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Fundamental scattering quantities for the determination of reflectance and transmittance

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Abstract: The bidirectional reflectance distribution function (BRDF) and the bidirectional scattering - surface reflectance distribution function (BSSRDF), which relate radiance at the surface to irradiance and radiant flux, respectively, are regarded as the most fundamental scattering quantities used to determine the reflectance of objects. However, for materials where the optical radiation is transmitted under the surface, this radiance depends not only on irradiance and radiant flux, but also on the size of the irradiated area of the surface. This article provides insight into such dependence under the special condition in which the radiance is evaluated within the irradiated area and, consequently, is produced by both the insurface reflection and the subsurface scattering, in contrast to the situation in which the radiance is evaluated at non-irradiated areas and only subsurface scattering contributes. By explicitly considering both contributions, two other scattering quantities are defined: one that accounts exclusively for the insurface reflection and the other that accounts for subsurface scattering. In this regard, these quantities might be considered more fundamental than the BRDF and the BSSRDF, although they are coincident with these two functions apart from the above-mentioned special condition and for materials with negligible subsurface scattering. In this work, the relevance of the proposed scattering quantities is supported by experimental data, practical considerations are given for measuring them, and their relation to the bidirectional transmittance distribution function (BTDF) is discussed.

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1. Introduction

According to the CIE International Lighting Vocabulary (ILV) [1], reflectance is defined as “the ratio of the reflected radiant flux or luminous flux to the incident flux in the given conditions”, while transmittance is defined as “the ratio of the transmitted radiant flux or luminous flux to the incident flux in the given conditions”. Both quantities describe the interaction of the radiant flux (Φ_i) with the objects, but for *given* conditions. These conditions include the irradiated area of the surface (A_i), the collection area of the surface for which the optical radiation is evaluated (A_r), the irradiation and collection solid angles (ω_i and ω_r) (Fig. 1, Table 1), and other conditions not addressed in this work such as polarization. Therefore, when reporting the value of reflectance or transmittance, these conditions must be specified explicitly. Quantities such as diffuse reflectance, specular (regular [1]) reflectance, diffuse transmittance, regular transmittance or others [2] are defined to implicitly include well-established geometrical conditions. Many other quantities might be defined, that would provide very different results, making the measurement

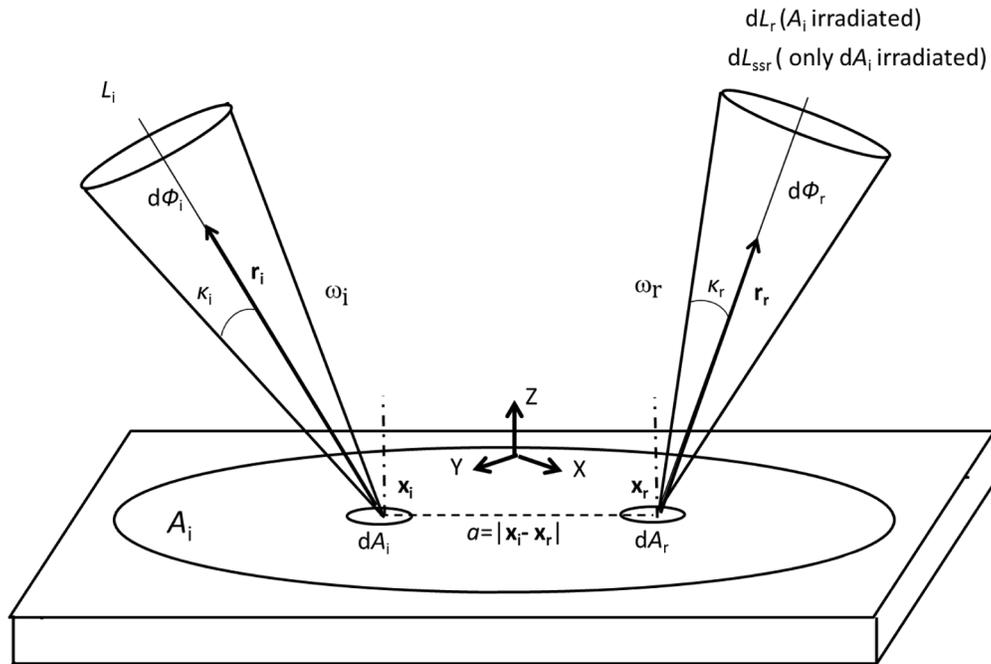


Fig. 1. Variables used in the definition of the scattering quantities. Quantities and symbols are given in detail in Table 1 for a better understanding of the equations.

of reflectance and transmittance in given conditions far from being a general characterization of the interaction of the radiant flux with objects.

In order to obtain a more general description, quantities must be defined that will allow the reflectance and the transmittance to be derived for any condition. Nicodemus *et al.* [3–5] defined the Bidirectional Reflectance Distribution Function (BRDF) with the purpose of describing such a fundamental quantity for reflectance. This function describes the bidirectional reflectance for infinitesimal solid angles and for any irradiation (\mathbf{r}_i) and collection (\mathbf{r}_r) directions (Fig. 1, Table 1). The BRDF should allow reflectance to be calculated for any condition via integration, as thoroughly described in Ref. [5], where it is defined as:

$$f_r(\mathbf{r}_i; \mathbf{r}_r) = \left. \frac{dL_r(\mathbf{r}_i; \mathbf{r}_r)}{dE(\mathbf{r}_i)} \right|_{A_i \text{ large enough}} \quad (1)$$

where E is the irradiance ($= d\Phi_i/dA_i$), which must be uniform on the sample, and L_r is the emerging radiance resulting from irradiation. The condition of being irradiated with a large enough area A_i will be commented on later; at this point, it is convenient to define the quantity:

$$f_r^*(\mathbf{r}_i; \mathbf{x}_r; \mathbf{r}_r; A_i) = \frac{dL_r(\mathbf{r}_i; \mathbf{x}_r; \mathbf{r}_r; A_i)}{dE(\mathbf{r}_i)}, \quad (2)$$

where the only difference from Eq. (1) is that no condition is imposed on the size of the irradiated area. In this case, the dependence on the collection position on the surface \mathbf{x}_r is explicitly specified, where both f_r^* and L_r depend on the position with respect to the center of the irradiated area in objects allowing transmission of optical radiation under the surface, in contrast with f_r , where the “large enough area” condition is applied precisely to avoid this dependence.

