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Finite-Time Estimation Algorithms for LPV Discrete-Time Systems with Application to Output Feedback Stabilization

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Abstract

This paper deals with new finite-time estimation algorithms for Linear Parameter Varying (LPV) discrete-time systems and their application to output feedback stabilization. Two exact finite-time estimation schemes are proposed. The first scheme provides a direct and explicit estimation algorithm based on the use of delayed outputs, while the second scheme uses two combined asymptotic observers, connected by a condition of invertibility of a certain time-varying matrix, to recover solution of the LPV system in a finite-time. Furthermore, two stabilization strategies are proposed. The first strategy, called Delayed Inputs/Outputs Feedback (DIOF) stabilization method, is based on the use of the explicit estimation algorithm. The second technique, called Two Connected Observers Feedback (2-COF) stabilization method, is based on the use of two combined observers providing exact finite-time estimation. A numerical example is given to show the validity and effectiveness of the proposed algorithms by simulation.

Key words: Estimation; observer design; LMI approach; LPV systems; output feedback stabilization.

1 Introduction

State estimation has many applications in control systems design. It is needed for the implementation of control laws and also for fault diagnosis (Gao and Ho, 2006), (Alcorta-Garcia and Frank, 1997), (Marino and Tomei, 1995), (Arcak and Kokotovic, 2001). Nonlinear estimation is more complex than the linear one; it lacks a general and systematic methodology. Although several methods have been proposed, recently, in the literature to improve the observer design and to cover a wide class of nonlinear systems, it still remains a challenge to design a nonlinear observer (Zemouche *et al.*, 2017), (Alessandri and Rossi, 2015), (Açikmese and Corless, 2011), (Wang *et al.*, 2017), (Kao *et al.*, 2015).

Among the widely used and investigated class of nonlinear systems is the Linear Parameters Varying (LPV) class (Heemels *et al.*, 2010), (Wu *et al.*, n.d.), (Wu, 2001), (Song and Yang, 2011). This particular structure of nonlinearity attracts the automatic control community for two reasons:

- Some real-world models in the literature can be represented under the form of LPV systems by tacking into ac-

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count the measurement of some state variables (Sename *et al.*, 2013). For instance, a vehicle lateral model can be written under an LPV form due to the availability for measurement of the longitudinal velocity (Wang *et al.*, 2019).
- Stabilizing LPV systems from an observer-based feedback control point of view is not an easy task due to the difficulty in obtaining non conservative sufficient conditions ensuring closed-loop exponential convergence (Jetto and Orsini, 2010), (Heemels *et al.*, 2010). For instance, from a Linear Matrix Inequality (LMI) point of view, all the available design methods for this class of systems provide conservative LMI design conditions. Although a recent and new technique has been proposed in (Bibi *et al.*, 2017) to reduce the conservatism of these LMIs, the issue is far from being well solved.

In this paper, we focus first on the exact and finite-time state estimation for a class of LPV systems in discrete-time. Then, we provide an extension to output feedback stabilization. Based on the ideas given in (Engel and Kreisselmeier, 2002; Sauvage *et al.*, 2007; Mazenc *et al.*, 2015) for linear continuous-time systems, we propose a generalization to LPV systems, which is not obvious in the continuous-time case. Indeed, in continuous-time case, it is difficult to integrate differential equations and get explicit solutions of the system where the complexity of the computation will lead to differentiating the time-varying parameters of the LPV system. The parameters' derivatives require additional and conservative assumptions and constraints on the system (such as boundedness of derivatives of the parameters). However, in discrete-time systems, only delayed values of the LPV parameters appear in the derivation of the explicit solution. This paper proposes two finite-time estimation algorithms:

- *A direct and explicit estimation:* The proposed algorithm for the three above mentioned classes of systems provides explicit state estimation in finite-time and is based on the use of delayed outputs to recover the solution of the system.
- *Two observers-based estimation:* This technique consists in combining two asymptotic state observers to reconstruct the solution of the system in finite-time.

Moreover, two numerical algorithms are proposed to design parameters of the above two estimation methods. The first algorithm is based on poly-quadratic stability, which consists in determining the parameters by solving a set of LMIs, while the second method is based on the use of pole assignment.
To demonstrate the value of the proposed exact finite-time estimation algorithms, an extension to output feedback stabilization is provided where two direct stabilization methods are proposed. The first method is based on the use of delayed inputs/outputs of the system (DIOF), while the second approach uses two connected observers feedback (2-COF). It is shown that both stabilization methods avoid solving Bilinear Matrix Inequalities (BMIs), which are computationally unsuitable for numerical solvers.
Compared to the short conference version in (Chaib-Draa *et al.*, 2019), this extended journal version contains several further contributions, namely detailed proofs; additional remarks and comments on the estimation algorithms; a detailed section on design of the parameters where two design algorithms are proposed; an additional output feedback stabilization technique; and extended numerical illustrations of both estimation and stabilization algorithms.
The advantages of the proposed approach, compared to other observer design methods existing in the literature for LPV systems, is the exact estimation in finite-time. Such class of systems is widely investigated in the literature, where numerous methods have been proposed (Wang *et al.*, 2019), (Pandey and de Oliveira, 2018), (Efimov *et al.*, 2013), (Kose and Jabbari, 1999). These methods provide asymptotic or exponential convergence of the observer. In case of the use of such observers in close-loop control systems, the results may be conservative because BMI conditions are encountered. As shown in Section 4, the exact estimation in finite-time allows getting a kind of separation principle when the estimation is combined with output feedback. This avoids solving BMI constraints often encountered in the output stabilization problem for LPV systems. On other hand, one of the estimation algorithm proposed in this paper (Algorithm 1) combines two standard asymptotic observers to get exact and finite-time estimation. Then, the proposed algorithm can exploit all the advantages of standard asymptotic observers established in the literature for this class of LPV systems (Sename *et al.*, 2013), (Pandey and de Oliveira, 2018), (Chadli *et al.*, 2008).

The rest of this paper is organized as follows: Section 2 is devoted to the development of two exact finite-time estimation algorithms. Section 3 provides two numerical procedures to show how to design the parameters of the two estimation methods presented in Section 2. Extension and application of the proposed estimation algorithms are presented in Section 4. To show validity and effectiveness of the proposed design algorithms, an illustrative example is given in Section 5. Finally, Section 6 concludes this work.

2 Exact Finite-Time Estimation of LPV systems

This section is dedicated to development of two exact finite-time estimation algorithms for a class of LPV systems. We consider the class of LPV systems defined by the following set of equations:

$$x_{k+1} = \mathbb{A}(\rho_k)x_k + Bu_k \tag{1a}$$
$$y_k = Cx_k, \tag{1b}$$

where $x_k \in \mathbb{R}^n$ is the state vector, $y_k \in \mathbb{R}^p$ is the output measurement and $u_k \in \mathbb{R}^m$ is the control input vector, $\rho_k \in \Theta \subset \mathbb{R}^r$ is a bounded time-varying parameter. B and C are constant matrices of appropriate dimensions. The matrix $\mathbb{A}(\rho_k)$ is affine in ρ_k.

We introduce the following assumptions:

- for any $k \in \mathbb{N}$, ρ_k is known and bounded. Furthermore, the set Θ is independent from k;
- the matrix $\mathbb{A}(\rho_k)$ can be written in the form $\mathbb{A}(\rho_k) = A_0 + \Delta A(\rho_k)$, where $\Delta A(\rho_k) \in \text{Co}(A_1, \cdots, A_{n_\rho})$, namely

$$\mathbb{A}(\rho_k) = A_0 + \sum_{i=1}^{n_\rho} \xi^i(\rho_k) A_i \tag{2}$$

  where for any $k \geq 1$, $\xi^i(\rho_k) \geq 0$ and $\sum_{i=1}^{n_\rho} \xi^i(\rho_k) = 1$;
- the pairs (A_i, C) are observable for all $i = 0, \ldots, n_\rho$.

The parameters $\xi^i(\rho_k)$ (functions of ρ_k) come from rewriting the bounded parameters ρ_k^i in a convex combination of their constant bounds by using some mathematical developments issued from convexity principle and convex sets. An example to show how we compute $\xi^i(\rho_k)$ is given in the illustrative example provided in Section 5.

Remark 2.1 *Even if* (2) *is needed to perform proofs and design the estimation schemes we propose in this paper, it is not restrictive because for bounded parameters ρ_k and since $\mathbb{A}(\rho_k)$ is affine (or piecewise affine) in ρ_k, we can rewrite* (1) *under the form* (2) *by using some mathematical transformations issued from convexity principle and convex sets.*

2.1 Explicit Solutions Using Delayed Outputs

In order to simplify the presentation, we introduce the following notations:

- $\xi^i(\rho_k) \triangleq \xi_k^i$ for $i \geq 0$ with $\xi^0(\rho_k) \triangleq 1$.
- For any set $\Sigma_M = \{M_0, M_1, \cdots, M_{n_\rho}\}$ of $n_\rho + 1$ matrices, set

$$\mathbb{M}(\rho_k) = M_0 + \sum_{i=1}^{n_\rho} \xi_k^i M_i = \sum_{i=0}^{n_\rho} \xi_k^i M_i.$$

Before proposing the first algorithm that provides an exact finite-time estimation of the state x_k, the following useful lemma is stated.

