\pi\hbar)^{2k}}. \quad (4.12)$$

We use the notation W_{12} and W_{123} respectively defined in (2.6) and (2.7) to denote the two and the three body operators. In the former case, this is with a slight abuse of notation: we replace $p^{\mathbf{A}}$ by the classical vector $p_1 + \mathbf{A}_e(x)$ in the phase space.

Lemma 4.3 (Energy in terms of the Diaconis-Freedman measure).

Let $m_N \in \mathcal{P}_{\text{sym}}(\mathbb{R}^{4N})$ be as in (4.11). Let $P_N^{\text{DF}} \in \mathcal{P}(\mathcal{P}(\mathbb{R}^4))$ be the associated Diaconis-Freedman measure, as in Theorem 4.1. Evaluating each term of (3.11) gives

$$\frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} (|p^{\mathbf{A}}|^2 + V(x)) m_{f, \Psi_N}^{(1)}(x, p) dx dp = \int_{\mathbb{R}^4} \int_{\mu \in \mathcal{P}(\mathbb{R}^4)} (|p^{\mathbf{A}}|^2 + V(x)) d\mu(x, p) dP_N^{\text{DF}}(\mu) \quad (4.13)$$

$$\frac{1}{(2\pi)^4} \int_{\mathbb{R}^8} W_{12} m_{f, \Psi_N}^{(2)}(x_{12}, p_{12}) dx_{12} dp_{12} = \int_{\mathbb{R}^8} \int_{\mu \in \mathcal{P}(\mathbb{R}^4)} W_{12} d\mu^{\otimes 2} dP_N^{\text{DF}}(\mu) - \frac{C}{NR^2} \quad (4.14)$$

$$\frac{1}{(2\pi)^6} \int_{\mathbb{R}^{12}} W_{123} dm_{f, \Psi_N}^{(3)} = \int_{\mathbb{R}^{12}} \int_{\mu \in \mathcal{P}(\mathbb{R}^4)} W_{123} d\mu^{\otimes 3} dP_N^{\text{DF}}(\mu) - \frac{C}{NR^2}. \quad (4.15)$$

Proof. To apply the Diaconis-Freedman theorem (4.1) to the measures $m_{f, \Psi_N}^{(i)}$, $i = 1, 2, 3$ they have to be normalized as in (4.12). We then use the measure $m_N^{(i)}$ instead of $m_{f, \Psi_N}^{(i)}$ in our expressions, keeping track of the adequate normalization factors. Denoting $Z_i = (x_i, p_i)$ and recalling

$$(2\pi\hbar)^{2k} N(N-1)\dots(N-k+1) m_N^{(k)} = m_{\Psi_N}^{(k)}$$

we calculate

$$\frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} |p_1^{\mathbf{A}}|^2 m_{f, \Psi_N}^{(1)}(x_1, p_1) dx_1 dp_1 = \int_{\mathbb{R}^4} \int_{\mu \in \mathcal{P}(\mathbb{R}^4)} |p_1^{\mathbf{A}}|^2 \mu(x_1, p_1) dx_1 dp_1 dP_N^{\text{DF}}(\mu)$$

and

$$\begin{aligned} & \frac{1}{(2\pi)^4} \int_{\mathbb{R}^8} (p_1^{\mathbf{A}}) \cdot \nabla^\perp w_R(x_1 - x_2) m_{f, \Psi_N}^{(2)}(Z_1, Z_2) dZ_1 dZ_2 \\ &= N(N-1) \hbar^4 \int_{\mathbb{R}^8} (p_1^{\mathbf{A}}) \cdot \nabla^\perp w_R(x_1 - x_2) m_N^{(2)}(Z_1, Z_2) dZ_1 dZ_2 \\ &= N^2 \hbar^4 \int_{\mathbb{R}^8} \int_{\mu \in \mathcal{P}(\mathbb{R}^4)} (p_1^{\mathbf{A}}) \cdot \nabla^\perp w_R(x_1 - x_2) d\mu^{\otimes 2} dP_N^{\text{DF}}(\mu) \\ &\quad - N \hbar^4 \int_{\mathbb{R}^8} (p_1^{\mathbf{A}}) \cdot \nabla^\perp w_R(x_1 - x_2) m_N^{(1)}(X_1) \delta_{Z_1=Z_2} dZ_1 dZ_2. \end{aligned} \quad (4.16)$$

Moreover

$$\begin{aligned} & \int_{\mathbb{R}^{12}} \nabla^\perp w_R(x_1 - x_2) \cdot \nabla^\perp w_R(x_1 - x_3) m_N^{(3)}(Z_1, Z_2, Z_3) dZ_1 dZ_2 dZ_3 \\ &= \frac{N^2}{(N-1)(N-2)} \int_{\mathbb{R}^{12}} \int_{\mu \in \mathcal{P}(\mathbb{R}^4)} \nabla^\perp w_R(x_1 - x_2) \cdot \nabla^\perp w_R(x_1 - x_3) d\mu^{\otimes 3} dP_N^{\text{DF}}(\mu) \\ &\quad - \frac{1}{(N-2)} \int_{\mathbb{R}^{12}} W_{123} \left(m_N^{(2)}(Z_1, Z_3) \delta_{Z_1=Z_2} + m_N^{(2)}(Z_2, Z_1) \delta_{Z_2=Z_3} + m_N^{(2)}(Z_3, Z_2) \delta_{Z_3=Z_1} \right) \end{aligned} \quad (4.17)$$

$$- \frac{1}{(N-1)(N-2)} \int_{\mathbb{R}^{12}} W_{123} m_N^{(1)}(Z_1) \delta_{Z_1=Z_2} \delta_{Z_2=Z_3}. \quad (4.18)$$

We will discard the error terms in (4.16) (4.17) and (4.18) using [24, Lemma 2.1]

$$\left\| \nabla^\perp w_R \right\|_{L^\infty} \leq R^{-1}$$

and the kinetic energy bound (3.1). The latter, combined with equation (3.3) gives

$$\iint_{\mathbb{R}^4} |p|^2 dm_{f, \Psi_N}^{(1)}(x, p) \leq \frac{C}{R^2}. \quad (4.19)$$

Applying the Cauchy-Schwarz inequality to the two particles term we obtain

$$\begin{aligned} & \left| \frac{1}{N} \int_{\mathbb{R}^8} (p_1^{\mathbf{A}}) \cdot \nabla^\perp w_R(x_1 - x_2) m_N^{(1)}(Z_1) \delta_{Z_1=Z_2} dZ_1 dZ_2 \right| \\ & \leq \frac{C \|\mathbf{A}_\epsilon\|_{L^\infty}}{NR} + \frac{1}{N} \left(\int_{\mathbb{R}^8} \left| \nabla^\perp w_R(x_1 - x_2) \right|^2 m_N^{(1)} \delta_{Z_1=Z_2} dZ_1 dZ_2 \right)^{1/2} \left(\int_{\mathbb{R}^4} |p|^2 dm_N^{(1)} \right)^{1/2} \\ & \leq \frac{C}{NR^2}. \end{aligned}$$

As for the three-particles term we have

$$\begin{aligned} \frac{1}{N} \left| \int_{\mathbb{R}^{12}} W_{123} \left(m_N^{(2)}(Z_1, Z_3) \delta_{Z_1=Z_2} \right) \right| &\leq \frac{C}{NR^2} \\ \frac{1}{N^2} \left| \int_{\mathbb{R}^{12}} W_{123} \left(m_N^{(1)}(Z_1) \delta_{Z_1=Z_2} \delta_{Z_2=Z_3} \right) \right| &\leq \frac{C}{N^2 R^2}. \end{aligned}$$

□

4.2. Quantitative semi-classical Pauli principle. Lemma (4.3) allows to write the energy as an integral of \mathcal{E}_V^R over $\mathcal{P}(\mathbb{R}^4)$ plus some negligible error terms. Assuming $R = N^{-\eta}$ for some $\eta < 1/4$, we get from Proposition 3.4 and the above lemma:

$$\frac{\langle \Psi_N, H_N^R \Psi_N \rangle}{N} \geq \int_{\mathcal{P}(\mathbb{R}^4)} \mathcal{E}_V^R[\mu] dP_{m_N}^{\text{DF}}(\mu) - o_N(1) \quad (4.20)$$

where

$$\mathcal{E}_V^R[\mu] = \int_{\mathbb{R}^4} |p + \mathbf{A}_e(x) + \beta \mathbf{A}^R[\rho](x)|^2 \mu(x, p) dx dp + \int_{\mathbb{R}^2} V(x) \rho(x) dx \quad (4.21)$$

with

$$\rho(x) = \int_{\mathbb{R}^2} \mu(x, p) dp. \quad (4.22)$$

Note that

$$\mathcal{E}_V^R \left[\frac{\mu}{(2\pi)^2} \right] = \mathcal{E}_{\text{Vla}}^R[\mu]$$

defined in (1.17). However, P_N^{DF} only charges empirical measures which do not satisfy the Pauli principle (3.6). To circumvent this issue we divide the phase-space

$$\cup_{m \in \mathbb{N}} \Omega_m = \mathbb{R}^4 \quad (4.23)$$

in hyperrectangles labeled $(\Omega_m)_{m \in \mathbb{N}}$. We take them to have side-length l_x in the space coordinates and l_p in the momentum coordinates, ensuring that

$$|\Omega_m| = l_x^2 l_p^2 = N^\beta. \quad (4.24)$$

The parameter $\beta > 0$ and the lengths l_x, l_p will be chosen in the sequel.

If each Ω_m contains less than $(1 + \varepsilon)(2\pi)^{-2} |\Omega_m|$ points (for some small ε) our measure will approximately satisfy the Pauli principle. We thus have to estimate the probability for a box to have the right density of points. This is the purpose of the following lemmas. We denote for a given measure μ :

$$\mathbb{P}_\mu(\Omega) = \int_\Omega d\mu. \quad (4.25)$$

Theorem 4.4 (Probability of violating the Pauli principle in a phase-space box).