Differentials are used to indicate that the irradiation is incident from a differential element of solid angle ω_i in direction of \mathbf{r}_i , with $dE(\mathbf{r}_i) = L_i(\mathbf{r}_i) \cos \theta_i d\omega_i$, where L_i and θ_i are the

Table 1. List of quantities and symbols.

| Symbol | Quantity | SI units |
|----------------|---|--------------------------------|
| a | Distance from \mathbf{x}_i to \mathbf{x}_r . | m |
| A_i | Irradiated area at the surface. Involved in the definition of the BRDF. | m^2 |
| A_r | Collection area at the surface for which the reflected radiant flux is evaluated. | m^2 |
| dA_i | Differential of irradiated area. Involved in the definition of the BSSRDF. | m^2 |
| ΔA_i | Increment of irradiated area. Used instead of dA_i when referring to measurement. | m^2 |
| dA_r | Differential of collection area. | m^2 |
| ΔA_r | Increment of collection area. Used instead of dA_r when referring to measurement. | m^2 |
| dL_r | Differential of radiance at \mathbf{x}_r produced by the irradiation on A_i . Involved in the definition of the BRDF. | $W \cdot m^{-2} \cdot sr^{-1}$ |
| dL_{ssr} | Differential of radiance at \mathbf{x}_r produced by the irradiation on dA_i at \mathbf{x}_i . Involved in the definition of the BSSRDF. | $W \cdot m^{-2} \cdot sr^{-1}$ |
| E | Irradiance on the surface. | $W \cdot m^{-2}$ |
| f_r | Bidirectional Reflectance Distribution Function (BRDF). | sr^{-1} |
| f_r^* | Derivative of L_r with respect to E . | sr^{-1} |
| f_{ssr} | Bidirectional Scattering - Surface Reflectance Distribution Function (BSSRDF). | $m^{-2} \cdot sr^{-1}$ |
| f_{ss} | Bidirectional Bipositional Subsurface Scattering function (BBSS). | $m^{-2} \cdot sr^{-1}$ |
| \hat{f}_{ss} | Measurement of f_{ss} . | $m^{-2} \cdot sr^{-1}$ |
| f_{is} | Bidirectional Insurface Scattering function (BIS). | sr^{-1} |
| \hat{f}_{is} | Measurement of f_{is} . | $m^{-2} \cdot sr^{-1}$ |
| f_t | Bidirectional Transmittance Distribution Function (BTDF). | sr^{-1} |
| L_i | Radiance of the incident optical radiation. | $W \cdot m^{-2} \cdot sr^{-1}$ |
| L_r | Radiance of the reflected optical radiation. | $W \cdot m^{-2} \cdot sr^{-1}$ |
| L_{is} | Radiance from insurface reflection. | $W \cdot m^{-2} \cdot sr^{-1}$ |
| L_{ss} | Radiance from subsurface scattering. | $W \cdot m^{-2} \cdot sr^{-1}$ |
| L_{ssr} | Radiance from subsurface scattering and insurface reflection. | $W \cdot m^{-2} \cdot sr^{-1}$ |
| L_t | Radiance of the transmitted optical radiation. | $W \cdot m^{-2} \cdot sr^{-1}$ |
| ω_i | Irradiation solid angle. | sr |
| ω_r | Collection solid angle. | sr |
| Φ_i | Incident radiant flux on the surface. | W |
| \mathbf{r}_i | Incidence direction. | (rad, rad) |
| \mathbf{r}_r | Collection direction. | (rad, rad) |
| θ_i | Incidence angle. | rad |
| \mathbf{x}_i | Incidence position on the object. | (m, m, m) |
| \mathbf{x}_r | Collection position on the object. | (m, m, m) |

incident radiance and the incidence angle, respectively. In this and in the following equations, Leibniz notation is used, for which quotients of differentials can be interpreted as derivatives (non-standard analysis). To our knowledge, this interpretation does not incur any contradictions in analogous expressions with radiometric quantities.

Nicodemus *et al.* [5] related the BRDF to a more fundamental distribution function, the Bidirectional Scattering - Surface Reflectance Distribution Function (BSSRDF, for which the

symbol f_{ssr} is used in this article; the subscript “ssr” stands for “Scattering - Surface Reflectance”):

$$f_{\text{ssr}}(\mathbf{x}_i, \mathbf{r}_i; \mathbf{x}_r, \mathbf{r}_r) = \frac{dL_{\text{ssr}}(\mathbf{x}_i, \mathbf{r}_i; \mathbf{x}_r, \mathbf{r}_r)}{d\Phi_i(\mathbf{x}_i, \mathbf{r}_i)}, \quad (3)$$

where L_{ssr} is the emerging radiance produced by the incident radiant flux Φ_i . Although both dL_r and dL_{ssr} are radiances at a specific position \mathbf{x}_r , the notation used in Eq. (3) differs from that used in Eqs. (1) and (2) because dL_r and dL_{ssr} refer to differentials of different orders. For dL_{ssr} , both radiance and BSSRDF depend explicitly on the positions of the irradiation (\mathbf{x}_i) and the collection (\mathbf{x}_r), and not only on the directions. This is relevant for addressing the spatial dimension of the scattering, as will be done in this article. It must be noted that the BSSRDF is a particular example of the even more general scattering function called S in Ref. [6], which also includes spectral and polarization variables. Note that, while the BSSRDF given in Ref. [5] refers to flat surfaces, which is the most feasible approach for metrologists, in this article, it is considered in a more generalized way for any kind of surface, such as in computer graphics [7], where the BSSRDF is interpreted in a way similar to the scattering function [6,8].

The reason why a scattering quantity that depends on \mathbf{x}_i and \mathbf{x}_r must be defined is the existence of subsurface scattering. The scattering caused by an object is often due to two optical phenomena: reflection at the air-matter interface (here referred to as *insurface reflection*), and scattering within the bulk (referred to as *subsurface scattering*). These two phenomena generate different lateral light propagation distances, which can be defined as the distance between the position where the incident radiant flux irradiates the object and the furthest position where a proportion of it may exit. Insurface reflection produces a very short lateral propagation due to inter-reflections within the concavities in the micro-rough structure of the surface, and approaches zero for flat surfaces. However, subsurface scattering can produce considerable lateral propagation, and the irradiation at \mathbf{x}_i gives rise to non-negligible radiance at non-irradiated positions \mathbf{x}_r . Very low scattering and absorption coefficients yield a *transparent* aspect, whereas, at the other extreme, very high scattering coefficients or absorption coefficients provide an *opaque* aspect. *Translucent* objects are those with appropriate intermediate scattering and absorption coefficients and thicknesses for permitting light to pass through but not allowing the objects on the opposite side to be clearly visible.

