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Structuring and destructuring effects of bending and gravity. An analytical approach to axisymmetric capillary distortions.

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Abstract This theoretical work for practical purposes presents an original approach to the problem of gravity distortions affecting axisymmetric capillary bridges under local bending stress. From the zero-gravity case knowledge (Delaunay surfaces), a forecasting method is developed for evaluating successively the weak or strong axisymmetric capillary distortions and sagging effects in the lower part. The main tool is to highlight exact first integrals for the classical or generalized Young-Laplace equations. These relationships (actually conservation laws) are taken advantage of to obtain all at once the exact expression of the varying inter-particle force, quantified effects of flexural strength, numerical iterative predictor-corrector methods to describe the distorted profiles and the values for various new parameters of capillary bridge behavior in gravity field. The method allows to identify some criteria specific to measure the relative importance of various causes of distortions. In the case of only bending effects, the true shape of the static bridges surface is described by parametric equations generalizing a Delaunay formula; the related Young-Laplace boundary value system is then solved as an inverse problem from experimental data.

Keywords: Distortion of capillary bridges · Mean and Gaussian curvatures impact · Generalized Young-Laplace equation · Bending and gravitational effects · Sag behavior.

MSC: 49N45 · 53A10 · 58E12 · 74F10 · 74G05 · 74G15

1 Preliminaries, scope and focus chosen for the topic

This study proposes a resolutely theoretical approach to the problem of gravity distortions affecting axisymmetric capillary bridges. Roughly speaking, from a certain volume of liquid

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at some critical size, the guiding principle is based on measuring the relative importance of gravitational forces and rival surface tension forces, under local bending stress. The goal is to model the structuring and destructuring effects of gravity, altering the constant mean curvature surfaces [8] and modifying the boundaries with dynamic phenomena of contact angle hysteresis. From then on, the profile of the distorted bridges is described by a generalized Young-Laplace equation, involving both mean and Gaussian curvatures, according to the work developed by J. Gaydos *et al.* [10]. We therefore consider some non-constant mean curvature surfaces resulting from a sag behavior.

1.1 A review of concepts about classical framework

Among various criteria signaling an established gravity regime, the capillary length λ_c defined as $\lambda_c = \sqrt{\frac{\gamma}{\Delta\rho g}}$ ($\Delta\rho$ is the difference in density of the two phases, g the gravitational acceleration, γ the surface tension) is a characteristic length scale for an interface beyond which gravity becomes important [11] (the capillary length is around $2,7\text{ mm}$ for clean water and air interfaces at thermodynamic standard conditions, when $\gamma = 72,86\text{ mNm}^{-1}$ at 20° C). This reference value can be estimated by equating the Laplace pressure $\frac{\gamma}{\lambda}$ and the hydrostatic pressure $\rho g\lambda$ at a depth λ in a liquid of density ρ submitted to earth's gravity g ([11], p. 33, Eq. 2.1).

This distance is generally of the order of a few mm , even for mercury-air interfaces (in fact, $\Delta\rho$ and $\gamma = 486,5\text{ mNm}^{-1}$ at 20° C are both large in this case³), except of course to be able to use two immiscible fluids with similar densities in the way of [15]. We can also [6] use water in a castor oil bath (density: 961 kg/m^3) [20]. The value of γ is considered constant for moderately curved interfaces at a given temperature, *i.e.* curvature-independent ([10], p.9).

The Eötvös or Bond number given by $B_0 = \frac{\Delta\rho g L^2}{\gamma}$, *i.e.* $B_0 = \frac{L^2}{\lambda_c^2}$, where L stands for a characteristic length of the bridge, is a dimensionless variant. Consequently, the B_0 values greater than 1 indicate the growing importance of gravity [16]. However, as a limitation on reliability, these criteria do not explicitly take into account the effect of bridge volume and the shape of the bridge *via* a mean curvature scale [21]. For a finer level of analysis, it is certainly necessary to establish a $3d$ map involving a concept for dimensionless bridge volume, a mean curvature scale and the Bond number B_0 in order to represent the occurrence of the different capillary regimes [1]. That identifies the need to increase our knowledge on experimental research of mechanisms triggering distortion: new phenomenological behavior laws are perhaps to be established for lack of a well-consistent theory about the gravitational distortion of liquid bridges. Simulations of shearing of capillary bridges are studied in [29], see also [30]. Note, in order to study all aspects of

³“it is a matter of experience that a droplet of mercury, with a very high surface tension, will be more nearly spherical than a drop a water of comparable size” [2], p.62.

the matter from the theoretical or applicative point of views, that the bending phenomena due to evaporation can have effects distorting the original constant mean curvature bridge shape [7], [31], [32],

The presentation first introduces, in a sense to be specified, a study of small capillary distortions under weak effects of gravity; we define, under restrictive conditions, a notion of mathematical perturbation for the resulting quantitative evaluations. These restrictions of use are not a crippling disadvantage because, as a matter of fact, the classical Young-Laplace theory is restricted to moderately curved liquid-fluid interfaces [10], p.vi, 25.

1.2 The original aspects of this theoretical approach

The second aspect, in practice covering a wide area, concerns the strong distortions for which the bending effects⁴ are modeled by an additional curvature-related term: the introduction of C_K , a multiplier coefficient of the Gaussian curvature K , at the dimension of a force and standing for the bending stress. Under appropriate boundary conditions, the shape of an interface between two non-solid substances is then described by the so-called generalized Young-Laplace equation, thus involving both mean and Gaussian curvatures. Indeed structurally analogous to the Gullstrand equation of geometrical optics, the resulting equation, at the downward vertical measurement x linked to the value Δp_0 at $x = 0$, comes in the following form [4], [10], [23] that will be detailed more specifically:

$$\gamma \left(\frac{1}{\rho_c} + \frac{1}{N} \right) + C_K \frac{1}{\rho_c N} = \Delta p_0 - \Delta \rho g x ,$$

where the force C_K divided by the area $\rho_c N$ stands for the bending stress, ρ_c and N for the principal radii of curvature (evaluated algebraically, positively when the curvature is turned into the interior of the capillary bridge) and the pressure deficiency is Δp_0 at $x = 0$. Thenceforth, a major difficulty is to estimate the influence of C_K on two determinant data: the modified contact angles and the spontaneous curvature $\frac{\Delta p_0}{\gamma}$ at $x = 0$ after distortion.

It is also reported [27] that in electro-capillarity, at the nanoscale, the presence of electric fields leads to an extra stress term to be added in the Young-Laplace equation.

As expected, the length $\frac{|C_K|}{\gamma}$ that occurs in the exact formulas that we will establish allows in a certain way to assess the relative importance of bending effects. In particular, for this purpose, the smallness or not of the dimensionless number $\frac{|C_K|}{2\gamma Y^*}$ appears significant,

Y^* being the gorge radius of the distorted bridge. In the form $\frac{\pi |C_K|}{2\pi\gamma Y^*}$, this number appears as the quotient of the contributions of the bending and liquid surface tension forces at the distorted bridge neck. Strictly speaking, the formulas obtained retain the

⁴bending the interface, *i.e.* changing its curvature, in a first approach.

value $\frac{\pi |C_K|}{2\pi\gamma Y^* - \Delta p_0 \pi Y^{*2}}$ as the most accurate criterion, taking then into account the contribution of the hydrostatic pressure.

It will also be noted that, in respect of certain theoretical issues, a capillary bridge may be considered as an optic system because it is composed of two interfaces [23].

The main tool is to highlight, in both cases, an exact first integral for the Young-Laplace equations, classical or generalized. These relationships, which are actually total energy conservation laws, are taken advantage of to obtain the theoretical expression of the varying inter-particle force, quantified effects of flexural strength. Numerical iterative predictor-corrector methods are developed to get both the consistent assessment of distorted profiles and values for various new parameters of capillary bridge behavior in gravity field. When considering the only bending effects, the method allows to easily obtain a parameterization of the profile by generalizing together a Delaunay formula related to constant mean curvature surfaces [5], p.313 and the resolution method of the Young-Laplace equation as an inverse problem developed in [8].

2 Analytical evaluation of small capillary distortions by weak gravity effects

Consider, first in the classical theory, an axisymmetric vertical liquid bridge (*i.e.* the x - axis is vertical and Δp_0 is the pressure difference through the interface at the neck level $x = 0$). I is an open interval on which we can define by Cartesian representation, say $x \rightarrow y(x)$, a portion of the Delaunay roulette strictly containing the convex profile of the bridge considered without taking into account the gravity (a zero or low gravity environment) [8]. So the shear stress is zero in the y direction and at first, we place ourselves in the relevant cases in which $y''(0) > 0$.

Taking then, if necessary [9], [11], [18] into account the effects of gravity, *via* an over-pressure, results conventionally in the modified nonlinear differential equation for the distorted profile $x \rightarrow Y(x)$, according to the volumic mass densities difference between the liquid and the surrounding fluid

$$\Delta\rho = \rho_{int} - \rho_{ext}$$

a quantitated balance between the surface tension and gravity forces:

$$(1) \quad \frac{Y''(x)}{(1 + Y'^2(x))^{3/2}} - \frac{1}{Y(x)\sqrt{1 + Y'^2(x)}} = -\frac{\Delta p_0}{\gamma} + \frac{g \Delta\rho}{\gamma} x$$

$$=: H + Bx, \quad x \in I.$$

As the only parameter of the disturbance, such an apparent density formula $\Delta\rho = \rho_{int} - \rho_{ext}$ can be discussed but, as a first approximation, remains conventional although the bridge fluid is not completely embedded in the surrounding fluid as for a wall-bound

pendant drop without frictional contact constraints on the low boundary, possibly strongly distorting ⁵. In continuum mechanics, this equation is obtained in the absence of motion when gravity is the only body force present. It is counterintuitive that the sign and the order of magnitude of the Gaussian curvature do not come into consideration for defining the distorted shape of the free capillary surface. This implicitly assumes that bending effects are neglected and that we are *de facto* limited here to studying rather moderate distortions. It is presumably a question of finding a balance between what is too simple and therefore necessarily false and the too complicated, unusable in the context of an experimental validation. By repeating the terms of [10], p. 33, "it is apparent that the free energy representation appropriate for nonmoderately curved capillary systems produces a modified form of the Laplace equation, containing the bending moment and the Gaussian curvature". The question of bending and its impact on the deformation will be thoroughly discussed below.