Let m_N be as in (4.11) and P_N^{DF} the associated Diaconis-Freedman measure. Recall the tiling (4.23) and assume (4.24) with $\beta < 1$. For any

$$0 < \delta < \frac{1 - \beta}{2}$$

we have, for constant $C_\delta > 0, c_\delta > 0$, that

$$\mathbb{P}_{P_N^{\text{DF}}} \left(\left\{ \text{Emp}_{Z_N}, \int_{\Omega_m} \text{Emp}_{Z_N} \geq \frac{(1+\varepsilon)}{(2\pi)^2} |\Omega_m| \right\} \right) \leq C_\delta e^{-c_\delta N^\delta \ln(1+\varepsilon)}. \quad (4.26)$$

Note that the condition $\beta < 1$ means $l_x^2 l_p^2 \geq N^{-1}$. Since the average interparticle distance in phase-space is $N^{-1/4}$, our hyperrectangles typically contain a large number of particles.

We start the proof of the above theorem with the

Lemma 4.5 (Expansion of the Diaconis-Freedman measure).

Let

$$\tilde{m}_N = \int_{\mathcal{P}(\mathbb{R}^4)} \mu^{\otimes N} dP_N^{\text{DF}}(\mu)$$

be associated to (4.11) as in Theorem 4.1, and $\tilde{m}_N^{(n)}$ the associated marginals. For any $\Omega \subset \mathbb{R}^4$ we have

$$\int_{\Omega^n} \tilde{m}_N^{(n)} \leq N^{-n} \sum_{k=1}^n \frac{|\Omega|^k S(n, k)}{(2\pi\hbar)^{2k}} \quad (4.27)$$

with $S(n, k)$ the Stirling number of the second kind, the number of ways to partition a set of n objects into k non-empty subsets.

We refer to [8] for combinatorial definitions and estimates.

Proof of Lemma 4.5. The measure $\tilde{\mu}_N$ constructed from a $\mu_N \in \mathcal{P}_{\text{sym}}(\mathbb{R}^{4N})$ in Theorem 4.1 satisfies (4.9). Hence

$$\tilde{\mu}_N^{(n)} = N^{-n} \sum_{k=1}^n \sum_{\gamma \in \Gamma_{n,k}} \int_{\mathbb{R}^{4(N-n)}} \mu_N(Z) \delta_{X_N=Z_\gamma} dZ \quad (4.28)$$

with $\Gamma_{n,k}$ the set of ordered samples of n indices amongst N (allowing repetitions) with exactly k distinct indices. Integrating on each side of (4.28) and using the symmetry of (recall Lemma 3.3)

$$\mu_N^{(k)} = m_N^{(k)} \leq \frac{1}{(2\pi\hbar)^{2k} N(N-1)\dots(N-k+1)}$$

we obtain

$$\int_{\Omega^n} \tilde{m}_N^{(n)} \leq N^{-n} \sum_{k=1}^n \frac{|\Gamma_{n,k}| |\Omega|^k}{(2\pi\hbar)^{2k} N(N-1)\dots(N-k+1)}.$$

Now we calculate the cardinal $|\Gamma_{n,k}|$. For a given k , choose k indices among N (without order) to, next, be distributed in n boxes. The first step gives us

$$\frac{N!}{(N-k)!k!}$$

choices. For each of these choices we have to distribute n balls in k boxes (boxes can be empty) which corresponds to the number of surjections from $\{1, 2, \dots, n\}$ to $\{1, 2, \dots, k\}$, $M(n, k)$ (with $k \leq n$). Then

$$|\Gamma_{n,k}| = \frac{N!}{(N-k)!k!} M(n, k)$$

with [8, Theorem 1.17]

$$M(n, k) = k!S(n, k)$$

and $S(n, k)$ is the Stirling number of the second kind [8, Lemma 1.16]. Thus

$$|\Gamma_{n,k}| = \frac{N!}{(N-k)!} S(n, k).$$

□

We now need a rough control of the sum appearing in (4.27) when the set Ω is not too small.

Lemma 4.6 (Control of the sum).

Take $|\Omega_m| = N^{-\beta}$ with $\beta < 1$. Choose some

$$0 < \delta < \frac{1-\beta}{2}$$

and define

$$n = \lfloor N^\delta \rfloor$$

the integer part of N^δ . We have that

$$(2\pi)^{2n} N^{-n} \sum_{k=1}^n \frac{|\Omega_m|^{(k-n)} S(n, k)}{(2\pi\hbar)^{2k}} \leq C N^{\delta'}$$

for some exponent $\delta' > 0$ depending only on δ .

Proof. We recall Stirling's formula

$$C_1 \sqrt{N} \left(\frac{N}{e}\right)^N \leq N! \leq C_2 \sqrt{N} \left(\frac{N}{e}\right)^N \quad (4.29)$$

and a bound for the Stirling number of the second kind from [53](see also [49, 33])

$$S(n, k) \leq C \binom{n}{k} k^{(n-k)}. \quad (4.30)$$

Inserting these bounds we have that

$$(2\pi)^{2n} N^{-n} \frac{|\Omega_m|^{(k-n)} S(n, k)}{(2\pi\hbar)^{2k}} \leq C \sqrt{\frac{n}{k(n-k)}} \exp(f_\beta(k))$$

with

$$\begin{aligned} f_\beta(k) &= (n-k) \left(\log k + (\beta-1) \log N + 2 \log(2\pi) \right) + n \log n - k \log k - (n-k) \log(n-k) \\ &= (n-k) \left(2 \log k + \log n + (\beta-1) \log N + 2 \log(2\pi) - \log(n-k) \right) + k \log n - n \log k \\ &= (n-k) \left(\log k + \log n + (\beta-1) \log N + 2 \log(2\pi) - \log(n-k) \right) + k \log \left(1 + \frac{n-k}{k} \right) \\ &\leq (n-k) \left(\log k + \log n + (\beta-1) \log N + 2 \log(2\pi) + 1 - \log(n-k) \right), \end{aligned}$$

using $\log(1+x) \leq x$. Since $k \leq n \sim N^\delta$ we find that the leading order of the above is

$$f_\beta(k) \lesssim (n-k)(2\delta + \beta - 1) \log N \leq 0$$

under our assumptions. Hence we find

$$(2\pi)^{2n} N^{-n} \sum_{k=1}^n \frac{|\Omega_m|^{(k-n)} S(n, k)}{(2\pi\hbar)^{2k}} \leq C \sum_{k=1}^n \sqrt{\frac{n}{k(n-k)}},$$

which yields the result. \square

Now we can complete the

Proof of Theorem 4.4. We call

$$\Gamma = \left\{ \text{Emp}_{Z_N}, \int_{\Omega_m} \text{Emp}_{Z_N} \geq \frac{(1+\varepsilon)}{(2\pi)^2} |\Omega_m| \right\} \quad (4.31)$$

the set of all empirical measure violating the Pauli principle in Ω_m by a finite amount. We have, using (4.3) that, for any $n \leq N$,

$$\frac{(1+\varepsilon)^n}{(2\pi)^{2n}} |\Omega_m|^n \mathbb{P}_{P_N^{\text{DF}}}(\Gamma) \leq \int_{\rho \in \Gamma} \left(\int_{\Omega_m} \rho^{\otimes n} \right) dP_N^{\text{DF}}(\rho) = \int_{(\Omega_m)^n} \tilde{\mu}_N^{(n)}.$$

Next, using Lemma 4.5 we obtain

$$\mathbb{P}_{P_N^{\text{DF}}}(\Gamma) \leq (1+\varepsilon)^{-n} (2\pi)^{2n} N^{-n} \sum_{k=1}^n \frac{|\Omega_m|^{(k-n)} S(n, k)}{(2\pi\hbar)^{2k}}.$$

Choosing n as in Lemma 4.6 and inserting the latter result we deduce

$$\mathbb{P}_{P_N^{\text{DF}}}(\Gamma) \leq C \frac{N^{\delta'}}{(1+\varepsilon)^n} \leq CN^{\delta'} e^{-N^\delta \ln(1+\varepsilon)}$$

and the result follows. \square

4.3. Averaging the Diaconis-Freedman measure. We divide \mathbb{R}^4 into hyperrectangles

$$\Omega_m = \Omega_{m_x} \times \Omega_{m_p}$$

as in (4.23). We set

$$|\Omega_m| = |\Omega_{m_x}| |\Omega_{m_p}| = l_x^2 l_p^2 = N^{-\beta} \quad (4.32)$$

for some $0 < \beta < 1$ to be fixed later. We know from Theorem 4.4 that the probability for one such hyperrectangle to violate the Pauli principle is exponentially small. This means that the contribution to (4.20) of empirical measures

$$\mu = \frac{1}{N} \sum_{j=1}^N \delta_{z_j}$$

with $Z_N = (z_1, \dots, z_j) \in \Gamma$ (see (4.31)) will be negligible (by the union bound), provided the number of hyperrectangles is not too large. The next steps of our proof are thus

- to reduce estimates to the contribution of a finite set of phase-space.
- in the latter set, to average empirical measures to replace μ in (4.20) by true Pauli-principle abiding measures (in the sense of (1.20)).

As for the first step, let $L > 0$ be a number of the form

$$L = nN^{-\frac{\beta}{4}} \quad (4.33)$$

for some $n \in \mathbb{N}$ and define the hypercube

$$S_L = [-L, L]^4$$

of side $2L$ centered at the origin. This way the number of Ω_m -hyperrectangles contained within S_L is $2^4 n^4$. We first discard the energetic contribution of S_L^c .

Lemma 4.7 (Reduction to the energy in the phase-space hypercube S_L).

