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ON THE KADOMTSEV-PETVIASHVILI HIERARCHY IN AN EXTENDED CLASS OF FORMAL PSEUDO-DIFFERENTIAL OPERATORS

JEAN-PIERRE MAGNOT¹ AND VLADIMIR ROUBTSOV^{2,3,4}

ABSTRACT. We study the existence and uniqueness of the Kadomtsev-Petviashvili (KP) hierarchy solutions in the algebra of $\mathcal{FCl}(S^1, \mathbb{K}^n)$ of formal classical pseudo-differential operators. The classical algebra $\Psi DO(S^1, \mathbb{K}^n)$ where the KP hierarchy is well-known appears as a subalgebra of $\mathcal{FCl}(S^1, \mathbb{K}^n)$. We investigate algebraic properties of $\mathcal{FCl}(S^1, \mathbb{K}^n)$ such as splittings, r-matrices, extension of the Gelfand-Dickii bracket, almost complex structures. Then, we prove the existence and uniqueness of the KP hierarchy solutions in $\mathcal{FCl}(S^1, \mathbb{K}^n)$ with respect to extended classes of initial values. Finally, we extend this KP hierarchy to complex order formal pseudo-differential operators and we describe their Hamiltonian structures similarly to previously known formal case.

Keywords: Formal pseudo-differential operators, Kadomtsev-Petviashvili hierarchy, almost complex structure, almost quaternionic structure.

MSC (2020): 37K10, 37K20, 37K30

INTRODUCTION

In the classical theory of the Kadomtsev-Petviashvili (KP) hierarchy, the considering algebra of pseudo-operators is

$$\Psi DO(S^1, \mathbb{K}) = C^\infty(S^1, \mathbb{K})((\partial^{-1}))$$

where ∂ is a derivation and $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or \mathbb{H} . Classically, $\partial = \frac{d}{dx}$, $x \in S^1$. It is well-known that this KP hierarchy is an integrable system, with existence and uniqueness of solutions with respect to a fixed initial value, (see e.g. [6] for a classical treatise). There exists various generalizations, or deformations, of the KP hierarchy which almost all satisfy the formal integrability condition, and solutions satisfy properties similar to the properties of the solutions of the (classical) KP hierarchy. Recently, well-posedness have been stated for these equations [10, 25]. Classical algebraic settings that arise in the theory of the KP hierarchy will be reviewed in section 1.3, in section 1.4 and in section 1.7.

Pseudo-differential operators appear also in some contexts other than the theory of integrable systems. In general, larger classes of such operators are studied, see e.g. [13, 30, 32, 33], starting from **non-formal** operators, i.e. operators acting on spaces of sections of a vector bundle. These non-formal operators, in particular **classical** pseudo-differential operators, have their own applications and one can build from them spaces of formal classical operators. The algebra of operators that we intend to use in this paper is the algebra of formal classical pseudo-differential operators $\mathcal{FCl}(S^1, \mathbb{K}^n)$ that are obtained from classical pseudo-differential operators acting

on smooth sections of the trivial vector bundle $S^1 \times \mathbb{K}^n$ over S^1 , for $K = \mathbb{C}$ or \mathbb{H} , see e.g. [13, 30]. In this algebra, it is possible to define functions of an elliptic positive operator that satisfy mild properties of the spectrum using a Cauchy-like formula [30, 32, 33]. In particular the square root of the Laplacian $|D| = \Delta^{1/2}$ is well-defined, as well as the sign of the Dirac operator $D = i \frac{d}{dx}$ defined by

$$\epsilon(D) = D|D|^{-1} = |D|^{-1}D.$$

This operator is not in $\Psi DO(S^1, \mathbb{K}^n)$. In fact, the algebra $\Psi DO(S^1, \mathbb{K})$ is the formal part of the so-called even-even class of (non-formal) classical pseudo-differential operators first defined, to our knowledge, by Kontsevich and Vishik [16, 17] and named as even-even class operators in [30, 32], mostly motivated by problems about renormalized determinants. As a consequence, $\Psi DO(S^1, \mathbb{K})$ is a subalgebra of $\mathcal{F}Cl(S^1, \mathbb{K}^n)$ which is noted in [30, 32] as $\mathcal{F}Cl_{ee}(S^1, \mathbb{K}^n)$. The necessary properties of these pseudo-differential operator algebras, both formal and non-formal, will be reviewed in section 1.1. The key properties of $\epsilon(D)$ that we use in our constructions are:

- the formal operator $\epsilon(D) \in \mathcal{F}Cl(S^1, \mathbb{K}^n)$ commutes with any formal operator $A \in \mathcal{F}Cl(S^1, \mathbb{K}^n)$,
- $\epsilon(D)^2 = Id$
- the composition on the left $A \mapsto \epsilon(D) \circ A$ is an endomorphism of the algebra $\mathcal{F}Cl(S^1, \mathbb{K}^n)$, which restricts to a bijective map from $\Psi DO(S^1, \mathbb{K}^n) = \mathcal{F}Cl_{ee}(S^1, \mathbb{K}^n)$ to an algebraic complement in $\mathcal{F}Cl(S^1, \mathbb{K}^n)$ noted as $\mathcal{F}Cl_{eo}(S^1, \mathbb{K}^n)$ following the terminology of [32]
- the restriction of the Wodzicki residue to $\Psi DO(S^1, \mathbb{K}^n) = \mathcal{F}Cl_{ee}(S^1, \mathbb{K}^n)$, which is similar to but not equal to the Adler functional, is vanishing.

Our first remarks are the following:

- The space $\mathcal{F}Cl(S^1, \mathbb{K}^n)$ splits in various ways: one is derived from the splitting of $T^*S^1 - S^1$ into two connected components (section 1.1.2), the splitting with respect to $\Psi DO(S^1, \mathbb{K}^n)$ as a subalgebra (section 1.1.3), and the extension of the splitting related to the classical Manin triple on $\Psi DO(S^1, \mathbb{K}^n)$ to $\mathcal{F}Cl(S^1, \mathbb{K}^n)$ (section 1.5.1).
- The operator $\epsilon(D)$ is in the center of $\mathcal{F}Cl(S^1, \mathbb{K}^n)$. It generates then a polarized Lie bracket using it as a \mathfrak{r} -matrix (section 1.6) and an integrable almost complex structure on $\mathcal{F}Cl(S^1, \mathbb{K}^n)$.

These technical features enables us to state the announced main results of this paper: existence and uniqueness of solutions of the KP hierarchy with various initial conditions (section 3.1) and KP hierarchy with complex powers (section 3.2).

The paper is organized as follows:

Section 1 is devoted to technical preliminaries: we remind and review some operator algebras, Poisson structures and Manin pairs. We give an overview of the classical method for solving the KP hierarchy. New results of this Section are concentrated in section 1.2 where formal operators of complex order that generalize operators in $\mathcal{F}Cl(S^1, \mathbb{K}^n)$, extending the definitions present in [11], [19], are described. In section 1.5, we explore some Manin pairs on $\mathcal{F}Cl(S^1, \mathbb{K}^n)$, and in section 1.6 we present some polarized brackets, inherited from the richer structure of $\mathcal{F}Cl(S^1, \mathbb{K}^n)$.

Section 2.1 is focused on the comparison of $\mathcal{F}Cl(S^1, \mathbb{K}^n)$ with $\Psi DO(S^1, \mathbb{K}^n)$. First, we develop various injections of $\Psi DO(S^1, \mathbb{K}^n)$ in $\mathcal{F}Cl(S^1, \mathbb{K}^n)$, beyond the

standard one described in section 1. Second, we describe three almost complex structures on $\mathcal{FCl}(S^1\mathbb{K}^n)$ J_1 , J_2 and J_3 such that each couple (J_1, J_2) , (J_1, J_3) and (J_2, J_3) form an almost quaternionic structure on $\mathcal{FCl}(S^1\mathbb{K}^n)$. We prove the integrability of J_1 , derived from $i\epsilon(D)$, and the non-integrability of the two others J_2 and J_3 .

Section 3 deals with various type of initial values for the KP system, which are derived from the various injections of $\Psi DO(S^1, \mathbb{K}^n)$ in $\mathcal{FCl}(S^1, \mathbb{K}^n)$, and ends up with a generalization to the KP hierarchy with operators of complex order. As it was announced, the existence and uniqueness of the solutions, depending on the initial value, is stated. We make few short remarks about well-posedness.

The final part of the paper extends the classical Hamiltonian formulations of the KP hierarchy from $\Psi DO(S^1, \mathbb{K}^n)$ to $\mathcal{FCL}(S^1, \mathbb{K}^n)$, using a generalized Adler-Gelfand-Dickii construction.

All technical and routine proofs are gathered and organized in the Appendix.

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0.2. Conflict of Interest. The authors declare that they have no conflicts of interest.

1. TECHNICAL PRELIMINARIES

1.1. Preliminaries on pseudo-differential operators.

1.1.1. *Description.* We shall start with a description of (non-formal!) pseudo-differential operator groups and algebras which we consider in this work. Throughout this section E denotes a complex finite-dimensional vector bundle over S^1 . We shall specialize below to the case $E = S^1 \times V$ in which V is a n -dimensional vector space. The following definition appears in [2, Section 2.1].

Definition 1.1. *The graded algebra of differential operators acting on the space of smooth sections $C^\infty(S^1, E)$ is the algebra $DO(E)$ generated by:*

- *elements of $End(E)$, the group of smooth maps $E \rightarrow E$ leaving each fibre globally invariant and which restrict to linear maps on each fibre. This group acts on sections of E via (matrix) multiplication;*
- *covariant derivation operators*

$$\nabla_X : g \in C^\infty(S^1, E) \mapsto \nabla_X g$$

where ∇ is a smooth connection on E and X is a smooth vector field on S^1 .

We assign as usual the order 0 to smooth function multiplication operators. The derivation operators and vector fields have the order 1. A differential operator of order k has the form $P(u)(x) = \sum p_{i_1 \dots i_r} \nabla_{x_{i_1}} \dots \nabla_{x_{i_r}} u(x)$, $r \leq k$, In local coordinates (the coefficients $p_{i_1 \dots i_r}$ can be matrix-valued). We denote by $DO^k(S^1), k \geq 0$, the differential operators of order less or equal than k . The algebra $DO(E)$ is filtered by the order. It is a subalgebra of the algebra of classical pseudo-differential operators $Cl(S^1, V)$ that we describe shortly hereafter, focusing

on its necessary aspects. This is an algebra that contains, for example, the square root of the Laplacian

$$(1) \quad |D| = \Delta^{1/2} = \int_{\Gamma} \lambda^{1/2} (\Delta - \lambda Id)^{-1} d\lambda,$$

where $\Delta = -\frac{d^2}{dx^2}$ is the positive Laplacian and Γ is a contour around the spectrum of the Laplacian, see e.g. [33, 30] for an exposition on contour integrals of pseudo-differential operators. $Cl(S^1, V)$ contains also the inverse of $Id + \Delta$, and all smoothing operators on $L^2(S^1, V)$. Among smoothing operators one can find the heat operator

$$e^{-\Delta} = \int_{\Gamma} e^{-\lambda} (\Delta - \lambda Id)^{-1} d\lambda.$$

pseudo-differential operators (maybe non-scalar) are linear operators acting on $C^\infty(S^1, V)$ which reads locally as

$$A(f) = \int e^{ix \cdot \xi} \sigma(x, \xi) \hat{f}(\xi) d\xi$$

where $\sigma \in C^\infty(T^*S^1, M_n(\mathbb{C}))$ satisfying additional estimates on its partial derivatives and \hat{f} means the Fourier transform of f . Basic facts on pseudo-differential operators defined on a vector bundle $E \rightarrow S^1$ can be found e.g. in [13].

Remark 1.2. *Since V is finite dimensional, there exists $n \in \mathbb{N}^*$ such that $V \sim \mathbb{C}^n$. Through this identification, a pseudo-differential operator $A \in Cl(S^1, V)$ can be identified with a matrix $(A_{i,j})_{(i,j) \in \mathbb{N}_n^2}$ with coefficients*

$$A_{i,j} \in Cl(S^1, \mathbb{C}).$$

In other words, the identification $V \sim \mathbb{C}^n$ that we fix induces the isomorphism of algebras

$$Cl(S^1, V) \sim M_n(Cl(S^1, \mathbb{C})).$$

This identification will remain true and useful in the successive constructions below, and will be recalled if appropriate. When it will not carry any ambiguity, we will use the notation $DO(S^1)$, $Cl(S^1)$, etc. instead of $DO(S^1, \mathbb{C})$, $Cl(S^1, \mathbb{C})$, etc. for operators acting on the space of smooth functions from S^1 to \mathbb{C} .

Pseudo-differential operators can be also described by their kernel

$$K(x, y) = \int_{\mathbb{R}} e^{i(x-y)\xi} \sigma(x, \xi) d\xi$$

which is off-diagonal smooth. Pseudo-differential operators with infinitely smooth kernel (or "smoothing" operators), i.e. that are maps: $L^2 \rightarrow C^\infty$ form a two-sided ideal that we note by $Cl^{-\infty}(S^1, V)$. Their symbols are those which are in the Schwartz space $\mathcal{S}(T^*S^1, M_n(\mathbb{C}))$. The quotient $\mathcal{F}Cl(S^1, V) = Cl(S^1, V)/Cl^{-\infty}(S^1, V)$ of the algebra of pseudo-differential operators by $Cl^{-\infty}(S^1, V)$ forms the algebra of formal pseudo-differential operators. Another algebra, which is actually known as a subalgebra of $\mathcal{F}Cl(S^1, V)$ following [26], is also called algebra of formal pseudo-differential operators. This algebra is generated by formal Laurent series

$$\Psi DO(S^1, V) = C^\infty(S^1, V)((\partial^{-1})) = \bigcup_{d \in \mathbb{Z}} \left\{ \sum_{k \leq d} a_k \partial^k \right\}$$

where each $a_k \in C^\infty(S^1, M_n(\mathbb{C}))$ and $\partial = \frac{d}{dx}$. Let us precise hereafter a short but complete description of basic correspondence between $\Psi DO(S^1, V)$ and $\mathcal{F}Cl(S^1, V)$.