According to Nicodemus *et al.* [5] [Eqs. (5)–(9) therein], the BRDF at a position \mathbf{x}_r can be calculated by integrating the BSSRDF via the following reasoning.

Let $dL_r(\mathbf{r}_i; \mathbf{x}_r, \mathbf{r}_r; A_i)$ be an element of radiance at position \mathbf{x}_r for given irradiation and collection directions ($\mathbf{r}_i, \mathbf{r}_r$), produced by irradiation of an area A_i . The area A_i can be considered a collection of differential elements of area dA_i , each with an element of incident radiant flux, $d\Phi_i(\mathbf{x}_i, \mathbf{r}_i)$ (see Fig. 1, Table 1). According to the definition in Eq. (3), $d\Phi_i(\mathbf{x}_i, \mathbf{r}_i)$ contributes to $dL_r(\mathbf{r}_i; \mathbf{x}_r, \mathbf{r}_r; A_i)$ with a proportion given by:

$$dL_{\text{ssr}}(\mathbf{x}_i, \mathbf{r}_i; \mathbf{x}_r, \mathbf{r}_r) = f_{\text{ssr}}(\mathbf{x}_i, \mathbf{r}_i; \mathbf{x}_r, \mathbf{r}_r) d\Phi_i(\mathbf{x}_i, \mathbf{r}_i) = f_{\text{ssr}}(\mathbf{x}_i, \mathbf{r}_i; \mathbf{x}_r, \mathbf{r}_r) dE(\mathbf{x}_i, \mathbf{r}_i) dA_i. \quad (4)$$

Thus, $dL_r(\mathbf{r}_i; \mathbf{x}_r, \mathbf{r}_r; A_i)$ is expressed as the integral of these elementary radiances, i.e.:

$$dL_r(\mathbf{r}_i; \mathbf{x}_r, \mathbf{r}_r; A_i) = \int_{A_i} dL_{\text{ssr}}(\mathbf{x}_i, \mathbf{r}_i; \mathbf{x}_r, \mathbf{r}_r) = dE(\mathbf{r}_i) \int_{A_i} f_{\text{ssr}}(\mathbf{x}_i, \mathbf{r}_i; \mathbf{x}_r, \mathbf{r}_r) dA_i, \quad (5)$$

where a uniform irradiation has been assumed; in this way, the irradiance dE becomes independent of the point of incidence. By rearranging terms, Eq. (5) can be written as:

$$f_r^*(\mathbf{r}_i; \mathbf{x}_r, \mathbf{r}_r; A_i) = \frac{dL_r(\mathbf{r}_i; \mathbf{x}_r, \mathbf{r}_r; A_i)}{dE(\mathbf{r}_i)} = \int_{A_i} f_{\text{ssr}}(\mathbf{x}_i, \mathbf{r}_i; \mathbf{x}_r, \mathbf{r}_r) dA_i. \quad (6)$$

According to Nicodemus *et al.* [5], this equation allows the relation between the BRDF and the BSSRDF to be established under the condition with a (as stated by the authors) “uniform

irradiance over a large enough area (A_i) of a uniform and isotropic surface". A "large enough area" A_i is required to make the BRDF invariant around \mathbf{x}_r . Because of the dependence of the BSSRDF on \mathbf{x}_i and \mathbf{x}_r , this condition is required even in the absence of local variations of the scattering properties of the surface, or under the assumption that any spatial inhomogeneity of the reflectance is spatially averaged by the detector. Therefore, due to this "large enough area" condition, the BRDF definition excludes situations faced with smaller irradiation areas; this leaves the following question unanswered: How can the radiance generated at an object's surface be described when the irradiated area is not "large enough"?

The radiance of the surface is produced under two different conditions. On the one hand, the radiance at $\mathbf{x}_r = \mathbf{x}_i$ is caused by both insurface reflection and subsurface scattering, while on the other hand, the radiance at $\mathbf{x}_r \neq \mathbf{x}_i$ is caused only by subsurface scattering. Therefore, a discontinuity in the BSSRDF is expected at $\mathbf{x}_r = \mathbf{x}_i$. This is a very special and important condition that needs to be closely examined.

This article gives insight into the condition in which the radiance is evaluated within the irradiated area and, consequently, is produced by both the insurface reflection and the subsurface scattering. By explicitly considering both contributions, two other scattering quantities are defined: one accounting exclusively for the insurface reflection and the other only for the subsurface scattering. In this regard, these quantities might be considered more fundamental than the BRDF and the BSSRDF, although they are coincident with these two functions apart from the above-mentioned special condition and for materials with negligible subsurface scattering.

2. Discontinuity and dependence on the irradiated area

As introduced above, the radiant flux $d\Phi_i(\mathbf{x}_i, \mathbf{r}_i)$ that irradiates each elementary area dA_i contributes to $dL_r(\mathbf{r}_i; \mathbf{x}_r, \mathbf{r}_r; A_i)$ with a proportion given by Eq. (4). In the following derivation, the discontinuity of $dL_{\text{ssr}}(\mathbf{x}_i, \mathbf{r}_i; \mathbf{x}_r, \mathbf{r}_r)$ at $\mathbf{x}_i = \mathbf{x}_r$ is explicitly taken into account.

The radiance dL_{ssr} is expressed as the sum of the radiance produced by the insurface reflection (dL_{is}) and the radiance produced by the subsurface scattering (dL_{ss}):

$$dL_{\text{ssr}}(\mathbf{x}_i, \mathbf{r}_i; \mathbf{x}_r, \mathbf{r}_r) = dL_{\text{ss}}(\mathbf{x}_i, \mathbf{r}_i; \mathbf{x}_r, \mathbf{r}_r) + dL_{\text{is}}(\mathbf{r}_i; \mathbf{r}_r), \quad (7)$$

where there is no spatial dependence on \mathbf{x}_r or on \mathbf{x}_i for dL_{is} because it describes the radiance that is exclusively due to the scattering at the air-matter interface and is therefore non-zero only for $\mathbf{x}_r = \mathbf{x}_i$.

