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Non-autonomous maximal regularity under Besov regularity in time

Mahdi Achache *

Abstract

We consider the maximal regularity problem for the non-autonomous Cauchy problems

$$u'(t) + \mathcal{A}(t)u(t) = f(t) \quad t\text{-a.e.}, \quad u(0) = u_0. \quad (0.1)$$

In this case, the time dependent operators $\mathcal{A}(t)$ are associated with a family of sesquilinear forms. We prove maximal L^p -regularity results with $p \leq 2$ under minimal regularity assumptions on the forms. Our main assumption is that $(\mathcal{A}(t))_{t \in [0, \tau]}$ are piecewise in the Besov space $B_p^{\frac{1}{2}, 2}$ with respect to the variable t . This regularity assumption is optimal and our result is the most general one.

Keywords: Besov space, maximal regularity, non-autonomous evolution equations, sesquilinear forms.

Mathematics Subject Classification (2010): 35K90, 35K45, 47D06.

1 Introduction

The present paper deals with maximal L^p -regularity for non-autonomous evolution equations in the setting of Hilbert spaces. Before explaining our results, we introduce some notations and assumptions.

Let $(\mathcal{H}, (\cdot, \cdot), \|\cdot\|)$ be a Hilbert space over \mathbb{R} or \mathbb{C} . We consider \mathcal{V} , another Hilbert space, which is densely and continuously embedded into \mathcal{H} . We denote by \mathcal{V}' the (anti-) dual space of \mathcal{V} , so that

$$\mathcal{V} \hookrightarrow_d \mathcal{H} \hookrightarrow_d \mathcal{V}'.$$

Let us called $\langle \cdot, \cdot \rangle$ the duality \mathcal{V} - \mathcal{V}' and noting that $\langle \psi, v \rangle = (\psi, v)$ if $\psi, v \in \mathcal{H}$. We consider a family of sesquilinear forms

$$\mathbf{a} : [0, \tau] \times \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{C}$$

such that

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- [H1]: $D(\mathbf{a}(t)) = \mathcal{V}$ (constant form domain),
- [H2]: $|\mathbf{a}(t, u, v)| \leq M\|u\|_{\mathcal{V}}\|v\|_{\mathcal{V}}$ (uniform boundedness),
- [H3]: $\operatorname{Re} \mathbf{a}(t, u, u) + \nu\|u\|^2 \geq \delta\|u\|_{\mathcal{V}}^2$ ($\forall u \in \mathcal{V}$) for some $\delta > 0$.

To each form $\mathbf{a}(t)$ we can associate two operators $A(t)$ and $\mathcal{A}(t)$ on \mathcal{H} and \mathcal{V}' , respectively. Recall that $u \in \mathcal{H}$ is in the domain $D(A(t))$ if there exists $h \in \mathcal{H}$ such that for all $v \in \mathcal{V}$: $\mathbf{a}(t, u, v) = (h, v)$. We then set $A(t)u := h$. The operator $\mathcal{A}(t)$ is a bounded operator from \mathcal{V} into \mathcal{V}' such that $\mathcal{A}(t)u = \mathbf{a}(t, u, \cdot)$. The operator $A(t)$ is the part of $\mathcal{A}(t)$ on \mathcal{H} . It is a classical fact that $-A(t)$ and $-\mathcal{A}(t)$ are both generators of holomorphic semigroups $(e^{-rA(t)})_{r \geq 0}$ and $(e^{-r\mathcal{A}(t)})_{r \geq 0}$ on \mathcal{H} and \mathcal{V}' , respectively. The semigroup $e^{-rA(t)}$ is the restriction of $e^{-r\mathcal{A}(t)}$ to \mathcal{H} . In addition, $e^{-rA(t)}$ induces a holomorphic semigroup on \mathcal{V} (see, e.g. Ouhabaz [26, Chapter 1]). We consider the non-homogeneous Cauchy problem

$$u'(t) + A(t)u(t) = f(t) \text{ } t\text{-a.e.}, \quad u(0) = u_0. \quad (\text{P})$$

The notion of maximal L^p -regularity for the above problem is defined as follows

Definition 1.1. *We say that the Cauchy problem (P) (or the family $(A(t))_{t \in [0, \tau]}$) has maximal L^p -regularity in a Banach space E if for every $f \in L^p(0, \tau; E)$, there exists a unique $u \in W^{1,p}(0, \tau; E)$ with $u(t) \in D(A(t))$ for a.e. $t \in [0, \tau]$ and u is a solution of (P) in the L^p -sense.*

The maximal regularity of an evolution equation on a Banach space E depends on the operators involved in the equation, the space E and the initial data u_0 . The initial data has to be in appropriate space. In the autonomous case, i.e. $\mathcal{A}(t) = \mathcal{A}(0)$ for all $t \in [0, \tau]$, maximal L^p -regularity is well understood and it is also known that u_0 has to be in the real interpolation space $(E, D(A(0)))_{1-\frac{1}{p}, p}$, see .

The maximal regularity is very useful for the study of quasilinear partial differential equations as it allows the application of powerful linearization techniques (see example).

A very famous and well-known result of J.L.Lions (see [?] (XVIII Chapter 3, p. 513)) states that the non-autonomous Cauchy problem (P) is well-posed with maximal L^2 -regularity in \mathcal{V}' , i.e. for every $f \in L^2(0, \tau; \mathcal{V}')$ and $u_0 \in \mathcal{H}$, there exists a unique $u \in H^1(0, \tau; \mathcal{V}') \cap L^2(0, \tau; \mathcal{V}) \hookrightarrow C([0, \tau]; \mathcal{H})$ that solves (P). In this result, only measurability of $t \rightarrow \mathbf{a}(t, \cdot, \cdot)$ respected to the time variable is required. Nevertheless, in applications to boundary valued problems this is not sufficient. One is more interested in maximal L^p -regularity in \mathcal{H} rather than in \mathcal{V}' , i.e. in solutions which belong to $W^{1,p}(0, \tau; \mathcal{H})$ rather than in $W^{1,p}(0, \tau; \mathcal{V}')$. Lions himself proved maximal L^2 -regularity in \mathcal{H} if the form $\mathbf{a}(t, \cdot, \cdot)$ is symmetric for all $t \in [0, \tau]$

and $t \rightarrow \mathbf{a}(t, u, v) \in C^1([0, \tau])$ for all $u, v \in \mathcal{V}$ (see [?]). Using a different approach, maximal L^p -regularity was established in [25], assuming that $t \rightarrow \mathbf{a}(t, u, v) \in C^\alpha([0, \tau])$ for all $u, v \in \mathcal{V}$, for some $\alpha > \frac{1}{2}$. This result was further improved in [18], where the Hölder condition is replaced by a weaker "Dini" condition for $\mathbf{a}(\cdot, \cdot, \cdot)$.

For maximal L^2 -regularity, this result was improved to the fractional Sobolev regularity $t \rightarrow \mathcal{A}(t) \in H^{\frac{1}{2}+\alpha}([0, \tau]; \mathcal{L}(\mathcal{V}, \mathcal{V}'))$ for $\alpha > \frac{1}{2}$ (see [15]). The proof is surprisingly elementary and based on the Lax-Milgram lemma. Furthermore, it is proved in [3] that maximal L^2 -regularity holds if $t \rightarrow \mathcal{A}(t) \in H^{\frac{1}{2}}([0, \tau]; \mathcal{L}(\mathcal{V}, \mathcal{V}'))$ (with some integrability conditions). Fackler [16] on the other hand was able to construct a symmetric non-autonomous form that is α -Hölder continuous for every $\alpha \leq \frac{1}{2}$ but does not have the maximal L^2 -regularity in \mathcal{H} . Fackler [?], generalized the result in [15] for any $p \in (1, \infty)$ by assuming fractional Sobolev regularity. In fact, he proved that maximal L^p -regularity is satisfied if

- (i) $\mathcal{A}(\cdot) \in \dot{H}^{\frac{1}{2}+\epsilon}(0, \tau; \mathcal{L}(\mathcal{V}, \mathcal{V}'))$ for $p \leq 2$
- (ii) $\mathcal{A}(\cdot) \in \dot{W}^{\frac{1}{2}+\epsilon, p}(0, \tau; \mathcal{L}(\mathcal{V}, \mathcal{V}'))$ for $p > 2$.

For more details and progresses on maximal regularity, we refer to the recent papers in [3] or [?].

The main purposes of the present paper is to extend the results in [?] for the case $p \leq 2$ by assuming less regularity as much possible. Our main result shows that for forms satisfying the uniform Kato square root property and if $t \rightarrow \mathcal{A}(t)$ is piecewise in the Besov space $B_p^{\frac{\gamma}{2}, 2}(0, \tau; \mathcal{L}(\mathcal{V}, \mathcal{V}'_\gamma))$, $\gamma \in [0, 1]$, then maximal L^p -regularity in \mathcal{H} is satisfied. Here, $\mathcal{V}_\gamma = [\mathcal{H}, \mathcal{V}]_\gamma$ is the interpolation space between \mathcal{H} and \mathcal{V} . The initial data u_0 is arbitrary in the interpolation space $(\mathcal{H}, D(A(0)))_{1-\frac{1}{p}, p}$.

The structure of paper is organized as follows: in the second Section we work with the autonomous first order Cauchy problem and prove some preparatory results. After that, we prove maximal L^p -regularity in Section 3 and discuss the optimality in Section 4, while the section 5 is an appendix.

Notation. We denote by $\mathcal{L}(E, F)$ (or $\mathcal{L}(E)$) the space of bounded linear operators from E to F (from E to E). The spaces $L^p(a, b; E)$ and $W^{1,p}(a, b; E)$ denote respectively the Lebesgue and Sobolev spaces of function on (a, b) with values in E . Recall that the norms of \mathcal{H} and \mathcal{V} are denoted by $\|\cdot\|$ and $\|\cdot\|_{\mathcal{V}}$. The scalar product of \mathcal{H} is (\cdot, \cdot) .

Finally, we denote by C, C' or $c\dots$ all inessential constants. Their values may change from line to line.

2 Main results

In this section we state explicitly our main results. We start by recalling the definition of vector-valued Besov spaces.

Definition 2.1. Let X be a Banach space, let $p, q \in [1, \infty]$, and $\alpha \in (0, 1)$, A Bochner measurable function $f : [0, \tau] \rightarrow X$, is in the homogeneous Besov space $\dot{B}_q^{\alpha,p}(0, \tau; X)$ if

$$\|f\|_{\dot{B}_q^{\alpha,p}(0,\tau;X)}^q = \int_0^\tau \frac{1}{l^{\alpha q}} \left(\int_l^\tau \|f(t) - f(t-l)\|_X^p dt \right)^{\frac{q}{p}} \frac{dl}{l} < \infty.$$

A function $f \in L^p(0, \tau; X)$ is in the Besov space $B_q^{\alpha,p}(0, \tau; X)$ if

$$\|f\|_{B_q^{\alpha,p}(0,\tau;X)}^q := \|f\|_{L^p(0,\tau;X)}^q + \|f\|_{\dot{B}_q^{\alpha,p}(0,\tau;X)}^q < \infty.$$

We shall say that f is piecewise in $B_q^{\alpha,p}(0, \tau; X)$ if there exists $0 = t_0 < t_1 < \dots < t_n = \tau$ such that the restriction of f to each sub-interval (t_i, t_{i+1}) is in $B_q^{\alpha,p}(t_i, t_{i+1}; X)$.

We shall need the following property.