Symbols σ project to formal symbols and there is an isomorphism between formal pseudo-differential operators and formal symbols. A detailed study can be found in [7, Tome VII]. Classical pseudo-differential operators are operators A which associated formal symbol $\sigma(A)$ reads as an asymptotic expansion

$$\sigma(A)(x, \xi) \sim \sum_{k \in \mathbb{Z}, k \leq o} \sigma_k(A)(x, \xi)$$

where the **partial symbol of order k**

$$\sigma_k(A) : (x, \xi) \in T^*S^1 \setminus S^1 \mapsto \sigma_k(A)(x, \xi) \in M_n(\mathbb{C})$$

is k -positively homogeneous in the ξ -variable, smooth on $T^*S^1 \setminus S^1 = \{(x, \xi) \in T^*S^1 \mid \xi \neq 0\}$ and such that $d \in \mathbb{Z}$ is the order of the operator A . The order of a smoothing operator we put equal to $-\infty$ and the formal symbol of a smoothing operator is 0.

The set $\mathcal{F}Cl(S^1, V)$ is not the same as the space of formal operators $\Psi DO(S^1, V)$ which naturally arises in the algebraic theory of PDEs, see e.g. [18] for an overview, but here the partial symbols $\sigma_k(A)$ of $A \in \Psi DO(S^1, V)$ are k -homogeneous. By the way one only has $\Psi DO(S^1, V) \subset \mathcal{F}Cl(S^1, V)$. Following the remarks given in [26], $\Psi DO(S^1, V)$ correspond to **even-even class** formal pseudo-differential operators that we describe in section 1.1.3. Two approaches for a global symbolic calculus of pseudo-differential operators have been described in [4, 34]. It is shown in these papers how the geometry of the base manifold M furnishes an obstruction to generalizing local formulas of symbol composition and inversion; we do not recall these formulas here since they are not involved in our computations. We assume henceforth (following e.g. [24], along the lines of the more general description of [13]), that S^1 is equipped with charts such that the changes of coordinates are translations. Under these assumptions,

$$\sigma(A \circ B) \sim \sum_{\alpha \in \mathbb{N}} \frac{(-i)^\alpha}{\alpha!} D_\xi^\alpha \sigma(A) D_x^\alpha \sigma(B), \quad \forall A, B \in Cl(S^1, V),$$

and specializing to partial symbols:

$$\forall k \in \mathbb{Z}, \sigma(A \circ B)_k = \sum_{\alpha \in \mathbb{N}} \sum_{m+n-\alpha=k} \frac{(-i)^\alpha}{\alpha!} D_\xi^\alpha \sigma_m(A) D_x^\alpha \sigma_n(B).$$

The composition $\sigma(A \circ B)$ for $A, B \in \Psi DO(S^1, V) \subset \mathcal{F}Cl(S^1, V)$ gives rise to a (unitary) associative algebra structure on $\Psi DO(S^1, V)$ and we shall write in this case (by abuse of notation)

$$(2) \quad A \circ B = \sum_{\alpha \in \mathbb{N}} \frac{(-i)^\alpha}{\alpha!} D_\xi^\alpha A D_x^\alpha B$$

Remark 1.3. *In such an "operator product" we shall always suppose so called "Wick order" which means that we write functions on $C^\infty(S^1)$ on (or "in front of") left-hand side of all degrees of D .*

Notations. We shall denote by $Cl^d(S^1, V)$ the vector space of classical pseudo-differential operators of order $\leq d$. We also denote by $Cl^*(S^1, V)$ the group

of invertible in $Cl(S^1, V)$ operators. We denote the sets of formal operators adding the script \mathcal{F} . The algebra of formal pseudo-differential operators, identified with formal symbols, is noted by $\mathcal{F}Cl(S^1, V)$, and its group of invertible element is $\mathcal{F}Cl^*(S^1, V)$, while formal pseudo-differential operators of order less or equal to $d \in \mathbb{Z}$ is noted by $\mathcal{F}Cl^d(S^1, V)$.

Remark 1.4. *Through identification of $\mathcal{F}Cl(S^1, V)$ with the corresponding space of formal symbols, the space $\mathcal{F}Cl(S^1, V)$ is equipped with the natural locally convex topology inherited from the space of formal symbols. A formal symbol σ_k is a smooth function in $C^\infty(T^*S^1 \setminus S^1, M_n(\mathbb{C}))$ which is k -homogeneous (for $k > 0$), and hence with an element of $C^\infty(S^1, M_n(\mathbb{C}))^2$ evaluating σ_k at $\xi = 1$ and $\xi = -1$. Identifying $Cl^d(S^1, V)$ with*

$$\prod_{k \leq d} C^\infty(S^1, M_n(\mathbb{C}))^2,$$

the vector space $Cl^d(S^1, V)$ is a Fréchet space, and hence

$$Cl(S^1, V) = \cup_{d \in \mathbb{Z}} Cl^d(S^1, V)$$

is a locally convex topological algebra.

We have to precise that the classical topology on non-formal classical pseudo-differential operators $Cl(S^1, V)$ is finer than the one obtained by pull-back from $\mathcal{F}Cl(S^1, V)$. A “useful” topology on $Cl(S^1, V)$ needs to ensure that partial symbols **and** off-diagonal smooth kernels converge. The topology on spaces of classical pseudo differential operators has been described by Kontsevich and Vishik in [16]; see also [5, 30, 32] for descriptions. This is a Fréchet topology on each space $Cl^d(S^1, E)$. However, passing to the quotients $\mathcal{F}Cl^d(S^1, E) = Cl^d(S^1, E)/Cl^{-\infty}(S^1, E)$, the push-forward topology coincides with the topology of $\mathcal{F}Cl^d(S^1, V)$ described at the beginning of this remark.

1.1.2. *The splitting with induced by the connected components of $T^*S^1 \setminus S^1$.* In this section, we define two ideals of the algebra $\mathcal{F}Cl(S^1, V)$, that we call $\mathcal{F}Cl_+(S^1, V)$ and $\mathcal{F}Cl_-(S^1, V)$, such that $\mathcal{F}Cl(S^1, V) = \mathcal{F}Cl_+(S^1, V) \oplus \mathcal{F}Cl_-(S^1, V)$. This decomposition is explicit in [15, section 4.4., p. 216], and we give an explicit description here following [21, 22].

Definition 1.5. *Let σ be a partial symbol of order o on E . Then, we define, for $(x, \xi) \in T^*S^1 \setminus S^1$,*

$$\sigma_+(x, \xi) = \begin{cases} \sigma(x, \xi) & \text{if } \xi > 0 \\ 0 & \text{if } \xi < 0 \end{cases} \quad \text{and} \quad \sigma_-(x, \xi) = \begin{cases} 0 & \text{if } \xi > 0 \\ \sigma(x, \xi) & \text{if } \xi < 0. \end{cases}$$

We define $p_+(\sigma) = \sigma_+$ and $p_-(\sigma) = \sigma_-$.

The maps $p_+ : \mathcal{F}Cl(S^1, V) \rightarrow \mathcal{F}Cl(S^1, V)$ and $p_- : \mathcal{F}Cl(S^1, V) \rightarrow \mathcal{F}Cl(S^1, V)$ are clearly smooth algebra morphisms (yet non-unital morphisms) that leave the order invariant and are also projections (since multiplication on formal symbols is expressed in terms of point-wise multiplication of tensors).

Definition 1.6. *We define $\mathcal{F}Cl_+(S^1, V) = \text{Im}(p_+) = \text{Ker}(p_-)$ and $\mathcal{F}Cl_-(S^1, V) = \text{Im}(p_-) = \text{Ker}(p_+)$.*

Since p_+ is a projection, we have the splitting

$$\mathcal{F}Cl(S^1, V) = \mathcal{F}Cl_+(S^1, V) \oplus \mathcal{F}Cl_-(S^1, V).$$

Let us give another characterization of p_+ and p_- . The operator $D = -i\frac{d}{dx}$ splits $C^\infty(S^1, \mathbb{C}^n)$ into three spaces :

- its kernel E_0 , built of constant maps
- E_+ , the vector space spanned by eigenvectors related to positive eigenvalues
- E_- , the vector space spanned by eigenvectors related to negative eigenvalues.

The L^2 -orthogonal projection on E_0 is a smoothing operator, which has null formal symbol. By the way, concentrating our attention on the formal symbol of operators, we can ignore this projection and hence we work on $E_+ \oplus E_-$. The following elementary result will be useful for the sequel.

Lemma 1.7. [21, 22]

- $\sigma(D) = \xi$, $\sigma(|D|) = |\xi|$
- $\sigma(\epsilon) = \frac{\xi}{|\xi|}$, where $\epsilon = D|D|^{-1} = |D|^{-1}D$ is the sign of D .
- Let p_{E_+} (resp. p_{E_-}) be the projection on E_+ (resp. E_-), then $\sigma(p_{E_+}) = \frac{1}{2}(Id + \frac{\xi}{|\xi|})$ and $\sigma(p_{E_-}) = \frac{1}{2}(Id - \frac{\xi}{|\xi|})$.

Let us now give an easy but very useful lemma:

Lemma 1.8. [21] Let $f : \mathbb{R}^* \rightarrow V$ be a 0-positively homogeneous function with values in a topological vector space V . Then, for any $n \in \mathbb{N}^*$, $f^{(n)} = 0$ where $f^{(n)}$ denotes the n -th derivative of f .

From this, we have the following result.

Proposition 1.9. [21, 22] Let $A \in \mathcal{FCl}(S^1, V)$. $p_+(A) = \sigma(p_{E_+}) \circ A = A \circ \sigma(p_{E_+})$ and $p_-(A) = \sigma(p_{E_-}) \circ A = A \circ \sigma(p_{E_-})$.

Notation. For shorter notations, we note by $A_\pm = p_\pm(A)$ the formal operators defined from another viewpoint by

$$\sigma(A_+)(x, \xi) \quad (\text{resp. } \sigma(A_-)(x, \xi)) = \begin{cases} \sigma(A)(x, \xi) & \text{if } \xi > 0 \quad (\text{resp. } \xi < 0) \\ 0 & \text{if } \xi < 0 \quad (\text{resp. } \xi > 0) \end{cases}$$

1.1.3. *The “odd-even” splitting.* We note by $\sigma(A)(x, \xi)$ the total formal symbol of $A \in \mathcal{FCl}(S^1, V)$. The following proposition is trivial:

Proposition 1.10. Let $\phi : \mathcal{FCl}(S^1, V) \rightarrow \mathcal{FCl}(S^1, V)$ defined by

$$\phi(A) = \frac{1}{2} \sum_{k \in \mathbb{Z}} \sigma_k(A)(x, \xi) - (-1)^k \sigma_k(A)(x, -\xi).$$

This map is smooth, and $\Psi DO(S^1, V) = \mathcal{FCl}_{ee}(S^1, V) = \text{Ker}(\phi)$.

Following [32], one can define **even-odd class** pseudo-differential operators

$$\mathcal{FCl}_{eo}(S^1, V) = \left\{ A \in \mathcal{FCl}(S^1, V) \mid \sum_{k \in \mathbb{Z}} \sigma_k(A)(x, \xi) + (-1)^k \sigma_k(A)(x, -\xi) = 0 \right\}.$$

Remark 1.11. This terminology is inherited from [32]. This reference is mostly concerned with non-formal operators. We have also to mention that the class of non formal even-even pseudo-differential operators was first described in [16, 17]. In these two references, even-even class pseudo-differential operators are called odd class pseudo-differential operators. By the way, following the terminology of [16, 17]

even-odd class pseudo-differential operators should be called even class. In this paper we prefer to fit with the terminology given in the textbooks[30, 32] even if the initial terminology given in [16, 17] and its natural extension would appear more natural to us.

Proposition 1.12. ϕ is a projection and $\mathcal{FCl}_{eo}(S^1, V) = \text{Im}\phi$.

By the way, we also have

$$\mathcal{FCl}(S^1, V) = \mathcal{FCl}_{ee}(S^1, V) \oplus \mathcal{FCl}_{eo}(S^1, V).$$

We have the following composition rules for the class of a formal operator $A \circ B$:

	A even-even class	A even class
B even-even class	$A \circ B$ even-even class	$A \circ B$ even-odd class
B even-odd class	$A \circ B$ even-odd class	$A \circ B$ even-even class

Example 1.13. $\epsilon(D)$ and $|D|$ are even-odd class, while we already mentioned that differential operators are even-even class.

Remark 1.14. The operator $\epsilon(D)$ satisfies the following properties:

- Since $\epsilon(D)^2 = \text{Id}$, the left composition $A \in \mathcal{FCl}(S^1, V) \mapsto \epsilon(D) \circ A$ is an involution on $\mathcal{FCl}(S^1, V)$
- Since $\epsilon(D) \in \mathcal{FCl}_{eo}(S^1, V)$, the restriction of $\epsilon(D) \circ (\cdot)$ to $\Psi\text{DO}(S^1, V) = \mathcal{FCl}_{ee}(S^1, V)$ is a bijection from $\mathcal{FCl}_{ee}(S^1, V)$ to $\mathcal{FCl}_{eo}(S^1, V)$.

One can also define the operator s on $\mathcal{FCl}(S^1, V)$ which extends the operator $s : T^*S^1 \rightarrow T^*S^1$ defined by $s(x, \xi) = (x, -\xi)$ by

$$s : \sum_n \sigma_n(x, \xi) \mapsto \sum_n (-1)^n \sigma_n(s(x, \xi)).$$

This operator obviously satisfies $s^2 = \text{Id}$, and we remark the following properties:

Proposition 1.15. • $s(\mathcal{FCl}_{\pm}(S^1, V)) = \mathcal{FCl}_{\mp}(S^1, V)$

- $\mathcal{FCl}_{ee}(S^1, V) = \text{Ker}(\text{Id} - s)$
- $\mathcal{FCl}_{eo}(S^1, V) = \text{Ker}(\text{Id} + s)$

Remark 1.16. One can consider also $s' : \sum_n \sigma_n(x, \xi) \mapsto \sum_n \sigma_n(s(x, \xi))$. We still have $s'^2 = \text{Id}$, $s'(\mathcal{FCl}_{\pm}(S^1, V)) = \mathcal{FCl}_{\mp}(S^1, V)$ but the two other properties are not fulfilled.