Obviously, predicting the shape of a bridge after distortion is more complicated than modeling the shape of a distorted bridge when observing *in situ* [24], [25], except at ignoring dynamic phenomena of contact angle hysteresis or at reporting a reliable theoretical model about this subject; the discrepancy (but not its sign) between the contact angles before and after the distortion is difficult to predict. Let us observe in a preliminary way a rather simplified situation: an adherent drop to a vertical wall under non-slip condition without bending effects; when the free capillary surface is representable *via* a function $\eta = \eta(x, y)$, $(x, y) \in \mathcal{D} \subset \mathbf{R}^2$, the doubled mean curvature is expressed using the outward-pointing unit normal; the mathematician is immediately confronted with a stationary nonlinear problem of the following divergent type (a second order elliptic partial differential equation involving a monotone operator in Banach space [12]; the "natural" framework is the space of functions of bounded variation (the weak first partial derivatives are Radon measures) and the Caccioppoli sets [13], *i.e.* the *BV* functions and sets of finite perimeter):

$$-\operatorname{div} \left(\frac{\nabla \eta}{\sqrt{1 + |\nabla \eta|^2}} \right) = H + Bx, \quad \eta = \eta(x, y), \quad (x, y) \in \mathcal{D} \subset \mathbf{R}^2,$$

an inverse problem, H being here an **unknown** spontaneous value to be identified by the data of an additional boundary condition as in [8] (see also the notion of sentinel introduced by J.-L. Lions [14] and [17] for mean curvature estimation methods).

The appropriate strengthened boundary conditions to determine the pair (H, η) are of the following type with varying contact angles δ , $\delta = \delta(x, y)$ along the triple line $\partial \mathcal{D}$ [28]:

$$\text{Dirichlet: } \eta = 0 \quad \text{on } \partial \mathcal{D} \quad \text{and Neumann: } \frac{\nabla \eta \cdot n}{\sqrt{1 + |\nabla \eta|^2}} = \cos \delta \quad \text{on } \partial \mathcal{D}.$$

⁵See in [6], p.780, figure 15, a stable pendent water drop in a bath of castor oil exhibiting inflection on the profile, also neck and bulge (artificial low gravity: $\Delta\rho = 39 \text{ kg/m}^3$).

The purpose of this study is to predict the evolution of a liquid bridge under the gravity effect. The parameter H to recognize plays a rather important role because knowing it allows to formulate mathematically a well-posed problem and physically or geometrically, to obtain the value of the mean curvature of the distorted bridge at the level of reference $x = 0$. The essential difficulty of the recognition problem which one considers here resides in the fact that the usual boundary conditions (contact points, contact angles) are not known *a priori* but result from the final equilibrium, with possible canthotaxis effects.

Given this theoretical difficulty and the improbable hope of getting explicit solutions (see [3] for new recent perspectives: the Plateau problem from the perspective of optimal transport), we define by pragmatism in a first approach, under restrictive conditions, a notion of **local mathematical perturbation**.

Let us note from the outset that the method used here can only have a strictly local character (in a restricted neighborhood of the neck, under strong conditions, for the further developments of the text in order to evaluate the position of the moved bridge neck). The method cannot be iterated to cover a given domain, due to the local nature of the condition. About that, reporting a fine remark from one of the reviewers, we indicate that the method cannot have a global character by the fact that the equation is not autonomous (*i.e.* here, by taking into account the gravity effects, the resulting nonlinear differential equation presents an explicit x dependence after nondimensionalizing the lengths).

We then use a two-steps **splitting method**.

The splitting method separates the original differential equation into two parts, each part having a specific role to play alternately by expressing different physics (here, H and B), in such a way that each sub-problem is simpler to deal with.

We first examine the influence of gravity on the mean curvature of the free surface at the level $x = 0$ *i.e.* on H , $H = H(B)$. This instantaneous mean curvature of the distorted bridge at level $x = 0$ is **an implicit unknown of the problem**; so, in a first step, we need to know the link between gravity effect and modification of the mean curvature at the location of the neck without gravity. Consequently, in a well-posed problem, there must be an additional constraint which ties the two quantities together so that the perturbed shape can be defined uniquely. For this, we could use theoretically a first integral of the (H, B) -Young-Laplace differential equation whose exact formulation in order to get a mathematically closed problem is proved independently (section 3) and which constitutes a principle of conservation. However, this procedure turns out to be complicated to carry out in practice, what remains subjective....

So, in the present approach; we first examine the resulting local distortion effect at the level $x = 0$ of the profile, H being transiently considered invariant: all the details are provided in section 4 on moving the bridge neck (X^*, Y^*) , explicit determination of the radius $R_S(0)$ of the osculating circle at level $x = 0$.

This makes it possible to estimate geometrically the unknown value $H(B)$ *via a recognition problem*; we obtain then a value of the spontaneous value of H in equation (1) under the following formulation (details in section 6):

$$H_g = H(B) \approx \frac{1}{R_S(0)} - \frac{1}{Y^*} - BX^*.$$

Secondly, we take into account the local resulting perturbation created by this new value. The result then is stated as follows:

To get a more accurate assessment of the distortion effects are introduced the representative quantities of the profiles, with or without gravitational effect,

$$Y_g = Y_{H_g, B} \quad , \quad y = Y_{H, 0} \quad ,$$

where H_g is the readjusted spontaneous mean curvature, observed after distortion effect at the level $x = 0$, to be introduced in differential equation (1).

It is then easily verified that we obtain, more precisely in this modeling, repeating the previous calculations for step 1, the correcting term for the asymmetrical distorted profile on an appropriate open I **restricted around the level** $x = 0$:

$$\begin{aligned} Y_g(x) &\simeq y(x) + (H_g - H + Bx) y^2(x) |\cos \phi(x)|, \quad x \in I, \\ &= y(x) + (H_g - H + Bx) \frac{2ay^3(x)}{|y^2(x) + \varepsilon b^2|}, \quad \varepsilon = \pm 1, \end{aligned}$$

where, by estimate, $H_g = \frac{1}{R_S(0)} - \frac{1}{Y^*} - BX^*$,

$$Y_g(0) \simeq y^* + (H_g - H) y^{*2} \quad , \quad Y'_g(0) \simeq B y^{*2}.$$

mathematical form where, more precisely,

$$Y = Y_{H, B}, \quad y = Y_{H, 0} \quad (\text{profile curve of a Delaunay free surface}),$$

$\phi(x)$ is the angle made at zero gravity by the tangent vector to the meniscus with the x -axis at $(x, y(x))$,

the case $\varepsilon = 1$, $a > b > 0$ (semi-axis of the associated conic) corresponds in zero gravity to an arc of Delaunay elliptic roulette (meridian of unduloid) whereas the case $\varepsilon = -1$, $a > 0$, $b > 0$ corresponds to an arc of hyperbolic roulette (*i.e.* a meridian of nodoid) with specifically:

$y^* = \sqrt{a^2 + b^2} - a \leq y(x) < b$, $x \in I$, (*cf.* [8] for explicit details, classification of the meridians and formulas).

These corrective formulas under restrictive conditions linking the validity interval I around the level $x = 0$ and the value of B , are valid if (this concept necessarily refers to a justification *a posteriori*):

$$(2) \quad \|\eta\|_{L^\infty(I)} \ll y^* \quad , \quad \|\eta'\|_{L^\infty(I)}^2 \ll 1, \quad \|\eta''\|_{L^\infty(I)} \ll H + \frac{1}{y^*} = y''(0)$$

which is always locally possible for B small enough since the corrective term of distortion is the product of B by a function of class C^2 on a compact set \bar{I} , independent of B . That creates a compatible specific reference set $\{(B, I), I = I_{B,y^*}\}$ for a local study of the gravity distortion, here therefore weak, in the neighbourhood of the neck. This partial information is also useful for what follows to obtain very technical *a priori* estimates in the immediate neighbourhood of the level $x = 0$.

However, the requirement $\eta'(0)^2 = B^2 y^{*4} \ll 1$, *i.e.* $\left(\frac{y^*}{\lambda_c}\right)^4 \ll 1$, shows the limits of this perturbation method and the need to change the paradigm to reach the general case. The exceptions are the liquid bridges with relatively large volumes and immersed in a second immiscible fluid with almost the same density (as liquid bridges of a di-n-butyl phthalate/liquid paraffin mixture suspended in water [15], silicone oil, castor oil and water [20]).

The proof of step 1 is given in the sections "Details on moving the capillary bridge neck" and "On the use of predictor–corrector schemes to describe capillary distortions by gravity effects" without there being a vicious circle.

Principle of the proof for step 2

Setting $Y = y + \eta$, it comes from differential equation (1), for $x \in I$,

$$\frac{y''(x) + \eta''(x)}{\left(1 + (y'(x) + \eta'(x))^2\right)^{3/2}} - \frac{1}{(y(x) + \eta(x)) \sqrt{1 + (y'(x) + \eta'(x))^2}} = H_g + Bx.$$

Assumptions on validity of the approximation argument (2) will afford to neglect certain terms and approximate as follows:

$$\frac{y''(x)}{\left(1 + (y'(x))^2\right)^{3/2}} - \frac{1}{(y(x) + \eta(x)) \sqrt{1 + (y'(x))^2}} = H_g + Bx.$$

According to (2), we may as well assume

$$\frac{1}{(y(x) + \eta(x))} \simeq \frac{1}{y(x)} - \frac{\eta(x)}{y^2(x)}$$

hence, for $x \in I$,

$$\frac{y''(x)}{(1+(y'(x))^2)^{3/2}} - \frac{1}{y(x)\sqrt{1+(y'(x))^2}} + \frac{\eta(x)}{y^2(x)\sqrt{1+(y'(x))^2}} = H + Bx.$$

Using the Young-Laplace equation corresponding to $x \rightarrow y(x)$ on I , we observe that

$$\eta(x) = (H_g - H + Bx) y^2(x) \sqrt{1+(y'(x))^2}, \quad x \in I.$$

The canonical nonlinear first order differential relationship (first integral) characterizing the Delaunay surfaces obtained by rotating the roulettes of the conics implies that [8], [9] :

$$1 + (y'(x))^2 = \frac{4a^2 y^2(x)}{(y^2(x) + \varepsilon b^2)^2}, \quad x \in I.$$

N.B. The object of the splitting method is also to be iterated and therefore, from the first estimated value of $H(B)$, we can reevaluate the position of the distorted neck and the local curvature of the distorted profile *via* the osculating circle at $x = 0$ and so deduce by a new iteration on the Young-Laplace equation a refined value of $H(B)$.