Given $\varepsilon > 0$, there exists $\tau_0 = 0 < \tau_1 < \dots < \tau_n = \tau$ such that

$$\sup_{t \in (\tau_{i-1}, \tau_i)} \int_{\tau_{i-1}}^{\tau_i} \frac{\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}_\gamma)}^2}{|t-s|^\gamma} ds < \varepsilon. \quad (2.1)$$

Note that this assumption is satisfied in many cases. Suppose for example that $\gamma = 1$, $t \mapsto \mathbf{a}(t, u, v)$ is C^α for some $\alpha > 0$ in the sense that

$$|\mathbf{a}(t, u, v) - \mathbf{a}(s, u, v)| \leq M|t-s|^\alpha \|u\|_{\mathcal{V}} \|v\|_{\mathcal{V}} \quad (2.2)$$

for some positive constant M and all $u, v \in \mathcal{V}$. Then clearly

$$\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V})} \leq M|t-s|^\alpha$$

and this implies (2.1). More generally, if ω_i denotes the modulus of continuity of $\mathcal{A}(\cdot)$ on the interval (τ_{i-1}, τ_i) then (2.1) is satisfied if

$$\int_{\tau_{i-1}}^{\tau_i} \frac{\omega_i(r)^2}{r} dr < \varepsilon. \quad (2.3)$$

For the case $\gamma < 1$ the assumption (2.1) is satisfied if

$$|\mathbf{a}(t, u, v) - \mathbf{a}(s, u, v)| \leq K_{t,s} \|u\|_{\mathcal{V}} \|v\|_{\mathcal{V}_\gamma}, u, v \in \mathcal{V}, K_{t,s} > 0.$$

Note that if (2.1) holds with $\gamma = 0$ then

$$\sup_{t \in (\tau_{i-1}, \tau_i)} \int_{\tau_{i-1}}^{\tau_i} \|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{H})}^2 ds < \varepsilon.$$

In particular this holds if

$$\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{H})} \leq M_0$$

for some positive constant M_0 and all $t, s \in [0, \tau]$. In that case $\mathcal{A}(t) - \mathcal{A}(s)$ defines a bounded operator from \mathcal{V} to \mathcal{H} . This implies in particular that the operators $\mathcal{A}(t)$ have the same domain $D(\mathcal{A}(t)) = D(\mathcal{A}(0))$.

The following is our main result.

Theorem 2.2. *Suppose that \mathcal{A} is piecewise in $B_p^{\frac{\gamma}{2}, 2}(0, \tau; \mathcal{L}(\mathcal{V}, \mathcal{V}'_\gamma))$, (2.1) and the uniform Kato square property (3.2) are satisfied. Then for all $u_0 \in (\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p} = (\mathcal{H}, D(A(0)))_{(1-\frac{1}{p}), p}$, with $p < 2$ and $f \in L^p(0, \tau; \mathcal{H})$ there exists a unique $u \in W^{2,p}(0, \tau; \mathcal{H}) \cap L^{\frac{2p}{2-p}}(0, \tau; \mathcal{V})$ be the solution to the Cauchy problem*

$$u'(t) + A(t)u(t) = f(t) \quad t - a.e., \quad u(0) = u_0. \quad (2.4)$$

Moreover, there exists a positive constant C independent of u_0 and f such that

$$\|A(\cdot)u(\cdot)\|_{L^p(0, \tau; \mathcal{H})} + \|u\|_{W^{1,p}(0, \tau; \mathcal{H})} \leq C \left[\|u_0\|_{(\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p}} + \|f\|_{L^p(0, \tau; \mathcal{H})} \right]. \quad (2.5)$$

Remark 2.3. - Applying the fixed point argument, one can easily extend this result to a more general equation

$$u'(t) + A(t)u(t) + P(t)u(t) = f(t) \quad t - a.e., \quad u(0) = u_0,$$

here $P(t) \in \mathcal{L}(\mathcal{V}, \mathcal{H})$ such that $\|P(t)\|_{\mathcal{L}(\mathcal{V}, \mathcal{H})} \leq C$.

- We refer to the next section for the definition of the uniform Kato square root property and for few more details on such property.

In the next result we prove continuity of the solution to (2.4) as a function with values in $(\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p}$, where we work with the same assumptions as in Theorem 2.2.

Theorem 2.4. *Assume that (3.2) and $t \mapsto \mathcal{A}(t) \in C([0, \tau], \mathcal{L}(\mathcal{V}, \mathcal{V}'_\beta))$ for some $\beta \in [0, 1[$ or $t \mapsto \mathcal{A}(t) \in C^\varepsilon([0, \tau], \mathcal{L}(\mathcal{V}, \mathcal{V}'))$, $\varepsilon > 0$. Then we have the following embedding $MR(p, \mathcal{H}) \hookrightarrow C([0, \tau], (\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p})$.*

Next we prove that maximal regularity holds without requiring any regularity in time for the operators (or the forms). More precisely we have

Proposition 2.5. *Suppose [H1]-[H3]. Suppose that (3.2) and (2.1) holds for $\gamma = 0$. Then (2.4) has maximal L^p -regularity in \mathcal{H} for all $u_0 \in (\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p}$. In addition, there exists a positive constant C independent of u_0 and f such that*

$$\|A(\cdot)u(\cdot)\|_{L^p(0, \tau; \mathcal{H})} + \|u\|_{W^{1,p}(0, \tau; \mathcal{H})} \leq C \left[\|u_0\|_{(\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p}} + \|f\|_{L^p(0, \tau; \mathcal{H})} \right]. \quad (2.6)$$

3 Preparatory lemmas

In this section we prove several estimates which will play an important role in the proof of the main results. We emphasize that one of the important points here is to prove estimates with constants which are independent of t .

Before we start let us point out that we may assume without loss of generality that assumption [H3] is satisfied with $\nu = 0$, that is the forms are coercive with constant $\delta > 0$ independent of t . The reason is that the maximal regularity of (P) is equivalent to the same property for

$$v'(t) + (A(t) + \nu)v(t) = g(t), \quad v(0) = u_0. \quad (3.1)$$

This can be seen by observing that for $g(t) := f(t)e^{-\nu t}$, then $v(t) = u(t)e^{-\nu t}$ and clearly $v \in W^{1,p}(0, \tau; \mathcal{H})$ if and only if $u \in W^{1,p}(0, \tau; \mathcal{H})$ (and obviously $f \in L^p(0, \tau; \mathcal{H})$ if and only if $g \in L^p(0, \tau; \mathcal{H})$).

When [H3] holds with $\nu = 0$ then clearly the operators $A(t)$ are invertible on \mathcal{H} .

The next lemma point out that the constants involved in the estimates of the resolvents and the semigroups are uniform with respect to t .

Lemma 3.1. *For any $t \in [0, \tau]$, the operators $-A(t)$ and $-\mathcal{A}(t)$ generate strongly continuous analytic semigroups of angle $\gamma = \frac{\pi}{2} - \arctan(\frac{M}{\delta})$ on \mathcal{H} and \mathcal{V}' , respectively. In addition, there exist constants $C_\theta, C_{\beta,\theta}, C_{\alpha,\beta}$ independent of t , such that*

$$1- \ \|(\lambda + A(t))^{-1}\|_{\mathcal{L}(Y)} \leq \frac{C_\theta}{1+|\lambda|} \text{ for all } \lambda \in \Sigma_{\pi-\theta} \text{ with fixed } \theta < \gamma. \text{ Here, } Y = \mathcal{H}, \mathcal{V} \text{ or } \mathcal{V}'.$$

$$2- \ \|(\lambda + \mathcal{A}(t))^{-1}\|_{\mathcal{L}(\mathcal{H}, [\mathcal{H}, \mathcal{V}]_\beta)} \leq \frac{C_{\beta,\theta}}{1+|\lambda|^{1-\frac{\beta}{2}}}, \lambda \in \Sigma_{\pi-\theta}, \beta \in [0, 1].$$

$$3- \ \|(\lambda + \mathcal{A}(t))^{-1}\|_{\mathcal{L}(\mathcal{V}'_\beta, \mathcal{H})} \leq \frac{C_{\beta,\theta}}{1+|\lambda|^{1-\frac{\beta}{2}}}, \lambda \in \Sigma_{\pi-\theta}, \beta \in [0, 1].$$

$$4- \ \|A(t)^\alpha e^{-rA(t)}x\| \leq \frac{C_{\alpha,\beta}}{r^{\frac{\beta}{2}+\alpha}} \|x\|_{\mathcal{V}'_\beta}, r \in (0, \infty), \alpha \in [0, 1].$$

$$5- \ \|e^{-rA(t)}x\|_{\mathcal{V}_\beta} \leq \frac{C_\beta}{r^{\frac{\beta}{2}}} \|x\|, r \in (0, \infty), x \in \mathcal{H}.$$

All of the previous estimates holds for the adjoint operator $A(t)^*$.

Following [3][Definition 3.4], we introduce the following definition

Definition 3.2. *We say that $(A(t))_{t \in [0, \tau]}$ satisfies the uniform Kato square root property if $D(A(t)^{\frac{1}{2}}) = \mathcal{V}$ for all $t \in [0, \tau]$ and there are $c_1, c^1 > 0$ such that for all $v \in \mathcal{V}$*

$$c_1 \|v\|_{\mathcal{V}} \leq \|A(t)^{\frac{1}{2}}v\| \leq c^1 \|v\|_{\mathcal{V}}. \quad (3.2)$$

The uniform Kato square root property is obviously satisfied for symmetric forms. It is also satisfied for uniformly elliptic operators (not necessarily symmetric)

$$A(t) = - \sum_{k,l=1}^d \partial_k(a_{kl}(t,x)\partial_l)$$

on $L^2(\mathbb{R}^d)$ since $\|\nabla u\|_2$ is equivalent to $\|A(t)^{\frac{1}{2}}u\|_2$ with constants depending only on the dimension and the ellipticity constants, see [8].

In the next lemma we show that (3.2) implies that $A(t)^{* \frac{1}{2}}$ is an isomorphism from \mathcal{V} to \mathcal{H} and $\mathcal{A}(t)^{\frac{1}{2}}$ is an isomorphism from \mathcal{H} to \mathcal{V}' .

Lemma 3.3. *Assume that (3.2). Then for all $t \in [0, \tau]$ we claim that*

(1)- $D(A(t)^{* \frac{1}{2}}) = \mathcal{V}$ and

$$\frac{\delta}{c_1} \|v\|_{\mathcal{V}} \leq \|A(t)^{* \frac{1}{2}}v\| \leq \frac{M}{c_1} \|v\|_{\mathcal{V}}, v \in \mathcal{V}.$$

(2)- $D(\mathcal{A}(t)^{\frac{1}{2}}) = D(\mathcal{A}(t)^{* \frac{1}{2}}) = \mathcal{H}$ and

$$\frac{\delta}{c_1} \|v\| \leq \|\mathcal{A}(t)^{\frac{1}{2}}v\|_{\mathcal{V}'} \leq \frac{M}{c_1} \|v\|, v \in \mathcal{H},$$

$$c_1 \|v\| \leq \|\mathcal{A}(t)^{* \frac{1}{2}}v\|_{\mathcal{V}'} \leq c_1 \|v\|, v \in \mathcal{H}.$$

Proof. Assume that (3.2) and let $x \in D(A(t)^*)$, $t \in [0, \tau]$. Using (3.2) we get

$$\begin{aligned} \|x\|_{\mathcal{V}}^2 &\leq \frac{1}{\delta} \operatorname{Re} (A(t)^{\frac{1}{2}}x, A(t)^{* \frac{1}{2}}x) \\ &\leq \frac{1}{\delta} \|A(t)^{\frac{1}{2}}x\| \|A(t)^{* \frac{1}{2}}x\| \\ &\leq \frac{c_1}{\delta} \|x\|_{\mathcal{V}} \|A(t)^{* \frac{1}{2}}x\|. \end{aligned}$$

The density of $D(A(t)^*)$ in $D(A(t)^{* \frac{1}{2}})$ gives

$$\|x\|_{\mathcal{V}} \leq \frac{c_1}{\delta} \|A(t)^{* \frac{1}{2}}x\|$$

for all $x \in D(A(t)^{* \frac{1}{2}})$. Then $D(A(t)^{* \frac{1}{2}}) \hookrightarrow \mathcal{V}$.