Lemma 1 *Assume that the pairs (A_j, C) are observable for all $j = 0, \ldots, n_\rho$. Then there exist L_j, K_j, $j = 0, \ldots, n_\rho$, such that the matrix*

$$\mathbb{E}_m(k) \triangleq \left[\prod_{i=1}^{m} \Big(\mathbb{A}(\rho_{k-i}) - \mathbb{L}(\rho_{k-i})C\Big)\right]^{-1} - \left[\prod_{i=1}^{m} \Big(\mathbb{A}(\rho_{k-i}) - \mathbb{K}(\rho_{k-i})C\Big)\right]^{-1}. \tag{3}$$

exists and is invertible for all $k \geq m$.

PROOF. The proof is given in Appendix A.

Consequently, we can provide a direct and exact estimation of the state x_k, presented in the following theorem.

Theorem 2 *Assume that there exist*

$$\mathbb{L}(\rho_k) = L_0 + \sum_{i=1}^{n_\rho} \xi_k^i L_i, \ \mathbb{K}(\rho_k) = K_0 + \sum_{i=1}^{n_\rho} \xi_k^i K_i$$

and $m \geq 1$ so that matrix $\mathbb{E}_m(k)$, defined in (3)*, exists and is invertible for all $k \geq m$. Then a direct and exact estimation of the state x_k can be computed as in* (4)*.*

PROOF. It is easy to show that iteratively x_k can be written under the forms (5) and (6). Then, by subtracting (6), after multiplication by $\left(\prod_{i=1}^{m} \Big(\mathbb{A}(\rho_{k-i}) - \mathbb{K}(\rho_{k-i})C\Big)\right)^{-1}$, from (5) multiplied by $\left(\prod_{i=1}^{m} \Big(\mathbb{A}(\rho_{k-i}) - \mathbb{L}(\rho_{k-i})C\Big)\right)^{-1}$, we get easily (4) by using the inverse of $\mathbb{E}_m(k)$.

Remark 2.2 *Since we are in discrete-time, continuity of the parameter ρ_k is not necessary. Indeed, in continuous-time, in certain cases, continuity may play an important role because the derivative of the parameter are used in the design. However, in discrete-time, such a property is not necessary and it does not make sense. For instance, the class of LPV systems studied in this paper contains the class of linear switched systems with known switching mode. It is clear that in switched systems, the switching mode parameter is not continuous and belongs to a finite set of integers. As for the boundedness of ρ_k, it is not a necessary condition. The necessary condition is the existence and invertibility of the matrix $\mathbb{E}_m(k)$ for $k \geq m$. If we can find gains L_i, K_i so that the matrix $\mathbb{E}_m(k)$ exists and is invertible for a non-bounded ρ_k, then the estimation algorithm remains valid. However, to guarantee existence and invertibility of $\mathbb{E}_m(k)$, we need boundedness of ρ_k, which justifies the introduction of such an assumption. The boundedness of ρ_k, or equivalently of ξ_k^i is the main key of the proof of Lemma 1 in Appendix A. As for the link between ρ_k and ξ_k^i, it can be deduced from the convexity principle; we refer the reader to (Sename* et al.*, 2013) for a deepen investigation of this issue. In addition to boundedness of ρ_k, the particular structure* (2) *plays an important role in the proposed estimation schemes because by combining it with the boundedness of ξ_k^i, we can guarantee existence and invertibility of the matrix $\mathbb{E}_m(k)$ for $k \geq m$. It is not so easy to consider implicit form* (1) *in LMI context or eigenvalues assignment.*

$$
\begin{aligned}
x_k = \mathbb{E}_m^{-1}(k) \sum_{j=1}^{m} \Bigg[& \left(\prod_{l=1}^{m} \left(\mathbb{A}(\rho_{k-l}) - \mathbb{L}(\rho_{k-l}) C \right) \right)^{-1} \left(\prod_{i=1}^{j-1} \left(\mathbb{A}(\rho_{k-i}) - \mathbb{L}(\rho_{k-i}) C \right) \right) \mathbb{L}(\rho_{k-j}) \\
& - \left(\prod_{l=1}^{m} \left(\mathbb{A}(\rho_{k-l}) - \mathbb{K}(\rho_{k-l}) C \right) \right)^{-1} \left(\prod_{i=1}^{j-1} \left(\mathbb{A}(\rho_{k-i}) - \mathbb{K}(\rho_{k-i}) C \right) \right) \mathbb{K}(\rho_{k-j}) \Bigg] y_{k-j} \\
+ \mathbb{E}_m^{-1}(k) \sum_{j=1}^{m} \Bigg[& \left(\prod_{l=1}^{m} \left(\mathbb{A}(\rho_{k-l}) - \mathbb{L}(\rho_{k-l}) C \right) \right)^{-1} \left(\prod_{i=1}^{j-1} \left(\mathbb{A}(\rho_{k-i}) - \mathbb{L}(\rho_{k-i}) C \right) \right) \\
& - \left(\prod_{l=1}^{m} \left(\mathbb{A}(\rho_{k-l}) - \mathbb{K}(\rho_{k-l}) C \right) \right)^{-1} \left(\prod_{i=1}^{j-1} \left(\mathbb{A}(\rho_{k-i}) - \mathbb{K}(\rho_{k-i}) C \right) \right) \Bigg] B u_{k-j}
\end{aligned} \tag{4}
$$

with the following convention for $j = 1$:

$$
\left(\prod_{i=1}^{j-1} \left(\mathbb{A}(\rho_{k-i}) - \mathbb{L}(\rho_{k-i}) C \right) \right)_{j=1} = \left(\prod_{i=1}^{j-1} \left(\mathbb{A}(\rho_{k-i}) - \mathbb{K}(\rho_{k-i}) C \right) \right)_{j=1} = \mathbb{I}_n .
$$

$$
\begin{aligned}
x_k = & \left(\prod_{j=1}^{m} \left(\mathbb{A}(\rho_{k-j}) - \mathbb{L}(\rho_{k-j}) C \right) \right) x_{k-m} \\
& + \sum_{j=1}^{m} \left(\prod_{i=1}^{j-1} \left(\mathbb{A}(\rho_{k-i}) - \mathbb{L}(\rho_{k-i}) C \right) \right) \left[\mathbb{L}(\rho_{k-j}) y_{k-j} + B u_{k-j} \right]
\end{aligned} \tag{5}
$$

and

$$
\begin{aligned}
x_k = & \left(\prod_{j=1}^{m} \left(\mathbb{A}(\rho_{k-j}) - \mathbb{K}(\rho_{k-j}) C \right) \right) x_{k-m} \\
& + \sum_{j=1}^{m} \left(\prod_{i=1}^{j-1} \left(\mathbb{A}(\rho_{k-i}) - \mathbb{K}(\rho_{k-i}) C \right) \right) \left[\mathbb{K}(\rho_{k-j}) y_{k-j} + B u_{k-j} \right]
\end{aligned} \tag{6}
$$

2.2 Estimation By Using Two Combined Observers

Unlike the previous section where a sum of delayed outputs weighted by powers of $A - LC$ and $A - KC$ have been used, this section is devoted to state estimation using two different asymptotic state observers. By using tools borrowed from the continuous-time results in (Engel and Kreisselmeier, 2002) and (Mazenc *et al.*, 2015), we get an exact estimation of the solution without using explicitly the delayed outputs. Indeed, the delayed output measurements are hidden and appear implicitly in states of the intermediate observers. This way to provide an exact estimation of the state x_k is more suitable from a practical point of view.

Considering the class of systems (1), then an exact estimation of x_k may be obtained by using two combined asymptotic observers, instead of using directly an explicit solution. The result is summarized in the following theorem.

Theorem 3 *Assume that the gain matrices L_i and K_i are selected such that:*

i) all the eigenvalues of $\left(A_i - L_i C\right)$ and $\left(A_i - K_i C\right)$ are non-zero and within the unit circle of the complex plane;
ii) there exists $m \geq 1$ so that the matrix $\mathbb{E}_m(k)$ exists and is invertible.

Then the extended state dynamic system

$$
\zeta_{k+1} = \mathbb{A}(\rho_k)\zeta_k + B u_k + \mathbb{L}(\rho_k)\left(y_k - C\zeta_k\right) \tag{7a}
$$

$$
\eta_{k+1} = \mathbb{A}(\rho_k)\eta_k + B u_k + \mathbb{K}(\rho_k)\left(y_k - C\eta_k\right) \tag{7b}
$$

$$\hat{x}_k = \mathbb{E}_m^{-1}(k)\left[\left(\prod_{j=1}^{m}\Big(\mathbb{A}(\rho_{k-j})-\mathbb{L}(\rho_{k-j})C\Big)\right)^{-1}\zeta_k - \left(\prod_{j=1}^{m}\Big(\mathbb{A}(\rho_{k-j})-\mathbb{K}(\rho_{k-j})C\Big)\right)^{-1}\eta_k + \eta_{k-m}-\zeta_{k-m}\right] \quad (7c)$$

is an observer for system (1)*, which converges in finite time* $m \geq 1$.