As an example of a somewhat academic illustration, it is argued that, around the neck, the distortion of a catenoidal minimal surface whose equation for the meridian is

$$y(x) = y^* \cosh \frac{x}{y^*}, \quad \text{the } x\text{-axis being vertical, } H = 0 \text{ by definition,}$$

is then given by the formula⁶, on a suitable interval I , with an asymmetric corrective term:

$$Y_g(x) \simeq y^* \cosh \frac{x}{y^*} (1 + y^* (H_g + Bx)), \quad x \in I.$$

We used the specific property that in this case

$$y(x) \cos \phi(x) = y^* \text{ at any point } (x, y(x)).$$

Indeed, in more detail, we get successively, by easy calculations,

$$R_S(0) \approx y^* \sqrt{1 + B^2 y^{*4}}$$

$$X^* \approx -B y^{*2} R_S(0), \quad Y^* \approx y^* - \frac{1}{2} R_S(0) B^2 y^{*4},$$

hence the readjusted values after the splitting method :

⁶As expected, note that if $B = 0$, then $H_g = H = 0$.

$$H_g = H(B) \approx \frac{1}{R_S(0)} - \frac{1}{Y^*} - BX^*$$

$$Y_g(x) \simeq y^* \cosh \frac{x}{y^*} + (H_g + Bx) y^2(x) |\cos \phi(x)|, \quad x \in I,$$

On the volume preservation condition. When considering, for simplicity, such axisymmetric liquid bridges between two parallel identical plates, the axis being vertical, we note that, by this method, the difference in volumes induced by the approximation is controlled by the relationship, where $Y = Y_{H,B}$, $y = Y_{H,0}$

$$0 \leq V_{ol}(Y) - V_{ol}(y) = O(B_0^2), \text{ (Bachmann-Landau notation)}$$

when the dimensionless Bond number B_0 (also called the Eötvös number), here defined by $B_0 = L^2 B$, is such that⁷ $B_0^2 \ll 1$, so this concerns a static equilibrium rather weakly dependent on gravitational forces.

2.1 Remark on the impact of the mean curvature variations

From a practical point of view, **the mean curvature recognition** *in situ* at the distorted neck, *i.e.* H_g , will be the subject of a subsequent paragraph; by taking advantage of an exact first integral for the modified Young-Laplace equation that we will explain, we will provide an expression usable by the observer (parameter identification).

Various **numerical adaptative refinements** to evaluate and predict the deformation behavior can be easily obtained in the neighborhood of the neck which would be observable and measurable without distortion effect (bridges exhibiting strictly negative Gaussian curvature).

The following example given serves as illustration only and is not intended to be exhaustive: we consider locally at the theoretical neck without gravity (at $x = 0$) that a distortion occurred, represented as

$$Y_g(x) \simeq y^* + \frac{x^2}{2} y''(0) + \eta(x) \quad \text{for sufficiently small values of } x,$$

where $y''(0) = \frac{1}{y^*} + H$, $y''(0) > 0$.

Note that solving the equation : $xy''(0) + \eta'(x) = 0$ would provide an estimate of the abscissa for the moved neck.

It follows, by taking up and deepening the previous steps, that

$$\eta''(x) + \frac{1}{y^{*2}} \eta(x) = (H_g - H + Bx) \quad \text{in a near vicinity of } x = 0,$$

⁷Where L is a characteristic length

ordinary differential equation whose general solution, depending on two arbitrary scalars measuring lengths, is given by

$$\eta(x) = y^{*2} (H_g - H + Bx) + \kappa \sin \frac{x}{y^*} + \rho \cos \frac{x}{y^*},$$

and so are introduced, for possible numerical readjustments, two corrective parameters κ and ρ dimensioned to a length such that

$$\begin{aligned} Y_g(0) &\simeq y^* + (H_g - H) y^{*2} + \rho, & \rho &= \rho(y^*, B), \\ Y'_g(0) &\simeq B y^{*2} + \frac{\kappa}{y^*}, & \kappa &= \kappa(y^*, B). \end{aligned}$$

The validity of the method is based on the (rather small) value of the neck radius y^* ; the proposed result is physically relevant under the conditions:

$$B y^{*2} + \frac{\kappa}{y^*} > 0, \quad \left(B y^{*2} + \frac{\kappa(y^*, B)}{y^*} \right)^2 \ll 1.$$

3 An exact energy invariant related to distorted bridges with neck, influenced by Earth gravity

3.1 A first integral of the (H, B) –Young-Laplace equation

We still find ourselves in the framework and notations of the previous subsection, concerning essentially any bridge with strictly negative Gaussian curvature K (the product of the two principal curvatures). The free surface is then saddle shaped.

It is particularly proposed to provide a theoretical justification for an extension of the conventional gorge method in order to evaluate the interparticle capillary force under gravitational perturbation at the neck level as a special case of an energy conservation principle. Unlike the situation of axisymmetric bridges with constant mean curvature, the capillary force is no longer constant at all points of the distorted profile. The analytic expression of the interparticle force $\mathcal{F}_{cap}(x)$ is given with exactness at the generic level x ; it can be used by direct calculation from observed data and takes into account the gravitational forces versus the upward buoyancy forces.

For other approaches, we can consult the Russian authors Myshkis, A. D., Babskii, V. G., Kopachevskii, N. D., Slobozhanin, L. A., and Tyuptsov, A. D. (2012). about low-gravity fluid mechanics.[19].

First, we introduce (X^*, Y^*) the coordinates of the moved neck (*i.e.* the point such that $Y(X^*) = Y^*$, $Y'(X^*) = 0$) and the two branches x^+ and x^- of Y^{-1} in the set-theoretical sense, respectively defined on $\{x \geq X^*\}$ and $\{x \leq X^*\}$, subsets of the vertical x - axis, with the convention that $x^+(Y) \geq 0$ if $Y \geq Y^*$, $x^+(Y^*) = 0$ in the Cartesian coordinate system linked to the neck level of the distorted bridge.

Keep in mind that the capillary bridge profile loses its symmetry: the gravitational perturbation modifies the localization of the contact points and hence, also the domain of definition for the modified nonlinear differential Young-Laplace equation; the associated boundary value problem does not admit locally symmetric solutions that are physically relevant.

According to the previous results, the mass of water is displaced toward the lower solid and in consequence, the real gorge radius is lower, *i.e.* more precisely,

$$X^* < 0, Y^* < y^* \text{ since } Y'(0) \approx B y^{*2} \text{ is strictly positive.}$$

Moreover, the upper boundary of the liquid thus slides over a wetted part of the solid, while the lower part spreads over a dry part, which should substantially affect the resulting values of the wetting angles; as is well known, the observed contact angle hysteresis depends on whether the liquid is advancing or receding on the surface. Let us add that the capillary phenomena are known to be highly sensitive to all types of microscopic non-uniformity (canthotaxis effects).

The **main result** is stated as follows:

Whatever the shape taken by the distorted axisymmetric bridge, we have in relation to the case where the effects of gravity are neglected, the following relationship which is a generalization of an energy conservation principle:

along each concerned branch of the profile, the two following functional expressions are **constant and equal**, at the dimension of a force, for any $x \geq X^*$, H being rigorously evaluated as the mean curvature at the neck of the distorted bridge:

$$\mathcal{F}^+ = 2\pi\gamma \left(\frac{Y(x)}{\sqrt{1+Y'^2(x)}} + \frac{H}{2}Y^2(x) + B \int_{Y^*}^{Y(x)} x^+(y) y dy \right)$$

and, if $x \leq X^*$,

$$\mathcal{F}^- = 2\pi\gamma \left(\frac{Y(x)}{\sqrt{1+Y'^2(x)}} + \frac{H}{2}Y^2(x) + B \int_{Y^*}^{Y(x)} x^-(y) y dy \right).$$

Moreover, the common value is

$$\begin{aligned} \mathcal{F}^+ &= \mathcal{F}^- = 2\pi\gamma Y^* + \pi\gamma H Y^{*2} \\ &= -\pi\Delta p_0 Y^{*2} + 2\pi\gamma Y^*, \end{aligned}$$

Δp_0 being evaluated, strictly speaking, at the neck of the distorted bridge.

The demonstration will be established at the end of the next subsection, after highlighting the important practical consequences of this abstract result.

These analytical expressions are generalizations of formulas obtained in [8], equation 14.

3.2 The general expression of the interparticle capillary force

The previous relationship opens the way for an **extension of the conventional gorge method** in order to evaluate the interparticle capillary force at the neck level. This result accurately quantifies the effect of gravitational forces only by the impact of the reduced neck diameter. These formulas are independent of the experimental device (horizontal plates, spheres, *etc.*) insofar as the distorted bridge keeps a saddle-shaped free capillary surface.