Now, since $D(A(t)^{\frac{1}{2}}) \hookrightarrow \mathcal{V}$. It follows that $A(t)^{-\frac{1}{2}} \in \mathcal{L}(\mathcal{H}, \mathcal{V})$ and

$$\|A(t)^{-\frac{1}{2}}\|_{\mathcal{L}(\mathcal{H}, \mathcal{V})} \leq \frac{1}{c_1}.$$

Let $x \in \mathcal{H}$ and write $\mathcal{A}(t)^{\frac{1}{2}}x = \mathcal{A}(t)A(t)^{-\frac{1}{2}}x$. We obtain

$$\|\mathcal{A}(t)^{\frac{1}{2}}x\|_{\mathcal{V}'} \leq \|\mathcal{A}(t)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}')} \|A(t)^{-\frac{1}{2}}x\|_{\mathcal{V}} \leq M \|A(t)^{-\frac{1}{2}}\|_{\mathcal{L}(\mathcal{H}, \mathcal{V})} \|x\| \leq \frac{M}{c_1} \|x\|.$$

The boundedness of norm implies $\mathcal{A}(t)^{\frac{1}{2}} \in \mathcal{L}(\mathcal{H}, \mathcal{V})$ and by duality we have $A(t)^{* \frac{1}{2}} \in \mathcal{L}(\mathcal{V}, \mathcal{H})$. Then $\mathcal{V} \subseteq D(A(t)^{* \frac{1}{2}})$ and we get for all $x \in \mathcal{V}$

$$\begin{aligned} \|A(t)^{* \frac{1}{2}} x\| &\leq \|A(t)^{* \frac{1}{2}}\|_{\mathcal{L}(\mathcal{V}, \mathcal{H})} \|x\|_{\mathcal{V}} \\ &\leq \frac{M}{c_1} \|x\|_{\mathcal{V}}. \end{aligned}$$

This shows (1) and by duality we have (2). \square

The following lemma shows that $D(A(t)^{\frac{1}{p}}) = [\mathcal{H}, \mathcal{V}]_{\frac{2}{p}}$ for all $t \in [0, \tau]$ and $p > 2$ with equivalent norms.

Lemma 3.4. *Assume that (3.2). For all $t \in [0, \tau]$, $p > 2$ we get*

$$c'_1 \|x\|_{[\mathcal{H}, \mathcal{V}]_{\frac{2}{p}}} \leq \|A(t)^{\frac{1}{p}} x\| \leq c^1 \|x\|_{[\mathcal{H}, \mathcal{V}]_{\frac{2}{p}}},$$

where $c'_1, c^1 > 0$ are independent of t .

Proof. Let $t \in [0, \tau]$. The reiteration theorem for the real method [29][1.10.3, Theorem 2] or property of power of positive operator [22][Theorem 4.3.11] shows that $D(A(t)^\alpha) = [\mathcal{H}, D(A(t))]_\alpha = [\mathcal{H}, D(A(t)^{\frac{1}{2}})]_{2\alpha}$, for all $\alpha \in [0, \frac{1}{2}]$. Since $D(A(t)^{\frac{1}{2}}) = \mathcal{V}$,

$$D(A(t)^{\frac{1}{p}}) = [\mathcal{H}, D(A(t)^{\frac{1}{2}})]_{\frac{2}{p}} = [\mathcal{H}, \mathcal{V}]_{\frac{2}{p}},$$

with constants of injection independent of t . \square

Lemma 3.5. *Suppose that (3.2). Let $p \geq 2$ and $g \in L^{\frac{p}{p-1}}(0, \infty; \mathcal{H})$. We have*

$$\|A(t)^{* \frac{1}{p}} \int_0^\infty e^{-sA(t)^*} g(s) ds\| \leq C \|g\|_{L^{\frac{p}{p-1}}(0, \infty; \mathcal{H})}.$$

Proof. Since $D(A(t)^{\frac{1}{2}}) = \mathcal{V}$, then $D(A(t)^{* \frac{1}{2}}) = \mathcal{V}$ and

$$c'_1 \|x\|_{[\mathcal{H}, \mathcal{V}]_{\frac{2}{p}}} \leq \|A(t)^{* \frac{1}{p}} x\| \leq c^1 \|x\|_{[\mathcal{H}, \mathcal{V}]_{\frac{2}{p}}}.$$

Then if $\int_0^\infty e^{-sA(t)^*} g(s) ds \in [\mathcal{H}, \mathcal{V}]_{\frac{2}{p}}$,

$$\|A(t)^{* \frac{1}{p}} \int_0^\infty e^{-sA(t)^*} g(s) ds\| \leq c^1 \left\| \int_0^\infty e^{-sA(t)^*} g(s) ds \right\|_{[\mathcal{H}, \mathcal{V}]_{\frac{2}{p}}}.$$

Let us define the operator $(Rg) := \int_0^\infty e^{-sA(t)^*} g(s) ds$. It easy to see that $R \in \mathcal{L}(L^1(0, \infty; \mathcal{H}), \mathcal{H})$ and by [3][Lemma 4.1] we get that $R \in \mathcal{L}(L^2(0, \infty; \mathcal{H}), \mathcal{V})$. Since $[L^1(0, \infty; \mathcal{H}), L^2(0, \infty; \mathcal{H})]_\alpha = L^{\frac{2}{2-\alpha}}(0, \infty; \mathcal{H})$, for all $\alpha \in [0, 1]$, so for $\alpha = \frac{2}{p}$ one has $[L^1(0, \infty; \mathcal{H}), L^2(0, \infty; \mathcal{H})]_\alpha = L^{\frac{p}{p-1}}(0, \infty; \mathcal{H})$.

Now we use an interpolation argument to obtain that $R \in \mathcal{L}(L^{\frac{p}{p-1}}(0, \infty; \mathcal{H}), [\mathcal{H}, \mathcal{V}]_{\frac{2}{p}})$.
Therefore

$$\|A(t)^{* \frac{1}{p}} \int_0^\infty e^{-sA(t)^*} g(s) ds\| \leq c \left\| \int_0^\infty e^{-sA(t)^*} g(s) ds \right\|_{[\mathcal{H}, \mathcal{V}]_{\frac{2}{p}}} \leq C \|g\|_{L^{\frac{p}{p-1}}(0, \infty; \mathcal{H})}.$$

□

In the next proposition we show a L^p -estimate for $A(t)$ with constant independent of t . Here we assume the uniform Kato square root property. L^p -estimates are an important tool in harmonic analysis and we will use them at several places in the proofs of maximal regularity.

Proposition 3.6. *For all $x \in \mathcal{H}, p \geq 2$ we have*

$$\int_0^\infty \|A(t)^{\frac{1}{p}} e^{-sA(t)} x\|^p ds \leq C \|x\|^p.$$

Here, $C > 0$ is independent of t .

Proof. We set $f(s) = A(t)^{\frac{1}{p}} e^{-sA(t)} x$, where $s \in (0, \infty), x \in \mathcal{H}$ and $t \in [0, \tau]$. Then for all $g \in L^{\frac{p}{p-1}}(0, \infty; \mathcal{H})$,

$$\begin{aligned} \left| \int_0^\infty (f(s), g(s)) ds \right| &= \left| \int_0^\infty (A(t)^{\frac{1}{p}} e^{-sA(t)} x, g(s)) ds \right| \\ &= \left| (x, A(t)^{* \frac{1}{p}} \int_0^\infty e^{-sA(t)^*} g(s) ds) \right| \\ &\leq \|A(t)^{* \frac{1}{p}}\|_{\mathcal{L}([\mathcal{H}, \mathcal{V}]_{\frac{2}{p}}, \mathcal{H})} \|x\| \|Rg\|_{[\mathcal{H}, \mathcal{V}]_{\frac{2}{p}}} \\ &\leq C \|x\| \|g\|_{L^{\frac{p}{p-1}}(0, \infty; \mathcal{H})}. \end{aligned}$$

Therefore $f \in L^p(0, \infty; \mathcal{H})$ and we claim the proof. □

Lemma 3.7. *Assume that (3.2). Let $0 \leq s \leq r \leq \tau$ and $g \in L^2(r, s; \mathcal{V}')$.*

We have

$$\left\| \int_s^r e^{-(t-s)A(s)^*} g(t) dt \right\| \leq C \|g\|_{L^2(s, r; \mathcal{V}')}.$$

Proof. Let $0 \leq r \leq s \leq \tau$ and $g \in L^2(r, s; \mathcal{V}'), x \in \mathcal{H}$. We have

$$\begin{aligned}
& \left\| \int_s^r e^{-(t-s)A(s)^*} g(t) dt \right\| \\
&= \sup_{x \in \mathcal{H}, \|x\|=1} \left| \left(\int_s^r e^{-(t-s)A(s)^*} g(t) dt, x \right) \right| \\
&= \sup_{x \in \mathcal{H}, \|x\|=1} \left| \int_s^r (A(s)^{\frac{1}{2}} e^{-(t-s)A(s)} x, \mathcal{A}(s)^{*-\frac{1}{2}} g(t)) dt \right| \\
&\leq \sup_{x \in \mathcal{H}, \|x\|=1} \left(\left\| \int_s^r A(s)^{\frac{1}{2}} e^{-(t-s)A(s)} x \right\|^2 dt \right)^{\frac{1}{2}} \\
&\quad \times \|\mathcal{A}(s)^{*-\frac{1}{2}} g\|_{L^2(s,r;\mathcal{V}')} \\
&\leq C \|\mathcal{A}(s)^{*-\frac{1}{2}}\|_{\mathcal{L}(\mathcal{V}',\mathcal{H})} \|g\|_{L^2(s,r;\mathcal{V}')} \\
&\leq C' \|g\|_{L^2(s,r;\mathcal{V}')} .
\end{aligned}$$

Where in the first inequality we have used the Cauchy-Schwarz inequality and Lemma 3.3, Proposition 3.6 in the second one. \square

For $0 \leq t \leq \tau, p \in (1, 2]$ define the operator

$$L_1(f)(t) = \int_0^t e^{-(t-r)A(t)} f(r) dr, f \in L^p(0, \tau; \mathcal{H}).$$

Proposition 3.8. *Let $f \in L^p(0, \tau; \mathcal{H})$. Assume that (2.1) and (3.2). Then $L_1 \in \mathcal{L}(L^p(0, \tau; \mathcal{H}); L^{\frac{2p}{2-p}}(0, \tau; \mathcal{V}))$ and $L_1(f)(t) \in (\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p}$ for all $t \in [0, \tau]$.*

Proof. By [3] [Lemma 4.1] we get that $L_1 \in \mathcal{L}(L^2(0, \tau; \mathcal{H}); L^\infty(0, \tau; \mathcal{V}))$. Now we prove $L_1 \in \mathcal{L}(L^1(0, \tau; \mathcal{H}); L^2(0, \tau; \mathcal{V}))$. Indeed, let $g \in L^2(0, \tau; \mathcal{V}')$ we obtain

$$\begin{aligned}
(L_1(f), g)_{L^2(0,\tau;\mathcal{V}) \times L^2(0,\tau;\mathcal{V}')} &= \int_0^\tau \int_0^t (e^{-(t-s)A(t)} f(s), g(t)) ds dt \\
&= \int_0^\tau (f(s), \int_s^\tau e^{-(t-s)A(t)^*} g(t) dt) ds.
\end{aligned}$$

Write

$$\begin{aligned}
\int_s^\tau e^{-(t-s)A(t)^*} g(t) dt &= \int_s^\tau [e^{-(t-s)A(t)^*} g(t) - e^{-(t-s)A(s)^*} g(t)] dt \\
&\quad + \int_s^\tau e^{-(t-s)A(s)^*} g(t) dt := I_1(s) + I_2(s).
\end{aligned}$$

Choose a contour Γ in the positive half-plane and write by the holomorphic functional calculus for the sectorial operators $\mathcal{A}^*(t), \mathcal{A}^*(s)$

$$e^{-t\mathcal{A}^*(t)} - e^{-t\mathcal{A}^*(s)} = \frac{1}{2\pi i} \int_\Gamma e^{-t\lambda} (\lambda I - \mathcal{A}^*(t))^{-1} (\mathcal{A}^*(t) - \mathcal{A}^*(s)) (\lambda I - \mathcal{A}^*(s))^{-1} d\lambda.$$

By taking the norm in \mathcal{H} we have by Lemma 3.1

$$\begin{aligned}
& \left\| \left[e^{-t\mathcal{A}^*(t)} - e^{-t\mathcal{A}^*(s)} \right] g(t) \right\| \\
& \leq C \int_0^\infty e^{-t|\cos \gamma||\lambda|} \|(\lambda I - \mathcal{A}^*(t))^{-1}\|_{\mathcal{L}(\mathcal{V}', \mathcal{H})} \\
& \quad \times \|(\lambda I - \mathcal{A}^*(s))^{-1}\|_{\mathcal{L}(\mathcal{V}', \mathcal{V}_\gamma)} d|\lambda| \|\mathcal{A}^*(t) - \mathcal{A}^*(s)\|_{\mathcal{L}(\mathcal{V}_\gamma, \mathcal{V}')} \|g(t)\|_{\mathcal{V}} \\
& \leq C_1 \int_0^\infty |\lambda|^{\frac{\gamma}{2}-1} e^{-t|\cos \gamma||\lambda|} d|\lambda| \|\mathcal{A}^*(t) - \mathcal{A}^*(s)\|_{\mathcal{L}(\mathcal{V}_\gamma, \mathcal{V}')} \|g(t)\|_{\mathcal{V}'}.
\end{aligned}$$