Under these properties, $\mathcal{FCl}_{ee}(S^1, V)$ and $\mathcal{FCl}_{eo}(S^1, V)$ appear respectively as eigen-spaces for the eigen values 1 and -1 of the symmetry s , and hence an operator $a \in \mathcal{FCl}(S^1, V) = \mathcal{FCl}_{ee}(S^1, V) \oplus \mathcal{FCl}_{eo}(S^1, V)$ decomposes as $a = a_{ee} + a_{eo}$ and

$$s[a, b] = [a, b]_{ee} - [a, b]_{eo} = ([a_{ee}, b_{ee}] + [a_{eo}, b_{eo}]) - ([a_{ee}, b_{eo}] + [a_{eo}, b_{ee}]).$$

1.2. Complex powers of a formal pseudo-differential operator. Following [12] inspired by [33], this is possible to define the complex power of an elliptic formal operator. Concerning formal operators, ellipticity is fully obtained by a condition on the principal symbol of the operator. This provides the possibility, when the algebra of functions R is e.g. a complete topological vector space with bounded addition and multiplication laws, to define complex powers A^α of a formal operator A for $\Re(\alpha) < 0$ via contour integrals similar to (1) and then extend it to arbitrary complex powers. Beyond these technical problems, for any formal \mathbb{C} -algebra of functions R with differentiation ∂ , it is possible to define the same complex powers of the Lax-type operators $L \in \Psi DO(R)$ present in the KP hierarchy, along the lines of [19] and [11]. Let $\alpha \in \mathbb{C}$ and let $\Psi DO^\alpha(R)$ be the affine space of formal series of the form

$$\sum_{k \in \mathbb{N}} a_{\alpha-k} \partial^{\alpha-k},$$

formally defined as $\Psi DO^\alpha(R) = \Psi DO^0(R) \cdot \partial^\alpha$. On the total space of formal pseudo-differential operators of complex order generated by the family $(\Psi DO^\alpha(R))_{\alpha \in \mathbb{C}}$, the same addition and multiplication rules as in $\Psi DO(R)$ holds true and consistent. Let $A \in \Psi DO^\alpha(R)$ with $a_\alpha \in \mathbb{R}_+^* \subset R$, one can define

$$\log(A) \in \alpha \log a \partial + \Psi DO^0(R)$$

such that $\exp(\log(A)) = A$ by standard rules of formal series.

Let $L \in \Psi DO^1(R)$ with principal symbol ∂ . We can then define the complex power L^α for $\alpha \in \mathbb{C}^*$, and following the notations of [11, 18], the affine space $\mathcal{L} = \partial + \Psi DO^0(R)$ has an affine isomorphism, for $\alpha \in \mathbb{C}^*$, with

$$\mathcal{L}^\alpha = \partial^\alpha + \Psi DO^{\alpha-1}(R)$$

through the identification $L \in \mathcal{L} \mapsto L^\alpha = \exp(\alpha \log(L)) \in \mathcal{L}^\alpha$. From this construction on $\Psi DO(S^1, \mathbb{K})$, one can push forward complex powers on subalgebras of $\mathcal{FCl}(S^1, \mathbb{K})$ via the identifications already described. More precisely, one use heuristically the bijection $\Phi_{1,0} : \Psi DO(S^1, \mathbb{K}) \rightarrow \mathcal{FCl}_+(S^1, \mathbb{K})$ to define, for $A = \frac{d}{dx}_+ + \sum_{k \leq 0} a_k \frac{d^k}{dx}_+ \in \mathcal{FCl}_+^1(S^1, \mathbb{K})$, first the logarithm

$$\log A = \log \frac{d}{dx}_+ + \sum_{k \leq 0} a_k \frac{d^k}{dx}_+$$

and the complex power $A^\alpha = \exp(\alpha \log A)$ which formal symbol vanishes for $\xi < 0$. Then we define

$$\mathcal{FCl}_+^\alpha(S^1, \mathbb{K}) = \mathcal{FCl}_+^0(S^1, \mathbb{K}) \frac{d^\alpha}{dx}_+$$

and (after these constructions) $\Phi_{1,0}$ extends naturally to a bijection from $\Psi DO^\alpha(S^1, \mathbb{K})$ to $\mathcal{FCl}_+^\alpha(S^1, \mathbb{K})$. The same construction holds to extend the identification of $\Psi DO(S^1, \mathbb{K})$ with $\mathcal{FCl}_-(S^1, \mathbb{K})$ to complex powers $\Phi_{0,1} : \Psi DO^\alpha(S^1, \mathbb{K}) \rightarrow \mathcal{FCl}_-^\alpha(S^1, \mathbb{K})$ and define

$$\mathcal{FCl}^\alpha(S^1, \mathbb{K}) = \mathcal{FCl}_+^\alpha(S^1, \mathbb{K}) \oplus \mathcal{FCl}_-^\alpha(S^1, \mathbb{K}) = (\Phi_{1,0} \times \Phi_{0,1}) (\Psi DO^\alpha(S^1, \mathbb{K})^2).$$

One can also understand $\mathcal{FCl}^\alpha(S^1, \mathbb{K})$ as $\mathcal{FCl}^\alpha(S^1, \mathbb{K}) = \mathcal{FCl}^0(S^1, \mathbb{K}) |D|^\alpha$ where $|D|^\alpha = \Delta^{\frac{\alpha}{2}}$ is defined via Seeley's complex powers [33]. Alternatively, setting

$$\left(\frac{d}{dx} \right)_+^\alpha = \left(\frac{d}{dx} \right)_+^\alpha + \left(\frac{d}{dx} \right)_-^\alpha = i\epsilon(D) |D|^\alpha,$$

we get $\mathcal{FCl}^\alpha(S^1, \mathbb{K}) = \mathcal{FCl}^0(S^1, \mathbb{K}) \left(\frac{d}{dx}\right)^\alpha$. These spaces of complex powers contain the projections on formal operators (up to smoothing operators) of the classes of pseudo-differential operators of complex order defined in [16, section 3].

1.3. Lie-algebraic digression.

1.3.1. *Operator bialgebras and Manin pairs.* One can easily define a Lie algebra structure by antisymmetrisation of the associative product $[A, B] = A \circ B - B \circ A$. We remark that the vector field Lie algebra $\text{Vect}(S^1)$ and its semi-direct product with $C^\infty(S^1) = C^\infty(S^1, \mathbb{C})$ is a natural Lie subalgebra of the differential operator Lie algebra $DO(S^1)$ which is formed by the order 1 differential operators and the order less or equal to 1. This remark can be also deduced from Definition 1.1 by setting $E = S^1 \times \mathbb{C}$, i.e. $V = \mathbb{C}$. When $V = \mathbb{C}^n$ with $n \geq 2$, an operator $X \in \text{Vect}(S^1)$ can be identified with the degree 1 differential operator $X \otimes Id_{\mathbb{C}^n} \in DO(S^1, V)$ while order 0 differential operators coincide with multiplication operators in $C^\infty(S^1, M_n(\mathbb{C}))$. We also have that $C^\infty(S^1, M_n(\mathbb{C})) \rtimes \text{Vect}(S^1) \subset DO^1(S^1, V)$ as a Lie algebra, but the off-diagonal operator

$$A = \frac{d}{dx} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{d}{dx} \\ \frac{d}{dx} & 0 \end{pmatrix} \in DO^1(S^1, V)$$

is not an operator in the Lie algebra $C^\infty(S^1, M_n(\mathbb{C})) \rtimes \text{Vect}(S^1)$. One can always embed $DO(S^1) = DO(S^1, \mathbb{C})$ into $DO(S^1, V)$ by identifying $A \in DO(S^1)$ with $A \otimes Id_{\mathbb{C}^n} \in DO(S^1, V)$. This identification is a morphism of unital algebras and a morphism of Lie algebras. It is straightforward to check that the similar antisymmetrization of the product (2) gives a Lie algebra structure on $\Psi DO(S^1)$ and the algebra $DO(S^1)$ is a Lie subalgebra in it.

One of the most exciting properties of this pair of infinite-dimensional Lie algebras is an existence of a trace functional (which is quite atypical in the infinite-dimensional world). This functional is known as *Adler trace*

$$\text{Tr}(A) = \oint_{S^1} tr_n(a_{-1}(x))dx,$$

where tr_n is the classical trace of $n \times n$ matrices, and it defines a bilinear invariant symmetric form on $\Psi DO(S^1)$

$$(A, B) \rightarrow \text{Tr}(A \circ B), \quad A, B \in \Psi DO(S^1),$$

which is invariant with respect the multiplication: $(C \circ A, B) = (A, B \circ C)$ and also invariant with respect to the Lie bracket: $([C, A], B) = (A, [B, C])$ for any triple $A, B, C \in \Psi DO(S^1)$. This form is a non-degenerate and can be used to build an injective map from the algebra $\Psi DO(S^1)$ with its *dual*: to each $A \in \Psi DO(S^1)$ one can assign the linear functional $l_A \in (\Psi DO(S^1))^*$ such that $l_A(X) = \text{Tr}(A \circ X)$

Let $A \in \Psi DO(S^1)$ such that it contains only *negative degrees* of the symbol $D = \partial$:

$$A = \sum_{k=-\infty}^{-1} b_k(x) \partial^k.$$

Such "purely Integral" operators are also closed with respect to both operations \circ and $[\cdot, \cdot]$ and we shall denote this subalgebra in $\Psi DO(S^1)$ by $IO(S^1)$. It is easy to check that the subalgebra $DO(S^1)$ is dual to the subalgebra $IO(S^1)$ via the bilinear

invariant form $(-, -)$ and the "full" algebra $\Psi DO(S^1) = DO(S^1) \oplus IO(S^1)$. Both subalgebras are isotropic with respect to $(-, -)$. The algebra triple

$$(\Psi DO(S^1), DO(S^1), IO(S^1))$$

is known as a *Manin triple* and the algebra $DO(S^1)$ carries a structure of a *Lie bialgebra*. We should admit that strictly speaking this triple and this bialgebra are not a genuine example of both structures in view of the following remark:

Remark 1.17. *We should remark that while $DO(S^1) = (IO(S^1))^*$ the natural map $IO(S^1) \rightarrow (DO(S^1))^*$ is not surjective since not every continuous linear functional on $C^\infty(S^1)$ is of the form $F \rightarrow (F, f)$, $F \in C^\infty(S^1)$ ([8]).*

In what follows by abuse of the rigorous terminology ("pseudo-Manin triple", "pseudo-Lie bialgebra", "Khovanova triple" etc.) we shall call the operator triple above by Manin triple and refer $DO(S^1)$ as a Lie bialgebra.

1.3.2. *Differential and integral part.* We first remind that if $V = \mathbb{C}^n$ and use the notations

$$DO(S^1, V) = \bigcup_{o \in \mathbb{N}} \left\{ \sum_{0 \leq k \leq o} a_k \partial^k \right\}, \quad IO(S^1, V) = \left\{ \sum_{k \leq -1} a_k \partial^k \right\}$$

we get also the vector space decomposition

$$(3) \quad \Psi DO(S^1, V) = DO(S^1, V) \oplus IO(S^1, V).$$

such that any (matrix) order k pseudo-differential operator $A = \sum_{i=-\infty}^k a_i \partial^i$ is splitted in two components $A = A_+ + A_-$ with $A_+ = \sum_{i=0}^k a_i \partial^i$ and $A_- = \sum_{i=-\infty}^{-1} a_i \partial^i$. In that case, when $V = \mathbb{C}^n$ and with obvious extension of notations, the algebra triple $(\Psi DO(S^1, V), DO(S^1, V), IO(S^1, V))$ is known as a Manin triple and the algebra $DO(S^1, V)$ carries a structure of a Lie bialgebra. We shall use also (by abuse of notation) the notation $\text{Res}(A)$ for the *residue-matrix function*:

$$\text{Res} : M_n(\Psi DO(S^1, \mathbb{C})) \rightarrow C^\infty(S^1, M_n(\mathbb{C})), \quad A \mapsto a_{-1}(x)$$

Let A, B be some matrix-valued pseudo-differential operators, such that $A = \sum_{i=-\infty}^k a_i \partial^i$, $B = \sum_{j=-\infty}^l b_j \partial^j$ with a_j, b_j some matrix-valued functions. Then it is a straightforward exercise to check that there exists a matrix-valued function F such that

$$\text{Tr}([A, B]) = \oint tr_n(\text{Res}[A, B]) = \oint dF = 0.$$

Remark 1.18. *The same holds when we replace concrete algebras of functions $C^\infty(S^1)$ by an abstract associative algebra \mathcal{R} with unit element, equipped with integration properties, we refer to [28, 29] for a detailed description for the corresponding algebra of formal operators $\Psi DO(\mathcal{R})$. Then, in presence of a non-trivial one-form $\oint : \mathcal{R} \rightarrow \mathbb{C}$, one can define an analogous of the Adler map that we note also Tr by*

$$\text{Tr} : \sum_{k \in \mathbb{Z}} a_k \partial^k \mapsto \oint a_{-1}.$$

For example, when $\mathcal{R} = C^\infty(S^1, M_n(\mathbb{C}))$ for $n \geq 2$, i.e. when

$$\Psi DO(\mathcal{R}) = \Psi DO(S^1, \mathbb{C}^n) = M_n(\Psi DO(S^1, \mathbb{C})),$$

the natural 1-form \oint on \mathcal{R} is exactly $\oint_{S^1} \circ tr_n$ already described.