Let $\sigma > 0$ and μ a probability measure satisfying

$$\int_{\mathbb{R}^4} (|p|^2 + V(x)) \, d\mu(x, p) \leq D. \quad (4.34)$$

Its energy can be bounded from below as

$$\mathcal{E}_V^R[\mu] \geq (1 - \sigma) \int_{\mathbb{R}^4} (|p^{\mathbf{A}} + \beta \mathbf{A}^R[\mathbb{1}_{S_L}\mu]|^2 + V) \mathbb{1}_{S_L} \, d\mu(x, p) - C \frac{D^2}{\sigma R^2 \inf(L^4, L^{2s})} \quad (4.35)$$

where

$$\mathbf{A}^R[\mathbb{1}_{S_L}\mu] = \int_{\mathbb{R}^4} \nabla^\perp w_R(x - y) \mathbb{1}_{S_L}(y, p) \, d\mu(y, p) = \mathbf{A}^R[\rho \mathbb{1}_{S_L}\mu]$$

and

$$p^{\mathbf{A}} = p + \mathbf{A}_e(x).$$

The parameter σ will tend to 0 at the very end of the proof (after we take $N \rightarrow \infty, L \rightarrow \infty, R \rightarrow 0$ while ensuring $R^6 \inf(L^4, L^{2s}) \rightarrow \infty$).

Proof. First, the positivity of the integrand gives

$$\mathcal{E}_V^R[\mu] \geq \int_{S_L} (|p^{\mathbf{A}} + \mathbf{A}^R[\mu]|^2 + V) \, d\mu(x, p)$$

Then we split $\mathbf{A}^R[\mu]$ into a contribution from S_L and one from its complement,

$$\mathbf{A}^R[\mu] = \mathbf{A}^R[\mathbb{1}_{S_L}\mu] + \mathbf{A}^R[\mathbb{1}_{S_L^c}\mu],$$

obtaining

$$\begin{aligned} (p^{\mathbf{A}} + \mathbf{A}^R[\mathbb{1}_{S_L}\mu] + \mathbf{A}^R[\mathbb{1}_{S_L^c}\mu])^2 &= (p^{\mathbf{A}} + \mathbf{A}^R[\mathbb{1}_{S_L}\mu])^2 + |\mathbf{A}^R[\mathbb{1}_{S_L^c}\mu]|^2 \\ &\quad + 2(p^{\mathbf{A}} + \mathbf{A}^R[\mathbb{1}_{S_L}\mu]) \cdot \mathbf{A}^R[\mathbb{1}_{S_L^c}\mu]. \end{aligned}$$

Next we use that $ab \leq \frac{a^2}{2\sigma} + \frac{\sigma b^2}{2}$ to deduce

$$2(p^{\mathbf{A}} + \mathbf{A}^R[\mathbb{1}_{S_L}\mu]) \cdot \mathbf{A}^R[\mathbb{1}_{S_L^c}\mu] \geq -\sigma (p^{\mathbf{A}} + \mathbf{A}^R[\mathbb{1}_{S_L}\mu])^2 - \frac{1}{\sigma} |\mathbf{A}^R[\mathbb{1}_{S_L^c}\mu]|^2.$$

But

$$\begin{aligned}
\int_{S_L} |\mathbf{A}^R [\mathbf{1}_{S_L^c} \mu]|^2(x) d\mu(x, p) &\leq \|\mathbf{A}^R [\mathbf{1}_{S_L^c} \mu]\|_{L^\infty}^2 \\
&\leq \left\| \nabla^\perp w_R * \mathbf{1}_{S_L^c} \mu \right\|_{L^\infty}^2 \\
&\leq \|\mathbf{1}_{S_L^c} \mu\|_{L^1}^2 \left\| \nabla^\perp w_R \right\|_{L^\infty}^2 \\
&\leq \frac{C}{R^2} \|\mathbf{1}_{S_L^c} \mu\|_{L^1}^2
\end{aligned}$$

where we used Young's inequality combined with [24, Lemma 2.1]. The measure μ satisfies (4.34) so, using Assumption 1.1,

$$\int_{\mathbb{R}^4} \mathbf{1}_{S_L^c} \mu(x, p) dx dp \leq \frac{D}{\inf(L^2, L^s)}$$

which concludes the proof. \square

We now turn to the averaging procedure turning empirical measures based on configurations from Γ^c (recall (4.31)) into bounded measures satisfying the Pauli principle (1.20):

Definition 4.8 (Averaging map).

Let μ be a positive measure on the phase-space hypercube $S_L = [-L, L]^4$. We associate to it its local average with respect to the grid (4.23):

$$\text{Ave}[\mu] = \sum_{m=1}^{16n^4} \mathbf{1}_{\Omega_m} \int_{\Omega_m} \frac{\mu}{|\Omega_m|} \quad (4.36)$$

with the associated position-space density (recall that $\Omega_m = \Omega_{m_x} \times \Omega_{m_p}$)

$$\rho_{\text{Ave}[\mu]} := \int \text{Ave}[\mu] dp = \sum_{m=1}^{16n^4} \mathbf{1}_{\Omega_{m_x}} \int_{\Omega_m} \frac{\mu}{|\Omega_{m_x}|}. \quad (4.37)$$

Note that $\text{Ave}[\mu]$ lives on S_L by definition. We next quantify the energetic cost of the above procedure:

Theorem 4.9 (Averaging the empirical measure).

Assume $\beta < 1$ in (4.24) as well as

$$l_x \ll R, \quad l_p \ll 1 \text{ in the limit } N \rightarrow \infty. \quad (4.38)$$

Let $Z_N = (x_1, p_1, \dots, x_N, p_N)$ with $\text{Emp}_{Z_N} \in \Gamma_\varepsilon^c$,

$$\Gamma_\varepsilon = \left\{ \text{Emp}_{Z_N}, \exists \Omega_m \subset S_L \text{ such that } \int_{\Omega_m} \text{Emp}_{Z_N} \geq \frac{(1+\varepsilon)}{(2\pi)^2} |\Omega_m| \right\}. \quad (4.39)$$

Assume that (4.34) holds for $\mu = \text{Emp}_{Z_N}$. Then, for any $\sigma > 0$,

$$\mathcal{E}_V^R[\text{Emp}_{Z_N}] \geq (1-\sigma)(1-o_N(1)) \mathcal{E}_V^R[\text{Ave}[\text{Emp}_{Z_N}]] - o_N(1) - \frac{CD^2}{\sigma R^2 \inf(L^4, L^{2s})}. \quad (4.40)$$

Proof. In this proof we denote

$$\mu = \text{Emp}_{Z_N}$$

for brevity. We first use Lemma 4.7 to write

$$\mathcal{E}_V^R[\mu] \geq (1 - \sigma)\mathcal{E}_V^R[\mathbb{1}_{S_L}\mu] - C\frac{D^2}{\sigma R^2 \inf(L^4, L^{2s})}$$

and now estimate $|\mathcal{E}_V^R[\mathbb{1}_{S_L}\mu] - \mathcal{E}_V^R[\text{Ave}[\mu]]|$. We expand \mathcal{E}_V^R (4.21) and proceed term by term. Recall from (4.33) that

$$S_L = \cup_{m=1}^{2^4 n^4} \Omega_m.$$

We first calculate

$$\begin{aligned} \int_{S_L} ((p^{\mathbf{A}})^2 + V(x)) d(\mu - \text{Ave}[\mu]) &= \int_{S_L} ((p^{\mathbf{A}})^2 + V(x)) \text{Emp}_{Z_N} dx dp \\ &\quad - \frac{1}{N} \sum_{m=1}^{16n^4} \int_{\Omega_m} \frac{(p^{\mathbf{A}})^2 + V(x)}{|\Omega_m|} dx dp \int_{\Omega_m} \sum_{i=1}^N \delta_{z_i=(x,p)} dx dp. \end{aligned}$$

We denote

$$\begin{aligned} G_N &= \{i, z_i \in S_L\} \\ G_N^c &= \{i, z_i \in S_L^c\} \end{aligned}$$

and let $m(i)$ be the label of the box Ω_m particle z_i is in. This way

$$\sum_{i=1}^N \int_{\Omega_m} \delta_{z_i=(x,p)} dx dp = \sum_{i \in G_N} \delta_{m=m(i)}$$

and

$$\begin{aligned} \int_{S_L} ((p^{\mathbf{A}})^2 + V(x)) d(\mu - \text{Ave}[\text{Emp}_{Z_N}]) &= \frac{1}{N} \sum_{i \in G_N} ((p_i^{\mathbf{A}})^2 + V(x_i)) \\ &\quad - \int_{\Omega_{m(i)}} \frac{(p^{\mathbf{A}})^2 + V(x)}{|\Omega_{m(i)}|} dx dp. \end{aligned}$$