Since,

$$\int_0^\infty |\lambda|^{\frac{\gamma}{2}-1} e^{-t|\cos \gamma||\lambda|} d|\lambda| = \frac{1}{|\cos \gamma| t^{\frac{\gamma}{2}}},$$

we obtain

$$\left\| \left[e^{-(t-s)\mathcal{A}(t)^*} - e^{-(t-s)\mathcal{A}(s)^*} \right] g(t) \right\| \leq C \frac{\|\mathcal{A}^*(t) - \mathcal{A}^*(s)\|_{\mathcal{L}(\mathcal{V}_\gamma, \mathcal{V}')}}{(t-s)^{\frac{\gamma}{2}}} \|g(t)\|_{\mathcal{V}'}.$$

The Cauchy-Schwarz inequality gives

$$\|I_1(s)\| \leq C \left(\int_s^\tau \frac{\|\mathcal{A}^*(t) - \mathcal{A}^*(s)\|_{\mathcal{L}(\mathcal{V}_\gamma, \mathcal{V}')}^2}{(t-s)^\gamma} dt \right)^{\frac{1}{2}} \|g\|_{L^2(s, \tau; \mathcal{V}')}. \quad (3.3)$$

In light of Lemma 3.7 we get

$$\|I_2(s)\| \leq C \|g\|_{L^2(s, \tau; \mathcal{V}')}. \quad (3.4)$$

Combining (3.3) and (3.4) we have

$$\left\| \int_s^\tau e^{-(t-s)\mathcal{A}(t)^*} g(t) dt \right\| \leq C \left(\left(\int_s^\tau \frac{\|\mathcal{A}^*(t) - \mathcal{A}^*(s)\|_{\mathcal{L}(\mathcal{V}_\gamma, \mathcal{V}')}^2}{(t-s)^\gamma} dt \right)^{\frac{1}{2}} + 1 \right) \|g\|_{L^2(s, \tau; \mathcal{V}')}.$$

Therefore

$$\begin{aligned}
|(L_1(f), g)_{L^2(0, \tau; \mathcal{V}) \times L^2(0, \tau; \mathcal{V}')}| & \leq C \left(\sup_{s \in [0, \tau]} \left(\int_s^\tau \frac{\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}_\gamma)}^2}{(t-s)^\gamma} dt \right)^{\frac{1}{2}} + 1 \right) \\
& \quad \times \|f\|_{L^1(0, \tau; \mathcal{H})} \|g\|_{L^2(0, \tau; \mathcal{V}')}.
\end{aligned}$$

Then $L_1 \in \mathcal{L}(L^1(0, \tau; \mathcal{H}); L^2(0, \tau; \mathcal{V}))$, and so by interpolation we obtain

$$L_1 \in \mathcal{L}([L^1(0, \tau; \mathcal{H}), L^2(0, \tau; \mathcal{H})]_\alpha; [L^2(0, \tau; \mathcal{V}), L^\infty(0, \tau; \mathcal{V})]_\alpha)$$

with $\alpha \in [0, 1]$.

We know that

$$\begin{aligned}
[L^2(0, \tau; \mathcal{V}), L^\infty(0, \tau; \mathcal{V})]_\alpha & = L^{\frac{2}{1-\alpha}}(0, \tau; \mathcal{V}) \\
[L^1(0, \tau; \mathcal{H}), L^2(0, \tau; \mathcal{H})]_\alpha & = L^{\frac{2}{2-\alpha}}(0, \tau; \mathcal{H}),
\end{aligned}$$

see [12] [Theorem 2.2.6]. Therefore for $\alpha = 2\lceil\frac{p-1}{p}\rceil$ we have

$$L_1 \in \mathcal{L}(L^p(0, \tau; \mathcal{H}); L^{\frac{2p}{2-p}}(0, \tau; \mathcal{V})).$$

Note that from [3][Lemma 4.1] we get that for all $t \in [0, \tau]$, $L_1(\cdot)(t) \in \mathcal{L}(L^2(0, \tau; \mathcal{H}); \mathcal{V})$. Now let $f \in L^1(0, \tau; \mathcal{H})$, $t \in [0, \tau]$. We have

$$\begin{aligned} \|L_1(f)(t)\| &\leq \int_0^t \|e^{-(t-r)A(t)}\|_{\mathcal{L}(\mathcal{H})} \|f(r)\| dr \\ &\leq C \|f\|_{L^1(0, \tau; \mathcal{H})}. \end{aligned}$$

Hence, $L_1(\cdot)(t) \in \mathcal{L}(L^1(0, \tau; \mathcal{H}); \mathcal{H})$ and so by interpolation one has $L_1(\cdot)(t) \in \mathcal{L}(L^p(0, \tau; \mathcal{H}); (\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p})$. Therefore $L_1(f)(t) \in (\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p}$ for all $t \in [0, \tau]$. \square

For $f \in L^p(0, \tau; \mathcal{H})$, $p \in (1, \infty)$ and for almost every $t \in [0, \tau]$ we define the operator L by

$$L(f)(t) := A(t) \int_0^t e^{-(t-s)A(t)} f(s) ds.$$

Note that in the autonomous case (i.e. $A(t) = A(0)$, $t \in [0, \tau]$) L is called the maximal regularity operator.

It is proved in [18] that L is bounded on $L^p(0, \tau; \mathcal{H})$ for all $p \in (1, \infty)$ provided $t \mapsto \mathfrak{a}(t, \dots)$ is C^ϵ for some $\epsilon > 0$ (or similarly, $t \mapsto \mathcal{A}(t)$ is C^ϵ on $[0, \tau]$ with values in $\mathcal{L}(\mathcal{V}, \mathcal{V}')$). The proof for the case $p = 2$ is based on vector-valued pseudo-differential operators.

Lemma 3.9. *Assume that*

- For $p \leq 2$ the conditions (2.1) and (3.2) holds.
- For $p > 2$, $t \mapsto \mathfrak{a}(t, \dots)$ is C^ϵ , $\epsilon > 0$.

Then L is bounded on $L^p(0, \tau; \mathcal{H})$.

Proof. For the case $p \leq 2$ we refer to [3][Proposition 4.2], and in the case of $p > 2$ the result follows by [18] [Lemma 2.6]. \square

4 Key estimates

In this section we state and prove some other estimates we will need in the proof of the main result.

Lemma 4.1. *Let $\tau_0 = 0 < \tau_1 < \dots < \tau_n = \tau$ be a subdivision of $[0, \tau]$ and $v \in L^{\frac{2p}{2-p}}(\tau_{i-1}, \tau_i; \mathcal{V})$, $p < 2$, $i = 1, \dots, n$. We define the operator $Q_{(\tau_{i-1}, \tau_i)}$ by*

$$(Q_{(\tau_{i-1}, \tau_i)}v)(t) = \int_{\tau_{i-1}}^t e^{-(t-s)A(t)} (\mathcal{A}(t) - \mathcal{A}(s))v(s) ds,$$

where $t \in [\tau_{i-1}, \tau_i]$. Assume that $\mathcal{A}(\cdot)$ satisfies the condition (2.1). Then $Q_{(\tau_{i-1}, \tau_i)} \in \mathcal{L}(L^{\frac{2p}{2-p}}(\tau_{i-1}, \tau_i; \mathcal{V}))$ and we have

$$\|Q_{(\tau_{i-1}, \tau_i)}\|_{\mathcal{L}(L^{\frac{2p}{2-p}}(\tau_{i-1}, \tau_i; \mathcal{V}))} \leq C \sup_{t \in [\tau_{i-1}, \tau_i]} \left[\int_{\tau_{i-1}}^{\tau_i} \frac{\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}'_\gamma)}^2}{|t-s|^\gamma} ds \right]^{\frac{1}{2}}, C > 0.$$

Moreover for ε small enough in the condition (2.1) we get

$$\|Q_{\mathcal{C}_{(\tau_{i-1}, \tau_i)}}\|_{\mathcal{L}(L^{\frac{2p}{2-p}}(\tau_{i-1}, \tau_i; \mathcal{V}))} < 1.$$

In addition,

$$(Q_{(\tau_{i-1}, \tau_i)}v)(t) \in (\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p}, t \in [\tau_{i-1}, \tau_i].$$

Proof. By [3] [Proposition 4.5] we have $Q_{(\tau_{i-1}, \tau_i)} \in \mathcal{L}(L^\infty(\tau_{i-1}, \tau_i; \mathcal{V}))$ for all $i = 1, \dots, n$. Now we prove $Q_{(\tau_{i-1}, \tau_i)} \in \mathcal{L}(L^2(\tau_{i-1}, \tau_i; \mathcal{V}))$. Indeed, let $v \in L^2(\tau_{i-1}, \tau_i; \mathcal{V})$, $w \in L^2(\tau_{i-1}, \tau_i; \mathcal{V}')$. We obtain

$$\begin{aligned} & \left| \left(Q_{(\tau_{i-1}, \tau_i)}v, w \right)_{L^2(\tau_{i-1}, \tau_i; \mathcal{V}) \times L^2(\tau_{i-1}, \tau_i; \mathcal{V}')} \right| \\ &= \left| \int_{\tau_{i-1}}^{\tau_i} \int_{\tau_{i-1}}^t \left(e^{-(t-s)A(t)} (\mathcal{A}(t) - \mathcal{A}(s))v(s), w(t) \right) ds dt \right| \\ &= \left| \int_{\tau_{i-1}}^{\tau_i} \int_{\tau_{i-1}}^t \left(e^{-\frac{(t-s)}{2}A(t)} (\mathcal{A}(t) - \mathcal{A}(s))v(s), A(t)^{* \frac{1}{2}} e^{-\frac{(t-s)}{2}A(t)^*} \mathcal{A}(t)^{* - \frac{1}{2}} w(t) \right) ds dt \right| \\ &\leq \int_{\tau_{i-1}}^{\tau_i} \int_{\tau_{i-1}}^t \|e^{-\frac{(t-s)}{2}A(t)}\|_{\mathcal{L}(\mathcal{V}'_\gamma, \mathcal{H})} \|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}'_\gamma)} \\ &\quad \times \|v(s)\|_{\mathcal{V}} \|A(t)^{* \frac{1}{2}} e^{-\frac{(t-s)}{2}A(t)^*} \mathcal{A}(t)^{* - \frac{1}{2}} w(t)\| ds dt \\ &\leq \int_{\tau_{i-1}}^{\tau_i} \int_{\tau_{i-1}}^t \frac{\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}'_\gamma)}}{(t-s)^{\frac{\gamma}{2}}} \|v(s)\|_{\mathcal{V}} \\ &\quad \times \|A(t)^{* \frac{1}{2}} e^{-\frac{(t-s)}{2}A(t)^*} \mathcal{A}(t)^{* - \frac{1}{2}} w(t)\| ds dt \\ &\leq \left(\int_{\tau_{i-1}}^{\tau_i} \int_{\tau_{i-1}}^t \frac{\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}'_\gamma)}^2}{(t-s)^\gamma} \|v(s)\|_{\mathcal{V}}^2 ds dt \right)^{\frac{1}{2}} \\ &\quad \times \left(\int_{\tau_{i-1}}^{\tau_i} \int_{\tau_{i-1}}^t \|A(t)^{* \frac{1}{2}} e^{-\frac{(t-s)}{2}A(t)^*} \mathcal{A}(t)^{* - \frac{1}{2}} w(t)\|^2 ds dt \right)^{\frac{1}{2}} \\ &\leq_{(i)} K_2 \sup_{t \in [\tau_{i-1}, \tau_i]} \left[\int_{\tau_{i-1}}^{\tau_i} \frac{\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}'_\gamma)}^2}{|t-s|^\gamma} ds \right]^{\frac{1}{2}} \|v\|_{L^2(0, \tau; \mathcal{V})} \|A(\cdot)^{* - \frac{1}{2}} w\|_{L^2(0, \tau; \mathcal{H})} \\ &\leq_{(ii)} K_2 \frac{c^1}{\delta} \sup_{t \in [\tau_{i-1}, \tau_i]} \left[\int_{\tau_{i-1}}^{\tau_i} \frac{\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}'_\gamma)}^2}{|t-s|^\gamma} ds \right]^{\frac{1}{2}} \|v\|_{L^2(0, \tau; \mathcal{V})} \|w\|_{L^2(0, \tau; \mathcal{V}')}. \end{aligned}$$

Where in (i) we have used Fubini's theorem, Lemma 3.6 and Lemma 3.3 in (ii).