1.4. Poisson structures on matrix pseudo-differential operators. . In analogy with the "scalar" ($n = 1$) case one can define the first and the second Gelfand-Dikii Poisson structures in the framework of the formal Gelfand "variational" differential-geometric formalism in the infinite-dimensional setting. The results of this subsection are not new and are well-known since almost 30 years (see for example [3]). We define an infinite dimensional affine variety L_k whose points, monic differential operators of order k , are defined by k matrix function coefficients $\bar{u} = (u_1(x), \dots, u_k(x))$ such that $\forall j : 1 \leq j \leq k, u_j(x) \in C^\infty(S^1, M_n(\mathbb{C}))$:

$$L_k = \{L = \partial^k + u_1 \partial^{k-1} + \dots + u_k\}.$$

We consider a function algebra $\mathcal{C}(L_k)$ as a set of functionals $l : L_k \rightarrow \mathbb{C}$ of type

$$l[\bar{u}] := \oint \text{tr}(\text{pol}(\partial_x^\alpha(u_j))),$$

where $\text{pol}(\partial_x^\alpha(u_j))$ is a differential polynomial on $u_j(x)$. The tangent space to L_k consists of differential operators of order $k - 1$ and the cotangent space T^*L_k can be identified with the quotient $IO(S^1, V)/IO_{-k}(S^1, V)$: via the coupling $T^*L_k \times TL_k \rightarrow \mathcal{C}(L_k)$, $\langle X, V \rangle = \text{Tr}(X \circ V)$. Here $X \in T^*L_k$ is the set of "covectors" of the type $X = \sum_{j=1}^k \partial_x^{-j} \circ p_j$, $p_j \in \text{pol}(\partial_x^\alpha(u_j))$. We shall remind the definition of *variational derivative* of a functional $l[\bar{u}] \in \mathcal{C}(L)$:

$$\frac{\delta l[\bar{u}]}{\delta u_j}(x)_{pq} = \sum_{s=0}^{\infty} (-1)^s \frac{d^r}{dx^r} \left(\frac{\partial \text{tr}(\text{pol})(\bar{u})(x)}{\partial (u_j^{(s)})_{pq}} \right), 1 \leq p, q \leq n.$$

The variational derivative assigns to each functional $l[\bar{u}] \in \mathcal{C}(L)$ the pseudo-differential operator

$$X_l = \sum_{r=0}^k \partial^{-r} \left(\frac{\delta l[\bar{u}]}{\delta u_{k+1-r}} \right).$$

Let $X_{l_1, 2}$ be two such operators which can be interpreted as two covectors on T^*L_k . We define a family of brackets

$$\{-, -\}_\lambda : \mathcal{C}(L) \times \mathcal{C}(L) \rightarrow \mathcal{C}(L) :$$

$$\begin{aligned} \{l_1, l_2\}_\lambda(L) &= \oint \text{tr}_n(\text{Res}((L + \lambda)(X_{l_1}(L + \lambda))_+ X_{l_2} - ((L + \lambda)X_{l_1})_+(L + \lambda)X_{l_2})) = \\ &= \{l_1, l_2\}_2(L) + \lambda \{l_1, l_2\}_1(L) = \\ &= \oint \text{tr}_n(\text{Res}(L(X_{l_1}L)_+ X_{l_2} - (LX_{l_1})_+ LX_{l_2})) + \lambda \oint \text{tr}_n(\text{Res}([L, X_{l_1}]_+ X_{l_2})). \end{aligned}$$

Theorem 1.19. (*Adler-Gelfand-Dickey*)

- (1) The family $\{-, -\}_\lambda$ is a family of Poisson structures on $\mathcal{C}(L)$;
- (2) The corresponding Hamiltonian map $H_\lambda : T^*L_k \rightarrow TL_k$: is given by

$$H_\lambda(X) = (LX)_+ L - L(XL)_+ + \lambda[L, X]_+, \quad X \in T^*L_k, L \in L_k.$$

- (3) $H_\lambda(X) = H_2(X) + \lambda H_1(X)$ and each $V_i, i = 1, 2$ are Hamiltonian mappings.
- (4) The Hamiltonian maps H_i relate to the Poisson brackets via

$$\{l_1, l_2\}_\lambda(L) = H_\lambda(\delta l_1)(l_2).$$

(5) Covector fields T^*L_k carry a Lie algebra structure with the bracket

$$[X, Y] = [(XL)_+Y + (YL)_-X - X(LY)_- - Y(LX)_+ + H_2(X)Y - H_2(Y)(X)]_-$$

which will be called the second Gelfand-Dikii algebra GD_2 .

This structure relates in some sense to the Manin triple on $\Psi DO(S^1, V)$.

1.4.1. *Semenov-Tyan-Shansky r -matrix construction.* Let A_\pm two elements of the Lie algebra $\Psi DO(S^1, V)$ such that $A_+ \in DO(S^1, V)$ and $A_- \in IO(S^1, V)$. Then one can identify $\Psi DO(S^1, V) \otimes \Psi DO(S^1, V)$ with $\text{Hom}(\Psi DO(S^1, V), \Psi DO(S^1, V))$ using the inner product on $\Psi DO(S^1, V)$. Therefore, if we consider the bi-vector $\mathbf{r} \in \Lambda^2(\Psi DO(S^1, V))$ such that $\langle \mathbf{r}, A_+^* \wedge A_-^* \rangle = (A_+, A_-) = \text{Tr}(A_+ \circ A_-)$, where A^* is a dual to A with respect to the inner product., then we can identify it with the operator $\tilde{\mathbf{r}} \in \text{End}(\Psi DO(S^1, V))$ such that $\tilde{\mathbf{r}}|_{DO(S^1, V)} = 1$, $\tilde{\mathbf{r}}|_{IO(S^1, V)} = -1$.

1.4.2. *Analogues of splittings.* Back to $\mathcal{F}Cl(S^1, V)$, the maps

$$A \in \mathcal{F}Cl(S^1, V) \mapsto \sum_{k \in \mathbb{Z}} \sigma_k(A)(x, 1) \partial^k$$

and

$$A \in \mathcal{F}Cl(S^1, V) \mapsto \sum_{k \in \mathbb{Z}} \sigma_k(A)(x, -1) \partial^k,$$

identify $\Psi DO(S^1, V)$ with $\mathcal{F}Cl_+(S^1, V)$ for the first one and $\mathcal{F}Cl_-(S^1, V)$ for the second one.

Thus, there exists a decomposition $\mathcal{F}Cl_+(S^1, V) = \mathcal{F}Cl_{+,D}(S^1, V) \oplus \mathcal{F}Cl_{+,S}(S^1, V)$ and another $\mathcal{F}Cl_-(S^1, V) = \mathcal{F}Cl_{-,D}(S^1, V) \oplus \mathcal{F}Cl_{-,S}(S^1, V)$, and setting

$$\mathcal{F}Cl_D(S^1, V) = \mathcal{F}Cl_{+,D}(S^1, V) \oplus \mathcal{F}Cl_{-,D}(S^1, V),$$

$$\mathcal{F}Cl_S(S^1, V) = \mathcal{F}Cl_{+,S}(S^1, V) \oplus \mathcal{F}Cl_{-,S}(S^1, V),$$

we get the vector space decomposition analogous to (3):

$$\mathcal{F}Cl(S^1, V) = \mathcal{F}Cl_D(S^1, V) \oplus \mathcal{F}Cl_S(S^1, V),$$

1.5. Manin pairs on $\mathcal{F}Cl(S^1, V)$.

1.5.1. *Extension of the classical Manin triple to $\mathcal{F}Cl(S^1, V)$.* The Adler trace [1] defined by

$$\text{Tr} : A = \sum_{k \leq o} a_k \partial^k \mapsto \int_{S^1} \text{tr}(a_{-1})$$

is the only non trivial trace on $\Psi DO(S^1, V)$. Moreover, see e.g. [11] and [19],

Theorem 1.20. $(\Psi DO(S^1, V), IO(S^1, V), DO(S^1, V), \text{Tr})$ is a Manin triple.

The Wodzicki residue ([35], see e.g. [15]) is usually known as an ‘‘extension’’ of the Adler trace to $\mathcal{F}Cl(S^1, V)$ and hence to $Cl(S^1, V)$. For the sake of deeper insight on what is described in the rest of this paper, we need to precise that the space of traces on $\mathcal{F}Cl(S^1, V)$ is 2-dimensional, generated by two functionals:

$$\text{res}_+ : A \mapsto \int_{S^1} \sigma_{-1}(A)(x, 1) |dx|$$

and

$$res_- : A \mapsto \int_{S^1} tr(\sigma_{-1}(A))(x, -1)|dx|.$$

The functionals res_{\pm} are the only non-vanishing traces on $\mathcal{FCl}_{\pm}(S^1, V)$ (up to a scalar factor) and are vanishing on $\mathcal{FCl}_{\mp}(S^1, V)$. The (classical) Wodzicki residue reads as $res = res_+ + res_-$. Because the partial symbol $\sigma_{-1}(A)$ of an operator $A \in \Psi DO(S^1, V)$ is skew-symmetric in the ξ -variable, res is vanishing on $\Psi DO(S^1, V) = \mathcal{FCl}_{ee}(S^1, V)$, so that it is superficial to state that the Wodzicki residue is “simply” the extension of the Adler trace. However the two linear functionals already described, namely

$$A \in \mathcal{FCl}(S^1, V) \mapsto \sum_{k \in \mathbb{Z}} \sigma_k(A)(x, 1) \partial^k$$

and

$$A \in \mathcal{FCl}(S^1, V) \mapsto \sum_{k \in \mathbb{Z}} \sigma_k(A)(x, -1) \partial^k,$$

identity res_+ and res_- respectively with Tr . By the way, we can state:

Theorem 1.21. *We have three Manin triples:*

$$\begin{aligned} &(\mathcal{FCl}_+(S^1, V), \mathcal{FCl}_{+,S}(S^1, V), \mathcal{FCl}_{+,D}(S^1, V), res_+), \\ &(\mathcal{FCl}_-(S^1, V), \mathcal{FCl}_{-,S}(S^1, V), \mathcal{FCl}_{-,D}(S^1, V), res_-) \end{aligned}$$

and

$$(\mathcal{FCl}(S^1, V), \mathcal{FCl}_S(S^1, V), \mathcal{FCl}_D(S^1, V), res).$$

1.5.2. *A remark on two “non-invariant Manin triples”.* Following [11], given an operator \mathbf{r} acting on $\mathcal{FCl}(S^1, V)$ satisfying $\mathbf{r}^2 = Id$, one can form a $\mathcal{FCl}(S^1, V)$ -valued skew-symmetric bilinear form

$$[\cdot, \cdot]_{\mathbf{r}} = \frac{1}{2} ([\mathbf{r}(\cdot), \cdot] + [\cdot, \mathbf{r}(\cdot)]).$$

In what follows, we concentrate on the cases $\mathbf{r} = \epsilon(D) \circ (\cdot)$, $\mathbf{r} = s$ and also $\mathbf{r} = s'$. The corresponding brackets will be noted respectively by $[\cdot, \cdot]_{\epsilon(D)}$, $[\cdot, \cdot]_s$ and $[\cdot, \cdot]_{s'}$.

Let us define $(A, B)_{s'} = res(A, s'(B))$. By direct calculations, we find successively:

Lemma 1.22. *$(\cdot; \cdot)_{s'}$ is non degenerate and symmetric.*

Theorem 1.23. *On $\mathcal{FCl}(S^1, V) = \mathcal{FCl}_+(S^1, V) + \mathcal{FCl}_-(S^1, V)$, $(\cdot; \cdot)_{s'}$ is a non degenerate and symmetric bilinear form for which the Lie algebras $\mathcal{FCl}_+(S^1, V)$ and $\mathcal{FCl}_-(S^1, V)$ are isotropic.*

Let us define $(A, B)_s = res(A, s(B))$.

Lemma 1.24. *$(\cdot; \cdot)_s$ is non degenerate and skew-symmetric but neither invariant for $[\cdot, \cdot]$ nor for $[\cdot, \cdot]_{\epsilon(D)}$.*

Theorem 1.25. *On $\mathcal{FCl}(S^1, V) = \mathcal{FCl}_+(S^1, V) + \mathcal{FCl}_-(S^1, V)$, $(\cdot; \cdot)_s$ is a non degenerate and skew-symmetric bilinear form for which the Lie algebras $\mathcal{FCl}_+(S^1, V)$ and $\mathcal{FCl}_-(S^1, V)$ are isotropic.*

1.5.3. *Two other Manin pairs.* Let us consider the decomposition

$$\mathcal{FCl}(S^1, V) = \mathcal{FCl}_{ee}(S^1, V) + \mathcal{FCl}_{eo}(S^1, V),$$

that we equip with the classical Lie bracket $[\cdot, \cdot]$ or with $[\cdot, \cdot]_{\epsilon(D)}$. and with the bilinear form $(A, B) = \text{res}(AB)$.

Theorem 1.26. *res(AB) is a bilinear, non degenerate, symmetric and invariant form for both brackets, and $\mathcal{FCl}_{ee}(S^1, V)$ as well as $\mathcal{FCl}_{eo}(S^1, V)$ are isotropic vector spaces. Moreover,*

- for $[\cdot, \cdot]$, $\mathcal{FCl}_{ee}(S^1, V)$ is a Lie algebra
- for $[\cdot, \cdot]_{\epsilon(D)}$, $\mathcal{FCl}_{eo}(S^1, V)$ is a Lie algebra.

1.6. **Polarized Lie bracket.** The modified Yang–Baxter equation gives the condition on \mathbf{r} for making $[\cdot, \cdot]_{\mathbf{r}}$ a Lie bracket:

$$[\mathbf{r}X, \mathbf{r}Y] - \mathbf{r}([\mathbf{r}X, Y] + [X, \mathbf{r}Y]) = -[X, Y].$$

By direct computations, we get the following:

Theorem 1.27. *On the vector space $\mathcal{FCl}(S^1, V)$,*

- (1) $[\cdot, \cdot]_{\epsilon(D)}$ is a Lie bracket for which $[\mathcal{FCl}_{ee}(S^1, V), \mathcal{FCl}_{ee}(S^1, V)]_{\epsilon(D)} \subset \mathcal{FCl}_{eo}(S^1, V)$ and $[\mathcal{FCl}_{eo}(S^1, V), \mathcal{FCl}_{eo}(S^1, V)]_{\epsilon(D)} \subset \mathcal{FCl}_{eo}(S^1, V)$.
- (2) $[\cdot, \cdot]_s$ and $[\cdot, \cdot]_{s'}$ are not Lie brackets.

Remark 1.28. (Testing Rota-Baxter equations and Reynolds operators)
Testing by direct calculations the Rota-Baxter equations

$$R(u)R(v) - R(R(u)v) - R(uR(v)) = \lambda R(uv)$$

for a weight $\lambda \in \mathbb{C}$, one finds that $R = \epsilon(D) \circ (\cdot)$, $R = s$ and $R = s'$ do not satisfy the Rota-Baxter equations (i.e. don't define new associative algebra operations) The same calculations show that these are not Reynolds operators (i.e. they do not satisfy the condition $R(R(u)v) = R(u)R(v)$ for all u, v in the underlying associative algebra).