By the mean value theorem applied in each Ω_m there exists a point $\tilde{z}_i = (\tilde{x}_i, \tilde{p}_i) \in \Omega_{m(i)}$ such that

$$\int_{\Omega_{m(i)}} \frac{(p^{\mathbf{A}})^2 + V(x)}{|\Omega_{m(i)}|} dx dp = (\tilde{p}_i^{\mathbf{A}})^2 + V(\tilde{x}_i)$$

and we have, using Assumption 1.1 to control the variations of \mathbf{A}_e ,

$$\begin{aligned} &\left| \int_{S_L} ((p^{\mathbf{A}})^2 + V(x)) d(\text{Ave}[\text{Emp}_{Z_N}] - \mu) \right| \leq \frac{1}{N} \sum_{i \in G_N} |((p_i^{\mathbf{A}})^2 - (\tilde{p}_i^{\mathbf{A}})^2 + V(x_i) - V(\tilde{x}_i))| \\ &\leq \frac{C}{N} \sum_{i \in G_N} \left(\sup_{z=(x,p) \in \Omega_{m(i)}} l_p |p| + l_x \sup_{z=(x,p) \in \Omega_{m(i)}} |\nabla V(x)| \right) + Cl_x \\ &= \frac{C}{N} \sum_{\Omega_m \subset S_L} \left(N_m l_p \sup_{z=(x,p) \in \Omega_m} |p| + N_m l_x \sup_{z=(x,p) \in \Omega_m} |\nabla V(x)| \right) + Cl_x \end{aligned}$$

where we denote

$$N_m := \#\{z_i \in Z_N \cap \Omega_m\} \leq C|\Omega_m|. \quad (4.41)$$

Since our assumptions imply that for any $\Omega_m \subset S_L$

$$N_m \leq C|\Omega_m| \quad (4.42)$$

we may recognize Riemann sums and use Assumption 1.1 again to deduce

$$\begin{aligned} & \left| \int_{S_L} ((p^{\mathbf{A}})^2 + V(x)) \, d(\text{Ave} [\text{Emp}_{Z_N}] - \mu) \right| \\ & \leq Cl_p \int_{\mathbb{R}^4} |p| \, d\text{Ave} [\mu] + Cl_x \int_{\mathbb{R}^4} |x|^{s-1} \, d\text{Ave} [\mu] + Cl_p \\ & \leq Cl_p \int_{\mathbb{R}^4} |p|^2 \, d\text{Ave} [\mu] + Cl_x \int_{\mathbb{R}^4} V \, d\text{Ave} [\mu] + Cl_x + Cl_p \end{aligned} \quad (4.43)$$

using Young's inequality

$$r^{s-1} \leq \frac{s-1}{s} r^s + \frac{1}{s}.$$

We next treat the two body operator W_{12} defined in (2.6) (understanding now p as a momentum variable rather than an operator) in the same way. We set

$$dx = dx_1 dx_2, \quad dp = dp_1 dp_2.$$

By the mean-value theorem, for each $(x_i, p_i) \in \Omega_{m(i)}$ there exists some $\tilde{z}_i = (\tilde{x}_i, \tilde{p}_i) \in \Omega_{m(i)}$ such that

$$\begin{aligned} & \int_{\mathbb{R}^8} W_{12} \, d \left(\text{Ave} [\text{Emp}_{Z_N}]^{\otimes 2} - (\mathbb{1}_{S_L} \mu)^{\otimes 2} \right) = \int_{S_L \times S_L} W_{12} \left[\mu^{\otimes 2} - \text{Ave} [\text{Emp}_{Z_N}]^{\otimes 2} \right] \, dx dp \\ & = \frac{1}{N^2} \sum_{i \in G_N} \sum_{j \in G_N} \left[(p_i + \mathbf{A}_e(x_i)) \cdot \nabla^\perp w_R(x_i, x_j) - (\tilde{p}_i + \mathbf{A}_e(\tilde{x}_i)) \cdot \nabla^\perp w_R(\tilde{x}_i, \tilde{x}_j) \right]. \end{aligned} \quad (4.44)$$

Expanding the functions around the points \tilde{z}_i of the sum there exists a $c_i \in \Omega_{m(i)}$ such that

$$\begin{aligned} & \left| \nabla^\perp w_R(x_i) - \nabla^\perp w_R(\tilde{x}_i) \right| \leq J_{\nabla w_R}(c_i) \cdot (x_i - \tilde{x}_i) \\ & |p_i - \tilde{p}_i| \leq l_p \end{aligned}$$

with $J_{\nabla w_R}$ the Jacobian matrix of ∇w_R . We denote

$$W = \partial_u^2 w_R \quad (4.45)$$

where ∂_u is the radial derivative explicitly calculated in (B.8). We then use (B.1) and estimate

$$\begin{aligned} & \left| \int_{\mathbb{R}^8} W_{12} \, d \left(\text{Ave} [\text{Emp}_{Z_N}]^{\otimes 2} - (\mathbb{1}_{S_L} \mu)^{\otimes 2} \right) \right| \\ & \leq \frac{C}{N^2} \sum_{\Omega_m \subset S_L} \sum_{\Omega_q \subset S_L} l_x N_m N_q \sup_{z_m \in \Omega_m, z_q \in \Omega_q} |p_m| |W(x_m - x_q)| \\ & \quad + \frac{C}{N^2} \sum_{\Omega_m \subset S_L} \sum_{\Omega_q \subset S_L} N_m N_q l_p \sup_{z_m \in \Omega_m, z_q \in \Omega_q} \left| \nabla^\perp w_R(x_m - x_q) \right| \end{aligned} \quad (4.46)$$

where we used that \mathbf{A}_e and its derivatives are in L^∞ by assumption and use the convention $z_i = (x_i, p_i)$ systematically. We denote

$$\begin{aligned}\overline{W}_1(x_1 - x_2) &= \sup \{|W(y_1 - y_2)|, y_1 \in \Omega_{m_x}(x_1), y_2 \in \Omega_{m_x}(x_2)\} \\ \overline{W}_2(x_1 - x_2) &= \sup \{|\nabla^\perp w_R(y_1 - y_2)|, y_1 \in \Omega_{m_x}(x_1), y_2 \in \Omega_{m_x}(x_2)\} \\ P(p_1) &= \sup \{|p|, p \in \Omega_{m_p}(p)\}\end{aligned}$$

with $\Omega_{m_x}(x) \subset \mathbb{R}^2$ and $\Omega_{m_p}(p) \subset \mathbb{R}^2$ the square of our grid (4.32) that x belongs to (respectively $\Omega_{m_p}(p)$ the square p belongs to). Hence

$$\begin{aligned}& \left| \int_{\mathbb{R}^8} W_{12} d \left(\text{Ave} [\text{Emp}_{Z_N}]^{\otimes 2} - (\mathbf{1}_{S_L} \mu)^{\otimes 2} \right) \right| \\ & \leq Cl_x \iint_{\mathbb{R}^8} P(p_1) \overline{W}_1(x_1 - x_2) \text{Ave}[\mu](x_1, p_1) \text{Ave}[\mu](x_2, p_2) dx_1 dx_2 dp_1 dp_2 \\ & \quad + Cl_p \iint_{\mathbb{R}^8} \overline{W}_2(x_1 - x_2) \text{Ave}[\mu](x_1, p_1) \text{Ave}[\mu](x_2, p_2) dx_1 dx_2 dp_1 dp_2 =: \text{I} + \text{II}.\end{aligned}$$

Recognizing a Riemann sum similarly as in (4.43) yields, using the properties of W (see (B.8)),

$$\begin{aligned}\text{I} & \leq Cl_x \int_{\mathbb{R}^4} |p_1| \text{Ave}[\mu](x_1, p_1) \int_{\mathbb{R}^2} \left(\frac{1}{R} |\nabla w_0(x_1 - x_2)| + \frac{1}{R^2} \mathbf{1}_{|x_1 - x_2| \leq R} \right) \rho_{\text{Ave}[\mu]}(x_2) dx_2 dx_1 dp_1 \\ & \leq C \frac{l_x}{R} \|\rho_{\text{Ave}[\mu]}\|_{L^2} \int |p| \text{Ave}[\mu](x, p) dx dp \\ & \leq C \frac{l_x}{R} \|\rho_{\text{Ave}[\mu]}\|_{L^2} \left(\int |p|^2 \text{Ave}[\mu](x, p) dx dp \right)^{1/2}\end{aligned}$$

where we also used the weak Young and the Cauchy-Schwarz inequality. Similarly, since $\nabla w_R \in L^{2,w}$,

$$\text{II} \leq Cl_p \iint_{\mathbb{R}^4} |\nabla w_R(x_1 - x_2)| \rho_{\text{Ave}[\mu]}(x_1) \rho_{\text{Ave}[\mu]}(x_2) dx_1 dx_2 \leq Cl_p \|\rho_{\text{Ave}[\mu]}\|_{L^2}.$$

We next turn to the three-body term (2.7)

$$\text{III} := \left| \int_{\mathbb{R}^{12}} W_{123} d \left((\mathbf{1}_{S_L} \mu)^{\otimes 3} - \text{Ave} [\text{Emp}_{Z_N}]^{\otimes 3} \right) \right|$$

with

$$W_{123} = \nabla^\perp w_R(x_1 - x_2) \nabla^\perp w_R(x_1 - x_3).$$

We denote (with W as in (4.45))

$$\begin{aligned}G(x_1, x_2, x_3) &= |W(x_1 - x_2)| |\nabla^\perp w_R(x_1 - x_3)| + |\nabla^\perp w_R(x_1 - x_2)| |W(x_1 - x_3)| \\ \overline{G}(x_1, x_2, x_3) &= \sup \{G(y_1, y_2, y_3), y_1 \in \Omega_{m_x}(x_1), y_2 \in \Omega_{m_x}(x_2), y_3 \in \Omega_{m_x}(x_3)\}\end{aligned}$$

and observe that G gives a pointwise upper bound to gradients of W_{123} in any variable. Hence, arguing as above

$$\begin{aligned} \text{III} &\leq Cl_x \int_{R^6} \overline{G}(x_1, x_2, x_3) \rho_{\text{Ave}[\mu]}(x_1) \rho_{\text{Ave}[\mu]}(x_2) \rho_{\text{Ave}[\mu]}(x_3) dx_1 dx_2 dx_3 \\ &\leq C \frac{l_x}{R} \|\rho_{\text{Ave}[\mu]}\|_{L^2}^2 \end{aligned}$$

where we used the weak Young inequality twice. Collecting the previous inequalities we conclude

$$\begin{aligned} \mathcal{E}_V^R[\mu] &\geq (1 - \sigma) \mathcal{E}_V^R[\text{Ave}[\mu]] - C(l_x + l_p) \\ &\quad - C \frac{l_x}{R} \|\rho_{\text{Ave}[\mu]}\|_{L^2} \left(\|\rho_{\text{Ave}[\mu]}\|_{L^2} + \left(\int |p|^2 d\text{Ave}[\mu] \right)^{1/2} \right) \\ &\quad - Cl_p \|\rho_{\text{Ave}[\mu]}\|_{L^2} - C \frac{D^2}{\sigma R^2 \inf(L^4, L^{2s})} \\ &\geq (1 - \sigma) \mathcal{E}_V^R[\text{Ave}[\mu]] - o_N(1) - o_N(1) \int |p|^2 d\text{Ave}[\mu] - C \frac{D^2}{\sigma R^2 \inf(L^4, L^{2s})}, \end{aligned}$$