Indeed, the general analytical expression of the interparticle capillary force at the level x on $\{x \geq X^*\}$ for illustration, H being rigorously evaluated as the mean curvature at the neck of the distorted bridge,,is

$$\begin{aligned} \mathcal{F}_{cap}(x) = & 2\pi\gamma \left(\frac{Y(x)}{\sqrt{1+Y'^2(x)}} + \frac{H}{2}Y^2(x) \right) \\ & + 2\pi g\Delta\rho \left(\int_{Y^*}^{Y(x)} x^+(y) y dy - \frac{x-X^*}{2}Y^2(x) \right) \end{aligned}$$

and therefore, conjugate with the expression of the first integral, becomes in practice immediately usable, by the various mathematical relationships:

$$\mathcal{F}_{cap}(x) = 2\pi\gamma Y^* + \pi\gamma H Y^{*2} - \pi g\Delta\rho (x - X^*) Y^2(x).$$

In other words, on $\{x \geq X^*\}$, at the generic abscissa x , the influence of gravity is thus explicitly reflected by the weight of the right cylindrical column of radius $Y(x)$, height $(x - X^*)$ and apparent density $\Delta\rho$, but also implicitly by the modified profile representation $x \rightarrow Y(x)$. Consequently,

$$\begin{aligned} \mathcal{F}_{cap}(X^*) &= 2\pi\gamma Y^* + \pi\gamma H Y^{*2}, \\ \mathcal{F}_{cap}(x) &= \mathcal{F}_{cap}(X^*) - \pi g\Delta\rho (x - X^*) Y^2(x), \quad x \geq X^*, \\ \mathcal{F}_{cap}(0) &= \mathcal{F}_{cap}(X^*) + \pi\gamma B X^* Y^2(0), \quad X^* < 0, \\ &\approx \mathcal{F}_{cap}(X^*) + \pi\gamma B X^* y^{*2}. \end{aligned}$$

In addition to affecting the lateral surface area of the bridge (denoted by $|\Sigma(d)|$, d as distorted by gravity) and the corresponding capillary energy, distortion causes a surface energy interchange by the wetting modifications of the solids boundaries:

the contact lines are subject to a frictional force generating a dissipation mechanism; hence, quantities $|\Lambda_i|(d)_{i=1,2}$, the area of the wetted region on solid S_i , are among the unknown of the problem and the capillary surface energy functional at prescribed volume, taking into account the free boundaries contribution, is to be transcribed in the following revised formulation:

$$\begin{aligned}\mathcal{E}_{frictional}(d) &= -\gamma (\cos \delta_1(d) |\Lambda_1(d)| + \cos \delta_2(d) |\Lambda_2(d)|), \quad \delta_i(d) \in [0, \pi], \\ \mathcal{E}_{cap}(d) &= \gamma (|\Sigma(d)| - \cos \delta_1(d) |\Lambda_1(d)| - \cos \delta_2(d) |\Lambda_2(d)|),\end{aligned}$$

where $\delta_i(d)$ are the contact angles values, after distortion effects.

The marked distortion of a capillary bridge significantly alters its shape, its geometry and stability properties (varying main curvature) and its resulting capillary force.

Limiting us to the interesting case of the distortion of a nodoid with a convex meridian in $\{y \geq 0\}$ (*i.e.* intrinsically, with negative Gaussian curvature) and assuming that H varies very little, we observe that the positive capillary force decreases weakly at the moved neck under the effect of gravity, due to the decrease of the gorge radius. At the level x , on $\{x \geq X^*\}$, the force decreases continuously under the effect of the apparent weight of the cylindrical liquid column with radius $Y(x)$ and height $(x - X^*)$.

The integral involved in the expression of \mathcal{F}^+ and \mathcal{F}^- is linked to the external volume of the figure of revolution generated by the rotation about the x -axis of a profile arc from the moved neck, preparatory term to transcribe the Archimedes' principle. Note that this would be the internal volume, counted negatively, for the observation of a hypothetical convex bridge.

Despite appearances, this last expression of \mathcal{F}^+ and $\mathcal{F}^- = 2\pi\gamma Y^* + \pi\gamma H Y^{*2}$ depends on B since (X^*, Y^*) , the coordinates of the neck, depend on B .

The key to understanding how to get this first integral is to rewrite locally the (H, B) -modified nonlinear differential Young-Laplace equation in the following local form, separately in the two branches related to $\{x \geq X^*\}$ and $\{x \leq X^*\}$, **H being rigorously evaluated as the mean curvature at the neck of the distorted bridge:**

$$-\frac{1}{Y} \frac{d}{dY} \frac{Y}{\sqrt{1+Y'^2}} = H + B x^\pm(Y).$$

Hence, by quadrature, quantities \mathcal{F}^+ and \mathcal{F}^- are constant respectively on $\{x \geq X^*\}$ and $\{x \leq X^*\}$; write $\mathcal{F}^+ = C^+$ and $\mathcal{F}^- = C^-$.

A continuity argument at (X^*, Y^*) implies

$$C^+ = C^- = 2\pi\gamma Y^* + \pi\gamma H Y^{*2}.$$

The demonstration lends itself to various easy generalizations, especially when a surface inflection exists and the Gaussian curvature changes sign.

3.3 The mean curvature recognition *in situ* at the distorted neck

The visual observation of a distorted capillary bridge allows by some geometric measurements to calculate analytically, at any point, the expression of the mean curvature (here varying, *a priori* unknown according to the formula $H+Bx$ in the Young-Laplace equation, H being a spontaneous value and not a known data).

For this purpose, we will take advantage of the exact first integral.

It is assumed that by a photographic process on the meridian profile, we could measure the coordinates of the neck (X^*, Y^*) , of the lower point (X_l, Y_l) and of the midpoint whose ordinate is $\frac{1}{2}(Y^* + Y_l)$, more precisely the point $(x^+(\frac{1}{2}(Y^* + Y_l)), \frac{1}{2}(Y^* + Y_l))$ and ϕ_l , the angle made by the tangent vector to the meniscus with the x -axis at the lower point. According to the exact first integral applied on the interval $[Y^*, Y_l]$, we note that

$$Y_l \cos \phi_l + \frac{H}{2} Y_l^2 + B \int_{Y^*}^{Y_l} x^+(y) y dy = Y^* + \frac{H}{2} Y^{*2}.$$

Therefore

$$H = 2 \frac{Y^* - Y_l \cos \phi_l - B \int_{Y^*}^{Y_l} x^+(y) y dy}{Y_l^2 - Y^{*2}}.$$

Simpson's rule will provide an adequate approximation to the exact integral. This is valid for very small "discretizations"; namely, when the distance between the neck and the "lower point" $|Y_l - Y^*| \ll 1$. In this case, it comes more precisely, so immediately usable:

$$\int_{Y^*}^{Y_l} x^+(y) y dy \approx \frac{Y_l - Y^*}{6} \left(X^* Y^* + 2(Y^* + Y_l) x^+\left(\frac{Y^* + Y_l}{2}\right) + X_l Y_l \right).$$

Note here that the method is easily generalized when subsequently, bending effects will also be considered. The corresponding relationships are then written:

$$Y_l \cos \phi_l + \frac{H}{2} Y_l^2 - \frac{C_K}{2\gamma} \cos^2 \phi_l + B \int_{Y^*}^{Y_l} x^+(y) y dy = Y^* + \frac{H}{2} Y^{*2} - \frac{C_K}{2\gamma}.$$

$$H = 2 \frac{Y^* - Y_l \cos \phi_l - \frac{C_K}{2\gamma} \sin^2 \phi_l - B \int_{Y^*}^{Y_l} x^+(y) y dy}{Y_l^2 - Y^{*2}}.$$

4 Details on moving the capillary bridge neck

A question of both practical and theoretical nature is interesting (generalized gorge method, initialization of a numerical scheme) for any bridge with strictly negative Gaussian curvature: can we give a fine estimate for the position of the moved neck? In particular, for the different differential problems related to the determination of the two disymmetric branches of the distorted meridian, two boundary conditions are then available: $Y(X^*) = Y^*$, $Y'(X^*) = 0$.

It suffices for this to consider **the osculating circle** to the disturbed profile at the point $(0, Y(0))$ for which we have fairly accurate information:

$$Y(0) \approx y(0) = y^* \text{ (the gorge radius without gravitational perturbation), } Y'(0) \approx B y^{*2}$$

and a fine value of the radius of curvature, as will be shown. Let us also observe that the point $(0, Y(0))$ is certainly a point very close to the moved neck as soon as the slope $B y^{*2}$ is there very weak (a slenderness criterium).

Indeed, concerning the osculating circle, the radius of curvature $R_S(0)$ is

$$\frac{(1 + Y'^2(0))^{3/2}}{Y''(0)}$$

i.e. via the modified nonlinear differential Young-Laplace equation (1)

$$R_S(0) = \left(\frac{1}{Y(0) \sqrt{1 + Y'^2(0)}} - \frac{\Delta p_0}{\gamma} \right)^{-1} \simeq \left(\frac{1}{y^* \sqrt{1 + B^2 y^{*4}}} + H \right)^{-1}.$$

Taking here as a first approximation⁸ the known value of H at zero gravity, it is then easy to evaluate the coordinates of the center of curvature

$$C(0) = (x_C(0), y_C(0))$$

in the quadrant $\{x < 0, y > 0\}$, on the normal line at $(0, Y(0))$ (the slope is then $-\frac{1}{B y^{*2}}$); consequently, by considering the point of this osculating circle with vertical tangent and smallest ordinate, (X^*, Y^*) the coordinates of the moved neck become

$$X^* \approx x_C(0) = -\frac{B y^{*2} R_S(0)}{\sqrt{1 + B^2 y^{*4}}},$$

$$Y^* \approx y_C(0) - R_S(0) = y^* - R_S(0) \left(1 - \frac{1}{\sqrt{1 + B^2 y^{*4}}} \right).$$

Moreover, if $B^2 y^{*4} \ll 1$ (dimensionless relationship), it can therefore be considered that

$$X^* \approx -B y^{*2} R_S(0),$$

$$Y^* \approx y^* - \frac{1}{2} R_S(0) B^2 y^{*4}.$$

⁸a subsequent iteration will clarify the spontaneous value in gravity field

The displacement of the bridge neck can therefore then be evaluated by the following elementary relationships in a first approach:

$$X^* < 0, \quad Y^* < y^*, \quad \frac{Y^* - y^*}{X^*} \approx \frac{1}{2}By^{*2} \approx \frac{1}{2}Y'(0), \quad Y'(X^*) = 0.$$

5 Remarks on distortions of cylindrical liquid bridges

In the rather theoretical case of a right cylindrical bridge (surface of zero Gaussian curvature), the same methods lead to considering the perturbed profile according to the formula:

$$Y(x) \simeq -\frac{1}{H} + \frac{B}{H^2}x + \kappa \sin Hx + \rho \cos Hx, \quad x \in I,$$

where

$$H < 0, \quad y^* = -\frac{1}{H}, \quad \rho \neq 0,$$

the relationship between κ and ρ (both dimensioned to a length) and the data H , B being partly specified by the two realistic conditions that the liquid volume is constant and the mass of water is displaced toward the lower solid; this last constraint would imply that

$$Y'(0) \approx \frac{B}{H^2} + \kappa H > 0.$$

To fix ideas, assume (stability criterion) that the height h is less than one-half of the bridge circumference (*i.e.* $h|H| < \pi$).