Since dL_{is} is zero when $\mathbf{x}_r \neq \mathbf{x}_i$, Eq. (4) can be written as:

$$dL_{\text{ssr}}(\mathbf{x}_i, \mathbf{r}_i; \mathbf{x}_r, \mathbf{r}_r) = \begin{cases} f_{\text{ss}}(\mathbf{x}_i, \mathbf{r}_i; \mathbf{x}_r, \mathbf{r}_r)d\Phi_i + dL_{\text{is}}(\mathbf{r}_i; \mathbf{r}_r), & \text{for } \mathbf{x}_r = \mathbf{x}_i \\ f_{\text{ss}}(\mathbf{x}_i, \mathbf{r}_i; \mathbf{x}_r, \mathbf{r}_r)d\Phi_i, & \text{otherwise} \end{cases} \quad (8)$$

where the definition of the function f_{ss} is similar to that of the BSSRDF but excludes dL_{is} :

$$f_{\text{ss}}(\mathbf{x}_i, \mathbf{r}_i; \mathbf{x}_r, \mathbf{r}_r) = \frac{dL_{\text{ss}}(\mathbf{x}_i, \mathbf{r}_i; \mathbf{x}_r, \mathbf{r}_r)}{d\Phi_i(\mathbf{x}_i, \mathbf{r}_i)}. \quad (9)$$

The BSSRDF can be obtained directly from Eq. (8) as:

$$f_{\text{ssr}}(\mathbf{x}_i, \mathbf{r}_i; \mathbf{x}_r, \mathbf{r}_r) = \frac{dL_{\text{ssr}}(\mathbf{x}_i, \mathbf{r}_i; \mathbf{x}_r, \mathbf{r}_r)}{d\Phi_i(\mathbf{x}_i, \mathbf{r}_i)} = \begin{cases} f_{\text{ss}}(\mathbf{x}_i, \mathbf{r}_i; \mathbf{x}_r, \mathbf{r}_r) + \frac{f_{\text{is}}(\mathbf{r}_i; \mathbf{r}_r)}{dA_i}, & \text{for } \mathbf{x}_r = \mathbf{x}_i \\ f_{\text{ss}}(\mathbf{x}_i, \mathbf{r}_i; \mathbf{x}_r, \mathbf{r}_r), & \text{otherwise} \end{cases} \quad (10)$$

where f_{is} is defined by:

$$f_{\text{is}}(\mathbf{r}_i; \mathbf{r}_r) = \frac{dL_{\text{is}}(\mathbf{r}_i; \mathbf{r}_r)}{dE(\mathbf{r}_i)}, \quad (11)$$

which is similar to f_r^* but excludes dL_{ss} .

The differential dA_i at the denominator in the second term of Eq. (10) describes the fact that the variation of the radiance with respect to the radiant flux increases for smaller irradiation areas, and that this variation is not completely defined unless the second term of the equation is negligible with respect to the first.

On the other hand, f_r^* is obtained from Eq. (8) via integration over A_i to obtain $dL_T(\mathbf{r}_i; \mathbf{x}_r, \mathbf{r}_r; A_i)$. In analogy with Eq. (5),

$$\begin{aligned} dL_T(\mathbf{r}_i; \mathbf{x}_r, \mathbf{r}_r; A_i) &= \int_{A_i} dL_{SSr}(\mathbf{x}_i, \mathbf{r}_i; \mathbf{x}_r, \mathbf{r}_r) \\ &= \begin{cases} dE(\mathbf{r}_i) \int_{A_i} f_{ss}(\mathbf{x}_i, \mathbf{r}_i; \mathbf{x}_r, \mathbf{r}_r) dA_i + dL_{is}(\mathbf{r}_i; \mathbf{r}_r), & \text{for } \mathbf{x}_r \in A_i \\ dE(\mathbf{r}_i) \int_{A_i} f_{ss}(\mathbf{x}_i, \mathbf{r}_i; \mathbf{x}_r, \mathbf{r}_r) dA_i, & \text{otherwise} \end{cases} \end{aligned} \quad (12)$$

where a uniform irradiation has again been assumed. The integration over A_i involves the condition $\mathbf{x}_r = \mathbf{x}_i$ changing to $\mathbf{x}_r \in A_i$. Note also that $\int_{A_i} dL_{is}(\mathbf{r}_i; \mathbf{r}_r) = dL_{is}(\mathbf{r}_i; \mathbf{r}_r)$ because the L_{is} contribution is zero except for the elementary area at $\mathbf{x}_r = \mathbf{x}_i$.

Finally, f_r^* can be expressed as:

$$f_r^*(\mathbf{r}_i; \mathbf{x}_r, \mathbf{r}_r; A_i) = \frac{dL_T(\mathbf{r}_i; \mathbf{x}_r, \mathbf{r}_r; A_i)}{dE(\mathbf{r}_i)} = \begin{cases} \int_{A_i} f_{ss}(\mathbf{x}_i, \mathbf{r}_i; \mathbf{x}_r, \mathbf{r}_r) dA_i + f_{is}(\mathbf{r}_i; \mathbf{r}_r), & \text{for } \mathbf{x}_r \in A_i \\ \int_{A_i} f_{ss}(\mathbf{x}_i, \mathbf{r}_i; \mathbf{x}_r, \mathbf{r}_r) dA_i, & \text{otherwise.} \end{cases} \quad (13)$$

Whereas the first condition of the equation is related to the BRDF, the interpretation of the second one ($\mathbf{x}_r \notin A_i$) is not obvious. At the back of a planar sample with parallel interfaces, the second condition is related to the scattering quantity usually referred to as the Bidirectional Transmittance Distribution Function (BTDF) [9–11]. In such a case, it can be expressed as:

$$f_t^*(\mathbf{r}_i; \mathbf{x}_r, \mathbf{r}_r; A_i) = \frac{dL_t(\mathbf{r}_i; \mathbf{x}_r, \mathbf{r}_r; A_i)}{dE(\mathbf{r}_i)} = \int_{A_i} f_{ss}(\mathbf{x}_i, \mathbf{r}_i; \mathbf{x}_r, \mathbf{r}_r) dA_i \quad (14)$$

where the subscript “r” for “reflectance” in f_r^* and L_r was replaced with “t” for “transmittance”. As in the case of f_r^* , this quantity is invariant with respect to an area A_i beyond a certain limit area. In this limit area, there is a significant possibility that entering photons will exit through the collection area. Very favorable situations are present when the thickness of the object is much smaller than $\sqrt{A_i}$, or when it is almost transparent, provided that no significant light guiding takes place within the sample via successive total internal reflections. By analogy with the BRDF, this quantity should be called BTDF (f_t) only if this “large enough area” condition is fulfilled. When the radiance L_t must be characterized for any size of the irradiated area, f_{ss} (or f_{SSr} , which is equivalent in these conditions) is the appropriate quantity.