Therefore $Q_{(\tau_{i-1}, \tau_i)} \in \mathcal{L}(L^2(\tau_{i-1}, \tau_i; \mathcal{V}))$ and by interpolation $Q_{(\tau_{i-1}, \tau_i)} \in \mathcal{L}([L^2(\tau_{i-1}, \tau_i; \mathcal{V}), L^\infty(\tau_{i-1}, \tau_i; \mathcal{V})]_\alpha)$ for all $\alpha \in [0, 1]$. Hence, for $\alpha = 2[\frac{p-1}{p}]$ we get $Q_{(\tau_{i-1}, \tau_i)} \in \mathcal{L}(L^{\frac{2p}{2-p}}(\tau_{i-1}, \tau_i; \mathcal{V}))$, with

$$\|Q_{(\tau_{i-1}, \tau_i)}\|_{\mathcal{L}(L^{\frac{2p}{2-p}}(\tau_{i-1}, \tau_i; \mathcal{V}))} \leq C \sup_{t \in [\tau_{i-1}, \tau_i]} \left[\int_{\tau_{i-1}}^{\tau_i} \frac{\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}'_\gamma)}^2 ds}{|t-s|^\gamma} \right]^{\frac{1}{2}}, C > 0.$$

Next we prove

$$(Q_{(\tau_{i-1}, \tau_i)}v)(t) \in (\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p}, t \in [\tau_{i-1}, \tau_i], v \in L^{\frac{2p}{2-p}}(\tau_{i-1}, \tau_i; \mathcal{V}).$$

Indeed, for $v \in L^\infty(\tau_{i-1}, \tau_i; \mathcal{V})$ one has by [3][(4.11)] that $(Q_{(\tau_{i-1}, \tau_i)}v)(t) \in \mathcal{V}$ and

$$\|(Q_{(\tau_{i-1}, \tau_i)}v)(t)\|_{\mathcal{V}} \leq C \left[\int_{\tau_{i-1}}^t \frac{\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}'_\gamma)}^2 ds}{|t-s|^\gamma} \right]^{\frac{1}{2}} \|v\|_{L^\infty(\tau_{i-1}, \tau_i; \mathcal{V})},$$

$t \in [\tau_{i-1}, \tau_i], C > 0$.

Similarly we have for $v \in L^2(\tau_{i-1}, \tau_i; \mathcal{V})$ that $(Q_{(\tau_{i-1}, \tau_i)}v)(t) \in \mathcal{H}$ and so by interpolation $(Q_{(\tau_{i-1}, \tau_i)}v)(t) \in (\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p}, t \in [0, \tau]$ for all $v \in L^{\frac{2p}{2-p}}(\tau_{i-1}, \tau_i; \mathcal{V})$. Moreover, we have for $t \in [\tau_{i-1}, \tau_i]$

$$\|(Q_{(\tau_{i-1}, \tau_i)}v)(t)\|_{(\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p}} \leq C \left[\int_{\tau_{i-1}}^t \frac{\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}'_\gamma)}^2 ds}{|t-s|^\gamma} \right]^{\frac{1}{2}} \|v\|_{L^{\frac{2p}{2-p}}(\tau_{i-1}, \tau_i; \mathcal{V})}. \quad (4.1)$$

□

Lemma 4.2. *Assume that (2.1). Then for all $u_0 \in (\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p}$ with $p < 2$ we have*

$$T \in \mathcal{L}((\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p}; L^{\frac{2p}{2-p}}(0, \tau; \mathcal{V})),$$

where $(Tu_0)(t) = e^{-tA(t)}u_0$ for $t \in [0, \tau]$.

Remark 4.3. *If $r \in (0, \tau)$ and $(Tu_0)(t) = e^{-(t-r)A(t)}u_0$ for $t \in [r, \tau]$, we get $Tu_0 \in L^{\frac{2p}{2-p}}(r, \tau; \mathcal{V})$.*

Proof. Noting that, since the semigroup $(e^{-rA(t)})_{r \geq 0}$ is uniformly bounded in \mathcal{V} we get that $T \in \mathcal{L}(\mathcal{V}; L^\infty(0, \tau; \mathcal{V}))$. We write

$$(Tu_0)(t) = [e^{-tA(t)} - e^{-tA(0)}]u_0 + e^{-tA(0)}u_0 := (T_1u_0)(t) + (T_2u_0)(t).$$

Let now $u_0 \in \mathcal{H}$. We get by Lemma 3.6

$$\begin{aligned} \|T_2u_0\|_{L^2(0, \tau; \mathcal{V})}^2 &= \int_0^\tau \|e^{-tA(0)}u_0\|_{\mathcal{V}}^2 dt \\ &\leq \frac{1}{c_1} \int_0^\tau \|A(0)^{\frac{1}{2}}e^{-tA(0)}u_0\|^2 dt \\ &\leq \frac{K_2}{c_1} \|u_0\|^2. \end{aligned}$$

This gives that $T_2 \in \mathcal{L}(\mathcal{H}; L^2(0, \tau; \mathcal{V}))$. Write by the holomorphic functional calculus for the sectorial operators $A(t), A(0)$

$$e^{-tA(t)} - e^{-tA(0)} = \frac{1}{2\pi i} \int_{\Gamma} e^{-t\lambda} (\lambda I - A(t))^{-1} (\mathcal{A}(t) - \mathcal{A}(s)) (\lambda I - \mathcal{A}(0))^{-1} d\lambda.$$

By taking the norm in \mathcal{V} we have by Lemma 3.1

$$\begin{aligned} & \| [e^{-tA(t)} - e^{-tA(0)}] u_0 \|_{\mathcal{V}} \\ & \leq C \int_0^{\infty} e^{-t|\cos \gamma||\lambda|} \| (\lambda I - A(t))^{-1} \|_{\mathcal{L}(\mathcal{V}'_t, \mathcal{V})} \\ & \quad \times \| (\lambda I - \mathcal{A}(0))^{-1} \|_{\mathcal{L}(\mathcal{H}; \mathcal{V})} d|\lambda| \| \mathcal{A}(t) - \mathcal{A}(s) \|_{\mathcal{L}(\mathcal{V}, \mathcal{V}'_t)} \| u_0 \| \\ & \leq C_1 \int_0^{\infty} |\lambda|^{\frac{\gamma}{2}-1} e^{-t|\cos \gamma||\lambda|} d|\lambda| \| \mathcal{A}(t) - \mathcal{A}(s) \|_{\mathcal{L}(\mathcal{V}, \mathcal{V}'_t)} \| u_0 \|. \end{aligned}$$

Since,

$$\int_0^{\infty} |\lambda|^{\frac{\gamma}{2}-1} e^{-t|\cos \gamma||\lambda|} d|\lambda| = \frac{1}{|\cos \gamma| t^{\frac{\gamma}{2}}},$$

we obtain

$$\| [e^{-tA(t)} - e^{-tA(0)}] u_0 \|_{\mathcal{V}} \leq C \frac{\| \mathcal{A}(t) - \mathcal{A}(s) \|_{\mathcal{L}(\mathcal{V}, \mathcal{V}'_t)}}{(t-s)^{\frac{\gamma}{2}}} \| u_0 \|.$$

Hence, $T_1 \in \mathcal{L}(\mathcal{H}; L^2(0, \tau; \mathcal{V}))$ and

$$\| T_1 u_0 \|_{L^2(0, \tau; \mathcal{V})}^2 \leq C \int_0^{\tau} \frac{\| \mathcal{A}(t) - \mathcal{A}(s) \|_{\mathcal{L}(\mathcal{V}, \mathcal{V}'_t)}^2}{(t-s)^{\gamma}} ds \| u_0 \|^2.$$

Therefore, $T \in \mathcal{L}(\mathcal{H}; L^2(0, \tau; \mathcal{V}))$ and so by interpolation one has

$$T \in \mathcal{L}((\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p}; L^{\frac{2p}{2-p}}(0, \tau; \mathcal{V})).$$

This finishes the proof. \square

We now prove the uniqueness of solution.

Proposition 4.4. *Assume that $t \mapsto \mathbf{a}(t)$ is measurable and let $[a, b] \subseteq [0, \tau], p \in (1, \infty)$ and $f \in L^p(a, b; \mathcal{V}'), u_a \in \mathcal{V}'$. If $u \in W^{1,p}(a, b; \mathcal{V}') \cap L^p(a, b; \mathcal{V})$ solve*

$$u'(t) + \mathcal{A}(t)u(t) = f(t) \quad (t \in (a, b)), \quad u(a) = u_a, \quad (4.2)$$

then this solution is unique.

Proof. Let $[a, b] \subseteq [0, \tau], p \in (1, \infty)$ and $f \in L^p(a, b; \mathcal{V}'), u_a \in \mathcal{V}'$. Assume that there exists two solution u_1, u_2 to (4.16). Set $w = u_1 - u_2$. Obviously, w satisfies

$$w'(t) + \mathcal{A}(t)w(t) = 0 \quad (t \in (a, b)), \quad w(a) = 0, \quad (4.3)$$

and by Lions' theorem we have $w = 0$. This gives $u_1 = u_2$. Hence, u is unique. \square

The next proposition shows that in many cases it is sufficient to show maximal L^p -regularity for initial value zero. This is well-known in the autonomous case.

Proposition 4.5. *Assume that the Cauchy problem*

$$w'(t) + \mathcal{A}(t)w(t) = f(t) \quad (t \in [0, \tau]), \quad w(0) = 0. \quad (4.4)$$

has maximal L^p -regularity for $p \in (1, \infty)$ and arbitrary $\tau > 0$. Then for all $u_0 \in (\mathcal{H}, D(A(0)))_{1-\frac{1}{p}, p}$ the Cauchy problem

$$u'(t) + \mathcal{A}(t)u(t) = f(t) \quad (t \in [0, \tau]), \quad u(0) = u_0. \quad (4.5)$$

has maximal L^p -regularity. Moreover, $u(t) \in (\mathcal{H}, D(A(t)))_{1-\frac{1}{p}, p}$ for all $t \in [0, \tau]$ and there exists a positive constant C such that

$$\begin{aligned} & \|u\|_{W^{1,p}(0, \tau; \mathcal{H})} + \|A(\cdot)u\|_{L^p(0, \tau; \mathcal{H})} + \sup_{t \in [0, \tau]} \|u(t)\|_{(\mathcal{H}, D(A(t)))_{1-\frac{1}{p}, p}} \\ & \leq C \left[\|f\|_{L^p(0, \tau; \mathcal{H})} + \|u_0\|_{(\mathcal{H}, D(A(0)))_{1-\frac{1}{p}, p}} \right]. \end{aligned}$$

Proof. Assume that (4.4) has maximal L^p -regularity. We set $\mathcal{B}(t) = \mathcal{A}(0)$ for $t \in [0, \tau]$ and $\mathcal{B}(t) = \mathcal{A}(t - \tau)$ for $t \in [\tau, 2\tau]$ and $g(t) = v'(t) + \mathcal{A}(0)v(t)$, $t \in [0, \tau]$, $g(t) = f(t - \tau)$ for $t \in [\tau, 2\tau]$, where $v(t) = \phi(t)e^{-(\tau-t)A(0)}u_0$, $\phi(0) = 0$, $\phi(\tau) = 1$ such that $\phi \in C^1([0, \tau])$ (we can take $\phi(t) = \frac{t}{\tau}$). By the characterization of real interpolation spaces via the trace method we obtain $v \in W^{1,p}(0, \tau; \mathcal{H}) \cap L^p(0, \tau; D(A(0)))$ and

$$\|v' + \mathcal{A}(0)v\|_{L^p(0, \tau; \mathcal{H})} \leq C \|u_0\|_{(\mathcal{H}, D(A(0)))_{1-\frac{1}{p}, p}}.$$

Remark that $v(\tau) = u_0$, $v(0) = 0$ and $g \in L^p(0, 2\tau; \mathcal{H})$. Let $z \in W^{1,p}(0, 2\tau; \mathcal{H})$ be the unique solution to

$$z'(t) + \mathcal{B}(t)z(t) = g(t) \quad (t \in [0, 2\tau]), \quad z(0) = 0. \quad (4.6)$$