This result is obtained by linearizing the modified differential Young-Laplace equation in the following form, where $Y(x) = -\frac{1}{H} + \eta(x)$:

$$\eta''(x) - \frac{1}{-\frac{1}{H} + \eta(x)} = H + Bx, \quad \text{if } \|\eta'^2\|_{L^\infty(I)} \ll 1,$$

and then, if furthermore, $|H| \|\eta\|_{L^\infty(I)} \ll 1$, it follows that

$$\eta''(x) + H^2\eta(x) = Bx, \quad x \in I,$$

equation whose general solution is given by

$$\eta(x) = \frac{B}{H^2}x + \kappa \sin Hx + \rho \cos Hx,$$

It nevertheless remains that there is a free boundary problem to solve when developing a predictive approach, with strong sensitivity to boundary conditions!

The complete calculations are particularly simple in the case of a stable cylindrical bridge stretched between horizontal parallel plates. Thenceforward

$$|x| \leq \frac{h}{2} < \frac{\pi}{2|H|}.$$

Some methodological elements are developed to determine the parameters κ and ρ . Several scenarios are possible depending on the evolution of the contact angles in the forward and backward movements.

The relationship traducting the fact that the enclosed liquid volume is constant implies that ρ is necessarily non-zero (profile without any symmetry) and the point (ρ, κ) lies on an ellipse, in the explicit form; therefore the coefficient ρ marks the actual shifting from the point $(0, -\frac{1}{H})$ to the point $(0, -\frac{1}{H} + \rho)$. It will be seen below that this is a receding point.

The modified profile has an inflection point at $x = \frac{1}{|H|} \arctan \frac{\rho}{\kappa}$ if $\kappa \neq 0$ (under this condition, a surface inflection exists and the Gaussian curvature changes sign). The value $\kappa = 0$ does not give rise to a possible point of inflection due to the prior criterion of stability $h|H| < \pi$.

Roughly speaking, the perturbed profile then vaguely resembles that of a suitable unduloid surface portion with one inflection point but here, the deformed surface of revolution **does not have a constant mean curvature**. Gaussian curvature changes from zero to strictly positive values under the inflection point and strictly negative above, due to gravity effects. Therefore, the bending effects can be considered within the general framework that will be developed later.

It is also possible to choose the parameters κ and ρ so that the equilibrium configuration satisfies the boundary condition of 90° contact angle with the lower plate (homogeneous Neumann boundary condition). This gives

$$\frac{B}{H^3} + \kappa \cos H \frac{h}{2} - \rho \sin H \frac{h}{2} = 0.$$

The same contact angle with the upper plate can then no longer be imposed *a priori* since the value of $Y' \left(-\frac{h}{2} \right)$ is then necessarily

$$Y' \left(-\frac{h}{2} \right) = 2\rho H \sin H \frac{h}{2} \neq 0 \text{ as } \rho \neq 0 \text{ and } 0 < h|H| < \pi$$

according to the stability criterion for the right circular cylinder.

The advancing contact angle would increase in moving and the receding contact angle would decrease. The observation of a receding contact angle at $x = -\frac{h}{2}$ should realistically impose that

$$Y' \left(-\frac{h}{2}\right) < 0$$

and in consequence, $\rho < 0$, which would lead to the existence of a local neck in the upper half of the distorted bridge, according to the theorem of the intermediate value applied to Y' on $[-\frac{h}{2}, 0]$.

According to the second Pappus's centroid theorem, the geometric centroid ordinate of the bridge cross-section under gravitational perturbation, *i.e.*

$$\left\{ (x, y), -\frac{h}{2} \leq x \leq \frac{h}{2}, 0 \leq y \leq Y(x) \right\}$$

can be expressed as

$$Y_G = \frac{h}{2|H| \left(h - 2\rho \sin H \frac{h}{2} \right)}$$

(instead of $y_G = \frac{1}{2|H|}$ in the absence of gravity),

while the abscissa is given without computational difficulties by the formula

$$x_G = \frac{\int_0^{\frac{h}{2}} \left(\frac{B}{H^2} x^2 + \kappa x \sin Hx \right) dx}{\int_0^{\frac{h}{2}} \left(-\frac{1}{H} + \rho \cos Hx \right) dx}$$

The lateral surface area of the perturbed bridge is, within this framework,

$$\mathcal{A} = 2\pi \int_{-\frac{h}{2}}^{\frac{h}{2}} Y(x) \sqrt{1 + Y'^2(x)} dx$$

instead of $\frac{2\pi h}{|H|}$ in the absence of gravity.

Yet note for example that the noteworthy relationship, generalization of an **energy conservation principle**, is valid :

for any $x \geq 0$, the following functional expression is constant, expressed at the dimension of a force:

$$\mathcal{F}_B = 2\pi\gamma \left(\frac{Y(x)}{\sqrt{1+Y'^2(x)}} + \frac{H}{2}Y^2(x) + B \int_{Y(0)}^{Y(x)} Y^{-1}(y) y dy \right)$$

and its value is

$$\mathcal{F}_B = 2\pi\gamma \left(\frac{-\frac{1}{H} + \rho}{\sqrt{1 + \left(\frac{B}{H^2} + \kappa H\right)^2}} + \frac{H}{2} \left(-\frac{1}{H} + \rho\right)^2 \right).$$

(instead of $\mathcal{F}_0 = \frac{\pi\gamma}{|H|}$ in the absence of gravity)).

It follows that the general expression of the interparticle capillary force at the level x , on $\{x \geq 0\}$ for illustration where the force weakens, is explicitly given by:

$$\mathcal{F}_{cap}(x) = \mathcal{F}_B - \pi\gamma B x Y^2(x).$$

The formula does not remain valid on the interval $\{x \leq 0\}$.

It should not be forgotten that distortion causes a surface energy interchange by the wetting of the plates when the contact angles are different from $\frac{\pi}{2}$.

Note that, obviously, the Vogel's stability criterion at fixed contact lines does not apply here; actually, as is well known, the bridge becomes unstable from the limit case $h|H| = \frac{h}{y^*} = \pi$, critical aspect ratio and lower bound for the instability of right cylindrical bridges.

Such a capillary bridge is unstable if its volume is smaller than $\frac{h^3}{\pi}$, which leads to highlighting again the critical value $h|H| = \pi$.

6 On the use of predictor–corrector schemes to describe capillary distortions by gravity effects

The preceding results propose both an approximate predictive analytical expression of the profile distorted by the gravitational effects and an exact invariant in the form of a first integral for the modified nonlinear differential Young-Laplace equation. This opens the way to build different numerical schemes of the predictor-corrector type generalizable with great flexibility of use to any bridge with strictly negative Gaussian curvature. The first iteration of the process provides a new predictor, *a priori* more accurate, and so on for an iterative numerical method.

Recall briefly with previous notations that for B small enough, it comes on an appropriate open I , predictively:

$$\begin{aligned} Y_{pred}(x) &= y(x) + (H_g - H + Bx) y^2(x) |\cos \phi(x)|, \quad x \in I, \\ &= y(x) + (H_g - H + Bx) \frac{2ay^3(x)}{|y^2(x) + \varepsilon b^2|}, \quad \varepsilon = \pm 1, \end{aligned}$$

where, more precisely, Y_{pred} is a predictor for $Y = Y_{H,B}$, $y = Y_{H,0}$ and $\phi(x)$ is the angle made by the tangent vector to the meniscus with the x -axis at $(x, y(x))$.

Otherwise, along each concerned branch of the distorted profile, the two following functional expressions are constant and equal to $2\pi\gamma Y^* + \pi\gamma H Y^{*2}$:

$$\begin{aligned} &2\pi\gamma \left(\frac{Y(x)}{\sqrt{1+Y'^2(x)}} + \frac{H}{2} Y^2(x) + B \int_{Y^*}^{Y(x)} x^+(y) y dy \right) \text{ for any } x \geq X^*, \\ &2\pi\gamma \left(\frac{Y(x)}{\sqrt{1+Y'^2(x)}} + \frac{H}{2} Y^2(x) + B \int_{Y^*}^{Y(x)} x^-(y) y dy \right) \text{ if } x \leq X^*. \end{aligned}$$

Let's develop, as an illustration, the general method by limiting ourselves to the distorted branch $\{x \geq X^*\}$.

Considering axisymmetric volumes, let us first note that

$$2\pi \int_{Y^*}^{Y(x)} x^+(y) y dy = \pi (x - X^*) Y^2(x) - \pi \int_{X^*}^x Y^2(u) du, \text{ for any } x \geq X^*,$$

and consequently, the exact relation on the first integral is rewritten, for any $x \geq X^*$,

$$\begin{aligned} &\frac{Y(x)}{\sqrt{1+Y'^2(x)}} + \frac{H}{2} Y^2(x) + \frac{B}{2} \left((x - X^*) Y^2(x) - \int_{X^*}^x Y^2(u) du \right) \\ &= Y^* + \frac{1}{2} H Y^{*2}, \text{ for any } x \geq X^*. \end{aligned}$$

We restrict ourselves for the approximation of the following implicit integral problem to explicit discretization with a uniform space increment $h > 0$ on the subdivision $x_k = X^* + kh$, $k = 1, 2, \dots$ (one could also use intervals of varying length):

$$\begin{aligned} &\frac{Y(x_k)}{\sqrt{1+Y'^2(x_k)}} + \frac{H}{2} Y^2(x_k) + \frac{B}{2} \left((x_k - X^*) Y^2(x_k) - \int_{X^*}^{x_k} Y^2(u) du \right) \\ &= Y^* + \frac{1}{2} H Y^{*2}, \text{ for any } k = 1, 2, \dots \end{aligned}$$

(X^*, Y^*) the coordinates of the moved neck having been evaluated precisely in a previous paragraph or observed and measured *in situ*. By performing a simple iteration from the Young-Laplace equation, we obtain an estimate of the spontaneous value of H under the formulation:

$$(3) \quad H = H(B) \approx \frac{1}{R_S(0)} - \frac{1}{Y^*} - BX^*.$$

Consequently, if Y_k denotes an appropriate approximation of the exact solution $Y(x_k)$, we can consider, in order to anticipate the expected inflection of the profile, that

$$\frac{Y(x_k)}{\sqrt{1 + Y'^2(x_k)}} \approx Y_k |\cos \Psi(x_{k+1})|$$

where $\Psi(x_k)$ is the easily calculable angle made by the tangent vector to the meniscus predictor with the x -axis at $(x_k, Y_{pred}(x_k))$, the curve $x \rightarrow Y_{pred}(x)$ being explicitly defined and drawn exactly.