PROOF. The proof exploits the explicit solution technique. Indeed, by analogy to (5) and (6), we get (8) and (9). Hence, by substituting (8) and (9) in (4) and using the definition of $\mathbb{E}_m(k)$, we get from (7c) that $\hat{x}_k = x_k, \forall k \geq m$.

3 Design of the Estimation Parameters

This section is devoted to the numerical implementation of the proposed exact finite-time estimation methods. We provide a structured algorithm for computing the values of L_i and K_i for $i = 0, \ldots, n_\rho$.

Since the LPV parameter ρ_k (or equivalently ξ_k) is known, then designing a single observer under the form (7a) or (7b) is not a difficult task. We can use some observer design techniques for LPV systems available in the literature. In this section, we use the well known poly-quadratic stability (Pandey and de Oliveira, 2018), (Bara *et al.*, 2001), (Daafouz and Bernussou, 2001). However, we need to slightly modify the standard LMIs to get different gains L_i and K_i for each i. The goal is to add a scalar parameter to place the eigenvalues of $A_i - L_iC$ and $A_i - K_iC$, respectively, in different regions in order to get $m \geq 1$ satisfying the invertibility of the matrix $\mathbb{E}_m(k)$ defined in (3), for all $k \geq m$.

3.1 Poly-quadratic stability based algorithm

To have a consistent and well-structured algorithm, we propose to design the gains L_i (or K_i) stabilizing asymptotically the error $\zeta_k - x_k$ (or $\eta_k - x_k$). The overall LMI synthesis conditions providing the observer gains are given in the following proposition.

Proposition 1 *Assume that there exist symmetric positive definite matrices* $\mathbb{P}_i$*, square matrices* $\mathbb{X}_i$*, and matrices* $\mathbb{Z}_i$*,* $i = 0, \cdots, n_\rho$*, of appropriate dimensions such that the following LMI conditions hold:*

$$\mathbb{LMI}_L^{i,j}(\mathbb{P},\mathbb{X},\mathbb{Z}) < 0, \forall i, j = 0, \ldots, n_\rho. \quad (10)$$

where

$$\mathbb{LMI}_L^{i,j}(\mathbb{P},\mathbb{X},\mathbb{Z}) \triangleq \begin{bmatrix} \mathbb{P}_j - \mathbb{X}_i - \mathbb{X}_i^\top & 2\Big(\mathbb{X}_iA_i - \mathbb{Z}_iC\Big) \\ 2\Big(\mathbb{X}_iA_i - \mathbb{Z}_iC\Big)^\top & -\mathbb{P}_i \end{bmatrix}. \quad (11)$$

Then the estimation error $\zeta_k - x_k$ *converges asymptotically to zero for* $L_i = \mathbb{X}_i^{-1}\mathbb{Z}_i$.

PROOF. For $k \geq m$, the dynamics of the estimation error $\tilde{x}_k = \zeta_k - x_k$ is expressed as follows:

$$\tilde{x}_{k+1} = \Big(A(\rho_k) - \mathbb{L}(\rho_k)C\Big)\tilde{x}_k.$$

To get global asymptotic stability of the estimation error, we exploit (Pandey and de Oliveira, 2018) by using the poly-quadratic Lyapunov function

$$\vartheta_k = \tilde{x}_k^\top \mathbb{P}(\rho_k)\tilde{x}_k$$

with $\mathbb{P}(\rho_k) \triangleq \sum_{i=0}^{n_\rho} \xi_k^i \mathbb{P}_i$ and $\mathbb{P}_i = \mathbb{P}_i^\top > 0$. If such Lyapunov function exists and satisfies $\Delta\vartheta \triangleq \vartheta_{k+1} - \vartheta_k < 0, \forall \tilde{x}_k \neq 0$, then $\tilde{x}_k$ converges asymptotically to zero. By developing $\Delta\vartheta$, we get

$$\Delta\vartheta = \tilde{x}_k{}^\top \Big[\Big(\mathbb{A}(\rho_k) - \mathbb{L}(\rho_k)C\Big)^\top \mathbb{P}(\rho_{k+1}) \Big(\mathbb{A}(\rho_k) - \mathbb{L}(\rho_k)C\Big) - \mathbb{P}(\rho_k)\Big]\tilde{x}_k. \quad (12)$$

Using Schur Lemma, it follows that $\Delta\vartheta < 0, \forall \tilde{x}_k \neq 0$ if and only if the following inequality holds,

$$\begin{bmatrix} -\mathbb{P}(\rho_k) & \Big(\mathbb{A}(\rho_k) - \mathbb{L}(\rho_k)C\Big)^\top \mathbb{P}(\rho_{k+1}) \\ (\star) & -\mathbb{P}(\rho_{k+1}) \end{bmatrix} < 0, \quad (13)$$

for all $k \in \mathbb{N}$. Since $\sum_{i=0}^{n_\rho} \xi_k^i = \sum_{j=0}^{n_\rho} \xi_{k+1}^j = 2$, for all k, then (13) can be rewritten under the form:

$$\sum_{i=0}^{n_\rho}\xi_k^i \sum_{j=0}^{n_\rho}\xi_{k+1}^j \begin{bmatrix} -\frac{1}{2}P_i & \Big(A_i - L_iC\Big)^\top P_j \\ P_j\Big(A_i - L_iC\Big) & -\frac{1}{2}P_j \end{bmatrix} < 0, \quad (14)$$

which is satisfied if

$$\zeta_k = \left(\prod_{j=1}^{m}\left(\mathbb{A}(\rho_{k-j}) - \mathbb{L}(\rho_{k-j})C\right)\right)\zeta_{k-m} + \sum_{j=1}^{m}\left(\prod_{i=1}^{j-1}\left(\mathbb{A}(\rho_{k-i}) - \mathbb{L}(\rho_{k-i})C\right)\right)\left[\mathbb{L}(\rho_{k-j})y_{k-j} + Bu_{k-j}\right], \tag{8}$$

$$\eta_k = \left(\prod_{j=1}^{m}\left(\mathbb{A}(\rho_{k-j}) - \mathbb{K}(\rho_{k-j})C\right)\right)\eta_{k-m} + \sum_{j=1}^{m}\left(\prod_{i=1}^{j-1}\left(\mathbb{A}(\rho_{k-i}) - \mathbb{K}(\rho_{k-i})C\right)\right)\left[\mathbb{K}(\rho_{k-j})y_{k-j} + Bu_{k-j}\right]. \tag{9}$$

$$\begin{bmatrix} -\frac{1}{2}P_i & \left(A_i - L_iC\right)^\top P_j \\ P_j\left(A_i - L_iC\right) & -\frac{1}{2}P_j \end{bmatrix} < 0, \forall i, j = 0, \ldots, n_\rho \tag{15}$$

or equivalently

$$\begin{bmatrix} -P_i & 2\left(A_i - L_iC\right)^\top \\ 2\left(A_i - L_iC\right) & -P_j^{-1} \end{bmatrix} < 0, \forall i, j = 0, \ldots, n_\rho. \tag{16}$$

By multiplying (16) on the right by $\begin{bmatrix} 0 & I \\ \mathbb{X}_i & 0 \end{bmatrix}$ and its transpose on the left, and by using the inequality

$$-\mathbb{X}_i^\top P_j^{-1}\mathbb{X}_i \leq P_j - \mathbb{X}_i^\top - \mathbb{X}_i \tag{17}$$

one get inequality (11). Indeed, inequality (17) is obtained by applying the Young's relation

$$X^\top Y + Y^\top X \leq X^\top S^{-1}X + Y^\top SY$$

with $X = \mathbb{X}_i$, $Y = I$ and $S = P_j$. Hence, inequality (11) implies (14), which means that $\Delta\vartheta < 0, \forall \tilde{x}_k \neq 0$. This ends the proof of Proposition 1.