inserting our assumptions on the various parameters involved $l_x^2 = N^{-\beta+\varepsilon} \ll R^2$ and $l_p^2 = N^{-\varepsilon}$. We have also used the fact that, since

$$\text{Ave}[\mu] \leq \frac{1 + \varepsilon}{2\pi^2}, \quad (4.47)$$

the bathtub principle [35, Theorem 1.14] implies that

$$\int |p|^2 d\text{Ave}[\mu] \geq C_\varepsilon \|\rho_{\text{Ave}[\mu]}\|^2.$$

Finally, arguing as in Appendix A below (using (4.47) again),

$$\mathcal{E}_V^R[\text{Ave}[\mu]] \geq \frac{1}{2} \int |p|^2 d\text{Ave}[\mu] - C \|\rho_{\text{Ave}[\mu]}\|^2$$

and we can absorb an error term in the main term to obtain

$$\mathcal{E}_V^R[\mu] \geq (1 - \sigma)(1 - o_N(1)) \mathcal{E}_V^R[\text{Ave}[\mu]] - o_N(1) - o_N(1) \|\rho_{\text{Ave}[\mu]}\|^2 - C \frac{D^2}{\sigma R^2 \inf(L^4, L^{2s})}.$$

A last use of the bathtub principle gives

$$\mathcal{E}_V^R[\text{Ave}[\mu]] \geq C \|\rho_{\text{Ave}[\mu]}\|^2.$$

Thus the proof is concluded by absorbing one last error term in the main expression. \square

4.4. Convergence of the energy. Going back to our lower bound (4.20), assuming $R = N^{-\eta}$, $0 < \eta < 1/4$, we have

$$\begin{aligned} \frac{E^R(N)}{N} &= \frac{\langle \Psi_N, H_N^R \Psi_N \rangle}{N} \\ &\geq \int_{\mathcal{P}(\mathbb{R}^4)} \mathcal{E}_V^R[\mu] dP_{m_N}^{\text{DF}}(\mu) - o_N(1) \\ &\geq \int_{\Gamma_\varepsilon^c \cap \Xi_\tau} \mathcal{E}_V^R[\mu] dP_{m_N}^{\text{DF}}(\mu) - o_N(1) \end{aligned} \quad (4.48)$$

where

$$\mathcal{E}_V^R[\mu] = \int_{\mathbb{R}^4} \left(|p^{\mathbf{A}} + \beta \mathbf{A}^R[\mu](x)|^2 + V(x) \right) d\mu, \quad (4.49)$$

Γ^ε is as in (4.31),

$$\Xi_\tau = \left\{ \mu \in \mathcal{P}(\mathbb{R}^4) \int_{\mathbb{R}^4} (|p|^2 + V(x)) d\mu(x, p) \leq \frac{\tau}{R^2} \right\},$$

and we have used the positivity of the integrand to go to the second line of (4.48). We next pick l_x, l_p to enforce

$$l_x \ll R, \quad l_p \ll 1, \quad l_x^2 l_p^2 = N^{-\beta}$$

for some $\beta < 1$. This is certainly compatible with our other requirements, since $R \gg N^{-1/4}$ by assumption. We can then use Theorem 4.9 to deduce

$$\frac{E^R(N)}{N} \geq (1 - \sigma)(1 - o_N(1)) \int_{\Gamma_\varepsilon^c \cap \Xi_\tau} \mathcal{E}_V^R[\text{Ave}[\mu]] dP_{m_N}^{\text{DF}}(\mu) - o_N(1) - C \frac{\tau^2}{\sigma R^6 \inf(L^4, L^{2s})}. \quad (4.50)$$

Next, denote, for some $\gamma > 0$

$$\Theta_\gamma = \left\{ \mu \in \mathcal{P}(\mathbb{R}^4) \text{ such that } \int_{S_L^c} \mu < \gamma \right\} \quad (4.51)$$

the set of measures whose mass lie mostly in S_L and note that, as per Lemmas A.3 and A.4,

$$\mathcal{E}_V^R[\text{Ave}[\mu]] \geq (1 - CR)(1 - 2\varepsilon)(1 - \gamma)e_{\text{TF}}$$

for any $\mu \in \Gamma_\varepsilon^c \cap \Theta_\gamma$. Clearly, if we set

$$\gamma = c \frac{\tau}{R^2 \inf(L^2, L^{2s})} \quad (4.52)$$

with a suitable $c > 0$ we have that

$$\Theta_\gamma \subset \Xi_\tau.$$

Hence, with this choice,

$$\begin{aligned}
\frac{E^R(N)}{N} &\geq (1 - CR)(1 - 2\varepsilon)(1 - \gamma)(1 - \sigma)(1 - o_N(1))e_{\text{TF}} \int_{\Gamma_\varepsilon^c \cap \Xi_\tau} dP_{m_N}^{\text{DF}}(\mu) - o_N(1) \\
&\quad - \frac{C\tau^2}{\sigma R^6 \inf(L^2, L^{2s})} \\
&\geq (1 - 2\varepsilon)(1 - \gamma)(1 - \sigma)(1 - o_N(1))e_{\text{TF}} (1 - \mathbb{P}_{m_N}^{\text{DF}}(\Gamma_\varepsilon) - \mathbb{P}_{m_N}^{\text{DF}}(\Xi_\tau^c)) \\
&\quad - \frac{C\tau^2}{\sigma R^6 \inf(L^2, L^{2s})}, \tag{4.53}
\end{aligned}$$

with a self-explanatory notation for $\mathbb{P}_{m_N}^{\text{DF}}$. There remains to estimate the probability of the “bad events” in the last inequality. First, there is a $\delta > 0$ such that

$$\begin{aligned}
\mathbb{P}_{m_N}^{\text{DF}}(\Gamma_\varepsilon) &= \mathbb{P}_{m_N}^{\text{DF}}\left(\cup_{S_K \in S_L} \left\{ \text{Emp}_{Z_N}, \int_{\Omega_k} \text{Emp}_{Z_N} \geq \frac{(1 + \varepsilon)}{(2\pi)^2} |\Omega_k| \right\}\right) \\
&\leq \sum_{k=1}^{16n^4} \mathbb{P}_{m_N}^{\text{DF}}\left(\left\{ \text{Emp}_{Z_N}, \int_{\Omega_k} \text{Emp}_{Z_N} \geq \frac{(1 + \varepsilon)}{(2\pi)^2} |\Omega_k| \right\}\right) \\
&\leq C_\delta L^4 N^\beta e^{-c_\delta N^\delta \ln(1 + \varepsilon)}. \tag{4.54}
\end{aligned}$$

We used the union bound, the bound on the probability of violating the Pauli principle in a single box from Theorem 4.4 and the relation (4.33).

On the other hand, using Markov’s inequality, (3.1), Lemma 3.2 and (4.6) we find

$$\begin{aligned}
\mathbb{P}_{m_N}^{\text{DF}}(\Xi_\tau) &\leq R^2 \tau^{-1} \int_{\mathcal{P}(\mathbb{R}^4)} \left(\int_{\mathbb{R}^4} (|p|^2 + V(x)) d\mu(x, p) \right) dP_{m_N}^{\text{DF}}(\mu) \\
&= CR^2 \tau^{-1} \int_{\mathbb{R}^4} (|p|^2 + V(x)) dm_N^{(1)}(x, p) \\
&\leq C\tau^{-1}. \tag{4.55}
\end{aligned}$$

There remains to insert (4.54) and (4.55) in (4.53). We can now set $L = N^\kappa$ with $\kappa > 0$ a suitably large, fixed number and $\tau = N^{\kappa'}$ with κ' a suitably small, fixed number. This makes all errors depending on N negligible because, crucially, (4.54) contains an exponentially small term in N . In particular, recalling (4.52), $\gamma \rightarrow 0$ with such a choice of a parameters. Hence we can pass to the limit in (4.53), first letting $N, R^{-1}, L \rightarrow \infty$ with the previously specified scalings, and finally let $\varepsilon, \sigma \rightarrow 0$, obtaining the energy lower bound claimed in 1.1, matching the upper bound from Section 2.

4.5. Convergence of states. We turn to the proof of Theorem 1.4. Coming back to (4.50) and inserting the previous choice of parameters we have that

$$e^{\text{TF}} \geq (1 - \sigma)(1 - o_N(1)) \int_{\mathcal{P}(\mathbb{R}^4)} \mathbb{1}_{\Xi_\tau}(\mu) \mathbb{1}_{\Gamma_\varepsilon}(\mu) \mathcal{E}_V[\text{Ave}[\mu]] dP_{m_N}^{\text{DF}}(\mu) - o_N(1)(1 + \sigma^{-1}). \tag{4.56}$$

It follows from (4.56) and results of Appendix A that $m_N^{(1)}$ is tight. Applying the last part of Theorem 4.1 we thus obtain that the Diaconis-Freedman measure $P_{m_N}^{\text{DF}}$ weakly converges to a probability measure $P \in \mathcal{P}(\mathcal{P}(\mathbb{R}^4))$ (this is the Hewitt-Savage measure). It follows from Theorem 4.4 (or [21, Theorem 2.6]) that P is concentrated on probability measures

satisfying $\mu \leq (2\pi)^{-2}$. Combining with Lemma 3.3 and (4.12) we have that, for any fixed $k \geq 0$

$$\frac{1}{(2\pi)^{2k}} m_{\Psi_N}^{(k)} \xrightarrow{N \rightarrow \infty} \int_{\mathcal{P}(\mathbb{R}^4)} \mu^{\otimes k} dP(\mu)$$

weakly as measures. We now identify the limit measure P to conclude the proof of Theorem 1.4.