The uniqueness of solutions in the autonomous case imply $z = v$ on $[0, \tau]$ and so $z(\tau) = u_0$. Now we set $u(t) = z(t + \tau)$, $t \in [0, \tau]$. Therefore u is the

unique solution to (4.5) and we obtain

$$\begin{aligned}
& \|u\|_{W^{1,p}(0,\tau;\mathcal{H})} + \|A(\cdot)u\|_{L^p(0,\tau;\mathcal{H})} \\
& \leq \|z\|_{W^{1,p}(0,2\tau;\mathcal{H})} + \|B(\cdot)z\|_{L^p(0,2\tau;\mathcal{H})} \\
& \leq C\|g\|_{L^p(0,2\tau;\mathcal{H})} \\
& \leq C\left[\|f\|_{L^p(0,\tau;\mathcal{H})} + \|v' + \mathcal{A}(0)v\|_{L^p(0,\tau;\mathcal{H})}\right] \\
& \leq C\left[\|f\|_{L^p(0,\tau;\mathcal{H})} + \|u_0\|_{(\mathcal{H},D(A(0)))_{1-\frac{1}{p},p}}\right].
\end{aligned}$$

Now let $t \in [0, \tau]$ and set $g(s) = u'(s) + A(s)u(s)$, $s \in [0, t]$ and $g(s) = 0$, $s \in [t, 2\tau]$. We put $C(s) = A(s)$, $t \in [0, t]$ and $C(s) = A(t)$, $s \in [t, 2\tau]$. Let z be the solution to

$$z'(s) + C(s)z(s) = g(s) \quad (s \in [0, 2\tau]), \quad z(0) = u_0. \quad (4.7)$$

By the uniqueness of solution we get that $z(s) = u(s)$ on $[0, t]$. Hence, $z(t) = u(t)$. Since $z \in W^{1,p}(t, 2\tau; \mathcal{H}) \cap L^p(t, 2\tau; D(A(t)))$ we have by [4] [Theorem III 4.10.2] that $z \in C([t, \tau]; (\mathcal{H}, D(A(t)))_{1-\frac{1}{p},p})$. Therefore, $z(t) = u(t) \in (\mathcal{H}, D(A(t)))_{1-\frac{1}{p},p}$. Moreover,

$$\begin{aligned}
\|u(t)\|_{(\mathcal{H},D(A(t)))_{1-\frac{1}{p},p}} & \leq C\left[\|z\|_{W^{1,p}(0,2\tau;\mathcal{H})} + \|C(\cdot)z\|_{L^p(0,2\tau;\mathcal{H})}\right] \\
& \leq C\left[\|f\|_{L^p(0,\tau;\mathcal{H})} + \|u_0\|_{(\mathcal{H},D(A(0)))_{1-\frac{1}{p},p}}\right].
\end{aligned}$$

This finishes the proof. \square

Proposition 4.6. *Let $f \in L^p(0, \tau; \mathcal{H})$, $p \in (1, 2]$ and assume that the assumptions (3.2), (2.1) holds. Then there exists a unique u be the solution of the Cauchy problem (2.4) such that $u \in L^{\frac{2p}{2-p}}(0, \tau; \mathcal{V})$. Moreover, $u(t) \in (\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}),p}$ for all $t \in [0, \tau]$.*

Proof. Let $0 \leq s \leq t \leq \tau_1 \leq \tau$ and u_1 be the solution to (2.4) in $[0, \tau_1]$. Set $v(s) = e^{-(t-s)A(t)}u_1(s)$. We remark that $v(t) = u_1(t)$, $v(0) = e^{-tA(t)}u_1(0) = 0$ and

$$v'(s) = e^{-(t-s)A(t)}(\mathcal{A}(t) - \mathcal{A}(s))u_1(s) + e^{-(t-s)A(t)}f(s).$$

Since $v(t) = v(0) + \int_0^t v'(s) ds$, we have

$$\begin{aligned}
u_1(t) & = \int_0^t e^{-(t-s)A(t)}(\mathcal{A}(t) - \mathcal{A}(s))u_1(s) ds + \int_0^t e^{-(t-s)A(t)}f(s) ds \\
& = (Q_{(0,\tau_1)}u_1)(t) + L_1(f)(t).
\end{aligned} \quad (4.8)$$

Due to Lemma 4.2 and Proposition 3.8 we get that $L_1(f)$ is bounded in $L^{\frac{2p}{2-p}}(0, \tau; \mathcal{V})$. Moreover $L_1(f)(t) \in (\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}),p}$ for all $t \in [0, \tau]$. In order

to continue we wish to take the inverse $I - Q_{(0, \tau_1)}$. It follows from Lemma 4.1 that $Q_{(0, \tau_1)}$ is bounded in $L^{\frac{2p}{2-p}}(0, \tau_1; \mathcal{V})$ and

$$\|Q_{(0, \tau_1)}\|_{\mathcal{L}(L^{\frac{2p}{2-p}}(0, \tau_1; \mathcal{V}))} \leq C \sup_{s \in [0, \tau_1]} \left[\int_0^{\tau_1} \frac{\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}'_\gamma)}^2}{|t-s|^\gamma} dt \right]^{\frac{1}{2}}.$$

Using (2.1), let $\varepsilon > 0$ and choose τ_1 such that

$$C \sup_{s \in [0, \tau_1]} \left[\int_0^{\tau_1} \frac{\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}'_\gamma)}^2}{|t-s|^\gamma} dt \right]^{\frac{1}{2}} < \varepsilon < 1.$$

Hence, $\|Q_{(0, \tau_1)}\|_{\mathcal{L}(L^{\frac{2p}{2-p}}(0, \tau_1; \mathcal{V}))} \leq \varepsilon < 1$ and so $I - Q_{(0, \tau_1)}$ is invertible in $L^{\frac{2p}{2-p}}(0, \tau_1; \mathcal{V})$. Therefore, $u_1 = (I - Q_{(0, \tau_1)})^{-1}(L_1)(f)$ and we obtain that $u_1 \in L^{\frac{2p}{2-p}}(0, \tau_1; \mathcal{V})$ and

$$\|u_1\|_{L^{\frac{2p}{2-p}}(0, \tau_1; \mathcal{V})} \leq C \|f\|_{L^p(0, \tau_1; \mathcal{H})}. \quad (4.9)$$

Moreover, from (4.8) we have for $t \in [0, \tau_1]$

$$\begin{aligned} \|u_1(t)\|_{(\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p}} &\leq \|(Q_{(0, \tau_1)}u_1)(t)\|_{(\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p}} + \|(L_1)(f)(t)\|_{(\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p}} \\ &\leq C \left[\|u_1\|_{L^{\frac{2p}{2-p}}(0, \tau_1; \mathcal{V})} + \|f\|_{L^p(0, \tau_1; \mathcal{H})} \right] \\ &\leq C' \|f\|_{L^p(0, \tau_1; \mathcal{H})}. \end{aligned} \quad (4.10)$$

Where in the second inequality we used Proposition 3.8, Lemma 4.1 and (4.9) in the last one. Let $\tau_1 \leq \tau_2 \leq \tau$ and u_2 be the solution of

$$u_2'(t) + \mathcal{A}(t)u_2(t) = f(t) \quad (t \in (\tau_1, \tau_2)), \quad u_2(\tau_1) = u_1(\tau_1). \quad (4.11)$$

For $\tau_1 \leq s \leq t \leq \tau_2$, we set $v(s) = e^{-(t-s)A(t)}u_2(s)$. We remark that $v(t) = u_2(t)$, $v(\tau_1) = e^{-(t-\tau_1)A(t)}u(\tau_1)$ and

$$v'(s) = e^{-(t-s)A(t)}(\mathcal{A}(t) - \mathcal{A}(s))u_2(s) + e^{-(t-s)A(t)}f(s).$$

Since $v(t) = v(\tau_1) + \int_{\tau_1}^t v'(s) ds$, we have

$$\begin{aligned} u_2(t) &= e^{-(t-\tau_1)A(t)}u(\tau_1) + \int_{\tau_1}^t e^{-(t-s)A(t)}(\mathcal{A}(t) - \mathcal{A}(s))u_2(s) ds + \int_{\tau_1}^t e^{-(t-s)A(t)}f(s) ds \\ &= (Tu(\tau_1))(t) + (Q_{(\tau_1, \tau_2)}u_2)(t) + L_1(I_{(\tau_1, \tau_2)}f)(t). \end{aligned} \quad (4.12)$$

Thanks to Lemma 4.2 and Proposition 3.8, we obtain that $Tu(\tau_1), L_1(f)$ are bounded in $L^{\frac{2p}{2-p}}(\tau_1, \tau; \mathcal{V})$. Now repeat the same strategy. We choose τ_2 such that

$$\|Q_{(\tau_1, \tau_2)}\|_{\mathcal{L}(L^{\frac{2p}{2-p}}(\tau_1, \tau_2; \mathcal{V}))} \leq C \sup_{s \in [\tau_1, \tau_2]} \left[\int_{\tau_1}^{\tau_2} \frac{\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}'_\gamma)}^2}{|t-s|^\gamma} dt \right]^{\frac{1}{2}} < 1.$$

Hence, $I - Q_{(\tau_1, \tau_2)}$ is invertible in $L^{\frac{2p}{2-p}}(\tau_1, \tau_2; \mathcal{V})$. Therefore, $u_2(t) = (I - Q_{(\tau_1, \tau_2)})^{-1}(Tu(\tau_1) + L_1(f))(t)$ and we obtain that $u_2 \in L^{\frac{2p}{2-p}}(\tau_1, \tau_2; \mathcal{V})$. In addition,

$$\begin{aligned} \|u_2\|_{L^{\frac{2p}{2-p}}(\tau_1, \tau_2; \mathcal{V})} &\leq C \left(\|Tu(\tau_1)\|_{L^{\frac{2p}{2-p}}(\tau_1, \tau_2; \mathcal{V})} + \|L_1(f)\|_{L^{\frac{2p}{2-p}}(\tau_1, \tau_2; \mathcal{V})} \right) \\ &\leq C' \left(\|u(\tau_1)\|_{(\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p}} + \|f\|_{L^p(\tau_1, \tau_2; \mathcal{H})} \right). \end{aligned} \quad (4.13)$$

Combining (4.10) and (4.13) we get

$$\|u_2\|_{L^{\frac{2p}{2-p}}(\tau_1, \tau_2; \mathcal{V})} \leq C \|f\|_{L^p(0, \tau_2; \mathcal{H})}. \quad (4.14)$$

Moreover, from (4.12), (4.13) and Proposition 3.8, Lemma 4.1 we have for all $t \in [\tau_1, \tau_2]$

$$\begin{aligned} &\|u_2(t)\|_{(\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p}} \\ &\leq \|(Tu(\tau_1))(t)\|_{(\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p}} + \|(Q_{(\tau_1, \tau_2)}u_2)(t)\|_{(\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p}} + \|(L_1)(f)(t)\|_{(\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p}} \\ &\leq C \left[\|u(\tau_1)\|_{(\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p}} + \|u_2\|_{L^{\frac{2p}{2-p}}(\tau_1, \tau_2; \mathcal{V})} + \|f\|_{L^p(\tau_1, \tau_2; \mathcal{H})} \right] \\ &\leq C \|f\|_{L^p(0, \tau_2; \mathcal{H})}. \end{aligned} \quad (4.15)$$

Now repeat the same strategy. We use (2.1) and we work on $[\tau_{i-1}, \tau_i]$ we search u_i be solution to

$$u_i'(t) + \mathcal{A}(t)u_i(t) = f(t) \quad (t \in (\tau_{i-1}, \tau_i)), \quad u_i(\tau_{i-1}) = u_{i-1}(\tau_{i-1}). \quad (4.16)$$

and argue exactly as before. We obtain (4.14) and (4.15) on each sub-intervals $[\tau_{i-1}, \tau_i]$. Now we set $u = u_i$ in $[\tau_{i-1}, \tau_i]$. Hence by using Proposition 4.4, u is the unique solution to (2.4). This implies (4.14) and (4.15) on $[0, \tau]$ for arbitrary $\tau > 0$ and finishes the proof. \square

5 Proof of the main results

After the auxiliary results of the last two sections we are now ready to give the proofs of the main results of this paper. Note that we may assume without loss of generality that [H3] holds with $= 0$, see the beginning of Section 3.