The same LMIs (11) ensure asymptotic stability of $\eta_k - x_k$ with $K_i = \mathbb{X}_i^{-1}\mathbb{Z}_i$, since ζ_k and η_k have the same dynamics. However, to achieve exact finite-time estimation, we need invertibility of the matrix $\mathbb{E}_m(k)$, which may not be satisfied if the eigenvalues of $A_i - L_iC$ and $A_i - K_iC$ are close to each other. On other hand, even if $\mathbb{E}_m(k)$ is invertible, the main issue is its condition number ($\kappa(\mathbb{E}_m(k)) = \|\mathbb{E}_m(k)\|.\|\mathbb{E}_m^{-1}(k)\|$). By construction of the matrix $\mathbb{E}_m(k)$, if the eigenvalues of $A_i - L_iC$ and $A_i - K_iC$ are close, then even if $\mathbb{E}_m(k)$ is invertible, its condition number will be high, which leads to numerical problems and bad estimation. Indeed, from numerical analysis viewpoint, if the condition number of a matrix is high, then there will be numerical issues in inverting the matrix and solving linear system of equations. To overcome this issue and augment the possibility to get invertibility of $\mathbb{E}_m(k)$ with a small condition number, we propose to slightly modify (11) by including a positive parameter $\alpha_i < 1, i = 0, \ldots, n_\rho$. That is, LMI (11) is replaced by the following one:

$$\begin{bmatrix} \mathbb{P}_j - \mathbb{X}_i - \mathbb{X}_i^\top & (\alpha_i)^{-1}\left(\mathbb{X}_iA_i - \mathbb{Z}_iC\right) \\ (\alpha_i)^{-1}\left(\mathbb{X}_iA_i - \mathbb{Z}_iC\right)^\top & -\mathbb{P}_i \end{bmatrix} < 0. \tag{18}$$

Hence, if we need the eigenvalues of $A_i - L_iC$ to be lower than α_i, we should solve LMIs (18). Indeed, if λ_i^α is an eigenvalue of $\frac{1}{\alpha_i}\left(A_i - L_iC\right) \triangleq A_i^\alpha$, then it is well known in discrete-time case that when LMIs (18) are feasible we have $|\lambda_i^\alpha| < 1$. On the other hand, we know that

$$\det\left(\lambda_i^\alpha\mathbb{I}_n - A_i^\alpha\right) = \frac{1}{\alpha_i^n}\det\left(\overbrace{\alpha_i\lambda_i^\alpha}^{\lambda_i}\mathbb{I}_n - \left(A_i - L_iC\right)\right) = 0,$$

which means that $\lambda_i \triangleq \alpha_i\lambda_i^\alpha$ is an eigenvalue of $\left(A_i - L_iC\right)$. It follows that since $|\lambda_i^\alpha| < 1$ from LMIs (18), we have $|\lambda_i| < \alpha_i$.

Moreover, if we need the eigenvalues of $A_i - K_iC$ to be greater than α_i while ensuring poly-quadratic convergence, we should solve

$$\mathbb{LMI}_K^{i,j}(\mathbb{P}, \mathbb{Y}, \mathbb{S}) < 0, \forall i, j = 0, \ldots, n_\rho \tag{19}$$

with respect to the new variables $\mathbb{Y}_i$ and $\mathbb{S}_i$ with $K_i = \mathbb{Y}_i^{-1}\mathbb{S}_i$, together with the quadratic matrix inequality (QMI) for $\beta_i > \alpha_i$:

$$\Big(A_i - K_iC\Big)^\top \mathbb{P}_i^{-1}\Big(A_i - K_iC\Big) - \beta_i^2\mathbb{P}_i^{-1} > 0, \qquad (20)$$

which characterizes the QMI-region, see e.g. (Ebihara *et al.*, 2015, Lemma 2.6):

$$\mathbb{QMI}_i \triangleq \{\lambda \in \mathbb{C};\ |\lambda| > \beta_i\}. \qquad (21)$$

Note that this quadratic constraint allows guaranteeing only that the eigenvalues of $(A_i - K_iC)$ belong to an LMI region; which can ensure invertibility of the matrix $\mathbb{E}_m(k)$. However, since (21) is a non-convex set then one can not convert (20) to an equivalent LMI. As a fallback solution, we propose to consider the two LMI sub-regions:

$$\mathscr{R}^1_{\beta_i} = \{\lambda \in \mathbb{C};\ |\lambda| < 1; \mathrm{Re}(\lambda) < -\beta_i\} \subset \mathbb{QMI}_i, \qquad (22)$$

$$\mathscr{R}^2_{\beta_i} = \{\lambda \in \mathbb{C};\ |\lambda| < 1; \mathrm{Re}(\lambda) > \beta_i\} \subset \mathbb{QMI}_i, \qquad (23)$$

which lead to the following two LMI constraints:

$$2\beta_i\mathbb{Y}_i - \Big(\mathbb{Y}_iA_i - \mathbb{S}_iC\Big) - \Big(\mathbb{Y}_iA_i - \mathbb{S}_iC\Big)^\top < 0 \qquad (24)$$

$$-2\beta_i\mathbb{Y}_i + \Big(\mathbb{Y}_iA_i + \mathbb{S}_iC\Big) + \Big(\mathbb{Y}_iA_i - \mathbb{S}_iC\Big)^\top < 0. \qquad (25)$$

Then, it is sufficient to solve (11) together with (24) or (25). Both sub-regions (22) and (23) guarantee that

$$\min_{j=0,\cdots,n_\rho} |\lambda_j(A_i - K_iC)| > \beta_i.$$

Once the gains L_i and K_i are computed, one checks existence and invertibility of the matrix $\mathbb{E}_m(k)$, for $1 \le m \le m^\star$, where $m^\star$ is a prescribed integer representing the maximum of the desired finite-time to achieve exact estimation. If $\mathbb{E}_m(k)$ is not invertible for all $m \le m^\star$, then the values of $\alpha_i, \beta_i, i = 0,\ldots,n_\rho$, must be changed. The numerical design procedure is summarized in the following Algorithm.

Remark 3.1 *Algorithm 1 uses sufficient LMI conditions related to poly-quadratic stability, which may lead to non-invertible matrix $\mathbb{E}_m(k)$. However, after including a second parameter β_i to conveniently separate the eigenvalues and by using QMI-regions, we reduced the conservatism of the algorithm and we augmented the possibility to get invertible matrix $\mathbb{E}_m(k)$. In addition, Algorithm 1 plays an important role because it guarantees asymptotic stability of the estimation errors $\hat{x}_t - \eta_t$ and $\hat{x}_t - \zeta_t$. This is one of the reason for which it is introduced. This property affects the estimation before finite-time convergence (for $k = 1,\ldots,m-1$). For instance, in case of systems with high dimension, the finite-time convergence $m \ge 1$ may be high. Hence if the asymptotic convergence of $\hat{x}_t - \eta_t$ and $\hat{x}_t - \zeta_t$ is not ensured, the estimation may be bad for $k = 1,\ldots,m-1$. On the other hand, the fact that Algorithm 1 cannot systematically ensure invertibility of $\mathbb{E}_m(k)$ has motivated us to introduce a second algorithm, namely Algorithm 2 to be stated in the next Subsection 3.2.*

Algorithm 1: Poly-quadratic stability based algorithm

Step 1. Choose $m^\star \ge 1$ and small values $\alpha_i, \beta_i, , i = 1,\ldots,n_\rho$, with $\alpha_i < \beta_i$.
Step 2. Solve LMIs (18) and compute the gains:
- $L_i = \mathbb{X}_i^{-1}\mathbb{Z}_i$.

Step 3. Solve jointly LMIs (19) and (24)
if *LMIs* (19) *and* (24) *are feasible* **then**
  Compute $K_i = \mathbb{Y}_i^{-1}\mathbb{S}_i$;
else
  Solve LMIs (19) and (25) and compute
  - $K_i = \mathbb{Y}_i^{-1}\mathbb{S}_i$.

Step 4. Check invertibility of the matrix $\mathbb{E}_\ell(k)$:
for $\ell \leftarrow 1$ *to* $m^\star$ *and* $k \ge 1$ **do**
if $\mathbb{E}_\ell(k)$ *is invertible* **then**
  return $m \leftarrow \ell$; **break**;
else
  Increase the values of α_i, β_i and go to **Step 2** to generate new observer gains L_i and K_i.

3.2 Pole placement based algorithm

The previous poly-quadratic stability based algorithm is based on feasibility of the sufficient LMIs (19) and (24) or LMIs (19) and (25), which are not always easy to tune more suitable eigenvalues of $A_i - L_iC$ and $A_i - K_iC$. To overcome this obstacle, we proposed a second algorithm based directly on fixing the eigenvalues of $A_i - L_iC$ and $A_i - K_iC$ in such a way that they will not be close to each other, according to the proof of Lemma 1. To this end, we propose to introduce scalar variables $\delta_L < 1$, $\delta_K < 1$. Then, we compute the eigenvalues of $A_i - L_iC$ and $A_i - K_iC$, respectively, so that:

$$\max_{1\le i\le n_\rho} \lambda_{\max}(A_i - L_iC) < \delta_L < \lambda_{\min}(A_0 - L_0C); \qquad (26)$$

$$\max_{1\le i\le n_\rho} \lambda_{\max}(A_i - K_iC) < \delta_K \le \lambda_{\min}(A_0 - K_0C); \qquad (27)$$

$$\max_{1\le i\le n_\rho} \lambda_{\max}(A_i - L_iC) + \lambda_{\max}(A_0 - L_0C) < \lambda_{\min}(A_0 - K_0C) - \max_{1\le i\le n_\rho} \lambda_{\max}(A_i - K_iC) \qquad (28)$$

with $\lambda_{\max}(.) = \max\{|\lambda|,\ \lambda \in Sp(.)\}$ and $\lambda_{\min}(.) = \min\{|\lambda|, \lambda \in Sp(.)\}$, guaranteeing the invertibility of $\mathbb{E}_m(k)$.