1.7. **Preliminaries on the KP hierarchy.** Let R be an algebra of functions equipped with a derivation ∂ . For us, $R = C^\infty(S^1, \mathbb{K})$ with $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and \mathbb{H} , and $\partial = \frac{d}{dx}$. In this context, where algebras of functions R are Fréchet algebras, a natural notion of differentiability occurs, making addition, multiplication and differentiation smooth. By the way, considering addition and multiplication in $\Psi DO(S^1, \mathbb{K})$, one can say that addition and multiplication in $\Psi DO(S^1, \mathbb{K})$ by understanding, under this terminology, that, if $A = \sum_{n \in \mathbb{Z}} a_n \partial^n$ and $B = \sum_{n \in \mathbb{Z}} b_n \partial^n$, setting $A + B = C = \sum_{n \in \mathbb{Z}} c_n \partial^n$ and $AB = D = \sum_{n \in \mathbb{Z}} d_n \partial^n$ the map

$$((a_n)_{n \in \mathbb{Z}}, (b_n)_{n \in \mathbb{Z}}) \mapsto ((c_n)_{n \in \mathbb{Z}}, (d_n)_{n \in \mathbb{Z}})$$

is smooth in the relevant infinite product. We make these precisions in other to circumvent the technical tools recently developed in [10, 25] where a fully rigorous framework for smoothness on these objects is described and used. Let $T = \{t_n\}_{n \in \mathbb{N}^*}$ be an infinite set of formal (time) variables and let us consider the algebra of formal series $\Psi DO(S^1, \mathbb{K})[[T]]$ with infinite set of formal variables t_1, t_2, \dots with T -valuation $\text{val}_T(t_n) = n$ [28]. One can extend naturally on $\Psi DO(S^1, \mathbb{K})[[T]]$

the notion of smoothness from the same notion on $\Psi DO(S^1, \mathbb{K})$, see [25] for a more complete description. The Kadomtsev-Petviashvili (KP) hierarchy reads

$$(4) \quad \frac{dL}{dt_k} = [(L^k)_D, L], \quad k \geq 1,$$

with initial condition $L(0) = L_0 \in \partial + \Psi^{-1}(R)$. The dependent variable L is chosen to be of the form $L = \partial + \sum_{\alpha \leq -1} u_\alpha \partial^\alpha \in \Psi^1(S^1, \mathbb{K})[[T]]$. A standard reference on (4) is L.A. Dickey's treatise [6], see also [19, 28, 29]. In order to solve the KP hierarchy, we need the following groups (see e.g. [25] for a latest adaptation of Mulase's construction [28, 29]):

$$\bar{G} = 1 + \Psi DO^{-1}(S^1, \mathbb{K})[[T]],$$

$$\bar{\Psi} = \left\{ P = \sum_{\alpha \in \mathbb{Z}} a_\alpha \partial^\alpha \in \Psi(S^1, \mathbb{K})[[T]] : \text{val}_T(a_\alpha) \geq \alpha \text{ and } P|_{t=0} \in 1 + \Psi DO^{-1}(S^1, \mathbb{K}) \right\}$$

and

$$\bar{\mathcal{D}} = \left\{ P = \sum_{\alpha \in \mathbb{Z}} a_\alpha \partial^\alpha : P \in \bar{\Psi}(A_t) \text{ and } a_\alpha = 0 \text{ for } \alpha < 0 \right\}.$$

We have a matched pair $\bar{\Psi} = \bar{G} \bowtie \bar{\mathcal{D}}$ which is smooth under the terminology we gave before. The following result, from [25], gives a synthesised statement of main results on the KP hierarchy (4) and states smooth dependence on the initial conditions in the case where R is commutative (i.e. $R = C^\infty(S^1, \mathbb{R})$ or $R = C^\infty(S^1, \mathbb{C})$ in this work).

Theorem 1.29. [25] *Consider the KP hierarchy 4 with initial condition $L(0) = L_0$. Then,*

- (1) *There exists a pair $(S, Y) \in \bar{G} \times \bar{\mathcal{D}}$ such that the unique solution to Equation (4) with $L|_{t=0} = L_0$ is $L(t_1, t_2, \dots) = Y L_0 Y^{-1} = S L_0 S^{-1}$.*
- (2) *The pair (S, Y) is uniquely determined by the smooth decomposition problem*

$$\exp\left(\sum_{k \in \mathbb{N}} \tau_k L_0^k\right) = S^{-1} Y$$

and the solution L depends smoothly on the initial condition L_0 .

- (3) *The solution operator L is smoothly dependent on the initial value L_0 .*

We now describe the case $\mathbb{K} = \mathbb{H} = \mathbb{R} + i\mathbb{R} + j\mathbb{R} + k\mathbb{R}$. The algebra $\Psi DO(S^1, \mathbb{H})$ is constructed from the non commutative Fréchet algebra

$$C^\infty(S^1, \mathbb{H}) = C^\infty(S^1, \mathbb{R}) \oplus iC^\infty(S^1, \mathbb{R}) \oplus jC^\infty(S^1, \mathbb{R}) \oplus kC^\infty(S^1, \mathbb{R}).$$

All the constructions before remain valid following [20, 14], setting $V = \mathbb{H}$ as a 4-dimensional \mathbb{R} -algebra, and the algebraic description of the solutions of the KP hierarchy (4) with $L_0 \in \Psi DO^1(S^1, \mathbb{H})$ and $L \in \Psi^1 DO(S^1, \mathbb{H})[[T]]$ as before can be completed by stating that the coefficients of the T -series of L depend smoothly on the initial value L_0 from [10].

2. INJECTING ΨDO INTO \mathcal{FCl} .

2.1. Injecting $\Psi DO(S^1, \mathbb{K})$ in $\mathcal{FCl}(S^1, \mathbb{K})$. We already mentioned the identification of $\Psi DO(S^1, \mathbb{K})$ with $\mathcal{FCl}_{ee}(S^1, \mathbb{K})$, present when $\mathbb{K} = \mathbb{R}$ or \mathbb{C} in [26]. We claim here that this identification also applies straightway when $\mathbb{K} = \mathbb{H}$. We denote by Φ_{ee} this identification, that can be generalized to

$$\Phi_{ee, \lambda} : \sum_{k \in \mathbb{Z}} a_k \left(\frac{d}{dx} \right)^k \in \Psi DO(S^1, \mathbb{K}) \mapsto \sum_{k \in \mathbb{Z}} a_k \left(\lambda \frac{d}{dx} \right)^k \in \mathcal{FCl}_{ee}(S^1, \mathbb{K}).$$

Similar to this identification, we have other injections for $\lambda \in \mathbb{R}^*$:

$$\Phi_{\epsilon(D), \lambda} : \sum_{k \in \mathbb{Z}} a_k \left(\frac{d}{dx} \right)^k \in \Psi DO(S^1, \mathbb{K}) \mapsto \sum_{k \in \mathbb{Z}} a_k \left(\lambda \epsilon(D) \frac{d}{dx} \right)^k \in \mathcal{FCl}(S^1, \mathbb{K}), \text{ and}$$

$$\Phi_{\lambda, \mu} : \sum_{k \in \mathbb{Z}} a_k \left(\frac{d}{dx} \right)^k \in \Psi DO(S^1, \mathbb{K}) \mapsto \sum_{k \in \mathbb{Z}} a_k \left(\lambda^k \left(\frac{d}{dx} \right)_+^k + \mu^k \left(\frac{d}{dx} \right)_-^k \right) \in \mathcal{FCl}(S^1, \mathbb{K})$$

for $(\lambda, \mu) \in \mathbb{C}^2 \setminus \{(0; 0)\}$, with unusual convention $0^k = 0 \forall k \in \mathbb{Z}$.

Remark 2.1. $\Phi_{1,1} = \Phi_{ee}$ and $\Phi_{1,-1} = \Phi_{\epsilon(D),1}$.

Remark 2.2. $Im \Phi_{1,0} = \mathcal{FCl}_+(S^1, \mathbb{K})$ and $\Phi_{1,0}$ is a isomorphism of algebras from $\Psi DO(S^1, \mathbb{K})$ to $\mathcal{FCl}_+(S^1, \mathbb{K})$. The same way, $\Phi_{0,1}$ identifies the algebras $\Psi DO(S^1, \mathbb{K})$ and $\mathcal{FCl}_-(S^1, \mathbb{K})$.

Remark 2.3. We have also to say that the maps $\Phi_{\lambda, \mu}$ are not algebra morphisms unless $(\lambda, \mu) \in \{(1; 0), (0; 1), (1; 1)\}$. For example, let $\lambda \in \mathbb{C} - \{0; 1\}$. the map $\Phi_{\lambda, 0}$ pushes forward the multiplication on $\Psi DO(S^1, \mathbb{K})$ to a deformed composition $*_k$ on $\mathcal{FCl}_+(S^1, \mathbb{K})$ that reads as $\sigma(A) *_k \sigma(B) = \sum_{\alpha \in \mathbb{N}} \frac{(-i)^\alpha}{\alpha! k^\alpha} D_x^\alpha \sigma(A) D_\xi^\alpha \sigma(B)$.

Let us now give some sample images:

$A \in \Psi DO(S^1, \mathbb{C})$	1	$\frac{d}{dx}$	$-\frac{d}{dx} = \Delta$	$\left(1 + \frac{d}{dx}\right)^{-1}$
$\Phi_{\epsilon(D), 1}(A)$	1	$\epsilon(D) \frac{d}{dx} = i D $	Δ	$\left(1 + \epsilon(D) \frac{d}{dx}\right)^{-1}$
$\Phi_{ee, -1}(A)$	1	$-\frac{d}{dx}$	Δ	$\left(1 - \frac{d}{dx}\right)^{-1}$
$\Phi_{1, 0}(A)$	1_+	$\left(\frac{d}{dx}\right)_+$	Δ_+	$\left(\left(1 + \frac{d}{dx}\right)^{-1}\right)_+$

From our previous remarks, we get:

Theorem 2.4. The map

$$\Phi_{1,0} \times \Phi_{0,1} : \Psi DO(S^1, \mathbb{K})^2 \rightarrow \mathcal{FCl}_+(S^1, \mathbb{K}) \times \mathcal{FCl}_-(S^1, \mathbb{K}) = \mathcal{FCl}(S^1, \mathbb{K})$$

is an isomorphism of algebra.

We also remark a new subalgebra of $\mathcal{FCl}(S^1, \mathbb{K})$:

Definition 2.5. Let $\mathcal{FCl}_\epsilon(S^1, \mathbb{K})$ be the image of $\Phi_{\epsilon(D),1}$ in $\mathcal{FCl}(S^1, \mathbb{K})$.

We have the obvious identification $\mathcal{FCl}_\epsilon(S^1, \mathbb{K}) = C^\infty(S^1, \mathbb{K})(i|D|^{-1})$ as a vector space.

2.2. Identification of $\mathcal{FCl}(S^1, \mathbb{C})$ with $\Psi DO(S^1, \mathbb{H})$. Let

$$i\epsilon(D) = \left(\frac{d}{dx} \right) \cdot |D|^{-1} = |D|^{-1} \cdot \left(\frac{d}{dx} \right).$$

We define the operator $J_1 = i\epsilon(D) \circ (\cdot)$ on $\mathcal{FCl}(S^1, V)$.

Theorem 2.6. The operator J_1 defines an integrable almost complex structure on $\mathcal{FCl}(S^1, V)$. $\mathcal{FCl}(S^1, V) = \Psi DO(S^1, V) \otimes \mathbb{C}$ as a real algebra, identifying $\mathcal{FCl}_\epsilon(S^1, V)$ with $\Psi DO(S^1, V)$ (real part) and $\mathcal{FCl}_\epsilon(S^1, V)$ with $i\Psi DO(S^1, V)$ (imaginary part).

We now identify two other almost complex structures: $J_2 = is(\cdot)$, $J_3 = is'$ and Clearly, $\forall i \in \{2; 3\}$, $J_i^2 = -Id$ and we have also:

Proposition 2.7. $J_1 \circ J_2 = -J_2 \circ J_1$

Theorem 2.8. The operator J_2 defines a non integrable almost complex structure on $\mathcal{FCl}(S^1, V)$. Hence, gathering all these results, we get that the almost quaternionic structure (J_1, J_2) is non integrable.

Proposition 2.9. $J_1 \circ J_3 = -J_3 \circ J_1$

Proposition 2.10. $J_3 \circ J_2 = J_2 J_3 \neq -J_2 \circ J_3$

Theorem 2.11. The operator J_3 defines a non integrable almost complex structure on $\mathcal{FCl}(S^1, V)$. The almost quaternionic structure (J_1, J_3) is non integrable.

Let us now define $J_4 = J_1 J_3$.

Proposition 2.12. We have:

- $J_4^2 = -Id$.
- $J_2 J_4 = -J_4 J_2$.
- $J_1 J_4 = -J_4 J_1$.

3. KP HIERARCHY WITH INTEGER AND COMPLEX ORDER LAX OPERATORS IN $\mathcal{FCl}(S^1, \mathbb{C})$ AND $\Psi DO(S^1, \mathbb{H})$.

3.1. Multiple classical KP hierarchies on $\mathcal{FCl}(S^1, \mathbb{K})$. The (classical) KP hierarchy on $\Psi DO(S^1, \mathbb{K})$ can then push-forward on \mathcal{FCl} -classes of operators by various ways:

- via identifications of $\Psi DO(S^1, \mathbb{K})$ with subalgebras or ideals of $\mathcal{FCl}(S^1, \mathbb{K})$, for $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or \mathbb{H} .
- by changing the standard multiplication of $\mathcal{FCl}(S^1, \mathbb{K})$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or \mathbb{H} , by “twisting it” by the operator $\epsilon(D)$ or $i\epsilon(D)$.
- via the almost quaternionic structures that we identified on $\mathcal{FCl}(S^1, \mathbb{C})$ in order to identify it with $\Psi DO(S^1, \mathbb{H})$

Let us describe in a detailed way these different approaches.

Push-Forward via $\Phi_{\lambda,\mu}$ maps. Let $\mathbb{K} = \mathbb{C}$ or \mathbb{H} . For each choice of $(\lambda, \mu) \in \mathbb{C}^2 \setminus \{0; 0\}$ identifies $\frac{d}{dx} \in \Psi DO(S^1, \mathbb{K})$ with an operator in $\mathcal{FCl}(S^1, \mathbb{K})$ with the same algebraic properties.

Notation: $\partial_{\lambda,\mu} = \Phi_{\lambda,\mu} \left(\frac{d}{dx} \right)$ and $\mathcal{FCl}_{\lambda,\mu}(S^1, \mathbb{K}) = \text{Im} \Phi_{\lambda,\mu}$.