Proof of Theorem 2.2. First we consider the case $u_0 = 0$. Indeed, let $f \in L^p(0, \tau; \mathcal{H})$ and $t \in (0, \tau)$. Then by Proposition 4.6 there exists a unique

$u \in L^{\frac{2p}{2-p}}(0, \tau; \mathcal{V})$ be the solution to (2.4). We have from (4.12)

$$\begin{aligned} A(t)u(t) &= A(t) \int_0^t e^{-(t-s)A(t)} (\mathcal{A}(t) - \mathcal{A}(s))u(s) ds \\ &\quad + A(t) \int_0^t e^{-(t-s)A(t)} f(s) ds \\ &:= (Su)(t) + (Lf)(t). \end{aligned} \tag{5.1}$$

Due to Lemma 3.9 we obtain that $L \in \mathcal{L}(L^p(0, \tau; \mathcal{H}))$.

Next we prove that $S \in \mathcal{L}(L^{\frac{2p}{2-p}}(0, \tau; \mathcal{V}), L^p(0, \tau; \mathcal{H}))$. Indeed, take $g \in L^{p'}(0, \tau; \mathcal{H})$ where $p' = \frac{p}{p-1}$ is the conjugate of p . Then

$$\begin{aligned} &\sup_{\|g\|_{L^{p'}(0, \tau; \mathcal{H})} = 1} |\langle Su, g \rangle_{L^p(0, \tau; \mathcal{H}) \times L^{p'}(0, \tau; \mathcal{H})}| \\ &= \sup_{\|g\|_{L^{p'}(0, \tau; \mathcal{H})} = 1} \left| \int_0^\tau \int_0^t ((A(t))^{\frac{1}{p}} e^{-\frac{(t-s)}{2}A(t)} \mathcal{A}(t) - \mathcal{A}(s))u(s), A(t)^{\frac{1}{p'}} e^{-\frac{(t-s)}{2}A(t)} g(t) ds dt \right| \\ &\leq \sup_{\|g\|_{L^{p'}(0, \tau; \mathcal{H})} = 1} \int_0^\tau \int_0^t \|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}'_\gamma)} \|u(s)\|_{\mathcal{V}} \|A(t)^{\frac{1}{p}} e^{-\frac{(t-s)}{2}A(t)}\|_{\mathcal{L}(\mathcal{V}'_\gamma, \mathcal{H})} \\ &\quad \times \|A(t)^{\frac{1}{p'}} e^{-\frac{(t-s)}{2}A(t)} g(t)\| ds dt \\ &\leq C_1 \sup_{\|g\|_{L^{p'}(0, \tau; \mathcal{H})} = 1} \int_0^\tau \int_0^t \frac{\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}'_\gamma)}}{|t-s|^{\frac{\gamma}{2} + \frac{1}{p}}} \|u(s)\|_{\mathcal{V}} \\ &\quad \times \|A(t)^{\frac{1}{p'}} e^{-\frac{(t-s)}{2}A(t)} g(t)\| ds dt \\ &\leq C_1 \sup_{\|g\|_{L^{p'}(0, \tau; \mathcal{H})} = 1} \left(\int_0^\tau \int_0^t \frac{\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}'_\gamma)}^p}{|t-s|^{\frac{p\gamma}{2} + 1}} \|u(s)\|_{\mathcal{V}}^p ds dt \right)^{\frac{1}{p}} \\ &\quad \times \left(\int_0^\tau \int_0^t \|A(t)^{\frac{1}{p'}} e^{-(t-s)A(t)} g(t)\|^{p'} ds dt \right)^{\frac{1}{p'}} \\ &\leq C_2 \left(\int_0^\tau \frac{1}{l^{\frac{p\gamma}{2} + 1}} \int_l^\tau \|\mathcal{A}(t) - \mathcal{A}(t-l)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}'_\gamma)}^p \|u(t-l)\|_{\mathcal{V}}^p dt dl \right)^{\frac{1}{p}} \\ &\quad \times \sup_{\|g\|_{L^{p'}(0, \tau; \mathcal{H})} = 1} \|g\|_{L^{p'}(0, \tau; \mathcal{H})} \\ &\leq C \left(\int_0^\tau \frac{1}{l^{\frac{p\gamma}{2} + 1}} \left(\int_l^\tau \|\mathcal{A}(t) - \mathcal{A}(t-l)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}'_\gamma)}^2 dt \right)^{\frac{p}{2}} \frac{dl}{l} \right)^{\frac{1}{p}} \|u\|_{L^{\frac{2p}{2-p}}(0, \tau; \mathcal{V})} \\ &= C \|\mathcal{A}(\cdot)\|_{\dot{B}^{\frac{\gamma}{2}, 2}(0, \tau; \mathcal{L}(\mathcal{V}, \mathcal{V}'_\gamma))} \|u\|_{L^{\frac{2p}{2-p}}(0, \tau; \mathcal{V})}. \end{aligned}$$

This gives that $S \in \mathcal{L}(L^{\frac{2p}{2-p}}(0, \tau; \mathcal{V}), L^p(0, \tau; \mathcal{H}))$. Then from (5.1) we have

$$\begin{aligned} \|A(\cdot)u\|_{L^p(0, \tau; \mathcal{H})} &\leq \|Su\|_{L^p(0, \tau; \mathcal{H})} + \|Lf\|_{L^p(0, \tau; \mathcal{H})} \\ &\leq C \left[\|u\|_{L^{\frac{2p}{2-p}}(0, \tau; \mathcal{V})} + \|f\|_{L^p(0, \tau; \mathcal{H})} \right] \\ &\leq C' \|f\|_{L^p(0, \tau; \mathcal{H})}. \end{aligned}$$

Since $u' = f - A(\cdot)u$ we get that $u \in W^{1,p}(0, \tau; \mathcal{H})$. \square

Proof of Theorem 2.4. Let $u \in MR(p, \mathcal{H})$ and set $f = u' + A(\cdot)u$. One has $f \in L^p(0, \tau; \mathcal{H})$ and by Proposition 4.6 we get that $u \in L^{\frac{2p}{2-p}}(0, \tau; \mathcal{V})$, $u(t) \in (\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p}$ for all $t \in [0, \tau]$. For $0 \leq s \leq l \leq t \leq \tau$ we set

$$v(l) = e^{-(t-l)A(t)}u(l).$$

We remark that $v(t) = u(t)$, $v(s) = e^{-(t-s)A(t)}u(s)$, $v'(l) = e^{-(t-l)A(t)}(\mathcal{A}(t) - \mathcal{A}(l))u(l) + e^{-(t-l)A(t)}f(l)$. Since $v(t) = v(s) + \int_s^t v'(l) dl$ we get

$$\begin{aligned} u(t) - u(s) &= \left[e^{-(t-s)A(t)}u(s) - u(s) \right] + \int_s^t e^{-(t-l)A(t)}[\mathcal{A}(t) - \mathcal{A}(l)]u(l) dl \\ &\quad + \int_s^t e^{-(t-l)A(t)}f(l) dl \\ &:= W(t, s) + (Q_{(t,s)}u)(t) + (L_1(1_{(s,t)}f))(t). \end{aligned}$$

We see easily that

$$\begin{aligned} W(t, s) &= \left[e^{-(t-s)A(t)}u(s) - e^{-(t-s)A(s)}u(s) \right] \\ &\quad + \left[e^{-(t-s)A(s)}u(s) - u(s) \right]. \end{aligned}$$

Choose a contour Γ in the positive half-plane and write by the holomorphic functional calculus for the sectorial operators $A(t), A(s)$

$$e^{-(t-s)A(t)} - e^{-(t-s)A(s)} = \frac{1}{2\pi i} \int_{\Gamma} e^{-(t-s)\lambda} (\lambda I - \mathcal{A}(t))^{-1} (\mathcal{A}(t) - \mathcal{A}(s)) (\lambda I - \mathcal{A}(s))^{-1} d\lambda.$$

By taking the norm in $(\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p}$ we have

$$\begin{aligned} &\| [e^{-(t-s)A(t)} - e^{-(t-s)A(s)}] u(s) \|_{(\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p}} \\ &\leq C \int_0^\infty e^{-(t-s)|\cos \gamma||\lambda|} \| (\lambda I - \mathcal{A}(t))^{-1} \|_{\mathcal{L}(\mathcal{V}'_\beta, (\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p})} \\ &\quad \times \| (\lambda I - \mathcal{A}(s))^{-1} \|_{\mathcal{L}((\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p}; \mathcal{V})} d|\lambda| \| \mathcal{A}(t) - \mathcal{A}(s) \|_{\mathcal{L}(\mathcal{V}, \mathcal{V}'_\beta)} \| u(s) \|_{(\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p}} \\ &\leq_{(i)} C_1 \int_0^\infty (1 + |\lambda|)^{-\frac{3}{2} + \frac{\beta}{2}} e^{-(t-s)|\cos \gamma||\lambda|} d|\lambda| \| \mathcal{A}(t) - \mathcal{A}(s) \|_{\mathcal{L}(\mathcal{V}, \mathcal{V}'_\beta)} \| u(s) \|_{(\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p}}. \end{aligned}$$

Where in (i) we used

$$\begin{aligned} \|(\lambda I - \mathcal{A}(s))^{-1}\|_{\mathcal{L}((\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p}; \mathcal{V})} &\leq \frac{C}{(1 + |\lambda|)^{\frac{3}{2} - \frac{1}{p}}}, \\ \|(\lambda I - \mathcal{A}(t))^{-1}\|_{\mathcal{L}(\mathcal{V}'_{\beta}(\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p})} &\leq \frac{C}{(1 + |\lambda|)^{\frac{1}{p} - \frac{\beta}{2}}}. \end{aligned}$$

Therefore for $\beta < 1$ we have

$$\| [e^{-(t-s)A(t)} - e^{-(t-s)A(s)}] u(s) \|_{(\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p}} \leq C \| \mathcal{A}(t) - \mathcal{A}(s) \|_{\mathcal{L}(\mathcal{V}, \mathcal{V}'_{\beta})} \| u(s) \|_{(\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p}}. \quad (5.2)$$

For $\beta = 1$ we obtain

$$\begin{aligned} &\| [e^{-(t-s)A(t)} - e^{-(t-s)A(s)}] u(s) \|_{(\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p}} \\ &\leq C_{\varepsilon} \frac{\| \mathcal{A}(t) - \mathcal{A}(s) \|_{\mathcal{L}(\mathcal{V}, \mathcal{V}')}}{(t-s)^{\varepsilon}} \| u(s) \|_{(\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p}}, \varepsilon > 0. \end{aligned} \quad (5.3)$$

By (5.2) we obtain

$$\| (Q_{(t,s)} u)(t) \|_{(\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p}} \leq C \left[\int_s^t \frac{\| \mathcal{A}(t) - \mathcal{A}(s) \|_{\mathcal{L}(\mathcal{V}, \mathcal{V}'_{\beta})}^2}{|t-s|^{\beta}} ds \right]^{\frac{1}{2}} \| u \|_{L^{\frac{2p}{2-p}}(s,t; \mathcal{V})}. \quad (5.4)$$

In light of Lemma 3.8 we have

$$\| (L_1(1_{(s,t)} f))(t) \|_{(\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p}} \leq C \left[\left(\int_s^t \frac{\| \mathcal{A}(t) - \mathcal{A}(s) \|_{\mathcal{L}(\mathcal{V}, \mathcal{V}'_{\beta})}^2}{|t-s|^{\beta}} ds \right)^{\frac{1}{2}} + 1 \right] \| f \|_{L^p(s,t; \mathcal{H})}. \quad (5.5)$$

Combining (5.2), (5.3) and (5.4), (5.5) we have

$$\begin{aligned} \| u(t) - u(s) \|_{(\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p}} &\leq C \left[\| \mathcal{A}(\cdot) \|_{C^{\alpha_{\beta}}([s,t]; \mathcal{L}(\mathcal{V}, \mathcal{V}'_{\beta}))} + 1 \right] \| f \|_{L^p(s,t; \mathcal{H})} \\ &\quad + \| \mathcal{A}(\cdot) \|_{C^{\alpha_{\beta}}([s,t]; \mathcal{L}(\mathcal{V}, \mathcal{V}'_{\beta}))} \left[\| u(s) \|_{(\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p}} + \| u \|_{L^{\frac{2p}{2-p}}(s,t; \mathcal{V})} \right] \\ &\quad + \| e^{-(t-s)A(s)} u(s) - u(s) \|_{(\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p}}. \end{aligned}$$

Here, $\alpha_{\beta} = 0$ for $\beta < 1$ and $\alpha_1 = \varepsilon, \varepsilon > 0$.