Conditions (26), (27), and (28) provide a separation between the different eigenvalues of the matrices $A_i - L_iC$ and $A_i - K_iC$. They show clearly that the eigenvalues of $A_0 - K_0C$ are larger in module than those of other matrices. In order to ensure (A.7), we can introduce another separation between

the eigenvalues of the matrix $A_0 - L_0C$ and those of $A_i - K_iC$, $i = 1, \cdots, n_\rho$, by introducing a positive parameter $\delta_{L_0K} < 1$ so that

$$\lambda_{\max}(A_0 - L_0C) < \delta_{L_0K} \leq \min_{1 \leq i \leq n_\rho} \lambda_{\min}(A_i - K_iC) \quad (29)$$

Then, for (28) to be verified, it is sufficient to separate the eigenvalues of $A_i - K_iC$, for $i = 1, \cdots, n_\rho$ from those of $A_0 - K_0C$ by a minimum distance equal to

$$\delta^* \triangleq \delta_K - 2\delta_{L_0K} > 0. \quad (30)$$

Hence, the following separation

$$\max_{1 \leq i \leq n_\rho} \lambda_{\max}(A_i - K_iC) < \delta^* < \delta_K \leq \lambda_{\min}(A_0 - K_0C) \quad (31)$$

guarantees the required condition (28).

Note that since we study discrete-time systems, we have also $\max_{1 \leq i \leq n_\rho} \mid \lambda_{\max}(A_i - K_iC) \mid < 1$. The idea consists in assigning eigenvalues satisfying (26), (27), and (28) and such that for a prescribed $m^\star \geq 1$, there exists $m \leq m^\star$ for which $\mathbb{E}_m(k)$ is invertible. Therefore, if such properties are not satisfied, we propose to decrease $\lambda_j(A_i - L_iC)$ and increase $\lambda_j(A_i - K_iC)$ until $\mathbb{E}_m(k)$ is invertible, where $\lambda_j(A)$ is the j^{th} eigenvalue of the matrix A.

Hence, we are ready to propose a second algorithm, which is more easier and simpler than Algorithm 1.

Algorithm 2: Eigenvalues assignment based algorithm

Step 1. Choose $m^\star \geq 1$, $\delta_L < \frac{1}{2}$, $\delta_{L_0K} > \delta_L$, $\delta_K > 2\delta_{L_0K}$. Compute δ^* according to (30);
Step 2. Assign eigenvalues for $A_i - L_iC$ and $A_i - K_iC$, $i = 0, \cdots, n_\rho$ according to (26), (29), and (31);
Step 3. Compute the corresponding gains L_i and K_i, respectively;
Step 4. Check invertibility of the matrix $\mathbb{E}_\ell(k)$:
for $\ell \leftarrow 1$ *to* $m^\star$ *and* $k \geq 1$ **do**
if $\mathbb{E}_\ell(k)$ *is invertible* **then**
  return $m \leftarrow \ell$; **break**;
else
  Decrease the eigenvalues of $A_i - L_iC$ and increase those of $A_i - K_iC$ and go to **Step 3** to generate new observer gains L_i and K_i.

Remark 3.2 *It is well-known that, in a general way, the stability of each $A_i - L_iC$ is not sufficient to guarantee stability of the estimation error in an LPV form. However, what Section 3.2 proposes is an efficient design algorithm based on pole assignment, namely the separation of the eigenvalues of $A_i - L_iC$ from those of $A_i - K_iC$ so that there exists $m \geq 1$ for which the matrix $\mathbb{E}_m(k)$ is invertible for any $k \geq m$. Theorem 3 provides all the finite-time convergence conditions. This means that the errors $\hat{x}_t - \eta_t$ and $\hat{x}_t - \zeta_t$ are not necessary stable. The only conditions are the existence and invertibility of $\mathbb{E}_m(k)$. As for the convergence of the estimation errors $\hat{x}_t - \eta_t$ and $\hat{x}_t - \zeta_t$, it is guaranteed by the results of Section 3.1, namely the poly-quadratic stability based analysis.*

Remark 3.3 *As in the previous subsection, Algorithm 2 gives a global view of the numerical procedure to design parameters of the proposed finite-time exact estimator by using eigenvalues assignment. For instance, to increase and decrease eigenvalues of $A_i - L_iC$ and $A_i - K_iC$, we can introduce a small scalar parameter ε that we will increase at each iteration, and put*

$$\lambda_j(A_i - L_iC) \leftarrow \lambda_j(A_i - L_iC) - \varepsilon,$$

$$\lambda_j(A_i - K_iC) \leftarrow \lambda_j(A_i - K_iC) + \varepsilon.$$

4 Output Feedback Stabilization of LPV Systems

In this section, we propose two different output feedback stabilization methods. Both methods are based on the exact finite-time estimation methodologies proposed in the previous sections.

4.1 2-COF stabilization method

In this paper, due to the exact finite-time estimation of the system state, we will propose necessary and sufficient LMI conditions ensuring poly-quadratic stabilization of the system state.

Theorem 4 *Assume that the gain matrices L_i, K_i and F_i are selected such that:*

i) L_i are solutions of LMIs (18) *and K_i are solutions of* (11) *and* (24) *or* (11) *and* (25)*, respectively, for prescribed $\alpha_i, i = 0, \ldots, n_\rho$;*
ii) there exists $m \geq 1$ so that the matrix $\mathbb{E}_m(k)$ exists and invertible;
iii) there exist matrices $\mathbb{P}_i = \mathbb{P}_i^\top > 0, i = 0, \ldots, n_\rho$ and matrices $\mathbb{X}_i, i = 0, \ldots, n_\rho$ of appropriate dimensions such that the following LMI conditions hold:

$$\begin{bmatrix} -\mathbb{P}_j & 2(A_i\mathbb{P}_i - B\mathbb{X}_i) \\ 2(\mathbb{P}_iA_i^\top - \mathbb{X}_i^\top B^\top) & -\mathbb{P}_i \end{bmatrix} < 0, \forall i, j = 0, \ldots, n_\rho. \quad (32)$$

Then the following observer-based controller

$$\zeta_{k+1} = A(\rho_k)\zeta_k + Bu_k + \mathbb{L}(\rho_k)\Big(y_k - C\zeta_k\Big) \quad (33a)$$

$$\eta_{k+1} = A(\rho_k)\eta_k + Bu_k + \mathbb{K}(\rho_k)\Big(y_k - C\eta_k\Big) \quad (33b)$$

$$\hat{x}_k = \mathbb{E}_m^{-1}(k)\left[\left(\prod_{j=1}^{m}\Big(\mathbb{A}(\rho_{k-j}) - \mathbb{L}(\rho_{k-j})C\Big)\right)^{-1}\zeta_k - \left(\prod_{j=1}^{m}\Big(\mathbb{A}(\rho_{k-j}) - \mathbb{K}(\rho_{k-j})C\Big)\right)^{-1}\eta_k + \eta_{k-m} - \zeta_{k-m}\right] \tag{33c}$$

$$u_k = -\mathbb{F}(\rho_k)\hat{x}_k \tag{33d}$$

with

$$\mathbb{F}(\rho_k) \triangleq F_0 + \sum_{i=1}^{n_\rho}\xi_k^i F_i,\ F_i = \mathbb{X}_i\mathbb{P}_i^{-1} \tag{34}$$

stabilizes globally asymptotically the system (1).

PROOF. From Theorem 3, we know that if *i*) and *ii*) of Theorem 4 are satisfied, then (33c) provides an exact and finite-time estimation of x_k. that is $\hat{x}_k = x_k, \forall k \geq m$. It follows that for $k \geq m$, equation (33d) becomes

$$u_k = -\mathbb{F}(\rho_k)x_k.$$

Consequently, for $k \geq m$, system (1) can be rewritten, after using feedback control, as:

$$\begin{aligned} x_{k+1} &= \Big(A(\rho_k) - B\mathbb{F}(\rho_k)\Big)x_k \\ &= \sum_{j=0}^{n_\rho}\xi_k^j\big(A_j - BF_j\big)x_k \end{aligned} \tag{35}$$

which is globally asymptotically stable if there exists a Lyapunov function

$$\vartheta_k = \sum_{i=0}^{n_\rho}\xi_k^i x_k^\top \mathbb{P}_i^{-1}x_k$$

such that $\Delta\vartheta \triangleq \vartheta_{k+1} - \vartheta_k < 0, \forall x_k \neq 0$.