Then we can develop the KP hierarchy on $\mathcal{FCl}_{\lambda,\mu}(S^1, \mathbb{K})$. We first remark that, since each map $\Phi_{\lambda,\mu}$ is a degree 0 morphism of filtered algebras, each push-forward of the unique solotion L of the KP hierachy (4) generates a solution of the corresponding equation in $\mathcal{FCl}(S^1, \mathbb{K})$ which reads the same way:

$$\frac{dL}{dt_k} = [(L^k)_D, L], \quad k \geq 1,$$

where solutions operators now belong to $\mathcal{FCl}^1(S^1, \mathbb{K})[[T]]$ and where each initial value $\Phi_{\lambda,\mu}(L_0) \in \partial_{\lambda,\mu} + \mathcal{FCl}_{\lambda,\mu}^{-1}(S^1, \mathbb{K})$ with obvious extension of notations. Therefore, for any initial value $L_0 \in \Psi DO(S^1, \mathbb{K})$, we get a family of operators

$$L_{\lambda,\mu} \in \mathcal{FCl}_{\lambda,\mu}^1(S^1, \mathbb{K})[[T]] \subset \mathcal{FCl}^1(S^1, \mathbb{K})[[T]]$$

parametrized by the complex parameters λ and μ chosen as before, which satisfies the KP hierarchy in $\mathcal{FCl}(S^1, \mathbb{K})$ and with initial values $\Phi_{\lambda,\mu}(L_0)$.

Existence, uniqueness and well-posedness of the KP system in $\mathcal{FCl}(S^1, \mathbb{K})$.

We adapt here the r -matrix approach for the construction of the solutions, along the lines of [10] with the following specific choices:

- The algebra of smooth coefficients for formal pseudo-differential operators is $\mathcal{R} = C^\infty(S^1, M_n(\mathbb{K})) \oplus \epsilon(D)C^\infty(S^1, M_n(\mathbb{K}))$ with multiplication rules inherited from $Cl(S^1, \mathbb{K}^n)$.
- The differential operator is $\partial = \frac{d}{dx}$.

Proposition 3.1. $\Psi DO(\mathcal{R}) = \mathcal{FCl}(S^1, \mathbb{K}^n)$ and there is an identification of the Manin triples $(\Psi DO(\mathcal{R}), DO(\mathcal{R}), IO(\mathcal{R}))$ with $(\mathcal{FCl}(S^1, \mathbb{K}^n), \mathcal{FCl}_D(S^1, \mathbb{K}^n), \mathcal{FCl}_S(S^1, \mathbb{K}^n))$.

Hence, applying the main result of [31] completed, for well-posedness, by [10, Theorem 4.1] or by [25, Theorem 4.1] when $\mathcal{R} = C^\infty(S^1, \mathbb{K}) = M_1(C^\infty(S^1, \mathbb{K}))$ is a commutative algebra, we can state the following:

Proposition 3.2. *The Kadomtsev-Petviashvili (KP) hierarchy (4) on $\Psi DO(\mathcal{R})$ (resp. $\mathcal{FCl}(S^1, \mathbb{K}^n)$) with initial condition $L(0) = L_0 \in \partial + \Psi DO^{-1}(\mathcal{R})$ (resp. $\in \partial + \mathcal{FCl}^{-1}(S^1, \mathbb{K}^n)$) satisfies Theorem 1.29.*

Remark 3.3. *We have used here, intrinsically, the integrable almost complex structure J_1 . Indeed, $\mathcal{R} = C^\infty(S^1, M_n(\mathbb{K})) + J_1 C^\infty(S^1, M_n(\mathbb{K}))$ is an algebra.*

Remark 3.4. *There exists another way to justify Proposition 3.2. One can use alternatively the splitting*

$$\mathcal{FCl}(S^1, \mathbb{K}^n) = \mathcal{FCl}_+(S^1, \mathbb{K}^n) \oplus \mathcal{FCl}_-(S^1, \mathbb{K}^n).$$

Then Equation (4) on $\mathcal{FCl}(S^1, \mathbb{K}^n)$ splits into two independent equations, similar to Equation (4) on $\mathcal{FCl}_\pm(S^1, \mathbb{K}^n)$. Through the identification maps $\Phi_{1,0}$ and $\Phi_{0,1}$ of $\mathcal{FCl}_\pm(S^1, \mathbb{K}^n)$ with $\Psi DO(S^1, \mathbb{K}^n)$, we get existence, uniqueness and well-posedness for Equation (4) on $\mathcal{FCl}(S^1, \mathbb{K}^n)$ with initaial value $L_0 \in \partial + \mathcal{FCl}^{-1}(S^1, \mathbb{K}^n)$.

From this last remark, we can generalize the identification procedure, changing the maps Φ_{ee} , $\Phi_{1,0}$ and $\Phi_{0,1}$ by the family of maps $\Phi_{\lambda,\mu}$.

Theorem 3.5. *Let $(\lambda, \mu) \in (\mathbb{C}^*)^2$. Then the KP equation (4) in $\mathcal{FCl}(S^1, \mathbb{K}^n)$ with initial value $L_0 \in \partial_{\lambda, \mu} + \mathcal{FCl}^{-1}(S^1, \mathbb{K}^n)$ has an unique solution L in $\partial_{\lambda, \mu} + \mathcal{FCl}^{-1}(S^1, \mathbb{K}^n)[[T]]$ and the problem is well-posed: the solution L depends smoothly on L_0 .*

Twisted KP hierarchy. Let us now change the standard multiplication on $\mathcal{FCl}(S^1, \mathbb{K})$ by $(A, B) \mapsto \epsilon AB$ where $\epsilon = \epsilon(D)$ or $a\epsilon(D)$ for any $a \in \mathbb{C}^*$. Since $\epsilon(D)$ commutes with any element of $\mathcal{FCl}(S^1, \mathbb{K})$ for the standard multiplication, this new multiplication defines a new algebra structure on $\mathcal{FCl}(S^1, \mathbb{K})$. When necessary we note by \circ the standard multiplication, and by \circ_ϵ the twisted one. Associated to this multiplication, we get the deformed Lie bracket $[\cdot, \cdot]_\epsilon$. Then we get again an equation similar to (4)

$$(5) \quad \frac{dL}{dt_k} = [\epsilon^{k-1}(L^k)_D, L]_\epsilon = \epsilon^k [(L^k)_D, L] , \quad k \geq 1 ,$$

where powers in this equation are taken with respect to \circ .

Theorem 3.6. *Let L_0 such that $L_0 \in \partial_{\lambda, \mu} + \mathcal{FCl}^{-1}(S^1, \mathbb{K}^n)$, with $(\lambda, \mu) \in (\mathbb{C}^*)^2$. Then the ϵ -KP hierarchy (5) with initial value L_0 has an unique solution. Moreover, the problem is well-posed.*

3.2. KP hierarchies with complex powers. We finally extend all the constructions of the last section to complex powers, along the lines of [11]. Let $\mathbb{K} = \mathbb{C}$ or \mathbb{H} . We consider an operator L_0 of complex order α such that

$$(6) \quad L_0 \in \left(\frac{d}{dx} \right)^\alpha + \mathcal{FCl}^{\alpha-1}(S^1, \mathbb{K})$$

or

$$(7) \quad L_0 \in |D|^\alpha + \mathcal{FCl}^{\alpha-1}(S^1, \mathbb{K})$$

For each setting (6) and (7), we define the complex KP hierarchy on $\mathcal{FCl}^\alpha(S^1, \mathbb{K})$ by

$$(8) \quad \frac{dL}{dt_k} = \left[(L^{k/\alpha})_D, L \right]_\epsilon = - \left[(L^k)_S, L \right] , \quad k \geq 1 ,$$

where $L^{k/\alpha} = \exp\left(\frac{k}{\alpha} \log L\right)$ and the solution $L \in \mathcal{FCl}^\alpha(S^1, \mathbb{K})[[T]]$.

Theorem 3.7. *The KP hierarchy (8) with initial value L_0 defined along the lines of (6) or (7) has an unique solution in $\mathcal{FCl}^\alpha(S^1, \mathbb{K})[[T]]$. Moreover, the problem is well-posed.*

4. HAMILTONIAN APPROACHES

Now we consider $\mathcal{FCl}(S^1, \mathbb{K}^n)$ and we define the *regular dual space*

$$\mathcal{FCl}(S^1, \mathbb{K}^n)' = \{ \mu \in L(\mathcal{FCl}(S^1, \mathbb{K}^n), \mathbb{K}) : \mu = \langle P, \cdot \rangle \text{ for some } P \in \mathcal{FCl}(S^1, \mathbb{K}^n) \} .$$

We can adapt standard results described in section 1.4 of Hamiltonian mechanics as follows: let $f : \mathcal{FCl}(S^1, \mathbb{K}^n)' \rightarrow B$ be a polynomial function of the type

$$f(\mu) = \sum_{k=0}^n a_k \text{res}_+(P^k) + \sum_{k=0}^n b_k \text{res}_-(P^k) = \text{res} \left(\sum_{k=0}^n a_k P_+^k + b_k P_-^k \right)$$

with $\mu = \langle P, \cdot \rangle$. In our picture, the decomposition $\mathcal{FCl}(S^1, \mathbb{K}^n) = \mathcal{FCl}_+(S^1, \mathbb{K}^n) \oplus \mathcal{FCl}_-(S^1, \mathbb{K}^n)$ that we use extensively all along this work carry a residue trace on

each component of the decomposition. These are these two residues, res_+ and res_- , that replace Res in the constructions of section 1.4. Under these assumptions, we define the same way the functional derivative and the pairing $\langle \cdot, \cdot \rangle$ of $\mathcal{FCl}(S^1, \mathbb{K}^n)'$ with $\mathcal{FCl}(S^1, \mathbb{K}^n)$. The decomposition $\mathcal{FCl}(S^1, \mathbb{K}^n) = \mathcal{FCl}_D(S^1, \mathbb{K}^n) \oplus \mathcal{FCl}_S(S^1, \mathbb{K}^n)$ allows us to consider a new Lie bracket on the regular dual space $\mathcal{FCl}(S^1, \mathbb{K}^n)'$ given by $[P, Q]_0 = [P_D, Q_D] - [P_S, Q_S]$, This bracket determines a new Poisson structure $\{, \}_0$ on $\mathcal{FCl}(S^1, \mathbb{K}^n)'$, simply by replacing the original Lie product for $[,]_0$. Using again the non-degenerate pairing we get:

Lemma 4.1. *Let $H : \mathcal{FCl}(S^1, \mathbb{K}^n)' \rightarrow \mathbb{K}$ be a smooth function on $\mathcal{FCl}(S^1, \mathbb{K}^n)'$ such that*

$$(9) \quad \left\langle \mu \left| \left[\frac{\delta H}{\delta \mu}, \cdot \right] \right. \right\rangle = 0 \quad \text{for all } \mu \in \mathcal{FCl}(S^1, \mathbb{K}^n)' .$$

Then, as equations on $\mathcal{FCl}(S^1, \mathbb{K}^n)$, the Hamiltonian equations of motion with respect to the $\{, \}_0$ Poisson structure of $\mathcal{FCl}(S^1, \mathbb{K}^n)''$ are

$$(10) \quad \frac{dP}{dt} = \left[\left(\frac{\delta H}{\delta \mu} \right)_+, P \right] .$$

We now use some specific functions H . Let us recall the following results (see for example [6] or the more recent review [9]):

Proposition 4.2. *We define the functions $H_k(L) = Trace((L^k))$, $k = 1, 2, 3, \dots$, for $L \in \mathcal{FCl}(S^1, \mathbb{K}^n)$. Then, $\frac{\delta H_k}{\delta L} = kL^{k-1}$. In particular, the functions H_k satisfy (9).*

Thus, we can apply Lemma 4.1. It yields:

Proposition 4.3. *Let us equip the Lie algebra $\mathcal{FCl}(S^1, \mathbb{K}^n)$ with the non-degenerate pairing $(a, b) \mapsto res(ab)$. Write $\mathcal{FCl}(S^1, \mathbb{K}^n) = \mathcal{FCl}_D(S^1, \mathbb{K}^n) \oplus \mathcal{FCl}_S(S^1, \mathbb{K}^n)$ and consider the Hamiltonian functions*

$$(11) \quad \mathcal{H}_k(\mu) = \frac{1}{k} res_W((L^{k+1}))$$

for $\mu = \langle L, \cdot \rangle$. The corresponding Hamiltonian equations of motion with respect to the $\{, \}_0$ Poisson structure of $\mathcal{FCl}(S^1, \mathbb{K}^n)'$ are $\frac{dL}{dt_k} = [(L^k)_D, L]$.

Following now [11] we get the Hamiltonian formulation of the KP hierarchy with complex powers: For this, we need to generalize the Gelfand-Dickii structure either to

$$\mathcal{L}_\alpha = \left(\frac{d}{dx} \right)^\alpha + \mathcal{FCl}^{\alpha-1}(S^1, \mathbb{K}^n)$$

or to

$$\mathcal{L}'_\alpha = \left| \frac{d}{dx} \right|^\alpha + \mathcal{FCl}^{\alpha-1}(S^1, \mathbb{K}^n).$$

In both case, we specialize our computations to $\mathcal{FCl}_+^\alpha(S^1, \mathbb{K}^n)$, and with $\mathcal{FCl}_-^\alpha(S^1, \mathbb{K}^n)$ respectively, which both identify with $\Psi DO^\alpha(S^1, \mathbb{K}^n)$. Under these identifications, the computations described in [11, pp 55–57]:

Theorem 4.4. *On \mathcal{L}_α and on \mathcal{L}'_α , the Hamiltonian vector field associated to $H_k = \frac{\alpha}{k} res L^{k/\alpha}$ is $V = \left[L_D^{k/\alpha}, L \right]$.*

5. APPENDIX: PROOFS

We collect in the Appendix all routine and technical proofs which are often nothing but a straightforward verification. The end of the Appendix contains also technical proofs of some theorems about KP hierarchies which are very similar ideologically of our ancient proofs from [11].