3. Fundamental scattering quantities for reflectance and transmittance

The analysis in the previous section demonstrates the convenience of defining new fundamental quantities to explicitly emphasize the spatial distribution dependence of the irradiation and to completely determine the reflectance and the transmittance of objects without the restriction imposed by such dependence.

The function $f_{is}(\mathbf{r}_i; \mathbf{r}_r)$, which is referred to here as the Bidirectional Insurface Scattering function (BIS), is defined, in analogy to f_r^* , by Eq. (11) and has units sr^{-1} . In this equation, as commented on above, L_{is} refers to the radiance produced exclusively by insurface reflection. In contrast with the function f_r^* defined in Eq. (2), this function is independent of the irradiated area A_i because the radiance involved excludes by definition any radiance produced by subsurface scattering, which is proportional to the total radiant flux on the surface ($E \times A_i$).

The function $f_{ss}(\mathbf{x}_i, \mathbf{r}_i; \mathbf{x}_r, \mathbf{r}_r)$, which is referred to here as the Bidirectional Bipositional Subsurface Scattering function (BBSS), is defined, in analogy with the BSSRDF, by Eq. (9) and has units $m^{-2} \cdot sr^{-1}$. In this equation, L_{ss} refers to the radiance produced exclusively by the subsurface scattering. Unlike the BSSRDF, it excludes by definition any radiance produced by the insurface reflection.

The explicit dependence of the scattering quantities f_r^* , f_{ssr} and f_t^* on the irradiation area and on these proposed fundamental scattering quantities is given in Eq. (13), Eq. (10) and Eq. (14), respectively, where the “large enough area” condition is not applied.

Note that, for objects for which the optical radiation transmitted at the air-matter interface is negligible with respect to the reflected optical radiation, the following is obtained:

$$f_r^*(\mathbf{r}_i; \mathbf{r}_r; A_i) = f_r(\mathbf{r}_i; \mathbf{r}_r) = f_{is}(\mathbf{r}_i; \mathbf{r}_r), \quad (15)$$

regardless of the size of the irradiation area, and that for any object, the following identity can be used except for $\mathbf{x}_r = \mathbf{x}_i$:

$$f_{ssr}(\mathbf{x}_i, \mathbf{r}_i; \mathbf{x}_r, \mathbf{r}_r) = f_{ss}(\mathbf{x}_i, \mathbf{r}_i; \mathbf{x}_r, \mathbf{r}_r). \quad (16)$$

4. Relations between the proposed fundamental scattering quantities and f_r^*

The radiance produced by insurface reflection (L_{is}) and the radiance produced by subsurface scattering (L_{ss}) are not directly evaluable by measurement when $\mathbf{x}_r \in A_i$; as a consequence, neither f_{is} nor f_{ss} is directly measurable except in the case described by Eq. (15). However, they can be related to f_r^* as explained in the following.

Over a range of $|\mathbf{x}_i - \mathbf{x}_r|$ length values much smaller than the absorption mean free path, the optical radiation flux crossing the air-matter interface at position \mathbf{x}_i and then undergoing multiple scattering within the bulk can go back across the air-matter interface at any position within this range with a very similar probability. Therefore, assuming $\sqrt{A_i}$ to be much smaller than the absorption mean free path, f_{ss} can be regarded as independent of the spatial dimensions \mathbf{x}_i and \mathbf{x}_r [$f_{ss}(\mathbf{r}_i; \mathbf{r}_r) = f_{ss}(\mathbf{x}_i, \mathbf{r}_i; \mathbf{x}_r, \mathbf{r}_r)$]; for $\mathbf{x}_r \in A_i$, Eq. (13) can be written as:

$$f_r^*(\mathbf{r}_i; \mathbf{r}_r; A_i) = f_{ss}(\mathbf{r}_i; \mathbf{r}_r)A_i + f_{is}(\mathbf{r}_i; \mathbf{r}_r). \quad (17)$$

By taking the derivative with respect to A_i , the following is obtained:

$$f_{ss}(\mathbf{r}_i; \mathbf{r}_r) = \frac{df_r^*(\mathbf{r}_i; \mathbf{r}_r; A_i)}{dA_i}. \quad (18)$$

The dependence of f_{is} on f_r^* can be expressed from Eqs. (17) and (18) by:

$$f_{is}(\mathbf{r}_i; \mathbf{r}_r) = f_r^*(\mathbf{r}_i; \mathbf{r}_r; A_i) - A_i \frac{df_r^*(\mathbf{r}_i; \mathbf{r}_r; A_i)}{dA_i} \quad (19)$$

for an A_i small enough to make f_{ss} independent of spatial variables.

Equations (18) and (19) relate f_{ss} and f_{is} , respectively, to f_r^* . Equation (18) expresses the fact that the function f_{ss} is the derivative of f_r^* under the specific condition $\mathbf{x}_r = \mathbf{x}_i$. Since, at this condition, the radiance L_{ss} is not always directly measurable because both the insurface reflection and the subsurface scattering contribute to the reflected radiant flux, it can be more practical to define f_{ss} as:

$$f_{ss}(\mathbf{x}_i, \mathbf{r}_i; \mathbf{x}_r, \mathbf{r}_r) = \begin{cases} \frac{df_r^*(\mathbf{r}_i; \mathbf{r}_r; A_i)}{dA_i}, & \text{for } \mathbf{x}_r = \mathbf{x}_i \\ \frac{dL_{ssr}(\mathbf{x}_i, \mathbf{r}_i; \mathbf{x}_r, \mathbf{r}_r)}{d\Phi_1(\mathbf{x}_i, \mathbf{r}_i)}, & \text{otherwise.} \end{cases} \quad (20)$$

For measurements, Eq. (18) and Eq. (19) can be approximated as:

$$\hat{f}_{ss} = \frac{f_r^*(A_{i,2}) - f_r^*(A_{i,1})}{A_{i,2} - A_{i,1}}, \quad (21)$$

and

$$\hat{f}_{is} = f_r^*(A_{i,1}) - A_{i,1}\hat{f}_{ss} = f_r^*(A_{i,2}) - A_{i,2}\hat{f}_{ss}, \quad (22)$$

where \hat{f}_{ss} and \hat{f}_{is} are the measurements of f_{ss} and f_{is} , respectively, and $A_{i,1}$ and $A_{i,2}$ are irradiated areas of different sizes, small enough to make Eq. (17) valid.