Observe that $e^{-(t-s)A(s)}$ is strongly continuous on $(\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p}$. Therefore $\| u(t) - u(s) \|_{(\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p}} \rightarrow 0$ as $t \rightarrow s$ (u is right continuous in $(\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p}$).

Now for $0 \leq t \leq l \leq s \leq \tau$, we set $v(l) = e^{-(l-t)A(t)} u(l)$. We remark that $v(t) = u(t), v(s) = e^{-(s-t)A(t)} u(s)$ and

$$v'(l) = -e^{-(l-t)A(t)} [\mathcal{A}(t) - \mathcal{A}(l)] u(l) + e^{-(l-t)A(t)} [f(l) - 2\mathcal{A}(l)u(l)].$$

Hence,

$$\begin{aligned}
u(t) - u(s) &= [e^{-(s-t)A(t)}u(s) - e^{-(s-t)A(s)}u(s)] \\
&\quad + [e^{-(s-t)A(s)}u(s) - u(s)] \\
&\quad + \int_t^s e^{-(l-t)A(t)}[\mathcal{A}(t) - \mathcal{A}(l)]u(l) dl \\
&\quad - \int_t^s e^{-(l-t)A(t)}[f(l) - 2A(l)u(l)] dl. \tag{5.6}
\end{aligned}$$

We use the same argument and we proceed similarly as the proof of the right continuous to get $\|u(t) - u(s)\|_{(\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p}} \rightarrow 0$ as $s \rightarrow t$ (u is left continuous in $(\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p}$). \square

The following proposition gives a characterization of the trace space $Tr(p, \mathcal{H})$.

Proposition 5.1. *Assume the same hypothesis of Theorem 2.4 we get*

$$Tr(p, \mathcal{H}) = (\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p}, \text{ with equivalent norms.}$$

Proof. First we prove the injection $(\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p} \hookrightarrow Tr(p, \mathcal{H})$. Indeed, let $u_0 \in (\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p}$ then by Theorem 2.4 there exists $u \in MR(p, \mathcal{H})$ be the unique solution to $u' + A(\cdot)u = 0$ such that $u(0) = u_0$. Hence, $u_0 \in Tr(p, \mathcal{H})$ and by (2.5) we get

$$\|u_0\|_{Tr(p, \mathcal{H})} \leq \|u\|_{MR(p, \mathcal{H})} \leq C\|u_0\|_{(\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p}}$$

and so the first injection holds.

For the second injection " \leftarrow " let us take $u_0 \in Tr(p, \mathcal{H})$. Then there exists $u \in MR(p, \mathcal{H})$ such that $u(0) = u_0$. Since by Theorem 2.4

$$u \in C([0, \tau]; (\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p}),$$

we have $u(0) = u_0 \in (\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p}$. One has

$$\begin{aligned}
\|u_0\|_{(\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p}} &\leq \inf\{\|u\|_{C([0, \tau]; (\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p})} : u \in MR(p, \mathcal{H}), u(0) = u_0\} \\
&\leq C \inf\{\|u\|_{MR(p, \mathcal{H})} : u \in MR(p, \mathcal{H}), u(0) = u_0\} \\
&= \|u_0\|_{Tr(p, \mathcal{H})}.
\end{aligned}$$

Therefore $Tr(p, \mathcal{H}) \hookrightarrow (\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}), p}$. \square

6 Applications

This section is devoted to some applications of the results given in the previous sections. We give examples illustrating the theory without seeking for generality. Here we study the maximal L^p -regularity for $p \leq 2$.

– ***Elliptic operators on \mathbb{R}^n .***

Let $\mathcal{H} = L^2(\mathbb{R}^n)$ and $\mathcal{V} = H^1(\mathbb{R}^n)$ and define the sesquilinear forms

$$\mathfrak{a}(t, u, v) = \sum_{k,l=1}^n \int_{\mathbb{R}^n} c_{kl}(t, x) \partial_k u \overline{\partial_l v} dx, \quad u, v \in \mathcal{V}.$$

We define the operator $P(t) \in \mathcal{L}(\mathcal{V}, \mathcal{H})$ by $P(t)u = \sum_{j=1}^n b_j(t) \partial_j u$, where $u \in \mathcal{V}$ and $t \in [0, \tau]$.

We assume that the matrix $C(t, x) = (c_{kl}(t, x))_{1 \leq k, l \leq n}$ satisfies the usual ellipticity condition. Next we assume that $C \in B_p^{\frac{1}{2}, 2}(0, \tau; L^\infty(\mathbb{C}^{n^2}))$ and $b_j \in L^p(0, \tau; L^\infty(\mathbb{R}^n))$ where $j \in \{1, \dots, n\}$. We note that

$$\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}')} \leq M' \|C(t, \cdot) - C(s, \cdot)\|_{L^\infty(\mathbb{C}^{n^2})}$$

for some constant M' . This implies that $\mathcal{A} \in B_p^{\frac{1}{2}, 2}(0, \tau; \mathcal{L}(\mathcal{V}, \mathcal{V}'))$.

We are now allowed to apply Theorem 2.2. We obtain maximal L^p -regularity and apriori estimate for the parabolic problem

$$\begin{cases} u'(t) + A(t)u(t) + P(t)u(t) = f(t) \\ u(0) = 0. \end{cases}$$

That is, for every $f \in L^p(0, \tau; L^2(\mathbb{R}^n))$ there is a unique solution

$$u \in W^{1,p}(0, \tau; L^2(\mathbb{R}^n)) \cap L^p(0, \tau; H^1(\mathbb{R}^n)).$$

– ***Schrödinger operators with time-dependent potentials.***

Let $0 \leq m_0 \in L_{\text{loc}}^1(\mathbb{R}^n)$ and $m : [0, \tau] \times \mathbb{R}^n \rightarrow \mathbb{R}$ be a measurable function for which there exist positive constants α_1, α_2 and M such that for a.e. x and all $t \in [0, \tau]$

$$\alpha_1 m_0(x) \leq m(t, x) \leq \alpha_2 m_0(x).$$

We define the form

$$\mathfrak{a}(t, u, v) = \int_{\mathbb{R}^n} \nabla u \nabla \overline{v} dx + \int_{\mathbb{R}^n} m(t, x) u \overline{v} dx,$$

with domain

$$\mathcal{V} = \{u \in H^1(\mathbb{R}^n) : \int_{\mathbb{R}^n} m_0(x) |u|^2 dx < \infty\}.$$

It is clear that \mathcal{V} is a Hilbert space for the norm $\|u\|_{\mathcal{V}}$ given by

$$\|u\|_{\mathcal{V}}^2 = \int_{\mathbb{R}^n} |\nabla u|^2 dx + \int_{\mathbb{R}^n} m_0(x)|u|^2 dx.$$

In addition, \mathfrak{a} is \mathcal{V} -bounded and coercive. Its associated operator on $L^2(\mathbb{R}^n)$ is formally given by

$$A(t) = -\Delta + m(t, \cdot)$$

with domain

$$D(A(t)) = \{u \in \mathcal{V} \text{ s.t. } -\Delta u + m(t, \cdot)u \in L^2(\mathbb{R}^n)\}.$$

Next we assume that $t \rightarrow m(t, \cdot)m_0(\cdot)^{-1} \in B_p^{\frac{1}{2}, 2}(0, \tau; L^\infty(\mathbb{R}^n))$, with $p \leq 2$. We have

$$\begin{aligned} & \|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}')} \\ &= \sup_{\|u\|_{\mathcal{V}}=1, \|v\|_{\mathcal{V}}=1} |\mathfrak{a}(t, u, v) - \mathfrak{a}(s, u, v)| \\ &\leq \sup_{\|u\|_{\mathcal{V}}=1, \|v\|_{\mathcal{V}}=1} \int_{\mathbb{R}^n} |m(t, x) - m(s, x)| |u| |v| dx \\ &\leq \|(m(t, \cdot) - m(s, \cdot))m_0^{-1}(\cdot)\|_{L^\infty(\mathbb{R}^n)} \sup_{\|u\|_{\mathcal{V}}=1, \|v\|_{\mathcal{V}}=1} \int_{\mathbb{R}^n} m_0(x)|u| |v| dx \\ &\leq \|(m(t, \cdot) - m(s, \cdot))m_0^{-1}(\cdot)\|_{L^\infty(\mathbb{R}^n)}. \end{aligned}$$

Then we get $\mathcal{A} \in B_p^{\frac{1}{2}, 2}(0, \tau; \mathcal{L}(\mathcal{V}, \mathcal{V}'))$.

Given $f \in L^p(0, \tau; L^2(\mathbb{R}^n))$, we apply Theorem 2.2 and obtain a unique solution $u \in W^{1,p}(0, \tau; L^2(\mathbb{R}^n)) \cap L^p(0, \tau; \mathcal{V})$ of the evolution equation

$$\begin{cases} u'(t) - \Delta u(t) + m(t, \cdot)u(t) = f(t) \text{ a.e.} \\ u(0) = 0. \end{cases}$$

– **Robin boundary conditions.**

Let Ω be a bounded domain of \mathbb{R}^d with Lipschitz boundary $\partial\Omega$. We denote by Tr the classical trace operator. Let $\beta : [0, \tau] \times \partial\Omega \rightarrow [0, \infty)$ be bounded and such that

$$\int_0^\tau \frac{1}{l^{\alpha p}} \left(\int_l^\tau \|\beta(t, \cdot) - \beta(t-l, \cdot)\|_{L^\infty(\partial\Omega)}^2 dt \right)^{\frac{p}{2}} \frac{dl}{l} < \infty \quad (6.1)$$

for some $\alpha > \frac{1}{4}$. In particular, $\beta(\cdot, x) \in \dot{B}_p^{\alpha, 2}$.

We define the forms

$$\mathfrak{a}(t, u, v) := \int_{\Omega} \nabla u \cdot \nabla v dx + \int_{\partial\Omega} \beta(t, \cdot) \text{Tr}(u) \text{Tr}(v) d\sigma,$$

for all $u, v \in \mathcal{V} := H^1(\Omega)$.

Formally, the associated operator $A(t)$ is (minus) the Laplacian with the time dependent Robin boundary condition given by

$$\frac{\partial u}{\partial n} + \beta(t, \cdot)u = 0 \text{ on } \partial\Omega.$$

Here $\frac{\partial u}{\partial n}$ denotes the normal derivative.

Note that for any $\varepsilon > 0$

$$\begin{aligned} & |\mathfrak{a}(t; u, v) - \mathfrak{a}(s; u, v)| \\ &= \left| \int_{\partial\Omega} [\beta(t, \cdot) - \beta(s, \cdot)] \text{Tr}(u) \text{Tr}(v) d\sigma \right| \\ &\leq \|\beta(t, \cdot) - \beta(s, \cdot)\|_{L^\infty(\partial\Omega)} \|u\|_{H^{\frac{1}{2}+\varepsilon}(\Omega)} \|v\|_{H^{\frac{1}{2}+\varepsilon}(\Omega)}, \end{aligned}$$

where we used the fact that the trace operator is bounded from $H^{\frac{1}{2}+\varepsilon}(\Omega)$ into $L_2(\partial\Omega)$. The forms considered here are symmetric and therefore the uniform Kato square root property can be checked easily. Now assumption (6.1) allows us to apply Theorem 2.2 with $\gamma = 2\alpha = \frac{1}{2} + \varepsilon$ and obtain maximal L^p -regularity for the corresponding evolution equation with initial data $u_0 \in (L^2(\Omega), H^1(\Omega))_{2(1-\frac{1}{p}), p} = B_p^{2(1-\frac{1}{p}), 2}(\Omega)$. The forms considered here are symmetric and therefore the uniform Kato square root property can be checked easily.

Maximal L^p -regularity for time dependent Robin boundary condition with $\beta(\cdot, x) \in C^\alpha$ for some $\alpha > \frac{1}{4}$ was previously proved in [6] and [24]. In the latter reference maximal L^p -regularity for all $p \in (1, \infty)$ is proved.

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