5.1. Proofs of section 1.

Lemma 1.22. Since res is non degenerate then $(.;.)_{s'}$ is non degenerate. Moreover, identifying $\mathcal{FCl}_+(S^1, V)$ and $\mathcal{FCl}_-(S^1, V)$ as two copies of $\Psi DO(S^1, V)$, writing by Tr the Adler trace on the latter one, we have that

$$res(A, s'(B)) = Tr(A_+ B_-) + Tr(A_- B_+) = Tr(B_+ A_-) + Tr(B_- A_+) = res(B, s'(A))$$

which proves symmetry. \square

Lemma 1.24. Since res is non degenerate then $(.;.)_s$ is non degenerate. Moreover,

$$\begin{aligned} res(A, s(B)) &= res((A_{ee} + A_{eo}) + (B_{ee} - B_{eo})) \\ &= res(A_{eo} B_{ee}) - res(A_{ee} B_{eo}) \\ &= -res(B, s(A)) \end{aligned}$$

which proves skewsymmetry.

$$\begin{aligned} res(A, s([B, C])) &= res((A_{ee} + A_{eo}) + ([B, C]_{ee} - [B, C]_{eo})) \\ &= res(A_{eo} B_{ee} C_{ee}) + res(A_{eo} B_{eo} C_{eo}) - res(A_{ee} B_{ee} C_{eo}) - res(A_{ee} B_{eo} C_{ee}) \\ &\quad - res(A_{eo} C_{ee} B_{ee}) - res(A_{eo} C_{eo} B_{eo}) + res(A_{ee} C_{ee} B_{eo}) + res(A_{ee} C_{eo} B_{ee}) \end{aligned}$$

while, with the same calculations,

$$\begin{aligned} res([A, C], s(B)) &= res(A_{eo} C_{ee} B_{ee}) + res(A_{ee} C_{eo} B_{ee}) - res(A_{ee} C_{ee} B_{eo}) - res(A_{eo} C_{eo} B_{eo}) \\ &\quad - res(A_{eo} B_{ee} C_{ee}) - res(A_{ee} B_{ee} C_{eo}) + res(A_{ee} B_{eo} C_{ee}) + res(A_{eo} B_{eo} C_{eo}) \end{aligned}$$

Let us investigate the same properties with $[\cdot, \cdot]_{\epsilon(D)}$:

$$\begin{aligned} res(A, s([B, C]_{\epsilon(D)})) &= res((A_{ee} + A_{eo})(-\epsilon(D)[B, C]_{ee} + \epsilon(D)[B, C]_{eo})) \\ &= res(\epsilon(D)A_{eo}B_{ee}C_{eo}) + res(\epsilon(D)A_{eo}B_{eo}C_{ee}) \\ &\quad - res(\epsilon(D)A_{ee}B_{eo}C_{eo}) - res(\epsilon(D)A_{ee}B_{ee}C_{ee}) \\ &\quad - res(\epsilon(D)A_{eo}C_{eo}B_{ee}) - res(\epsilon(D)A_{eo}C_{ee}B_{eo}) \\ &\quad + res(\epsilon(D)A_{ee}C_{eo}B_{eo}) + res(\epsilon(D)A_{ee}C_{ee}B_{ee}) \end{aligned}$$

while

$$\begin{aligned} res([A, C], s(B)) &= res(\epsilon(D)A_{ee}C_{ee}B_{ee}) + res(\epsilon(D)A_{eo}C_{eo}B_{ee}) \\ &\quad - res(\epsilon(D)A_{eo}C_{ee}B_{eo}) - res(\epsilon(D)A_{ee}C_{eo}B_{eo}) \\ &\quad - res(\epsilon(D)A_{ee}B_{ee}C_{ee}) - res(\epsilon(D)A_{eo}B_{ee}C_{eo}) \\ &\quad + res(\epsilon(D)A_{eo}B_{eo}C_{ee}) + res(\epsilon(D)A_{ee}B_{eo}C_{eo}) \end{aligned}$$

\square

Theorem 1.26. First, $\mathcal{FCl}_{ee}(S^1, V)$ is itself a subalgebra of $\mathcal{FCl}(S^1, V)$ hence $(\mathcal{FCl}_{ee}(S^1, V), [\cdot, \cdot])$ is a Lie subalgebra of $\mathcal{FCl}_{ee}(S^1, V)$ on which $res(A, B)$ satisfies the same well-known properties: bilinear and symmetric. Moreover, it is well-known that res is

non-degenerate on $\mathcal{FCl}(S^1, V)$. From [16], one can deduce by considering only formal operators that $\mathcal{FCl}_{ee}(S^1, V)$ is isotropic for res . Secondly, since $\epsilon(D)$ commutes with any element of $\mathcal{FCl}(S^1, V)$, we have that

$$\begin{aligned} \forall A, B, C \in \mathcal{FCl}_{eo}(S^1, V), \quad & [A, [B, C]_{\epsilon(D)}]_{\epsilon(D)} + [C, [A, B]_{\epsilon(D)}]_{\epsilon(D)} + [B, [C, A]_{\epsilon(D)}]_{\epsilon(D)} \\ & = \epsilon(D)^2 ([A, [B, C]] + [C, [A, B]] + [B, [C, A]]) = 0, \end{aligned}$$

which proves that $(\mathcal{FCl}_{eo}(S^1, V), [.,.]_{\epsilon(D)})$ is a Lie bracket. Moreover, $A \mapsto \epsilon(D)A$ is a vector space isomorphism from $\mathcal{FCl}_{eo}(S^1, V)$ to $\mathcal{FCl}_{ee}(S^1, V)$, which implies, with $\epsilon(D)^2 = 1$, that

$$\forall (A, B) \in \mathcal{FCl}_{eo}(S^1, V), \quad res(AB) = res((\epsilon(D)A)(\epsilon(D)B))$$

and shows that $\mathcal{FCl}_{eo}(S^1, V)$ is isotropic for $res(AB)$. Let us finish the proof with invariance on res . Invariance with respect to $[.,.]$ in $\mathcal{FCl}(S^1, V)$ is well-known since res is tracial. Again since $\epsilon(D)$ commutes, we have that $\forall (A, B) \in \mathcal{FCl}(S^1, V)$, $[A, B]_{\epsilon(D)} = [\epsilon(D)A, B] = [A, \epsilon(D)B]$ hence for $(A, B, C) \in \mathcal{FCl}(S^1, V)^3$,

$$\begin{aligned} res([A, B]_{\epsilon(D)}C) & = res([\epsilon(D)A, B]C) \\ & = res(B[C, \epsilon(D)A]) \\ & = -res(B[A, C]_{\epsilon(D)}). \end{aligned}$$

□

5.2. Proofs of section 2.2.

5.2.1. *Proof of Theorem 2.6.* The operator $i\epsilon(D)$ commutes with any operator $u \in \mathcal{FCl}(S^1, V)$. By the way, we simplify the relation that can be found e.g. in [27] the following way: $[u, J_1v] + [J_1u, v] = 2J_1[u, v]$ and

$$J_1[u, v] - J_1[J_1u, J_1v] = J_1[u, v] - J_1^3[u, v] = 2J_1[u, v]$$

Hence

$$[u, J_1(v)] + [J_1(u), v] = J_1[u, v] - J_1[J_1(u), J_1(v)]$$

which proves integrability.

Lemma 5.1. *We have $J_1(\mathcal{FCl}_{ee}(S^1, V)) = \mathcal{FCl}_{eo}(S^1, V)$ and $J_1(\mathcal{FCl}_{eo}(S^1, V)) = \mathcal{FCl}_{ee}(S^1, V)$*

Proof. Since $J = i\epsilon(D) \circ (\cdot)$ it follows from the composition rules between even-even and even-odd class already described. □

Identifying $\mathcal{FCl}_{eo}(S^1, V)$ with $\epsilon(D)\mathcal{FCl}_{ee}(S^1, V)$, we recall that

$$\mathcal{FCl}(S^1, V) = \mathcal{FCl}_{ee}(S^1, V) \oplus \epsilon(D)\mathcal{FCl}_{ee}(S^1, V),$$

we get the complexification result.

5.2.2. *Other proofs.*

Proposition 2.7. By straightforward computations, we check first that $s(\epsilon(D)) = -\epsilon(D)$. Then, since composition of symbols by $\epsilon(D)$ is only pointwise multiplication, we get, for $a \in \mathcal{FCl}(S^1, V)$,

$$\begin{aligned}
J_2 \circ J_1(a)(x, \xi) &= -s(\epsilon(D) \circ a)(x, \xi) \\
&= -\epsilon(D)(x, -\xi) \left(\sum_{k \in \mathbb{Z}} (-1)^k a_k(x, -\xi) \right) \text{ (pointwise multiplication)} \\
&= \epsilon(D)(x, \xi) s(a)(x, \xi) \text{ (pointwise multiplication)} \\
&= -i\epsilon(D) \circ (is(a))(x, \xi) \\
&= -J_1 \circ J_2(a)(x, \xi)
\end{aligned}$$

□

Theorem 2.8.

$$\begin{aligned}
[u, J_2 v] + [J_2 u, v] &= i[u, v_{ee}] - i[u, v_{eo}] + i[u_{ee}, v] - i[u_{eo}, v] \\
&= i([u_{ee}, v_{ee}] + [u_{eo}, v_{ee}] - [u_{ee}, v_{eo}] - [u_{eo}, v_{eo}]) \\
&\quad + [u_{ee}, v_{ee}] + [u_{ee}, v_{eo}] - [u_{eo}, v_{ee}] - [u_{eo}, v_{eo}] \\
&= 2i([u_{ee}, v_{ee}] - [u_{eo}, v_{eo}])
\end{aligned}$$

and

$$\begin{aligned}
J_2[u, v] - J_2[J_2 u, J_2 v] &= i([u_{ee}, v_{ee}] + [u_{eo}, v_{eo}]) - i([u_{ee}, v_{eo}] + [u_{eo}, v_{ee}]) \\
&\quad + i([u_{ee}, v_{ee}] + [-u_{eo}, -v_{eo}]) - i([u_{ee}, -v_{eo}] + [-u_{eo}, v_{ee}]) \\
&= 2i([u_{ee}, v_{ee}] + [u_{eo}, v_{eo}])
\end{aligned}$$

As a counter-example, let $X = f(x)\partial$ and let $Y = g(x)\partial$ be two vector fields over S^1 such that $[X, Y] \neq 0$. Let $u = \epsilon(D)X \in \mathcal{FCl}_{eo}(S^1, \mathbb{R})$ and let $v = \epsilon(D)Y \in \mathcal{FCl}_{eo}(S^1, \mathbb{R})$. Then $J_2[u, v] - J_2[J_2 u, J_2 v] = 2i[X, Y]$ while $[u, J_2 v] + [J_2 u, v] = -2i[X, Y]$. □

Proposition 2.9. By straightforward computations, we check first that $s'(\epsilon(D)) = -\epsilon(D)$. Then, since s' is a morphism of algebra,

$$\begin{aligned}
J_3 \circ J_1(a) &= -s'(\epsilon(D) \circ (a_+, a_-)) = -(-a_-, a_+) \\
&= -\epsilon(D) \circ (-a_-, -a_+) = \epsilon(D) \circ s'(a_+, a_-) = -J_1 \circ J_3(a)
\end{aligned}$$

□

Proposition 2.10. By straightforward computations, we check first that $ss' = s's$. Then, $J_2 J_3 = J_3 J_2$. □

Theorem 2.11. $[u, J_3 v] + [J_3 u, v] = i([u_+ v_-] + [u_- v_+], [u_+ v_-] + [u_- v_+])$ and $J_2[u, v] - J_2[J_2 u, J_2 v] = i([u_+, v_+] + [u_-, v_-], [u_+, v_+] + [u_-, v_-])$. As a counter-example, let $X = f(x)\partial$ and let $Y = g(x)\partial$ be two vector fields over S^1 such that $[X, Y] \neq 0$. Let $u = X_+ \in \mathcal{FCl}_+(S^1, \mathbb{R})$ and let $v = Y_+ \in \mathcal{FCl}_+(S^1, \mathbb{R})$. Then $J_3[u, v] - J_2[J_3 u, J_3 v] = i[X, Y]_+$ while $[u, J_3 v] + [J_3 u, v] = 0$. □

Proposition 2.12. • $J_4^2 = J_1 J_3 J_1 J_3 = -J_3 J_1 J_1 J_3 = -Id$
• $J_2 J_4 = J_2 J_1 J_3 = -J_1 J_2 J_3 = -J_1 J_3 J_2 = -J_4 J_2$

$$\bullet J_1 J_4 = J_1 J_1 J_3 = -J_1 J_3 J_1 = -J_3 J_1$$

□

5.3. Proofs of section 3.

Proof of Proposition 3.1. From

$$\mathcal{FCl}(S^1, \mathbb{K}^n) = \mathcal{FCl}_{ee}(S^1, \mathbb{K}^n) \oplus \mathcal{FCl}_{eo}(S^1, \mathbb{K}^n) = \mathcal{FCl}_{ee}(S^1, \mathbb{K}^n) \oplus i\epsilon(D)\mathcal{FCl}_{ee}(S^1, \mathbb{K}^n)$$

we get, for $A = A_{ee} + A_{eo} \in \mathcal{FCl}_{ee}(S^1, \mathbb{K}^n) \oplus \mathcal{FCl}_{eo}(S^1, \mathbb{K}^n)$, and for $k \in \mathbb{Z}$,

$$\begin{aligned} \sigma_k A &= \sigma_k(A_{ee}) + \sigma_k(A_{eo}) = a_{k,ee} \frac{d^k}{dx} + a_{k,eo} i\epsilon(D) \frac{d^k}{dx} \\ &= (a_{k,ee} + i\epsilon(D)a_{k,eo}) \partial^k \end{aligned}$$

(where $(a_{k,ee}, a_{k,eo}) \in C^\infty(S^1, M_n(\mathbb{K}))$) which ends the identification of $\mathcal{FCl}(S^1, \mathbb{K}^n)$ with $\Psi DO(\mathcal{R})$. Since the order of partial symbols is conserved, we get the same identifications between $\mathcal{FCl}_D(S^1, \mathbb{K}^n)$ and $DO(\mathcal{R})$, and between $\mathcal{FCl}_S(S^1, \mathbb{K}^n)$ and $IO(\mathcal{R})$. □