According to Eq. (10), f_{ssr} (usually obtained as L_r/Φ_i) equals f_{ss} when it is much larger than f_{is}/dA_i , that is, when the condition $dA_i \gg f_{is}/f_{ss}$ is fulfilled. This condition, which involves an infinitesimal element, may be confusing, but makes sense in a BSSRDF measurement, where ΔA_i instead of dA_i would denote the finite-size irradiated area. The condition implies that $L_{ss} \gg L_{is}$ and that most of the total radiance is produced by the subsurface scattering term. On the other hand, according to Eq. (17), f_r^* (usually obtained as L_r/E) equals f_{is} for small enough values of A_i , i.e., when $A_i \ll f_{is}/f_{ss}$ ($L_{is} \gg L_{ss}$).

From these considerations, a very simple formula can be given for the measurement of f_{is} and f_{ss} at the condition $\mathbf{x}_r \in A_i$:

$$\hat{f}_{is} = \frac{L_r}{E}, \text{ if } A_i \ll f_{is}/f_{ss} \quad (23)$$

$$\hat{f}_{ss} = \frac{L_r}{\Phi_i}, \text{ if } \Delta A_i \gg f_{is}/f_{ss}. \quad (24)$$

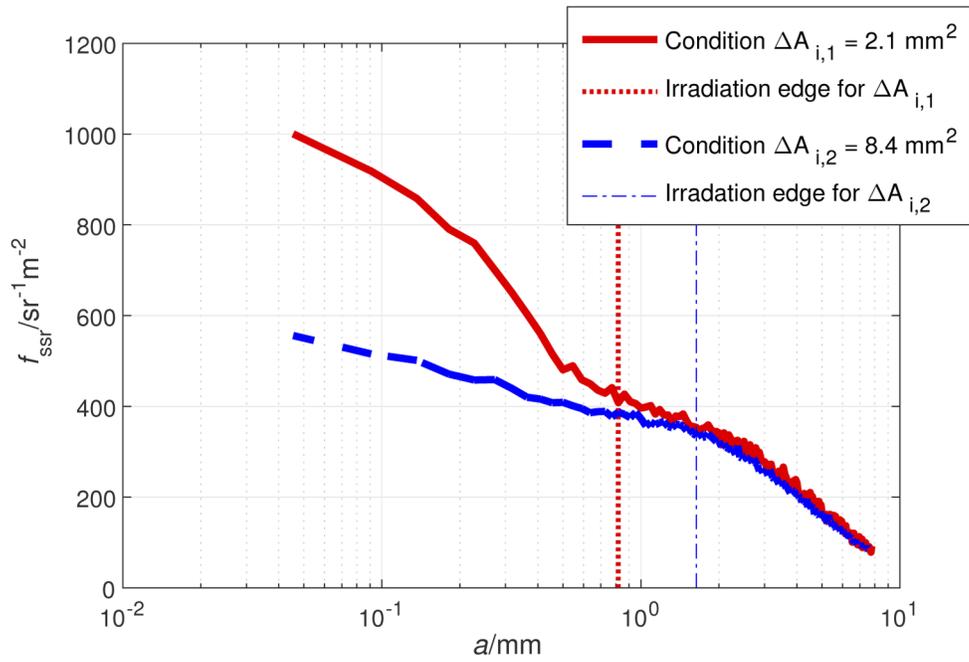
Here, the increment ΔA_i is used instead of the infinitesimal dA_i because finite intervals are involved in the measurements.

Notice that Eqs. (23) and (24) require at least rough previous knowledge of the order of magnitude of the values of f_{is} and f_{ss} to select the adequate irradiated area. Otherwise, an iterative procedure can be used.

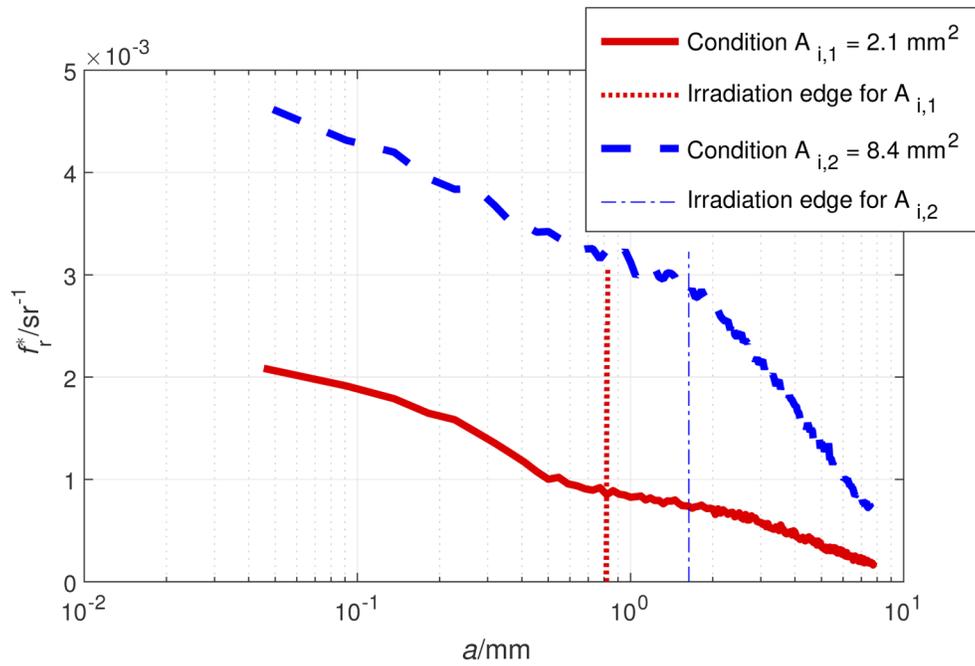
One may question why a value of ΔA_i that is larger than A_i needs to be used for measuring f_{ss} , since dA_i is included in A_i (Fig. 1, Table 1). However, it should be noted that f_{is} is not related to f_{ss} , and that both measurements described in Eqs. (23) and (24) are independent. The difference in the nomenclature of both irradiated areas simply allows these equations to be related to Eqs. (10) and (17).