By proceeding as in the proof of Proposition 1, it follows that $\Delta\vartheta < 0, \forall x_k \neq 0$ if the following inequalities hold, $\forall i, j = 0, \ldots, n_\rho$:

$$\begin{bmatrix} -P_i^{-1} & 2\Big(A_i - BF_i\Big)^\top \\ 2\Big(A_i - L_iC\Big) & -P_j \end{bmatrix} < 0 \tag{36}$$

or equivalently

$$\begin{bmatrix} -P_j & 2\Big(A_i - BF_i\Big) \\ 2\Big(A_i - L_iC\Big)^\top & -P_i^{-1} \end{bmatrix} < 0. \tag{37}$$

Pre- and post-multiplying (37) by $\begin{bmatrix} I & 0 \\ 0 & P_i \end{bmatrix}$ leads to (32) with $\mathbb{X}_i = F_i\mathbb{P}_i$. This ends the proof.

Remark 4.1 *The proof of Theorem 4 is more straightforward than that of Proposition 1 because it does not need the introduction of a new slack variable when applying the congruence principle to linearize the inequalities. In addition, in Theorem 4, we used a Lyapunov function with matrices P_i^{-1} instead of P_i, in order to avoid the introduction of a new variable $S_i \triangleq P_i^{-1}$.*

Remark 4.2 *It is worth to notice that LMIs* (32) *are necessary and sufficient conditions for the global poly-quadratic stabilization of system* (1)*, however, they are only sufficient for its global asymptotic stabilization. Indeed, according to (Daafouz and Bernussou, 2001, Definition 2), the notion of poly-quadratic stability is stronger than asymptotic stability. Poly-quadratic stability is basically, by definition, a sufficient criterion to ensure asymptotic stability.*

Remark 4.3 *In the presence of uncertainties, the proof of convergence is different and the LMIs* (32) *are not sufficient to ensure poly-quadratic stability of the system. This issue is one of the future work we aim to tackle. Especially, we aim to investigate the class of LPV systems with inexact parameters. Indeed, to cover nonlinear systems, we need to extend the results of this paper to systems with unknown parameters or to quasi-LPV systems. When the nonlinearity is globally Lipschitz, it has been demonstrated in Zemouche and Boutayeb (2013) that the reformulation of the Lipschitz condition allows rewriting the system as a quasi-LPV one with unknown parameter. This leads to the case where $\rho_k = \rho_k^0 + \Delta\rho_k$, with ρ_k^0 is known and $\Delta\rho_k$ is unknown but bounded. Solving this problem allows to generalize the methodology to a class of nonlinear systems.*

4.2 DIOF stabilization method

This section is dedicated to a new stabilization technique, called DIOF stabilization method, which allows overcoming the issue of static output feedback (SOF) stabilization problem. Although SOF controller is simple, from LMI point of view, it is not so obvious because of resulting Bilinear Matrix Inequalities (BMIs), which are not easy to solve (from complexity point of view) by using available convex optimization algorithms. SOF controller consists in stabilizing (1) by using $u_k = -\mathbb{F}(\rho_k)y_k$, which leads to

$$x_{k+1} = \sum_{j=0}^{n_\rho}\xi_k^j\big(A_j - BF_jC\big)x_k. \tag{38}$$

However, by following the steps in Section 4.1, equation (38) leads to the BMIs

$$\begin{bmatrix} -\mathbb{P}_j & 2\Big(A_i - BF_iC\Big)\mathbb{P}_i \\ 2\mathbb{P}_i\Big(A_i - BF_iC\Big)^\top & -\mathbb{P}_i \end{bmatrix} < 0, \quad \forall i,j = 0,\ldots,n_\rho, \quad (39)$$

which are not easy to linearize. Several techniques have been proposed in the literature, but the challenge of obtaining less conservative LMIs is still open. To have a precise idea on the difficulty of this problem in linear case, we refer to (Huynh *et al.*, 2019) and the references therein. On the other hand, by exploiting the first exact estimation methodology proposed in Section 2.1, we are able to stabilize (1) by using delayed inputs/outputs. Indeed, to overcome the BMIs (39), we propose the following output feedback controller, which uses only measured quantities:

$$u_k = -\mathbb{F}(\rho_k)x_k, \text{ with } x_k \text{ given by (4)}, \forall k \geq m; \quad (40)$$
$$u_k = u_k^m, \text{ for } k = 0,\cdots,m-1, \quad (41)$$

where u_k^m is a given input vector for the initial values of $k \leq m-1$, and to be chosen by the user. It is introduced because (40) gives only u_k for $k \geq m$.

It follows that with (40), instead of system (38), we get (35). Consequently, instead of facing the complicated BMIs (39), we only need to solve LMIs (32).

Remark 4.4 *The explicit estimation technique proposed in this paper is applied to output feedback stabilization issue, nevertheless, we need the value of u_k^m for $k = 0,\cdots,m-1$, which is necessary from* (40)-(41)*. We propose to fix these values by putting $u_k^m = -F\zeta_k$, where ζ_k is defined in* (7a)*. Since generally, we have small value of m (we can fix small values of m for appropriate L and K), hence we can also select $u_k^m = 0$ for $k = 0,\cdots,m-1$.*

Remark 4.5 *The aim of Subsection 4.2 is to consolidate Section 4. Although the synthesis of the gains $F_i, i = 0,\ldots,n_\rho$ is the same in both subsections and is based on solving LMIs* (32)*, Subsection 4.2 has two objectives:*

(1) to offer the possibility for users to stabilize systems under the use of only delayed outputs according to the first estimation algorithm (direct and explicit estimation);
(2) to show that the use of a sliding window of weighted outputs leads to avoid a complex BMI problem encountered in static output feedback stabilization problem.

Remark 4.6 *Notice that both stabilization methods DIOF and 2-COF use the same synthesis conditions to determine the parameters L_j, K_j, and F_j because both techniques give $u_k = -\mathbb{F}(\rho_k)x_k$ for $k \geq m$.*

5 Illustrative Example

This section is devoted to illustrate the theoretical contributions presented in the previous sections. Due to lack of space, only the methodology based on the use of two combined observers will be illustrated.

5.1 System description

As an example, consider the LPV system described by the following equations (Heemels *et al.*, 2010):

$$x_{k+1} = \begin{bmatrix} 0.25 & 1 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.6+\rho_k \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u_k \quad (42a)$$

$$y_k = \begin{bmatrix} 1 & 0 & 2 \end{bmatrix} x_k \quad (42b)$$

with $\rho_k \in \begin{bmatrix} 0 & 0.5 \end{bmatrix}, k \in \mathbb{N}$. This model can be rewritten under the form (2) with

$$A_0 = \begin{bmatrix} 0.05 & 0.5 & 0 \\ 0 & 0.05 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}, \; A_1 = \begin{bmatrix} 0.2 & 0.5 & 0 \\ 0 & 0.05 & 0 \\ 0 & 0 & 0.5 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.2 & 0.5 & 0 \\ 0 & 0.05 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; \xi_k^1 = \frac{(0.5-\rho_k)}{0.5}, \; \xi_k^2 = \frac{\rho_k}{0.5}.$$

5.2 Estimation without feedback stabilization

By using Algorithm 1, we obtain the following solutions:

$$L_0 = \begin{bmatrix} -0.0156 \\ 0.0002 \\ 0.2484 \end{bmatrix}, L_1 = \begin{bmatrix} 0.0474 \\ -0.0007 \\ -0.3637 \end{bmatrix}, L_2 = \begin{bmatrix} 0.0134 \\ 0.0009 \\ -0.6372 \end{bmatrix},$$

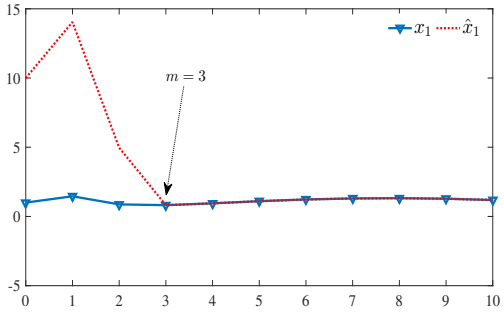
$$K_0 = \begin{bmatrix} -1.5234 \\ 0.8312 \\ -0.0383 \end{bmatrix}, K_1 = \begin{bmatrix} 0.0314 \\ 0.0010 \\ -0.3260 \end{bmatrix}, K_2 = \begin{bmatrix} 0.0638 \\ -0.0033 \\ -0.7513 \end{bmatrix}$$

with $m = 3$. The matrix $\mathbb{E}_m(k)$ in (3) exists and found invertible for any $k \geq 0$.