To this aim it is convenient to write the integral above as

$$\int_{\mathcal{P}(\mathbb{R}^4)} \mathbb{1}_{\Xi_\tau}(\mu) \mathbb{1}_{\Gamma_\varepsilon}(\mu) \mathcal{E}_V[\text{Ave}[\mu]] dP_N^{\text{DF}}(\mu) = \int_{\mathcal{P}(\mathbb{R}^4)} \mathcal{E}_V[\text{Ave}[\mu]] dQ_N(\mu) \quad (4.57)$$

with

$$dQ_N[\mu] := \mathbb{1}_{\Xi_\tau}(\mu) \mathbb{1}_{\Gamma_\varepsilon}(\mu) dP_N^{\text{DF}}(\mu). \quad (4.58)$$

We will see $\mathcal{E}_V[\text{Ave}[\mu]]$ as a lower semi-continuous function on the space $(\mathcal{P}(\mathbb{R}^4), d_1^W)$ where

$$d_1^W(\mu, \nu) = \sup_{\|\phi\|_{\text{Lip}} \leq 1} \left| \int_{\mathbb{R}^4} \phi d\mu - \int_{\mathbb{R}^4} \phi d\nu \right|$$

is the Monge-Kantorovitch-Wasserstein distance, known to metrize the weak convergence of measures [64, Theorem 6.9] (and we use Kantorovitch-Rubinstein duality [64, Remark 6.5]). We have

Lemma 4.10 (Limit of Ave).

Let $\mu \in \mathcal{P}(\mathbb{R}^4)$. In the d_1^W metric,

$$\text{Ave}[\mu] = \sum_m^{16n^4} \mathbb{1}_{\Omega_m} \int_{\Omega_m} \frac{d\mu}{|\Omega_m|} \rightarrow \mu, \quad (4.59)$$

when $N \rightarrow \infty$ and $L \rightarrow \infty$.

Proof. We pick a function ϕ , $\|\nabla\phi\|_{L^\infty} \leq 1$ and denote

$$a[\phi]_m = \int_{\Omega_m} \frac{\phi(x, p)}{|\Omega_m|} dx dp$$

its average over a box Ω_m . We have that, for any $x \in \Omega_m$,

$$|\phi(x) - a[\phi]_m| \leq Cl \|\nabla\phi\|_{L^\infty} \quad (4.60)$$

where $l = \max(l_x, l_p)$ is the largest dimension of the box Ω_m . We thus have

$$\int_{\mathbb{R}^4} \phi(x) d\text{Ave}[\mu] = \sum_m^{16n^4} \int_{\Omega_m} a[\phi]_m d\mu = \sum_m \int_{\Omega_m} \phi d\mu + o_N(1)$$

where the $o_N(1)$ is uniform in $\|\nabla\phi\|_{L^\infty}$ by (4.60). \square

It follows that, if we take a sequence $\mu_n \rightarrow \mu$ when $N \rightarrow \infty$,

$$d_1^W(\text{Ave}[\mu_n], \mu) \leq d_1^W(\text{Ave}[\mu_n], \mu') + d_1^W(\mu_n, \mu) \leq o_N(1).$$

Hence, combining with Lemma (A.1), we deduce

$$\liminf_{N \rightarrow \infty, \mu' \rightarrow \mu} \mathcal{E}_V[\text{Ave}[\mu']] \geq \mathcal{E}_V[\mu]. \quad (4.61)$$

We turn to the convergence of Q_N defined in (4.58). For any continuous bounded function $\Phi \in C_b(\mathcal{P}(\mathbb{R}^4))$ over probability measures,

$$\left| \int_{\mathcal{P}(\mathbb{R}^4)} \Phi(\mu) (dQ_N(\mu) - dP_{m_N}^{\text{DF}}(\mu)) \right| \xrightarrow{N \rightarrow \infty} 0$$

by (4.54) and (4.55). Hence, applying Theorem 4.1 we have that

$$Q_N \rightarrow P$$

as measures over $\mathcal{P}(\mathbb{R}^4)$, where P is the limit of the Diaconis-Freedman measure. Combining with (4.61) we may apply the improved Fatou Lemma of [20, Theorem 1.1] to (4.57), obtaining

$$\liminf_{N \rightarrow \infty} \int_{\mathcal{P}(\mathbb{R}^4)} \mathcal{E}_V[\text{Ave}[\mu]] dQ_N(\mu) \geq \int_{\mathcal{P}(\mathbb{R}^4)} \mathcal{E}_V[\mu] dP(\mu).$$

Combining with (4.56) and letting $\sigma \rightarrow 0$ after $N \rightarrow \infty$ we conclude

$$e^{\text{TF}} \geq \int_{\mathcal{P}(\mathbb{R}^4)} \mathcal{E}_V[\mu] dP(\mu).$$

Recall that P is concentrated on probability measures satisfying $\mu \leq (2\pi)^{-2}$ and that e^{TF} is the infimum of $\mathcal{E}_V[\mu]$ over those. Hence P must be concentrated on the unique minimizer of $\mathcal{E}_V[\mu]$, and this concludes the proof.

APPENDIX A. PROPERTIES OF THE VLASOV FUNCTIONAL

In this appendix we establish some of the fundamental properties of the functional \mathcal{E}_V and some useful bounds on the vector potential $\mathbf{A}^R[\rho]$ associated to a measure ρ .

Lemma A.1 (Lower semicontinuity of \mathcal{E}_V).

Let $(\mu_n)_{n \geq 0}$ be a sequence of positive measures on \mathbb{R}^4 satisfying

$$0 \leq \mu_n \leq (2\pi)^{-2}, \quad \int_{\mathbb{R}^4} \mu_n \leq C$$

with C independent of N . If μ_n converges to μ as measures we have

$$\liminf_{n \rightarrow \infty} \mathcal{E}_V[\mu_n] \geq \mathcal{E}_V[\mu]. \quad (\text{A.1})$$

Proof. We recall that the marginal in p of μ_n is

$$\rho_n(x) = \int_{\mathbb{R}^2} \mu_n(x, p) dp. \quad (\text{A.2})$$

It follows from applying the bathtub principle [35, Theorem 1.14] in the p variable that

$$\mathcal{E}_V[\mu_n] \geq \mathcal{E}_V^{\text{TF}}[\rho_n] := 2\pi \int_{\mathbb{R}^2} \rho_n^2(x) dx + \int_{\mathbb{R}^2} V(x) \rho_n(x) dx.$$

Hence we may assume that $\|\rho_n\|_{L^2}$ is uniformly bounded and that $\rho_n \rightharpoonup \rho$ weakly in $L^2(\mathbb{R}^2)$. We then deduce from the weak Young inequality [35, Chapter 4] that

$$\|\mathbf{A}^2[\rho_n] \rho_n\|_{L^1} \leq C \left\| \nabla^\perp w * \rho_n \right\|_{L^\infty}^2 \leq C \|\rho_n\|_{L^2}^2 \left\| \nabla^\perp w \right\|_{L^{2,w}}^2 \leq C.$$

We recall that the weak $L^p(\mathbb{R}^d)$ space is the set of functions

$$L^{w,2} = \left\{ f \mid \text{Leb} \left(\left\{ x \in \mathbb{R}^d : |f(x)| > \lambda \right\} \right) \leq \left(\frac{C}{\lambda} \right)^p \right\}$$

with the associated norm

$$\|f\|_{L^{w,2}} = \sup_{\lambda>0} \lambda \text{Leb} \left(x \in \mathbb{R}^d, |f(x)| > \lambda \right)^{1/2}$$

where Leb is the Lebesgue measure. We expand the energy

$$\mathcal{E}_V[\mu_n] = \int_{\mathbb{R}^4} \left((p^{\mathbf{A}})^2 + 2\beta p^{\mathbf{A}} \cdot \mathbf{A}[\rho_n] + \beta^2 \mathbf{A}^2[\rho_n] + V \right) \mu_n(x, p) dx dp \quad (\text{A.3})$$

and use that $ab \leq \frac{a^2}{2\sigma} + \frac{\sigma b^2}{2}$ to get

$$2(p^{\mathbf{A}}) \cdot \mathbf{A}[\rho_n] \geq -\sigma |p^{\mathbf{A}}|^2 - \frac{1}{\sigma} |\mathbf{A}[\rho_n]|^2$$

and hence

$$\mathcal{E}_V[\mu_n] + C \|\mathbf{A}^2[\rho_n] \rho_n\|_{L^1} \geq C \int_{\mathbb{R}^4} (|p^{\mathbf{A}}|^2 + V) d\mu_n.$$