5. Illustration by means of a measurement

Experimental results are shown in Fig. 2 to illustrate the conclusions from this article. The measurements were carried out by means of the goniospectrophotometer described in [12,13], with a camera for spatial resolution in the detection path. The radiance of the reflected optical radiation, the size of the irradiated area and the irradiance on the sample were measured, allowing both f_r^* and f_{ssr} to be obtained from the same measuring procedure. The irradiated area was uniform and always much larger than the collection area, which is given by the field-of-view area of the pixels of the camera ($\Delta A_r = 2 \times 10^{-3} \text{ mm}^2$). The data presented in Fig. 2(a) consist of BSSRDF measurements of a translucent sample made of a polycarbonate with organic scattering particles. The data was obtained at a fixed polar angle of incidence of 15° , a fixed polar angle of collection of 10° , fixed azimuth angles of incidence and collection of 0° with respect to the incidence plane, and a fixed wavelength of 600 nm. The figure represents the BSSRDF as a function of a , which is the distance from the center of the irradiation spot (\mathbf{x}_i) to the collection position (\mathbf{x}_r) (Fig. 1, Table 1). Two sets of measurements are shown for two different irradiated areas ($\Delta A_{i,1}$ and $\Delta A_{i,2}$), which are circular areas of 2.1 mm^2 and 8.4 mm^2 , respectively. Note that the a -axis is on a logarithmic scale to better show the results at the irradiated area. The vertical lines in the plot show the positions of the edges of the irradiation spots. It must be noted that, although the very small collection area allows better insight into the spatial distribution, only one integrated f_{ssr} value should be reported within the irradiation spot, which corresponds to the elementary irradiated area ΔA_i for assessing the BSSRDF. This f_{ssr} value at $\mathbf{x}_r = \mathbf{x}_i$ can be given as the ratio between the average radiance within the irradiated area and the total radiant flux.



(a) Results of f_{ssr} measurements.



(b) Results of f_r^* measurements.

Fig. 2. f_{ssr} (BSSRDF) and f_r^* measurements supporting the conclusions of this article.

The data presented in Fig. 2(b) consists of the BSSDRF data in Fig. 2(a) multiplied by the size of the irradiated area. Therefore, the importance of this additional figure lies in its absolute value. The data represents measurements of f_r^* and can only be interpreted as BRDF (f_r) measurements for “large enough” irradiated areas, and at irradiated positions ($\mathbf{x}_r \in A_i$), on the left of the vertical lines, but not when $\mathbf{x}_r \notin A_i$. Note that, to better relate the experimental results to the previous theoretical analysis, the irradiated areas are denoted as ΔA_i for BSSDRF measurements and as A_i for f_r^* measurements, although they represent the same area.

Several observations support the proposed formalism:

1. In Fig. 2(a), f_{ssr} is independent of ΔA_i outside the irradiation spot (right side of the vertical line) but not within the irradiation spot (left side), where the smaller spot produces a much higher result.
2. In Fig. 2(b), the f_r^* measurements show a clear dependence on A_i . The larger the irradiated area is, the larger the f_r^* value is, which corresponds to Eq. (13).
3. The two figures show that neither f_r^* nor f_{ssr} are completely independent of the irradiated areas.
4. The variation of f_{ssr} from condition $\mathbf{x}_r = \mathbf{x}_i$ to condition $\mathbf{x}_r \neq \mathbf{x}_i$ (within and outside the irradiated area) is larger for the smaller irradiated area [Fig. 2(a)]. That would mean that this abrupt variation tends to disappear for larger areas, as predicted by Eq. (10) ($f_{is}/\Delta A_i \rightarrow 0$ for very large values of ΔA_i).
5. By using the data shown in Fig. 2(b), an estimated value at $\mathbf{x}_r = \mathbf{x}_i$ of $f_{ss} \approx 413 \text{ m}^{-2} \cdot \text{sr}^{-1}$ was calculated in accordance with the approximation in Eq. (21). This value is coherent with the f_{ssr} value at the edge of the larger irradiation spot ($336 \text{ m}^{-2} \cdot \text{sr}^{-1}$), as shown in Fig. 2(a). This is expected because, as predicted by Eq. (10), f_{ss} approaches f_{ssr} for large irradiated areas.
6. By again using the data shown in Fig. 2(b), two estimated values of f_{is} were calculated from Eq. (22), one for each irradiated area. For this calculation, the f_r^* values at the center of the spot were used. Very similar values were obtained regardless of whether the large or the small irradiated area (0.0011 sr^{-1} and 0.0012 sr^{-1} , respectively) was used for the calculation. These values are in agreement with the difference between the f_r^* values for small irradiated areas at the center of the spot (0.0021 sr^{-1}) and outside and close to the edge (0.0009 sr^{-1}) as shown in Fig. 2(b). Equation (13) predicts that f_{is} approaches f_r^* for small irradiated areas; therefore, it is expected that, for smaller irradiated areas, the value of f_r^* at the center of the spots would approach f_{is} (around 0.001 sr^{-1} , as calculated above) and that its value outside the spots would approach zero. These trends can even be observed in the two curves for different areas shown in Fig. 2(b).
7. The previous estimations provide that $f_{is}/f_{ss} \approx 2.5 \text{ mm}^2$; according to Eqs. (23) and (24), this means that the larger area $\Delta A_{i,2}$ is more convenient for measuring f_{ss} (8.4 mm^2 is more than three times larger than 2.5 mm^2) than the smaller area $A_{i,1}$ is for measuring f_{is} (2.1 mm^2 , which is similar to 2.5 mm^2).

6. Discussion

The BRDF is well accepted as a fundamental scattering quantity for the determination of reflectance. However, for objects allowing the optical radiation to be transmitted under the surface, the BRDF needs to be defined with “large enough” irradiated areas because it is invariant for larger irradiated areas only at this condition. Consequently, the BRDF does not completely describe the reflectance of these objects for any irradiation condition. For the “large enough”

condition, the radiance caused by the subsurface scattering prevails over the radiance from insurface reflection. For the “small enough” condition, the insurface reflection would prevail over the subsurface scattering contribution. For arbitrary intermediate sizes, both contributions are important.