Let's comment on the numerical method to implement. It is then a question of interpolating the function $x \rightarrow Y^2(x)$ *via* the points $\{x_k, Y^2(x_k)\}_k$ in order to evaluate finely the volume integral.

The problem of obtaining stable methods of numerical integration is to be considered.

In that case therefore, the composite trapezoidal rule is an approximation by strong excess accordingly with the strictly negative Gaussian curvature of the bridge; it would be a source of numerical instability by overestimating progressively the gravity effects *via* the coefficient B .

Consequently, we will use the composite Simpson's rule; this method whose literal formulation depends on the parity of k must be explicitly applied alternately to k even and k odd. Specifically, this can be here stated as, if k is an **even** number:

$$\int_{X^*}^{x_k} Y^2(u) du \approx \frac{h}{3} \left(Y^{*2} + 2 \sum_{j=1, \dots, \frac{k}{2}-1} Y_{2j}^2 + 4 \sum_{j=1, \dots, \frac{k}{2}} Y_{2j-1}^2 + Y_k^2 \right).$$

where $Y_1 = Y_{pred}(X^* + h)$ to initialize the numerical process.

We are therefore led to successively solve the following quadratic equations in the unknown Y_k , for $k = 2, 4, \dots$, the sequence of positive ordinates $\{Y_k\}_{k \geq 1}$ to be realistically increasing (sagging effects in the lower part):

$$\begin{aligned}
& \left(\frac{H}{2} + \frac{B}{2} h \left(k - \frac{1}{3} \right) \right) Y_k^2 + Y_k |\cos \Psi(x_{k+1})| \\
& - h \frac{B}{6} \left(Y^{*2} + 2 \sum_{j=1, \dots, \frac{k}{2}-1} Y_{2j}^2 + 4 \sum_{j=1, \dots, \frac{k}{2}} Y_{2j-1}^2 \right) \\
& = Y^* + \frac{1}{2} H Y^{*2}.
\end{aligned}$$

If k is an **odd** number, we use the following strategy by writing

$$\int_{X^*}^{x_k} Y^2(u) du = \int_{X^*}^{x_{k-1}} Y^2(u) du + \int_{x_{k-1}}^{x_k} Y^2(u) du.$$

The treatment of the integral $\int_{X^*}^{x_{k-1}} Y^2(u) du$ falls under the previous case and for the other term, we use the adaptive Simpson's rule:

$$\int_{x_{k-1}}^{x_k} Y^2(u) du \approx h \left(-\frac{1}{12} Y_{k-2}^2 + \frac{2}{3} Y_{k-1}^2 + \frac{5}{12} Y_k^2 \right)$$

approximation exact for all quadratic polynomials, taking really into account the convexity of the profile.

We are therefore again led to successively solve quadratic equations in the unknown Y_k , for $k = 3, 5, \dots$

Discretized determination of the distorted profile on the interval $\{x \leq X^*\}$ is accomplished in the same way by this predictor-corrector method, with minor adjustments.

To improve the accuracy, it goes without saying that general integration methods such as adaptative Runge-Kutta methods can be applied to the recognition problem of the distorted profile ((associated to Lagrange or Hermite interpolation) and thus be used to evaluate finely the volume integral at

$$\{(x_k, Y^2(x_k))\}_k.$$

For example, in this context, Hermite interpolation matches the unknow function $x \rightarrow Y^2(x)$ in the values of parameters set $\{Y^2(x_k), (Y^2)'(x_k)\}$ at $\{x_k\}_{k=1,2,\dots}$, evaluated by $\{x_k, Y_k^2, 2Y_k \tan \Psi(x_{k+1})\}$ and to be identified by the first integral of the modified nonlinear differential Young-Laplace equation taking into account the effects of gravity...

7 On the generalized Young–Laplace equation. Effects of the Gaussian curvature at strong distortions

We set out here, for the convenience of the reader and in order to point out various directions of research, a generalization of the Young-Laplace equation presented in 1996 and which may be of particular interest in the case of strong capillary distortions.

The chapter 1 of the book "Applied Surface Thermodynamics" presents a generalized theory of capillarity [10]. The approach is entirely Gibbsian [4], based on the concept of the dividing surface, according to the statement of the authors, John Gaydos, Yehuda Rotenberg, Ladislav Boruvka, Pu Chen, and A. W. Neumann. However, it extends and differs from the classical theory in that it is not restricted to moderately curved liquid-fluid interfaces by taking into account the bending stress via mechanical curvature potentials along the interface (see also [2], p. 63-64, [22], [26]).

Various independent approaches lead to the generalized Young-Laplace equation involving both the mean curvature and the Gaussian curvature K , two intrinsic curvature measures of the free surface (in particular, bulk thermodynamics, minimization of an appropriate free energy function).

The shape of an interface between two non-solid substances is then described in the zero-gravity case by the **generalized Young–Laplace** equation:

$$\gamma \left(\frac{1}{\rho_c} + \frac{1}{N} \right) + C_K \frac{1}{\rho_c N} = \Delta p$$

where C_K , at the dimension of a force, stands for the account taken for the bending stress, ρ_c and N for the principal radii of curvature (evaluated algebraically). The new bending term discriminates the saddle-shaped interfaces (Gaussian curvature $K = \frac{1}{\rho_c N}$ is everywhere negative) and the synclastic interfaces on which Gaussian curvature is everywhere positive (locally convex axisymmetric bridges). A meridional profile of revolution with an inflection point juxtaposes the opposing effects of bending. It must be borne in mind that for very distorted profiles, the surface tension γ may be surface temperature and curvature-dependent, which severely complicates the mathematical treatment. It follows the specified formulation, with suitably dimensioned coefficients c_T and c_J , c_K ([10], p.9, eq. 27; here, interfacial tension equals local Gibbs free energy per non-planar surface area for chemically pure fluids):

$$\gamma = \gamma_0 + c_T T + c_J \left(\frac{1}{\rho_c} + \frac{1}{N} \right) + c_K \frac{1}{\rho_c N} .$$

Taking then into account simultaneously the combined effects of gravity and flexure results in the generalized highly nonlinear differential equation for the distorted profile $x \rightarrow Y(x)$ of axisymmetric capillary systems:

$$\begin{aligned} & \frac{Y''(x)}{(1+Y'^2(x))^{3/2}} - \frac{1}{Y(x)\sqrt{1+Y'^2(x)}} - \frac{C_K}{\gamma} \frac{Y''(x)}{Y(x)(1+Y'^2(x))^2} \\ & = -\frac{\Delta p_0}{\gamma} + \frac{g}{\gamma} \frac{\Delta \rho}{\gamma} x =: H + Bx, \quad x \in I. \end{aligned}$$

Mathematically isomorphic but with different variables and physical units, this differential equation has the same structure as the **Gullstrand equation** of geometrical optics, which relates the optic power P'_{op} of a thick lens (in dioptries, the reciprocal of the equivalent focal length) to its geometry and the properties of the media. For example, the superficial tension γ is equivalent to the refractivity $\frac{n_1}{n_2} - 1$, where n_i is a refractive index, C_K is analogous to the expression $-\left(\frac{n_1}{n_2} - 1\right)^2 \frac{n_2}{n_1} d$, d the lens thickness and Δp corresponds to P'_{op} .

Such an analogy is the subject of a thorough analysis in the publication of Rodriguez-Valverde, M. A., Cabrerizo-Vilchez, M. A. and Hidalgo-Alvarez,[23]. whose title is enlightening: The Young–Laplace equation links capillarity with geometrical optics.

8 A generalization of an exact energy invariant related to strongly distorted bridges with neck

The qualitative results elaborated in the framework of the constant mean curvature theory are essentially based on the existence of an exact invariant (in fact, a first integral for the second order nonlinear differential equation which reveals the conservation of the total energy of the free surface). With minor adaptations, they are immediately applicable to the situation where the Gaussian curvature and bending effects are taken into account. Indeed, as we will see, we still highlight in this case a first integral for the generalized Young-Laplace equation by limiting ourselves to a presentation concerning essentially any bridge with strictly negative Gaussian curvature.

For the spontaneous but a priori unknown value of H , $H = H_B$, the generalized equation can be rewritten, with the previous notations, in the differential form:

$$-\frac{1}{Y} \frac{d}{dY} \left(\frac{Y}{\sqrt{1+Y'^2}} - \frac{C_K}{2\gamma} \frac{1}{1+Y'^2} \right) = H + Bx^\pm(Y).$$

Hence, along each concerned branch of the strongly distorted profile, the two following functional expressions are constant and equal, at the dimension of a force:

for any $x \geq X^*$, (X^*, Y^*) being the coordinates of the moved neck,

$$\mathcal{F}_{C_K}^+ = 2\pi\gamma \left(\frac{Y(x)}{\sqrt{1+Y'^2(x)}} - \frac{C_K}{2\gamma} \frac{1}{1+Y'^2(x)} + \frac{H}{2} Y^2(x) + B \int_{Y^*}^{Y(x)} x^+(y) y dy \right)$$

and, if $x < X^*$,

$$\mathcal{F}_{C_K}^- = 2\pi\gamma\left(\frac{Y(x)}{\sqrt{1+Y'^2(x)}} - \frac{C_K}{2\gamma} \frac{1}{1+Y'^2(x)} + \frac{H}{2}Y^2(x) + B \int_{Y^*}^{Y(x)} x^-(y) y dy\right).$$

Moreover, by highlighting a continuous connection at the neck, the common value is

$$\begin{aligned}\mathcal{F}_{C_K}^+ &= \mathcal{F}_{C_K}^- = 2\pi\gamma Y^* - \pi C_K + \pi\gamma H Y^{*2} \\ &= -\pi\Delta p_0 Y^{*2} - \pi C_K + 2\pi\gamma Y^*,\end{aligned}$$

Δp_0 being evaluated, strictly speaking, at the neck of the distorted bridge.