Thus (μ_n) is tight and, up to extraction, converges strongly in $L^1(\mathbb{R}^4)$. In order to obtain the convergence of $\mathbf{A}^2[\rho_n] \mu_n$ we write

$$\begin{aligned} \rho_n(x_1) \rho_n(x_2) \mu_n(x_3, p_3) - \rho(x_1) \rho(x_2) \mu(x_3, p_3) &= (\rho_n(x_1) - \rho(x_1)) (\rho_n(x_2) - \rho(x_2)) \mu_n(x_3, p_3) \\ &\quad + (\rho_n(x_1) - \rho(x_1)) \rho(x_2) \mu_n(x_3, p_3) \\ &\quad + \rho(x_1) (\rho_n(x_2) - \rho(x_2)) \mu_n(x_3, p_3) \\ &\quad - \rho(x_1) \rho(x_2) (\mu_n(x_3, p_3) - \mu(x_3, p_3)) \end{aligned} \quad (\text{A.4})$$

and treat each resulting term separately. The first term yields

$$\|\mathbf{A}^2[\rho_n - \rho] \mu_n\|_{L^1} \leq \|\mu_n\|_{L^1} \left\| \nabla^\perp w * (\rho_n - \rho) \right\|_{L^\infty}^2 \leq \|\rho_n - \rho\|_{L^2}^2 \left\| \nabla^\perp w \right\|_{L^{2,w}}^2$$

using that $\|\mu_n\|_{L^1} \leq C$ and the weak Young inequality. For the second term of (A.4)

$$\|\mathbf{A}[\rho_n - \rho] \mathbf{A}[\rho] \mu_n\|_{L^1} \leq \|\mathbf{A}[\rho]\|_{L^\infty} \left\| \nabla^\perp w * (\rho_n - \rho) \right\|_{L^\infty} \leq \|\rho\|_{L^2} \|\rho_n - \rho\|_{L^2} \left\| \nabla^\perp w \right\|_{L^{2,w}}^2$$

and for the last one

$$\|\mathbf{A}^2[\rho] (\mu_n - \mu)\|_{L^1} \leq \|\mu_n - \mu\|_{L^1} \|\mathbf{A}^2[\rho]\|_{L^\infty} \leq \|\mu_n - \mu\|_{L^1} \left\| \nabla^\perp w \right\|_{L^{2,w}}^2 \|\rho\|_{L^2}$$

For the cross term of (A.3), $p^{\mathbf{A}} \cdot \mathbf{A}[\rho_n] \mu_n$ we observe that

$$\|\mathbf{A}[\rho - \rho_n]\|_{L^2} \leq \|\rho - \rho_n\|_{L^1} \left\| \nabla^\perp w \right\|_{L^{2,w}}$$

so $\mathbf{A}[\rho_n]$ converges strongly in L^2 . We have $\|(p^{\mathbf{A}})^2 \mu_n\|_{L^1} \leq C$ so $p^{\mathbf{A}} \mu_n$ converges weakly in L^2 and by weak-strong convergence we deduce that the cross term converges. We conclude

using Fatou's lemma for V and the kinetic term

$$\begin{aligned} \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^4} |p + \mathbf{A}_e(x)|^2 \mu_n(x, p) dx dp &\geq \int_{\mathbb{R}^4} |p + \mathbf{A}_e(x)|^2 \mu(x, p) dx dp \\ \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^2} V(x) \rho_n(x) dx &\geq \int_{\mathbb{R}^2} V(x) \rho(x) dx. \end{aligned}$$

□

Lemma A.2 (Existence of minimizers).

There exists a minimizer for the problem

$$e^{\text{TF}} = \inf \left\{ \mathcal{E}_V[\mu] \mid 0 \leq \mu \leq (2\pi)^{-2} \mid \int_{\mathbb{R}^4} \mu = 1 \right\}.$$

Proof. We consider a minimizing sequence $(\mu_n)_n$ converging to a candidate minimizer μ_∞ . By Lemma (A.1) we have

$$e^{\text{TF}} = \inf_{n \rightarrow \infty} \mathcal{E}_V[\mu_n] \geq \mathcal{E}_V[\mu_\infty].$$

We also found during the previous proof that $(\mu_n)_n$ must be tight, hence

$$\int_{\mathbb{R}^4} \mu_\infty = 1.$$

□

Lemma A.3 (Convergence to \mathcal{E}_V when $R \rightarrow 0$).

For any measure $\mu \leq (2\pi)^{-2}$ such that $\int_{\mathbb{R}^4} \mu \leq C$ we have that

$$|\mathcal{E}_V^R[\mu] - \mathcal{E}_V^0[\mu]| \leq CR (\mathcal{E}_V^0[\mu] + \mathcal{E}_V^R[\mu])$$

where

$$\mathcal{E}_V^R[\mu] = \int_{\mathbb{R}^4} (p^{\mathbf{A}} + \beta \mathbf{A}^R[\mu](x))^2 + V(x) d\mu.$$

Proof.

$$\begin{aligned} |\mathcal{E}_V^R[\mu] - \mathcal{E}_V^0[\mu]| &= \left| \|(p^{\mathbf{A}} + \beta \mathbf{A}^R[\mu](x)) \sqrt{\mu}\|_{L^2}^2 - \|(p^{\mathbf{A}} + \beta \mathbf{A}^0[\mu](x)) \sqrt{\mu}\|_{L^2}^2 \right| \\ &\leq \left| \|(p^{\mathbf{A}} + \beta \mathbf{A}^R[\mu](x)) \sqrt{\mu}\|_{L^2} - \|(p^{\mathbf{A}} + \beta \mathbf{A}^0[\mu](x)) \sqrt{\mu}\|_{L^2} \right| \\ &\quad \cdot \left(\mathcal{E}_V^R[\mu]^{1/2} + \mathcal{E}_V[\mu]^{1/2} \right) \\ &\leq \|(\mathbf{A}^R[\mu] - \mathbf{A}[\mu]) \sqrt{\mu}\|_{L^2} \left(\mathcal{E}_V^R[\mu]^{1/2} + \mathcal{E}_V[\mu]^{1/2} \right) \end{aligned}$$

where we have used the triangle inequality. Moreover we have that

$$\begin{aligned} \|(\mathbf{A}^R[\mu] - \mathbf{A}[\mu])^2 \rho\|_{L^1}^{1/2} &\leq \|\rho\|_{L^1}^{1/2} \left\| (\nabla^\perp w_R - \nabla^\perp w_0) * \rho \right\|_{L^\infty} \\ &\leq \|\rho\|_{L^2} \left\| \nabla^\perp w_R - \nabla^\perp w_0 \right\|_{L^2, w} \end{aligned}$$

by the weak Young inequality and because

$$\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} \frac{1}{|x|} \chi(u) du - \int_{B(0, R)} \frac{1}{|x|} \chi(u) du \right) dx \leq \int_{B(0, 2R)} \frac{1}{|x|} dx = CR.$$

Using that the minimizer in p

$$\mu^{\text{TF}} = (2\pi)^{-2} \mathbb{1} \left(|p^{\mathbf{A}} + \mathbf{A}[\rho]| \leq \sqrt{4\pi\rho} \right)$$

is explicit we have

$$\mathcal{E}_V[\mu] \geq \mathcal{E}_{\text{TF}}(\rho) = 2\pi \int_{\mathbb{R}^2} \rho^2(x) dx + \int_{\mathbb{R}^2} V(x)\rho(x) dx$$

and the minimization problem is now formulated in terms of

$$\rho(x) = \int_{\mathbb{R}^2} \mu(x, p) dp.$$

This gives

$$\mathcal{E}_V[\mu] \geq C \|\rho\|_{L^2}^2$$

and concludes the proof. \square

Lemma A.4 (Dependence on the upper perturbed constraint).

The infimum of \mathcal{E}_V does not depend on ε nor γ at first order

$$\inf \left\{ \mathcal{E}_V[\mu] \mid 0 \leq \mu \leq \frac{1+\varepsilon}{(2\pi)^2}, \int \mu = 1 - \gamma \right\} \geq (1 - 2\varepsilon)(1 - \gamma) e_{\text{TF}}.$$

Proof. We calculate the infimum with the Bathtub principle [35, Theorem 1.14]. This infimum is achieved for

$$\mu = \mu_\varepsilon^{\text{TF}} = (2\pi)^{-2} (1 + \varepsilon) \mathbb{1} (|p^{\mathbf{A}} + \beta \mathbf{A}[\rho]|^2 \leq s(x))$$

where $s(x) = \frac{4\pi\rho}{1+\varepsilon}$ because

$$\int_{\mathbb{R}^2} \mu^{\text{TF}}(x, p) dp = \rho(x) = \int_{\mathbb{R}^2} (1 + \varepsilon) \mathbb{1} (|p^{\mathbf{A}} + \beta \mathbf{A}[\rho]| \leq s(x)) dp = \pi(1 + \varepsilon)s(x).$$

So, evaluating the energy

$$\mathcal{E}_V[\mu_\varepsilon^{\text{TF}}] = \mathcal{E}_{\text{TF}}^\varepsilon[\rho] = 2\pi \int_{\mathbb{R}^2} \frac{\rho^2(x)}{(1 + \varepsilon)^2} dx + \int_{\mathbb{R}^2} \frac{V(x)\rho(x)}{(1 + \varepsilon)} dx \tag{A.5}$$

but we can see that

$$\mathcal{E}_{\text{TF}}^\varepsilon[\rho] \geq \mathcal{E}_{\text{TF}}[\rho](1 - 2\varepsilon) \tag{A.6}$$

where $\mathcal{E}_{\text{TF}}[\rho]$ is given in (1.21). \square

APPENDIX B. BOUNDS FOR \mathbf{A}^R

Lemma B.1 (Bounds linked to \mathbf{A}^R).