The BSSRDF can be regarded exclusively as a fundamental scattering quantity with the exception of the condition $\mathbf{x}_r = \mathbf{x}_i$. For this condition, the radiance is produced both by the insurface reflection (proportional to the irradiance) and by the subsurface scattering (proportional to the radiant flux).

Given these considerations, insurface and subsurface scattering quantities need to be given separately, as they are quantities that are more fundamental for reflectance than the BRDF and the BSSRDF.

For transmittance, the condition $\mathbf{x}_r = \mathbf{x}_i$ is always excluded. However, the BTDF, as well as the BRDF, needs to be defined with a large enough irradiation area to be invariant for larger irradiation areas and makes the Bidirectional Bipositional Subsurface Scattering function [BBSS, Eq. (9)] a more fundamental quantity for transmittance.

The proposed fundamental scattering quantities can be related to the radiance at the surface of an object in a more practical way as:

$$\frac{dL_{\text{ssr}}(\mathbf{x}_i, \mathbf{r}_i; \mathbf{x}_r, \mathbf{r}_r)}{d\Phi_i(\mathbf{x}_i, \mathbf{r}_i)} = f_{\text{ss}}(\mathbf{x}_i, \mathbf{r}_i; \mathbf{x}_r, \mathbf{r}_r) + f_{\text{is}}(\mathbf{r}_i, \mathbf{r}_r)\delta(\mathbf{x}_i - \mathbf{x}_r) \quad (25)$$

where the Dirac delta function $\delta(\mathbf{x}_i - \mathbf{x}_r)$ replaces the discontinuous function in Eq. (10). The Dirac delta function is adequate for integration over A_i , resulting in the value of the integrated function $f_{\text{is}}(\mathbf{r}_i; \mathbf{r}_r)$ as long as \mathbf{x}_r belongs to A_i , and zero otherwise. Note that $\delta(\mathbf{x}_i - \mathbf{x}_r)$ has the inverse dimension of its argument by definition. In this case, it has the dimension of inverse squared length because f_{is} is relative to a surface and \mathbf{x}_i and \mathbf{x}_r are regarded as two-dimensional variables. Equation (25) describes the variation of the radiance of an object with the incident radiant flux using the two fundamental scattering quantities defined in this article.

In this work, we conclude that the BRDF and the BSSRDF, as defined in Eqs. (1) and (3), can be regarded as fundamental scattering quantities under most of the conditions, but the following points must be considered:

1. The “large enough area” condition in the definition of the BRDF favors the inclusion of the subsurface scattering contribution, whereas a “small enough area” condition would make the insurface reflection contribution prevail. Under this latter condition, f_r^* coincides with the Bidirectional Insurface Scattering function, BIS.
2. The BSSRDF coincides with the Bidirectional Bipositional Subsurface Scattering function, BBSS, under the condition $\mathbf{x}_r \neq \mathbf{x}_i$, which includes only the subsurface scattering; however, when $\mathbf{x}_r = \mathbf{x}_i$, the insurface reflection is also included, meaning that the BSSRDF is no longer a fundamental scattering quantity. In such a case, Eqs. (9) and (18) are more adequate definitions for the subsurface scattering quantity.

To obtain representative measurements of samples, the size of the evaluated area must be selected such that the result is spatially invariant. In this article, it is assumed that the diameters of both irradiated and collection areas contain many times the wavelength of the optical radiation and the correlation length of any internal structure. This is fulfilled for the shown experimental results, where the material does not have a surface structure and the surface roughness is very small (the amplitude parameter R_a is around 5 nm).

In addition, uniform irradiation has been assumed, meaning that radiance and irradiance are well specified. In some practical cases, uniform irradiation on the scale required for irradiance and radiance measurements is not possible, and these quantities must be determined as the

average values of radiant fluxes. In such cases, the measurement equation for the BRDF and the BTDF is obtained from Eq. (2) as:

$$f_s = \frac{dL_s}{dE} = \frac{\Phi_s}{\omega_s A_s \cos \theta_s} / \frac{\Phi_i}{A_i} = \frac{1}{\omega_s \cos \theta_s} \left(\frac{\Phi_s}{\Phi_i} \right) \left(\frac{A_i}{A_s} \right), \quad (26)$$

where the subscript “r” (or “t”) is replaced by the more general “s” for “scattering” and θ_s is the scattering angle with respect to the surface normal. A_i is the irradiated area and A_s is the area including the full scattered radiant flux. When the object is non-translucent and there is no lateral scattering, both areas are coincident and Eq. (26) is usually simplified as [9–11]:

$$f_s = \frac{1}{\omega_s \cos \theta_s} \left(\frac{\Phi_s}{\Phi_i} \right). \quad (27)$$

However, when the object is translucent, this simplification could be wrong, under conditions in which a significant part of the scattered radiant flux emerges from an area larger than A_i . In these cases, such a measurement approach is not recommended; here the measurement of the BBSS is a better option, since it does not require a uniform irradiation but a known value of the radiant flux.

7. Conclusions

In this work, relating radiance to irradiation conditions at the surface of objects has been studied theoretically. The Bidirectional Reflectance Distribution Function (BRDF) is only invariant for “large enough” irradiated areas, and cannot be used to predict radiance for smaller irradiated areas. The Bidirectional Scattering - Surface Reflectance Distribution Function (BSSRDF) can include both subsurface and insurface scattering contributions when the radiance is evaluated at an irradiated position, which makes it dependent on the irradiation area. Therefore, the BRDF and the BSSRDF cannot be regarded as fundamental scattering quantities for determining reflectance and transmittance under every irradiation condition. We have proposed the definition of two different scattering quantities, one which exclusively describes insurface reflection and the other which exclusively describes subsurface scattering; these quantities are completely independent of the irradiation conditions and can therefore be regarded as more fundamental than the BRDF and the BSSRDF. Therefore, the proposed quantities should allow the reflectance and transmittance of an object to be calculated under any given irradiation condition. The conclusions drawn from the formalism presented have been supported by measurements. In addition, relating these quantities to the Bidirectional Transmittance Distribution Function (BTDF) has been examined and some practical measurement issues discussed. New procedures to provide traceability to measurements of BIS and BBSS need to be developed, and that will be part of future work.

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Disclosures

The authors declare no conflicts of interest.

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