Proof of Theorem 3.5. We analyze separately the equation on $\mathcal{FCl}_+(S^1, \mathbb{K})$ and on $\mathcal{FCl}_-(S^1, \mathbb{K})$. Let us work on $\mathcal{FCl}_+(S^1, \mathbb{K})$. The map $\Phi_{1,0}$ pulls-back the KP hierarchy on $\Psi DO(S^1, \mathbb{K})$ with initial value $L_0 \in \lambda\partial + \Psi DO(S^1, \mathbb{K})$. When $\lambda \neq 1$, the classical integration of the KP hierarchy is not achieved by the classical method. However, we use here the scaling first defined to our knowledge in [23]. Let $q = \lambda^{-1}$. Let $\tilde{L}_0 = qL_0$. Then $\tilde{L}_0 \in \lambda\partial + \Psi DO(S^1, \mathbb{K})$, and there exists a Sato operator $S_0 \in 1 + \Psi DO^{-1}(S^1, \mathbb{K})$ such that $\tilde{L}_0 = S_0 \partial S_0^{-1}$ and the KP-system with initial value \tilde{L}_0 has a unique solution \tilde{L} . We define a λ -scaling in time: $t_k \mapsto \lambda^k t_k$. Following [23],

$$\tilde{L}(t_1, t_2, \dots) \text{ is solution of (4)} \Leftrightarrow L(t_1, t_2, \dots) = \lambda \tilde{L}(\lambda t_1, \lambda^2 t_2, \dots) \text{ is solution of (4).}$$

The initial value of the solution L is L_0 , which proves existence, uniqueness and smooth dependence of L on L_0 . Then, we can push-forward the solution L of (4) on $\Psi DO(S^1, \mathbb{K})$ to the solution $L_+ = \Phi_{1,0}(L)$ on $\mathcal{FCl}_+(S^1, \mathbb{K})$. The same procedure holds to get the solution L_- on $\mathcal{FCl}_-(S^1, \mathbb{K})$, replacing the constant λ by the constant μ . The operator $L_+ + L_-$ furnishes the desired solution of (4) on $\mathcal{FCl}(S^1, \mathbb{K})$, which is unique and smoothly dependent on the initial value by construction. □

Proof of Theorem 3.6. Let us transform slightly Equation 5 for $k \in \mathbb{N}^*$:

$$\frac{dL}{dt_k} = \epsilon^k [(L^k)_D, L] \Leftrightarrow \epsilon \frac{dL}{dt_k} = \epsilon^{k+1} [(L^k)_D, L] \Leftrightarrow \frac{d(\epsilon L)}{dt_k} = [((\epsilon L)^k)_D, (\epsilon L)]$$

By the way, the field of operators ϵL , with initial value $\epsilon L_0 \in \partial_{a\lambda, -a\mu} + \mathcal{FCl}^{-1}(S^1, \mathbb{K}^n)$, is the unique solution of the KP hierarchy (4). Moreover, the map $L \mapsto \epsilon L$ is smooth, biunivoque, with smooth inverse, which ends the proof. □

Proof of Theorem 3.7. Let us first analyze case (6). Then

$$L_0^{1/\alpha} = \exp\left(\frac{1}{\alpha} \log L\right) \in \frac{d}{dx} + \mathcal{FCl}^0(S^1, \mathbb{K}).$$

We can then adapt [31] and define the dressing operator

$$U = \exp\left(\sum_{k \in \mathbb{N}^*} t_k (L_0^{1/\alpha})^k\right)$$

that we decompose into $U = U_+ + U_-$ where

$$U_{\pm} = \exp \left(\sum_{k \in \mathbb{N}^*} t_k (L_0^{1/\alpha})_{\pm}^k \right).$$

Working independently on the two components $\mathcal{FCl}_{\pm}(S^1, \mathbb{K})$, Mulase factorization holds, $U_{\pm} = S_{\pm}^{-1} Y_{\pm}$ and setting $L_{\pm} = Y_{\pm}(L_0)_{\pm} Y_{\pm}^{-1}$, which implies that $\forall k \in \mathbb{N}^*$, $L_{\pm}^{k/\alpha} = Y_{\pm}(L_0)_{\pm}^{k/\alpha} Y_{\pm}^{-1}$. We moreover have that $\forall \beta \in \mathbb{C}^*$, $L_0^{\beta} U = U L_0^{\beta}$ which implies that

$$L_{\pm} = Y_{\pm}(L_0)_{\pm} Y_{\pm}^{-1} = S_{\pm} S_{\pm}^{-1} Y_{\pm}(L_0)_{\pm} Y_{\pm}^{-1} S_{\pm} S_{\pm}^{-1} = S_{\pm}(L_0)_{\pm} S_{\pm}^{-1}$$

and similarly $L_{\pm}^{k/\alpha} = Y_{\pm}(L_0)_{\pm}^{k/\alpha} Y_{\pm}^{-1} = S_{\pm}(L_0)_{\pm}^{k/\alpha} S_{\pm}^{-1}$. We can now differentiate U

$$\frac{dU_{\pm}}{dt_k} = (L_0)_{\pm}^{k/\alpha} U_{\pm} = -S_{\pm}^{-1} \frac{dS_{\pm}}{dt_k} S_{\pm}^{-1} Y_{\pm} + S_{\pm}^{-1} \frac{dY_{\pm}}{dt_k}$$

which implies that

$$S_{\pm}(L_0)_{\pm}^{k/\alpha} S_{\pm}^{-1} = -\frac{dS_{\pm}}{dt_k} S_{\pm}^{-1} + \frac{dY_{\pm}}{dt_k} Y_{\pm}.$$

By the way, $(L_{\pm}^{k/\alpha})_D = \frac{dY_{\pm}}{dt_k} Y_{\pm}$ and hence

$$\begin{aligned} \frac{dL_{\pm}}{dt_k} &= \frac{dY_{\pm}(L_0)_{\pm} Y_{\pm}^{-1}}{dt_k} = \frac{dY_{\pm}}{dt_k} (L_0)_{\pm} Y_{\pm}^{-1} - Y_{\pm}(L_0)_{\pm} Y_{\pm}^{-1} \frac{dY_{\pm}}{dt_k} Y_{\pm}^{-1} \\ &= \left(L_{\pm}^{k/\alpha} \right)_D Y_{\pm}(L_0)_{\pm} Y_{\pm}^{-1} - Y_{\pm}(L_0)_{\pm} Y_{\pm}^{-1} \left(L_{\pm}^{k/\alpha} \right)_D \\ &= \left[\left(L_{\pm}^{k/\alpha} \right)_D, Y_{\pm}(L_0)_{\pm} Y_{\pm}^{-1} \right] \end{aligned}$$

Gathering the \pm parts, we get that $L = L_+ + L_-$ is a solution of (8) with initial condition (6). Let us now deal with initial condition (7). In that case,

$$L_0^{1/\alpha} \in |D| + \mathcal{FCl}_0(S^1, \mathbb{K}) = i\epsilon(D) \left(\frac{d}{dx} + \mathcal{FCl}_0(S^1, \mathbb{K}) \right) = i\epsilon(D) \frac{d}{dx} + \mathcal{FCl}_0(S^1, \mathbb{K}).$$

Let U be the dressing operator with respect to L_0 along the lines of the previous computations, that we decompose into the

$$U = U_+ + U_-$$

in the $\mathcal{FCl}_+(S^1, \mathbb{K})$ and $\mathcal{FCl}_-(S^1, \mathbb{K})$ -components. Then U_+ and U_- decompose in the Mulase decomposition and we can re-construct two operators S and Y in $\mathcal{FCl}(S^1, \mathbb{K})$ from this decomposition. We set $L = Y L_0 Y^{-1} = S L_0 S^{-1}$. Then, with the same computation as before,

$$\begin{aligned} \frac{dL}{dt_k} &= \frac{dY L_0 Y^{-1}}{dt_k} = \frac{dY}{dt_k} L_0 Y^{-1} - Y L_0 Y_{\pm}^{-1} \frac{dY}{dt_k} Y^{-1} \\ &= \left(L^{k/\alpha} \right)_D Y L_0 Y^{-1} - Y L_0 Y^{-1} \left(L^{k/\alpha} \right)_D = \left[\left(L^{k/\alpha} \right)_D, Y L_0 Y^{-1} \right] \end{aligned}$$

□

REFERENCES

- [1] Adler, M.; On a trace functional for formal pseudo-differential operators and the symplectic structure of Korteweg-de Vries type equations *Inventiones Math.* **50** 219-248 (1979)
- [2] Berline, N.; Getzler, E.; Vergne, M.; *Heat Kernels and Dirac Operators* Springer (2004)
- [3] Bilal, A.; Non-local matrix generalizations of W -algebras. *Comm. Math. Phys.* (1995), **170**, **1**, 117–150. ;
- [4] Bokobza-Haggiag, J.; Opérateurs pseudo-différentiels sur une variété différentiable; *Ann. Inst. Fourier, Grenoble* **19**,**1** 125-177 (1969)
- [5] Cardona, A.; Ducourtioux, C.; Magnot, J-P.; Paycha, S.; Weighted traces on pseudo-differential operators and geometry on loop groups; *Inf. Dimens. Anal. Quantum Probab. Relat. Top.* **5** no. 4 503-541 (2002)
- [6] Dickey, L.A.; *Soliton equations and Hamiltonian systems, second edition* (2003).
- [7] Dieudonné, J.; *Eléments d'analyse* Gauthier-Villars, Paris (1978)
- [8] Drinfeld, V.; On some unsolved problems in quantum group theory in: Kulish P.P. (eds) *IQuantum Groups*, Lecture Notes in Mathematics, **1510**, 1-8(1992), Springer, Berlin, Heidelberg. <https://doi.org/10.1007/BFb0101175>,
- [9] Eslami Rad, A.; Reyes, E. G.; The Kadomtsev-Petviashvili hierarchy and the Mulase factorization of formal Lie groups *J. Geom. Mech.* **5**, no 3 (2013) 345–363.
- [10] Eslami Rad, A.; Magnot, J.-P.; Reyes, E. G.; The Cauchy problem of the Kadomtsev-Petviashvili hierarchy with arbitrary coefficient algebra. *J. Nonlinear Math. Phys.* 24:sup1 (2017), 103–120.
- [11] Enriquez, B.; Khoroshkin, S.; Radul, A.; Rosly, A.; Rubtsov, V.; Poisson-Lie aspects of classical W -algebras; in: *The interplay between Differential geometry and differential equations Amer. Math. Soc. Transl. (2)* **167**, 37-59 (1995)
- [12] Figueroa-O'Farill, J.; Mas, J.; Ramos, E.; A one-parameter family of Hamiltonian structures for the KP Hierarchy and a Continuous deformation of the W_{KP} algebra. *Comm. Math. Phys.* **158** 17–43 (1993)
- [13] Gilkey, P.; *Invariance theory, the heat equation and the Atiyah-Singer index theorem* Publish or Perish (1984)
- [14] Mc Intosh, I.; The quaternionic KP hierarchy and conformally immersed 2-tori on the 4-sphere. *Tohoku Math. J.* **63** 183-215 (2011)
- [15] Kassel, Ch.; Le résidu non commutatif (d'après M. Wodzicki) Séminaire Bourbaki, Vol. 1988/89. *Astérisque* **177-178**, Exp. No. 708, 199-229 (1989)
- [16] Kontsevich, M.; Vishik, S.; Determinants of elliptic pseudo-differential operators Max Plank Institut fur Mathematik, Bonn, Germany, preprint n. 94-30 (1994)
- [17] Kontsevich, M.; Vishik, S.; Geometry of determinants of elliptic operators. Functional analysis on the eve of the 21st century, Vol. 1 (New Brunswick, NJ, 1993), *Progr. Math.* **131**,173-197 (1995)
- [18] Khesin, B.; Wendt, R.; *The Geometry of Infinite-Dimensional Groups* Springer Verlag (2009)
- [19] Khesin, B.A. and Zakharevich, I.; Poisson-Lie groups of pseudo-differential symbols, *Comm. Math. Phys.* **171** no. 3 (1995), 475–530.
- [20] Kupersmidt, B.A.; *KP or mKP: Mathematics of Lagrangian, Hamiltonian and Integrable Systems*. Math. Surv. Monographs **78** (2000)
- [21] Magnot, J-P.; The Kähler form on the loop group and the Radul cocycle on Pseudo-differential Operators; *GROUP'24: Physical and Mathematical aspects of symmetries*, Proceedings of the 24th International Colloquium on Group Theoretical Methods in Physics, Paris, France, 15-20 July 2002; Institut of Physic conferences Publishing **173**, 671-675, IOP Bristol and Philadelphia (2003)
- [22] Magnot, J-P.; Renormalized traces and cocycles on the algebra of S^1 -pseudo-differential operators; *Lett. Math. Phys.* **75** no2, 111-127 (2006)
- [23] Magnot, J-P.; Ambrose-Singer theorem on diffeological bundles and complete integrability of KP equations. *Int. J. Geom. Meth. Mod. Phys.* **10**, no 9 (2013) Article ID 1350043.
- [24] Magnot, J-P.; On $\text{Diff}(M)$ -pseudo-differential operators and the geometry of non linear Grassmannians. *Mathematics* **4**, 1; doi:10.3390/math4010001 (2016)
- [25] Magnot, J-P.; Reyes, E. G.; Well-posedness of the Kadomtsev-Petviashvili hierarchy, Mulase factorization, and Frölicher Lie groups. *Ann. H. Poincaré* (in press)

- [26] Magnot, J-P; Reyes E. G.; *Diff₊(S¹)*-pseudo-differential operators and the Kadomtsev-Petviashvili hierarchy [ArXiv:1808.03791](#)
- [27] Malgrange, B.; Sur l'intégrabilité des structures presque-complexes *Séminaire Jean Leray* **1**, 1-8 (1968-1969)
- [28] Mulase, M.; Complete integrability of the Kadomtsev-Petvishvili equation. *Advances in Math.* **54** (1984), 57–66.
- [29] Mulase, M.; Solvability of the super KP equation and a generalization of the Birkhoff decomposition. *Invent. Math.* **92** (1988), 1–46.
- [30] Paycha, S; *Regularised integrals, sums and traces. An analytic point of view.* University Lecture Series **59**, AMS (2012).
- [31] Reyman, A.G.; Semenov-Tian-Shansky, M.A.; Reduction of Hamiltonian Systems, Affine Lie Algebras and Lax Equations II, *Invent. Math.* **63** (1981), 423–432.
- [32] Scott, S.; *Traces and determinants of pseudo-differential operators*; OUP (2010)
- [33] Seeley, R.T.; Complex powers of an elliptic operator *AMS Proc. Symp. Pure Math.* **10**, 288-307 (1968)
- [34] Widom, H.; A complete symbolic calculus for pseudo-differential operators; *Bull. Sc. Math. 2e serie* **104** (1980) 19-63
- [35] Wodzicki, M.; Local invariants in spectral asymmetry *Inv. Math.* **75**, 143-178 (1984)

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