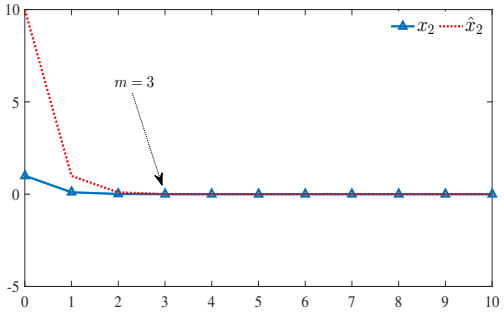
For simulations, we use

$$\rho_k = \frac{1}{2} \mid \sin(\frac{\pi}{10}k) \mid \text{ and } u_k = \sin(\frac{\pi}{15}k).$$

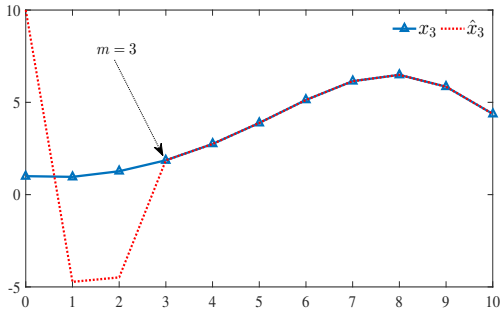
The initial state of the system is $x_0 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\top$. As for ζ_0 and η_0 in (7a)-(7b) are given by $\begin{bmatrix} 10 & 10 & 10 \end{bmatrix}^\top$ and $\begin{bmatrix} 5 & 5 & 5 \end{bmatrix}^\top$, respectively. We also use $\hat{x}_k = \zeta_k$ for $k = 0, \ldots, m-1$. It is quite clear from Figure 1 that the estimation $\hat{x}_k$ given by (7c) reaches exactly the solution x_k of (42a) in finite-time.



(a) x_1 and its exact estimation $\hat{x}_1$



(b) x_2 and its exact estimation $\hat{x}_2$



(c) x_3 and its exact estimation $\hat{x}_3$

Fig. 1. Behavior of the states and their estimates.

To show, by simulation, performance of the proposed estimation algorithm, we add a measurement noise. The output y_k is assumed to be disturbed by a Gaussian noise with mean zero and standard deviation $\sigma = 0.3$. The simulation results are depicted in Figure 2.

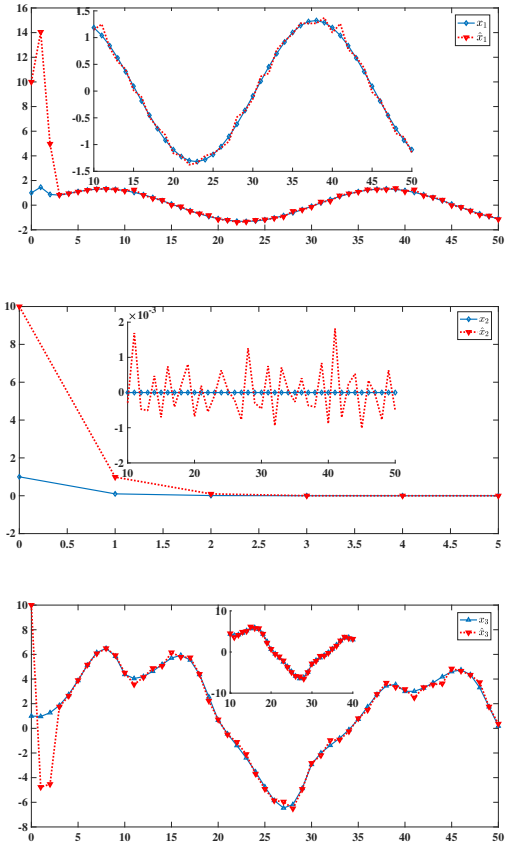


Fig. 2. Estimation results with measurement noise.

5.3 *Observer-based feedback stabilization*

This subsection is devoted to show effectiveness of the exact estimation based output feedback control method proposed in Section 4. The parameters related to the exact estimation are those obtained in the previous subsection devoted to estimation only. Furthermore, to compute the controller parameters $F_i, i = 1, \ldots, n_\rho$, we need to solve LMIs (32). Hence, by using Matlab LMI toolbox with Yalmip interface, we get the following solutions:

$$F_0 = \begin{bmatrix} 0.0075 \\ 0.0024 \\ 0.4411 \end{bmatrix}^\top, \; F_1 = \begin{bmatrix} 0.0356 \\ 0.0003 \\ 0.4422 \end{bmatrix}^\top, \; F_2 = \begin{bmatrix} 0.0297 \\ 0.0009 \\ 0.8510 \end{bmatrix}^\top.$$

It is quite clear from Figure 3 that with these parameters, the exact finite-time estimation based controller (33) ensures asymptotic stabilization of system (1). Real states and their estimations are depicted in Figure 3 to show, at the same time, that the exact estimation is done in finite-time with the

new control input. Additional simulations are presented in Figure 4 to show performance of the proposed stabilization scheme. Both the output y_k and each component of the system are assumed to be disturbed by a Gaussian noise with mean zero and standard deviation $\sigma = 0.1$.

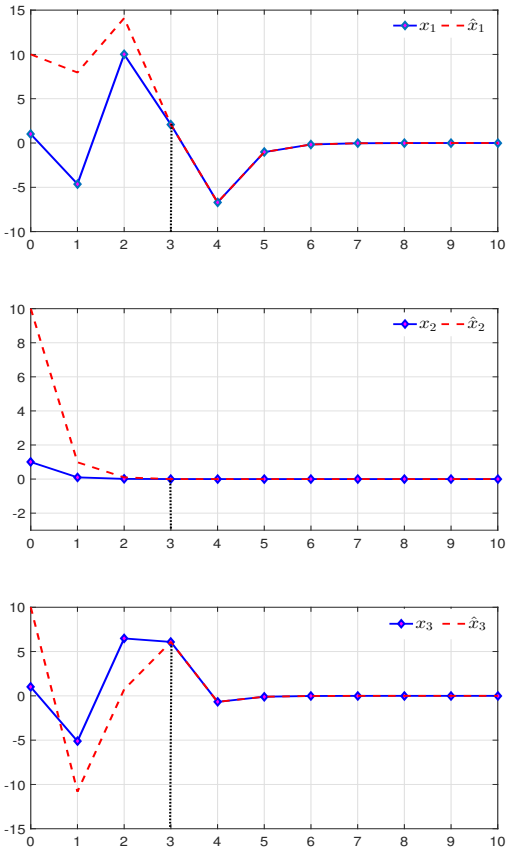


Fig. 3. Stabilized states.

6 Conclusion

This paper provides powerful state estimation algorithms for LPV discrete-time systems. Two new estimation procedures are proposed. The first one allows computing explicitly solution of the system through delayed outputs/inputs, while the second one uses the strategy of two connected asymptotic observers. Due to the exact estimation in finite-time, the problem of output feedback stabilization of LPV systems is solved by mean of simple and non conservative LMI conditions. Therefore, two novel control design strategies are proposed and two well-structured algorithms are given to design parameters of the estimation and stabilization schemes. A numerical example is provided to show effectiveness of the proposed exact finite-time estimation algorithms and their application to output feedback stabilization. As a future work, we aim to generalize the results, in this paper, to systems with unknown parameters ($\rho_k = \rho_k^0 + \Delta\rho_k$) in order to provide robust stabilization schemes.

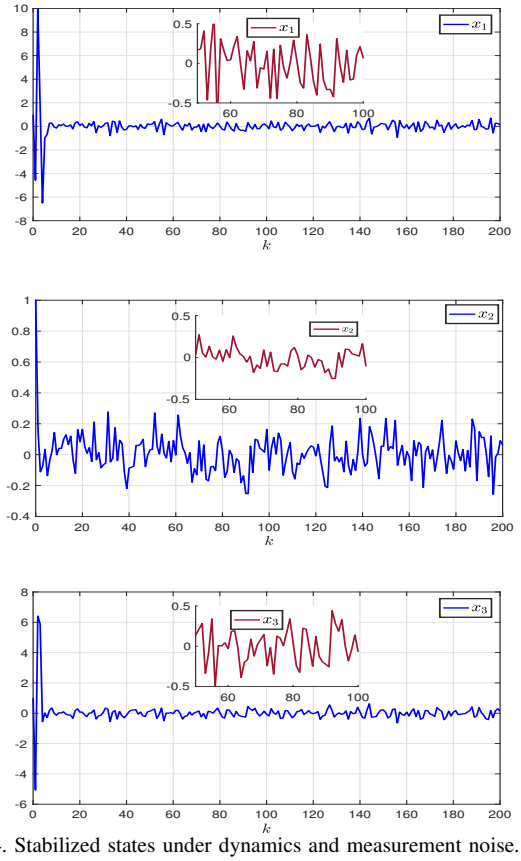


Fig. 4. Stabilized states under dynamics and measurement noise.

A Proof of Lemma 1

Here we give the proof of Lemma 1, which ensures existence and invertibility of the matrix $\mathbb{E}_m(k)$. First, for any $\sigma > 0$, we denote by $\mathscr{B}(0;\sigma)$ a ball in the complex plane centered at the origin with radius σ.