8.1 The special case of only bending effects

It is interesting to note that when considering the only bending effects (*i.e.* $C_K \neq 0$, $B = 0$), then for any axisymmetric capillary bridge, the interparticle capillary force

$$(4) \quad \mathcal{F}_{C_K}^{cap} = 2\pi\gamma \left(\frac{Y}{\sqrt{1+Y'^2}} - \frac{C_K}{2\gamma} \frac{1}{1+Y'^2} + \frac{H}{2}Y^2 \right)$$

is constant at all points of the profile (generalization of the proposition 1 *in* [8] on the conservation of the total energy of the liquid bridge free surface).

The evaluation of $\mathcal{F}_{C_K}^{cap}$ at the gorge radius Y^* leads to, rather than the "gorge method" :

$$\mathcal{F}_{C_K}^{cap} = 2\pi\gamma Y^* - \pi C_K + \pi\gamma H Y^{*2}.$$

As explicated at the beginning of the paper, this exact formula allows to assess the relative importance of bending effects.

Of course equivalently, this expression may be evaluated at one or the other triple line. More generally, in fact

$$\mathcal{F}_{C_K}^{cap} = 2\pi\gamma Y(x) \cos \Theta(x) - \pi C_K \cos^2 \Theta(x) + \pi\gamma H Y^2(x)$$

where $\Theta(x)$ is the easily calculable angle made by the tangent vector to the meniscus with the x -axis at the generic point $(x, Y(x))$.

8.2 Parameterization of the profile

This latter relationship is interesting because it allows to easily obtain a parameterization of the profile *via* Θ by generalizing formulas established in an other way by C.H. Delaunay in 1841 [5], p.313. By a direct calculation and solving a quadratic equation if $H \neq 0$, we get at $H > 0$, for example:

$$Y(\Theta) = Y\left(\Theta, \frac{C_K}{\gamma}\right), \text{ the positive root such that } Y\left(0, \frac{C_K}{\gamma}\right) = Y^*,$$

i.e.

$$Y(\Theta) = \frac{1}{H} \left(-\cos \Theta + \sqrt{(HY^* + 1)^2 - \left(1 + H \frac{C_K}{\gamma}\right) \sin^2 \Theta} \right),$$

$$x(\Theta) = X^* + \int_0^\Theta dY(\theta) \cot(\theta) d\theta, \quad \textit{i.e.},$$

$$x(\Theta) = X^* + \frac{1}{H} \left(\sin \Theta + \left(1 + H \frac{C_K}{\gamma}\right) \int_0^\Theta \frac{\cos^2 \theta d\theta}{\sqrt{(HY^* + 1)^2 - \left(1 + H \frac{C_K}{\gamma}\right) \sin^2 \theta}} \right).$$

It should be kept in mind that the value of the gorge radius Y^* measured by observation *in situ*, and the same goes for H , depends on $\frac{C_K}{\gamma}$. As a practical observation, for the experimenter, the value of the capillary pressure \bar{H} is *a priori* an implicit unknown of the problem (a spontaneous value) whose mode of identification will be detailed below.

Note in particular that

$$Y\left(\Theta, \frac{C_K}{\gamma}\right) = \frac{2Y^* - \frac{C_K}{\gamma} \sin^2 \Theta}{2 \cos \Theta} \quad \text{if } H = 0.$$

(bending effects on a catenoid; if, moreover, $C_K = 0$, then the quantity $Y(\Theta) \cos \Theta$ is proved to be constant, a well-known geometrical property for catenaries).

8.3 Missing data recovery using the first integral

When the three dimensioned constant terms $\mathcal{F}_{C_K}^{cap}$, C_K , H are considered as missing values to be restored (*a priori* unknowns of the problem as spontaneous values resulting *a posteriori* from a static equilibrium), the relation (4) makes it possible to obtain the result by solving a linear system. To do this *via* an inverse problem, it suffices to apply the specific invariance relation at three points of the profile (experimental data), taking care to verify that the corresponding determinant of the 3×3 matrix is not close to zero (numerical stability condition). For this purpose, considering, for example, the observation of three distinct points $Y(0) = Y^*$, $Y(\Theta_1) = Y_1$, $Y(\Theta_2) = Y_2$, we note that the explicit determinant of the involved system is expressed, in absolute value, by

$$|\sin^2 \Theta_1 (Y^{*2} - Y_2^2) - \sin^2 \Theta_2 (Y^{*2} - Y_1^2)|.$$

So, in that case, are generalized the analytical calculation methods of distorted capillary bridges properties deduced as an inverse problem from experimental data, developed in [8].

8.4 Gravity and bending effects on the interparticle capillary force

Renewing with the same principles the previous procedures by minor adaptations leads to the general analytical expression of the interparticle capillary force at any level .

Indeed, the interparticle capillary force at the level x , on $\{x \geq X^*\}$ for illustration, can be evaluated as follows:

$$\begin{aligned} \mathcal{F}_{cap,C_K,B} (x) = & 2\pi\gamma \left(\frac{Y(x)}{\sqrt{1+Y'^2(x)}} - \frac{C_K}{2\gamma} \frac{1}{1+Y'^2(x)} + \frac{H}{2} Y^2(x) \right) \\ & + 2\pi\gamma B \left(\int_{Y^*}^{Y(x)} x^+(y) y dy - \frac{x-X^*}{2} Y^2(x) \right) \end{aligned}$$

and therefore, conjugate with the expression of the first integral of the generalized Young-Laplace equation, becomes in practice immediately usable, by the mathematical relationships:

$$\begin{aligned} \mathcal{F}_{cap,C_K,B} (x) = & 2\pi\gamma Y^* - \pi C_K + \pi\gamma H Y^{*2} - \pi\gamma B (x - X^*) Y^2(x), \\ & \text{and, in other words, rather than the "gorge method",} \\ \mathcal{F}_{cap,C_K,B} (X^*) = & 2\pi\gamma Y^* - \pi C_K + \pi\gamma H Y^{*2}. \end{aligned}$$

It follows then that concisely, by referring to the moved gorge point,

$$\mathcal{F}_{cap,C_K,B} (x) = \mathcal{F}_{cap,C_K,B} (X^*) - \pi g \Delta\rho (x - X^*) Y^2(x), \quad x \geq X^*.$$

These formulas remain valid on $\{x \leq X^*\}$ insofar as the distorted bridge keeps a saddle-shaped free capillary surface and thus, $x \rightarrow \mathcal{F}_{cap,C_K,B} (x)$ is everywhere a decreasing function. They are independent of the experimental device (horizontal plates, spheres, *etc.*).

It must be kept in mind that the values Y^* , $Y(x)$ on $\{x \geq X^*\}$ and H depend henceforth on the value of the force C_K , besides B and the contact parameters.

This also opens the way to build different numerical schemes of the predictor-corrector type with great flexibility of use for any strongly distorted bridge with strictly negative Gaussian curvature.

8.5 Gravity and bending effects on the strongly distorted profiles

For principal information, it should be noted that the numerical predictor-corrector scheme, previously presented out of the bending context (in the $H - B$ Young-Laplace framework) and corresponding to this more general situation, reveals a new term involving C_K as:

$$\begin{aligned} & \left(\frac{H}{2} + Bh \left(k - \frac{1}{2} \right) \right) Y_k^2 + Y_k \cos \Psi (x_k) - hB \left(\frac{1}{2} Y^{*2} + \sum_{j=1, \dots, k-1} Y_j^2 \right) \\ & = Y^* + \frac{1}{2} H Y^{*2} - \frac{C_K}{2\gamma} \sin^2 \Psi (x_k) \quad , \quad x_k \geq X^* , \end{aligned}$$

(X^*, Y^*) the coordinates of the moved neck having been observed and measured *in situ* in the simplified case of a direct experience.

In situation of a behavior prediction from the zero-gravity case, the evaluation of the coordinates (X^*, Y^*) as initialization value is more complicated on the basis of the previously developed considerations, reconstructing the shape of the local curvature. We can again approximate (X^*, Y^*) by the coordinates of the point with smallest ordinate on the osculating circle at $(0, Y(0))$ whose radius $R_S(0)$ is given by the following relationship, according to the generalized Young–Laplace equation:

$$R_S(0) = \frac{Y(0) \sqrt{1 + Y'^2(0)} - \frac{C_K}{\gamma}}{HY(0) \sqrt{1 + Y'^2(0)} + 1}.$$

The point $(0, y(0))$ can be in first approximation considered as an invariant point of the distortion and therefore $(0, y(0)) = (0, y^*) = (0, Y(0))$; at this point, only vary the local profile shape and the angle formed by the tangent to the distorted meniscus and the x -axis, from value 0 to α_0 , $\alpha_0 > 0$. We can use a successive approximations method (each new approximation is calculated on the basis of the preceding approximation) and the choice of the initial approximation for α_0 is, to some contextual extent, arbitrary but rather to underestimate: it is interesting to build ordered profiles approximations for the sake of convergence of the method (the value α_0 such as $\tan \alpha_0 = B y^{*2}$ if $B^2 y^{*4} \ll 1$ corresponds to a really weak distortion).

Therefore, taking here as a first approximation the value of H at zero gravity,

$$\begin{aligned} R_S(0) & \approx \frac{y^* \cos \alpha_0 - \frac{C_K}{\gamma}}{H y^* \cos \alpha_0 + 1} \\ X^* & \approx -R_S(0) \sin \alpha_0 \\ Y^* & \approx y^* - R_S(0) (1 - \cos \alpha_0) . \end{aligned}$$

By performing then a simple iteration from the generalized Young–Laplace equation, we obtain an estimate of the spontaneous value of H under the formulation:

$$H = H \left(B, \frac{C_K}{\gamma} \right) \approx \frac{1}{R_S(0)} \left(1 - \frac{C_K}{\gamma Y^*} \right) - \frac{1}{Y^*} - B X^* .$$

So, for the profile of the lower part of the distorted bridge, we have just to successively solve quadratic equations in the unknown Y_k , for $k = 1, 2, \dots$, the sequence of positive ordinates $\{Y_k\}_{k \geq 1}$ to be realistically increasing. The sequence $\{Y_k\}$ obtained at the first iteration of the process can serve as a new predictor and so on, which then defines an iterative numerical method by successive approximations and readjustment of the neck position and the H -value at each iteration.