All second-order directional derivatives of the function

$$w_R(u_1, u_2) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

are bounded in absolute value by the radial derivative:

$$|\partial_{u_i} \partial_{u_j} w_R(\vec{u})| \leq C |\partial_u^2 w_R(u)| \tag{B.1}$$

for any $(i, j) \in \{1, 2\}^2$. We also have the estimates

$$\left\| \Delta \nabla^\perp w_R \right\|_{L^\infty} \leq \frac{C}{R^3}, \quad (\text{B.2})$$

$$\left\| \nabla^\perp w_R \right\|_{L^\infty} \leq \frac{C}{R}, \quad (\text{B.3})$$

$$\left\| \Delta \nabla^\perp w_R \right\|_2 \leq \frac{C}{R^2}, \quad (\text{B.4})$$

$$\left\| \nabla \left(\nabla^\perp w_R(u) \right)^i \right\|_{L^\infty} \leq \frac{C}{R^2}, \quad (\text{B.5})$$

$$\left\| \nabla \left(\nabla^\perp w_R(u) \cdot \nabla^\perp w_R(v) \right) \right\|_{L^\infty} \leq \frac{C}{R^3}. \quad (\text{B.6})$$

Proof. Recall that

$$w_R(x) = (\log |\cdot| * \chi_R)(x)$$

with χ_R defined as in (1.8). We call u_1 and u_2 the two components of the vector \vec{u} and u its norm. Using Newton's theorem [35, Theorem 9.7] we write

$$w_R(u) = \int_{\mathbb{R}^2} \log |\vec{u} - \vec{v}| \chi_R(v) dv = 2\pi \log(u) \int_0^u \chi_R(r) r dr + 2\pi \int_u^{+\infty} r \log(r) \chi_R(r) dr$$

hence

$$\begin{aligned} \partial_u w_R(u) &= \frac{1}{u} \int_{B(0,u)} \chi_R(v) dv \\ \partial_u^2 w_R(u) &= -\frac{1}{u^2} \int_{B(0,u)} \chi_R(v) dv + 2\pi \chi_R(u) \end{aligned} \quad (\text{B.7})$$

$$\partial_u^3 w_R(u) = \frac{1}{u^3} \int_{B(0,u)} \chi_R(v) dv + C \frac{\chi_R(u)}{u} + C \phi_R(u) \quad (\text{B.8})$$

where $\phi_R(u) = \partial_u \chi_R(u)$ is bounded with compact support. We observe that regardless of whether u is smaller or greater than $2R$ we have

$$|\partial_u w_R(u)| \leq \frac{C}{R} \quad \text{and} \quad |\partial_u^2 w_R(u)| \leq \frac{C}{R^2} \quad (\text{B.9})$$

and get (B.3). Moreover

$$\nabla^\perp w_R(u_1, u_2) = \nabla^\perp w_R(u) = \frac{\partial_u w_R(u)}{u} \vec{u}^\perp. \quad (\text{B.10})$$

We compute the two derivatives of the second component of (B.10)

$$\left| \partial_{u_1} \left(\frac{u_1}{u} \partial_u w_R \right) \right| = \left| \frac{\partial_u w_R}{u} + \frac{u_1^2}{u^2} \partial_u^2 w_R - \frac{u_1^2}{u^3} \partial_u w_R \right| \leq \frac{C}{u^2} \int_{B(0,u)} \chi_R(v) dv + C \chi_R(u) \quad (\text{B.11})$$

$$\left| \partial_{u_2} \left(\frac{u_1}{u} \partial_u w_R \right) \right| = \left| -\frac{u_1 u_2}{u^3} \partial_u w_R(u) + \frac{u_1 u_2}{u^2} \partial_u^2 w_R(u) \right| \leq \frac{C}{u^2} \int_{B(0,u)} \chi_R(v) dv + C \chi_R(u). \quad (\text{B.12})$$

We do the same with the first component of $\nabla^\perp w_R$ and get (B.1). If we differentiate once again we get

$$\left| \partial_{u_1}^2 \left(\frac{u_1}{u} \partial_u w_R \right) \right| \leq \frac{C}{u^3} \int_{B(0,u)} \chi_R(v) dv + C \frac{\chi_R(u)}{u} + C \phi_R(u)$$

which is also the case for the other component and derivative. We deduce

$$\left\| \Delta \nabla^\perp w_R \right\|_{L^\infty} \leq \frac{C}{R^3}.$$

We can also compute

$$\left\| \Delta \nabla^\perp w_R \right\|_{L^2}^2 \leq \frac{C}{R^4}$$

which gives (B.4). To get (B.5) we combine (B.9) and (B.12). For the third inequality (B.6) we expand a little bit our first expression

$$\nabla^\perp w_R(u) \cdot \nabla^\perp w_R(v) = (u_1 v_1 + u_2 v_2) \frac{\partial_u w_R(u) \partial_v w_R(v)}{uv}$$

so the first component of the gradient is

$$\partial_{u_1} \nabla^\perp w_R(u) \cdot \nabla^\perp w_R(v) = v_1 \frac{\partial_u w_R(u) \partial_v w_R(v)}{uv} + \frac{u_1}{u} \partial_u \left[\frac{\partial_u w_R(u) \partial_v w_R(v)}{uv} \right] (u_1 v_1 + u_2 v_2).$$

Now

$$\left\| \nabla \left(\nabla^\perp w_R(u) \cdot \nabla^\perp w_R(v) \right) \right\|_{L^\infty}^2 = \sum_{x=u_1, u_2, v_1, v_2} \left(\partial_x \nabla^\perp w_R(u) \cdot \nabla^\perp w_R(v) \right)^2$$

The rest of the proof consists of the computation of the above term using basic inequalities. \square

APPENDIX C. COMPUTATIONS FOR SQUEEZED COHERENT STATES

We give for completeness three proofs that we skipped in the main text.

Proof of Lemma 1.3.

For any $u \in L^2(\mathbb{R}^2)$,

$$\begin{aligned} \langle F_{x,p}, u \rangle &= \int_{\mathbb{R}^2} F_{\hbar x}(y-x) u(y) e^{-i \frac{p \cdot y}{\hbar}} dy \\ &= 2\pi \hbar \mathcal{F}_\hbar [F_{\hbar x}(\cdot - x) u(\cdot)](p) = 2\pi \hbar (\mathcal{F}_\hbar [F_{\hbar x}(\cdot - x)] * \mathcal{F}_\hbar [u(\cdot)])(p) \end{aligned} \quad (\text{C.1})$$

$$= \int_{\mathbb{R}^2} G_{\hbar p}(k-p) \mathcal{F}_\hbar [u](k) e^{-i \frac{k \cdot x}{\hbar}} dk = 2\pi \hbar \mathcal{F}_\hbar [G_{\hbar p}(\cdot - p) \mathcal{F}_\hbar [u]](x). \quad (\text{C.2})$$

It follows that

$$\begin{aligned}
\frac{1}{(2\pi\hbar)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \langle \psi, P_{x,p} \psi \rangle \, dx dp &= \frac{1}{(2\pi\hbar)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |\langle F_{x,p}, u \rangle|^2 dx dp \\
&= \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} |\mathcal{F}_\hbar [F_{\hbar_x}(\cdot - x)u(\cdot)](p)|^2 dp \right) dx \\
&= \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} |F_{\hbar_x}(y - x)u(y)|^2 dy \right) dx \\
&= \|F_{\hbar_x}\|_{L^2(\mathbb{R}^2)}^2 \|u\|_{L^2(\mathbb{R}^2)}^2 = \|u\|_{L^2(\mathbb{R}^2)}^2.
\end{aligned}$$

□

Fourier transform of F .

We need to calculate the Fourier Transform of the Gaussian

$$G_{\hbar_p}(p) = \frac{1}{2\pi\hbar} \frac{\sqrt{\hbar_x}}{\sqrt{\hbar_x}} \frac{1}{\sqrt{\pi}} \left(\int_{\mathbb{R}} e^{-\frac{u^2}{2} - \frac{ipu}{\sqrt{\hbar_p}}} du \right)^2 = \frac{1}{2\pi\sqrt{\hbar_p}} \frac{1}{\sqrt{\pi}} G_1^2(p)$$

with

$$G_1(p) = \int_{\mathbb{R}} e^{-\frac{u^2}{2} - \frac{ipu}{\sqrt{\hbar_p}}} du$$

but we also have

$$\frac{dG_1(p)}{dp} = -\frac{p}{\hbar_p} G_1(p)$$

which gives the result. □

Proof of Lemma 3.2. We use (C.2) and write for every fixed $y \in \mathbb{R}^{2(N-k)}$

$$\begin{aligned}
&\left\langle F_{x_1, p_1}(\cdot) \otimes \dots \otimes F_{x_k, p_k}(\cdot), \Psi_N(\cdot, z) \right\rangle_{L^2(\mathbb{R}^{2k})} \\
&= (2\pi\hbar)^{2k} \mathcal{F}_\hbar [F_{x_1, 0}(\cdot) \otimes \dots \otimes F_{x_k, 0}(\cdot) \Psi_N(\cdot, z)](p_1, \dots, p_k) \\
&= (2\pi\hbar)^{2k} \mathcal{F}_\hbar [G_{0, p_1}(\cdot) \otimes \dots \otimes G_{0, p_k}(\cdot) \mathcal{F}_\hbar [\Psi_N](\cdot, z)](x_1, \dots, x_k).
\end{aligned}$$

Next we sum over the p_j 's using (1.32):

$$\begin{aligned}
&\frac{1}{(2\pi)^{2k}} \int_{\mathbb{R}^{2k}} m_{\Psi_N}^{(k)}(x_1, p_1, \dots, x_k, p_k) dp_1 \dots dp_k \\
&= \frac{k!}{(2\pi)^{2k}} \binom{N}{k} \int_{\mathbb{R}^{2k}} dp_1 \dots dp_k \int_{\mathbb{R}^{2(N-k)}} \left| \left\langle F_{x_1, p_1}(\cdot) \otimes \dots \otimes F_{x_k, p_k}(\cdot), \Psi_N(\cdot, z) \right\rangle \right|^2 dz \\
&= k! \binom{N}{k} \int_{\mathbb{R}^{2k}} dp_1 \dots dp_k \int_{\mathbb{R}^{2(N-k)}} |\mathcal{F}_\hbar [F_{x_1, 0}(\cdot) \otimes \dots \otimes F_{x_k, 0}(\cdot) \Psi_N(\cdot, z)](p_1, \dots, p_k)|^2 dz \\
&= k! \binom{N}{k} \int_{\mathbb{R}^{2k}} |F_{\hbar_{x_1}}(y_1 - x_1) \dots F_{\hbar_{x_k}}(y_k - x_k) \Psi_N(y)|^2 dy_1 \dots dy_N \\
&= k! \hbar^{2k} \rho_{\Psi_N}^{(k)} * (|F_{\hbar_x}|^2)^{\otimes k}(x_1, \dots, x_k)
\end{aligned}$$

which gives (3.4). The proof of (3.5) is similar. □

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