For all $k \geq 1$, and $1 \leq m \leq k$, the matrix $\mathbb{E}_m(k)$ is well-defined. Indeed, the observability property allows chosing L_j and K_j, $j = 0, \cdots, n_\rho$, such that the eigenvalues of $(A_j - L_jC)$ and $(A_j - K_jC)$, for $j = 0, \cdots, n_\rho$, are inside $\mathscr{B}(0;\delta) - \{0\}$, for a given $\delta < 1$. To simplify the presentation, let us introduce the notation:

$$\mathscr{L}_j \stackrel{\Delta}{=} (A_j - L_jC);\ \mathscr{K}_j \stackrel{\Delta}{=} (A_j - K_jC);\ j = 0, \cdots, n_\rho.$$

Invertibility of $\prod_{i=1}^{m} \left(\sum_{j=0}^{n_\rho} \xi_{k-i}^{j} \mathscr{L}_j \right)$ is equivalent to the invertibility of $\sum_{j=0}^{n_\rho} \xi_{k-i}^{j} \mathscr{L}_j$, for all $k \geq 1$ and all $i \leq m$. On

the other hand, we can separate the eigenvalues of $\mathscr{L}_j$, (resp. $\mathscr{K}_j$) $j=1,\cdots,n_\rho$, , inside the ball $\mathscr{B}(0;\delta_L)$, (resp. $\mathscr{B}(0;\delta_K)$) with $\delta_L<\delta$ (resp. $\delta_K<\delta$) that is

$$\max_{1\le j\le n_\rho}\left\|\mathscr{L}_j\right\|<\delta_L<\left\|\mathscr{L}_0^{-1}\right\|^{-1}\le\|\mathscr{L}_0\|<\delta<1, \tag{A.1}$$

$$\max_{1\le j\le n_\rho}\left\|\mathscr{K}_j\right\|<\delta_K<\left\|\mathscr{K}_0^{-1}\right\|^{-1}\le\delta<1, \tag{A.2}$$

which lead to

$$\left\|\mathscr{L}_0^{-1}\right\|\left\|\mathscr{L}_j\right\|<1,\ \ \forall j=1,\cdots,n_\rho.$$

Now, using the boundedness and convexity properties of the sequence $(\xi_k^i)_k$, it follows that

$$\left\|\sum_{j=1}^{n_\rho}\xi_{k-i}^j\mathscr{L}_0^{-1}\mathscr{L}_j\right\|\le\sum_{j=1}^{n_\rho}\xi_{k-i}^j\left\|\mathscr{L}_0^{-1}\right\|\max_{1\le j\le n_\rho}\left\|\mathscr{L}_j\right\|<1. \tag{A.3}$$

From (A.3), we deduce that the matrix

$$\mathbb{I}_n+\sum_{j=1}^{n_\rho}\xi_{k-i}^j\mathscr{L}_0^{-1}\mathscr{L}_j$$

is invertibile, for any $k\ge1$ and any $i\le k$. The invertibility of the matrix $\sum_{j=0}^{n_\rho}\xi_{k-i}^j\mathscr{L}_j$ can be deduced from the factorization

$$\sum_{j=0}^{n_\rho}\xi_{k-i}^j\mathscr{L}_j=\mathscr{L}_0\left[\mathbb{I}_n+\sum_{j=1}^{n_\rho}\xi_{k-i}^j\mathscr{L}_0^{-1}\mathscr{L}_j\right]. \tag{A.4}$$

The previous arguments remain valid for the invertibility of $\prod_{i=1}^{m}\sum_{j=0}^{n_\rho}\xi_{k-i}^j\mathscr{K}_j$, for any $k\ge1$ and $m\le k$. Now, to show that $\mathbb{E}_m(k)$ is invertible, we use the factorization

$$\mathbb{E}_m(k)=\left[\prod_{i=1}^{m}\sum_{j=0}^{n_\rho}\xi_{k-i}^j\mathscr{L}_j\right]^{-1}\left(\mathbb{I}_n-\left[\prod_{i=1}^{m}\sum_{j=0}^{n_\rho}\xi_{k-i}^j\mathscr{L}_j\right]\left[\prod_{i=1}^{m}\sum_{j=0}^{n_\rho}\xi_{k-i}^j\mathscr{K}_j\right]^{-1}\right).$$

We have

$$\left\|\prod_{i=1}^{m}\sum_{j=0}^{n_\rho}\xi_{k-i}^j\mathscr{L}_j\right\|\le\prod_{i=1}^{m}\left(\|\mathscr{L}_0\|+\max_{1\le j\le n_\rho}\left\|\mathscr{L}_j\right\|\right)=\left(\|\mathscr{L}_0\|+\max_{1\le j\le n_\rho}\left\|\mathscr{L}_j\right\|\right)^m$$

and

$$\begin{aligned}&\left\|\left(\prod_{i=1}^{m}\sum_{j=0}^{n_\rho}\xi_{k-i}^j\mathscr{K}_j\right)^{-1}\right\|\\&=\left\|\left(\prod_{i=1}^{m}\mathscr{K}_0\left(\mathbb{I}_n+\sum_{j=1}^{n_\rho}\xi_{k-i}^j\mathscr{K}_0^{-1}\mathscr{K}_j\right)\right)^{-1}\right\| \qquad (A.5)\\&\le\prod_{i=1}^{m}\left\|\mathscr{K}_0^{-1}\right\|\left\|\left(\mathbb{I}_n+\sum_{j=1}^{n_\rho}\xi_{k-i}^j\mathscr{K}_0^{-1}\mathscr{K}_j\right)^{-1}\right\|\\&\le\left(\left\|\mathscr{K}_0^{-1}\right\|\right)^m\prod_{i=1}^{m}\left\|\left(\mathbb{I}_n+\sum_{j=1}^{n_\rho}\xi_{k-i}^j\mathscr{K}_0^{-1}\mathscr{K}_j\right)^{-1}\right\|\\&\le\left(\left\|\mathscr{K}_0^{-1}\right\|\right)^m\prod_{i=1}^{m}\left(1-\left\|\sum_{j=1}^{n_\rho}\xi_{k-i}^j\mathscr{K}_0^{-1}\mathscr{K}_j\right\|\right)^{-1}, \qquad (A.6)\end{aligned}$$

where the last inequality (A.6) is due to

$$\left\|\left(\mathbb{I}_n+\sum_{j=1}^{n_\rho}\xi_{k-i}^j\mathscr{K}_0^{-1}\mathscr{K}_j\right)^{-1}\right\|\le\left(1-\left\|\sum_{j=1}^{n_\rho}\xi_{k-i}^j\mathscr{K}_0^{-1}\mathscr{K}_j\right\|\right)^{-1}$$

since $\left\|\sum_{j=1}^{n_\rho}\xi_{k-i}^j\mathscr{K}_0^{-1}\mathscr{K}_j\right\|<1$. Hence, (A.6) becomes

$$\begin{aligned}\left\|\left(\prod_{i=1}^{m}\sum_{j=0}^{n_\rho}\xi_{k-i}^j\mathscr{K}_j\right)^{-1}\right\|&\le\left(\frac{\left\|\mathscr{K}_0^{-1}\right\|}{1-\left\|\mathscr{K}_0^{-1}\right\|\max_{1\le j\le n_\rho}\left\|\mathscr{K}_j\right\|}\right)^m\\&=\left(\left\|\mathscr{K}_0^{-1}\right\|^{-1}-\max_{1\le j\le n_\rho}\left\|\mathscr{K}_j\right\|\right)^{-m}.\end{aligned}$$

Summarizing, we get

$$\begin{aligned}&\left\|\prod_{i=1}^{m}\sum_{j=0}^{n_\rho}\xi_{k-i}^j\mathscr{L}_j\right\|\left\|\left(\prod_{i=1}^{m}\sum_{j=0}^{n_\rho}\xi_{k-i}^j\mathscr{K}_j\right)^{-1}\right\|\\&\qquad\le\left(\frac{\|\mathscr{L}_0\|+\max_{1\le j\le n_\rho}\left\|\mathscr{L}_j\right\|}{\left\|\mathscr{K}_0^{-1}\right\|^{-1}-\max_{1\le j\le n_\rho}\left\|\mathscr{K}_j\right\|}\right)^m.\end{aligned}$$

Therefore, by choosing the eigenvalues of $\mathscr{L}_j$ and $\mathscr{K}_j$, $j=0,\cdots,n_\rho$, in a convenient way such that

$$\|\mathscr{L}_0\|+\max_{1\le j\le n_\rho}\left\|\mathscr{L}_j\right\|<\left\|\mathscr{K}_0^{-1}\right\|^{-1}-\max_{1\le j\le n_\rho}\left\|\mathscr{K}_j\right\|, \tag{A.7}$$

we get

$$\left\|\prod_{i=1}^{m}\sum_{j=0}^{n_\rho}\xi_{k-i}^j\mathscr{L}_j\right\|\left\|\left(\prod_{i=1}^{m}\sum_{j=0}^{n_\rho}\xi_{k-i}^j\mathscr{K}_j\right)^{-1}\right\|<1$$

for every k, and $m \leq k$. This guarantees the invertibility of $\mathbb{E}_m(k)$ for all k and $m \leq k$. This ends the proof.

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