Failing to know a first approximation of the distortion, the configuration of the bridge by neglecting both gravity and flexion effects, when it is known (by a microgravity experiment in parabolic flight for example) can be used to initialize the iterative algorithm, with readjustment of the neck position and the H -value at each iteration.

9 Adaptations to the case of strictly positive Gaussian curvature bridges

Taking again the previous argument by adapting to the case of strictly positive Gaussian curvatures, it shows that formally, the expression of the interparticle capillary force, *a priori* repulsive, formally keeps the same expression at any level x , on $\{x \geq X^*\}$:

$$\mathcal{F}_{cap,C_K,B}(x) = 2\pi\gamma Y^* - \pi C_K + \pi\gamma H Y^{*2} - \pi g \Delta\rho (x - X^*) Y^2(x).$$

However, it must be kept in mind that the spontaneous and a priori unknown values X^* , Y^* , $Y(x)$ on $\{x \geq X^*\}$ and H (negative) depend on the value of the force C_K , on B and on the Gaussian curvature sign, besides the contact parameters. The physical interpretation of this relationship reminds us of the Pascal's experiment, known as "crève-tonneau" (barrel-buster).

The sign of variations of the interparticle capillary force depends on the expression:

$$\text{sign} \frac{d\mathcal{F}_{cap,C_K,B}}{dx}(x) = -\text{sign} (Y(x) + 2(x - X^*) Y'(x)).$$

In addition, one can still take advantage of the first integral on $\{x \geq X^*\}$, to fix ideas, still valid if $B = 0$,

$$\begin{aligned} & 2\pi\gamma \left(\frac{Y(x)}{\sqrt{1+Y'^2(x)}} - \frac{C_K}{2\gamma} \frac{1}{1+Y'^2(x)} + \frac{H}{2} Y^2(x) + B \int_{Y^*}^{Y(x)} x^+(y) y dy \right) \\ & = 2\pi\gamma Y^* - \pi C_K + \pi\gamma H Y^{*2}. \end{aligned}$$

This again opens the way to implement different numerical schemes of the predictor-corrector type with great flexibility of use for any strongly distorted bridge with strictly positive Gaussian curvature.

10 Conclusions

In summary of the key aspects, we have presented a forecasting method for evaluating the axisymmetric capillary distortions, from the zero-gravity case assumed to be known precisely. We took advantage of the knowledge of an exact first integral for the generalized Young-Laplace equation with missing data values and a double estimate of the position of the moved neck and the spontaneous value H after gravity and bending effects. It must of course be kept in mind that conduct such a predictive modeling for the motion of the contact lines by gravity and flexure effects is a problem considerably more difficult than to model the static distorted case, observed *in situ*. The isomorphic structure between the Gullstrand and generalized Young-Laplace equations may be thought to allow experimenters to consider a capillary bridge as an optical system; although the two physical phenomena seem *a priori* disjoined but intellectually close, it could be deduced new practices for curvature measurements and fast, effective parameters identification. Nevertheless, the combined effect of volume, bending and axial gravity on the axisymmetric liquid bridge stability is a broad research subject to explore. The considerations on the numerical treatment of the distortion problem are given here as an indication of a research direction necessary to the advancement of the topic; this rather new subject could be of interest to specialists in finite elements or spline functions techniques.

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References

- [1] Adams, M. J., Johnson, S. A., Seville, J. P., & Willett, C. D. (2002). Mapping the influence of gravity on pendular liquid bridges between rigid spheres. *Langmuir*, 18(16), 6180-6184.
- [2] Berg, J. C. (2010). *An introduction to interfaces & colloids: the bridge to nanoscience*. World Scientific.
- [3] Brezis, H., & Mironescu, P. (2019). The Plateau problem from the perspective of optimal transport. *Comptes Rendus Mathématique*, 357(7), 597-612.
- [4] Boruvka, L., & Neumann, A. W. (1977). Generalization of the classical theory of capillarity. *The Journal of Chemical Physics*, 66(12), 5464-5476.
- [5] Delaunay, C. H. (1841). Sur la surface de révolution dont la courbure moyenne est constante. *Journal de mathématiques pures et appliquées*, 309-314.
- [6] Finn, R. (1999). Capillary surface interfaces. *Notices AMS*, 46(7), 770-781.

- [7] Ferguson, A. (1912). LXXXIX. On the shape of the capillary surface formed by the external contact of a liquid with a cylinder of large radius. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 24(144), 837-844. and Nicolson, M. M., *Proc. Cambridge Phil. Soc.* 45,288 (1949).
- [8] Gagneux, G., & Millet, O. (2014). Analytic Calculation of Capillary Bridge Properties Deduced as an Inverse Problem from Experimental Data. *Transport in Porous Media*, Volume 105, Issue 1, p. 117-139.
- [9] Gagneux, G., Millet, O., Mielniczuk, B., & El Youssoufi, M. S. (2016). Theoretical and experimental study of pendular regime in unsaturated granular media. *European Journal of Environmental and Civil Engineering*, 1-14.
- [10] Gaydos, J., Boruvka, L., Rotenberg, Y., Chen, P. and Neumann, A. W. (1996). Applied Surface Thermodynamics. In *The Generalized Theory of Capillarity* (pp.1-52). Surfactant science series, vol. 63, edited by A. W Neumann and Jan K.Spelt, Marcel Dekker Inc., New York, Basel.
- [11] De Gennes, P. G., Brochard-Wyart, F., & Quéré, D. (2004). Capillarity and gravity. In *Capillarity and Wetting Phenomena* (pp. 33-67). Springer, New York, NY.
- [12] Gilbarg, D., & Trudinger, N. S. (2015). *Elliptic partial differential equations of second order*. Springer.
- [13] Giusti, E., & Williams, G. H. (1984). *Minimal surfaces and functions of bounded variation* (Vol. 2, No. 3). Boston: Birkhäuser.
- [14] Lions, J. L. (1992). *Sentinelles pour les systèmes distribués à données incomplètes* (Vol. 21). *Recherches en Mathématiques appliquées*. Masson.
- [15] Mason, G., & Clark, W. C. (1965). Liquid bridges between spheres. *Chemical Engineering Science*, 20(10), 859-866.
- [16] Mazzone, D. N., Tardos, G. I., & Pfeffer, R. (1986). The effect of gravity on the shape and strength of a liquid bridge between two spheres. *Journal of colloid and interface science*, 113(2), 544-556.
- [17] Mesmoudi, M. M., De Floriani, L., & Magillo, P. (2012). Discrete curvature estimation methods for triangulated surfaces. In *Applications of Discrete Geometry and Mathematical Morphology* (pp. 28-42). Springer, Berlin, Heidelberg.
- [18] Mielniczuk, B., Millet, O., Gagneux, G., & El Youssoufi, M. S. (2018). Characterisation of pendular capillary bridges derived from experimental data using inverse problem method. *Granular Matter*, 20(1), 14.

- [19] Myshkis, A. D., Babskii, V. G., Kopachevskii, N. D., Slobozhanin, L. A., & Tyuptsov, A. D. (2012). *Low-gravity fluid mechanics*. Springer-Verlag Berlin An.
- [20] Nguyen, H. N. G., Zhao, C. F., Millet, O., & Gagneux, G. (2020). An original method for measuring liquid surface tension from capillary bridges between two equal-sized spherical particles. *Powder Technology*, 363, 349-359.
- [21] Orr, F. M., Scriven, L. E., & Rivas, A. P. (1975). Pendular rings between solids: meniscus properties and capillary force. *Journal of Fluid Mechanics*, 67(04), 723-742.
- [22] Rey, A. D. (2007). Capillary models for liquid crystal fibers, membranes, films, and drops. *Soft Matter*, 3(11), 1349-1368.
- [23] Rodriguez-Valverde, M. A., Cabrerizo-Vilchez, M. A., & Hidalgo-Alvarez, R. (2003). The Young–Laplace equation links capillarity with geometrical optics. *European journal of physics*, 24(2), 159.
- [24] Sauvigny, F. (2016). Surfaces of prescribed mean curvature $H(x, y, z)$ with one-to-one central projection onto a plane. *Pacific Journal of Mathematics*, 281(2), 481-509.
- [25] Sauvigny, F. (2018). Solution of boundary value problems for surfaces of prescribed mean curvature $H(x, y, z)$ with 1–1 central projection via the continuity method. *Lithuanian Mathematical Journal*, 58(3), 320-328.
- [26] Scholtès, L., Hicher, P. Y., Nicot, F., Chareyre, B., & Darve, F. (2009). On the capillary stress tensor in wet granular materials. *International journal for numerical and analytical methods in geomechanics*, 33(10), 1289-1313.
- [27] Van Honschoten, J. W., Brunets, N., & Tas, N. R. (2010). Capillarity at the nanoscale. *Chemical Society Reviews*, 39(3), 1096-1114.
- [28] Wang, Y. J., Guo, S., Chen, H. Y., & Tong, P. (2016). Understanding contact angle hysteresis on an ambient solid surface. *Physical Review E*, 93(5), 052802.
- [29] Wiklund, H. S., & Uesaka, T. (2012). Simulations of shearing of capillary bridges. *The Journal of chemical physics*, 136(9), 094703.
- [30] Willett, C. D., Johnson, S. A., Adams, M. J., & Seville, J. P. (2007). Pendular capillary bridges. In *Handbook of Powder Technology* (Vol. 11, pp. 1317-1351). Elsevier Science BV.
- [31] Yang, S. (2018). *Capillary Water Behavior During Evaporation of Granular Media* (Doctoral dissertation, Duke University).
- [32] Yang, S., Mielniczuk, B., El Youssoufi, M. S., & Hueckel, T. (2018). A note on evolution of pressure and flow within an evaporating capillary bridge. *The European Physical Journal E*, 41(12), 140.